Bayesian Probability Estimation: The Case of Lifetime Batting Averages in Baseball

Adam Kapelner

Department of Mathematics, Queens College, CUNY

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Macaulay Honors College at CUNY
Data Science Curriculum Module

Learning Objectives

- Understand Frequentist probability estimation for the probability parameter in the binomial model
- Understand Bayesian probability estimation for the probability parameter in the binomial model
- Build Bayesian credible intervals for the probability parameter in the binomial model
- Use a Bayesian model to forecast future data
- Use R to create empirical Bayes estimate of the binomial parameters

Materials Required

- Access to the Internet where readings will be linked
- The R statistical software which can be downloaded here

Readings

- Each of the concepts can be wikipediad or googled. These are the best readings...

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Note: detailed knowledge of baseball is not a prereq for this module, but many of the problems are motivated by baseball.

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3. Use historical data to improve the estimates (empirical Bayes).

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- Empirical Bayes
- Estimating Batting Averages with R

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Technicalities

- The domain is only the powerset of the universe on discrete support random variables.
- Probability's mathematical definition is undisputed but its real-world definition is disputed by philosophers. This is beyond the scope of this module.
- We ignore outcomes, events and the universe Ω going forward...

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$$BA := \frac{1}{L} \sum_{i=1}^{L} \mathbb{1}_{\text{Hit } i\text{th at bat}}$$

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What is X exactly? It's a "probability model" for a future outcome, x. Note the capital letter and lowercase letter. The future outcome is also called a "realization" because the model gets "realized" (i.e. made to be real) when the random trial is completed.

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What is $\mathbb{S}\text{upp}\,[X]$? Since the only two legal things that could happen are the hit (1) and not a hit (0), $\mathbb{S}\text{upp}\,[X] = \{0, 1\}$.

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- The domain of the probability function is a subset of Ω, so technically the first query $\mathbb{P}(X=1)$ is an "abuse of notation" which is "shorthand" for $\mathbb{P}(\{\omega \in \Omega : X(\omega) = 1\}) = \mathbb{P}(\{\omega = \text{Hit}\}) = \theta$.
- x can take on only those values $\in \mathbb{S}\text{upp}[X]$. In the above, any value $\notin \mathbb{S}\text{upp}[X]$ gets probability 0. So e.g. $\mathbb{P}(X=17) = 0$ and $\mathbb{P}\left(X = \frac{2}{3}\right) = 0$, etc.
- Densities are only available for continuous r.v.'s (which we will see later in this module)

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Usually, the PMF and PDF are given different notation $p(x)$ and $f(x)$. We are calling them both $\mathbb{P}(X)$ here for simplicity as they will be both used within the Bayesian paradigm interchangably — this is a slight abuse of notation as sometimes you will not know if the r.v. is discrete or continuous from its notation.

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Back to our regularly scheduled program...

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4. use historical data to improve the estimation

We will be covering 1, 3 and 4.

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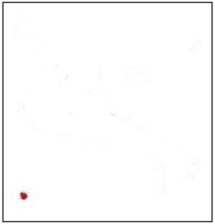

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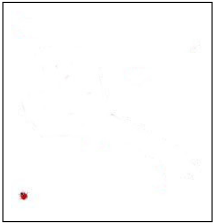

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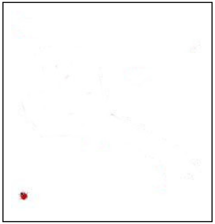

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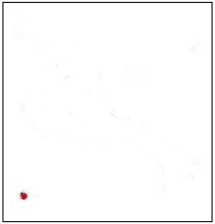

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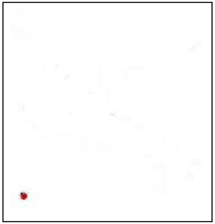

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This example is a bit confusing. Here, $AB = A$ since A is a proper subset of B. The original bug size is A which is also AB. Thus, $\mathbb{P}(A \mid B) = \mathbb{P}(A) / \mathbb{P}(B)$ only in this case though.

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Denote hit as H and not hit as O. There are 20 ways to make 3 hits occur out of 6:

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Where "`dbinom(3, 6, 0.25)`" is the `R` code to compute the answer.

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So how do we use data? We first begin with the popular tools of frequentist estimation and then move to the Bayesian tools which is the focus of this module.

Homework Problems

1. Draw the PMF for $X \sim \text{Bernoulli}(\theta)$.
2. Imagine two Bernoulli r.v.'s X_1 and X_2 which model two fair coin flips where Heads is mapped to 1 and tails is mapped to 0. The probability of heads is $1/2$. Explain using the definition of r.v. independence why these two r.v.'s are *dependent*.
3. Using the same two sorcery-controlled coins, explain using the definition of equality in distribution why or why not X_1 and X_2 are equally distributed.
4. Are $X_1, X_2 \overset{iid}{\sim} \text{Bernoulli}(\theta)$ if they are modeled by these two sorcery-controlled coins?
5. A new situation: the probability of a bundle of coins (scotch-taped together) landing on its side is $\mathbb{P}(S) = 1/11$. Heads and tail probability are $5/11$. Let's call landing on its side the event we seek, create a Bernoulli r.v. for this event indicate its support, its parameter and the parameter space.

Homework Problems

6 I flip the coin bundle once. What is the probability of the trial of interest being found? Write a probability statement.

7 Let's say we flip 10 times. What is the probability that we get one (and only one) success? I want to see a probability model. Write "$X \sim$" something below. Then I want to see a probability statement. Then I want to see a computation. Answer then in decimal rounded to two digits.

8 Let's say we flip 10 times. What is the probability that we get 5 (and only 5) successes?

9 Let's say we flip 10 times. What is the probability that we get 8 (and only 8) successes?

10 Let's say we flip 10 times. What is the probability we get one or two successes?

Homework Problems

11 Consider three NYC buildings: One World Trade Center (104 floors), the Empire State Building (86 occupied floors) and the Bank of America Tower (55 floors). Consider walking into one of them at random and picking a random floor. What is the probability of choosing the Empire State Building's 40th floor?

12 What is the probability he on the 30the floor?

13 If you know he's on the 40th floor, what is the probability he is in the empire state building?

14 Look up the Monte Hall game. Explain why switching is a good strategy based on conditional probability.

15 Is the $\overset{iid}{\sim}$ bernoulli model for at bats a good model in baseball? This is complicated.

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The L stands now for likelihood and the θ is now the free variable and the data X is the parameter. (but otherwise it is the same as the PMF!)

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This comes down to taking the derivative, setting it equal to zero and solving for θ just like in Calculus 101. Note that the maximum of the function is immune to monotonic transformations of the function. One such convenient monotonic transformation is the natural log, $\ell(\theta; X) := \ln(L(\theta; X))$, so we can now calculate the maximum likelihood estimator via

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Homework Problems

1. Rederive the MLE (estimator and estimate) for the Binomial likelihood model.
2. Try to derive it without using the monotonic transformation of the log of the likelihood.
3. Instead of the maximum likelihood, write an expression for the 90%ile of the likelihood.
4. Given 345 at bats and 132 hits, what is the maximum likelihood of the hit probability?
5. Is θ the same as lifetime batting average? Discuss.
6. Why is an MLE of 0.000 or 1.000 a bad thing? Discuss.
7. Devise a means to fix these two pathological situations without looking ahead in the module.

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The "proportionality" sign indicates it differs from the true PMF by only a constant which can be recovered by doing the sum.

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(since $1/\mathbb{P}(X)$ is clearly a constant and not a function of the free variable θ).

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This means no matter what the value of θ, we don't think any of them are more likely than any other!

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This is a very famous integral! It is known as the beta integral and its solution is the non-computable (but approximable) beta function.

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The Beta Distribution

Just like the beta function, this is the density of a famous continuous r.v. called the "beta",

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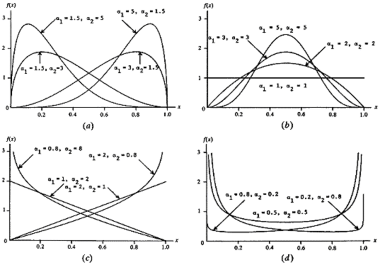

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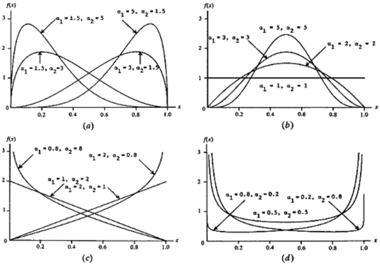

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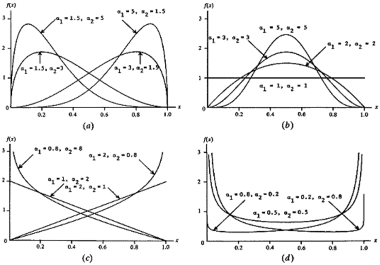

So, now we have moved from $\theta \sim U(0, 1) \to \theta \mid X \sim \mathrm{Beta}\left(x+1,\, n-x+1\right)$ using Bayesian conditionalism.

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where the integral can be figured out by numerical integration using R via the command `pbeta(0.2, x+1, n-x+1)`.

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We use the posterior expectation because it minimizes the squared error loss from the truth θ and this is usually an appropriate loss function unless there are other considerations.

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```
qbeta(0.025, x + 1, n - x + 1)
qbeta(0.975, x + 1, n - x + 1)
```

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If we have the likelihood be a binomial and the prior, a beta, then the posterior is also a beta! This is known as "conjugacy" and we say "the beta is the conjugate prior for the binomial model". Although priors are still arbitrary, this gives us a feeling of "naturalness" about having the beta prior.

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Homework Problems

1. What is the Bayesian paradigm shift? Compare and contrast the frequentist view to the Bayesian view.
2. Under Bayesian Conditionalism, what changes from before to after?
3. What is the update factor?
4. Derive the kernel of the Binomial r.v.
5. Derive the kernel of the Gaussian r.v.
6. Derive the kernel of the Beta r.v.
7. What does it mean that the kernel is "proportional" to the density?

Homework Problems

8. Prove that the posterior is proportional to the likelihood times the prior.
9. Why is the support of the prior the same as the parameter space of the likelihood?
10. What is an objective prior? Why does it make sense?
11. What is the objective prior for θ in the binomial model?
12. What is the kernel of the posterior of the binomial-uniform model?
13. Derive the posterior and show it is a beta.
14. Create an integral that will compute the probability theta is greater than 0.4.

Homework Problems

15 If $x = 4$ and $n = 10$, compute the probability from (14) in `R`.

16 Compute a credible Interval / Region for the data in (15) using `R`.

17 Show that a standard uniform is a Beta(1,1).

18 Show that under binomial likelihood and a general beta prior, the posterior is beta and find its posterior parameters.

19 Derive the posterior expectation under (18).

20 Explain how α and β can be interpreted as "pseudocounts".

21 Explain how the posterior expectation limits to the MLE.

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Empirical Bayes

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We now develop this idea using **this** blog post by David Robinson on 9/30/15. We first install necessary packages in R,

```
options(repos = structure(c(CRAN =
  "http://cran.revolutionanalytics.com/")))
tryCatch(library(dplyr),
  error = function(e){install.packages("dplyr")},
  finally = library(dplyr))
tryCatch(library(tidyr),
  error = function(e){install.packages("tidyr")},
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tryCatch(library(Lahman),
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?Batting
```

We see this has Lahman's Baseball Database, 1871-2014, 99,846 player stints measuring 22 baseball statistics (we will only make use of "hits" and "at bats").

Load Data

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career <- Batting %>%
  filter(AB > 0) %>%
  anti_join(Pitching, by = "playerID") %>%
  group_by(playerID) %>%
  summarize(H = sum(H), AB = sum(AB)) %>%
  mutate(average = H / AB)
career <- Master %>%
  tbl_df() %>%
  select(playerID, nameFirst, nameLast) %>%
  unite(name, nameFirst, nameLast, sep = " ") %>%
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head(career)
```

The first colum is the baseball player's full name, followed by hits, at bats and the MLE.

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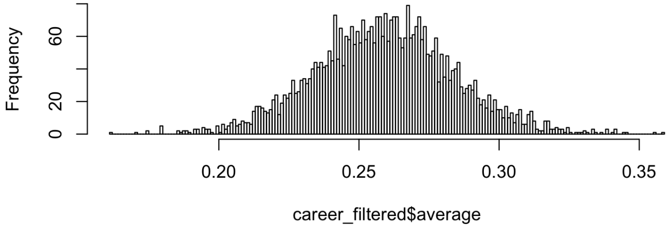
```
career_filtered <- career %>% filter(AB >= 500)
hist(career_filtered$average, br = 200)
```



Maximum Likelihood Estimates of α and β

We now fit a beta to the historical θ estimates, plot atop and obtain the MLE's of α and β:

```
m <- MASS::fitdistr(career_filtered$average, dbeta,
                    start = list(shape1 = 1, shape2 = 10))
alpha0 <- m$estimate[1]
beta0 <- m$estimate[2]
round(alpha0, 1) # 79.0
round(beta0, 1) #225.9
hist(rbeta(10000, alpha0, beta0), br = 200, col = "red", prob = TRUE)
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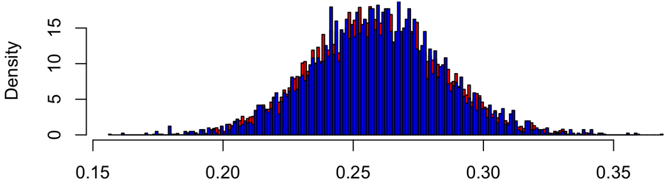

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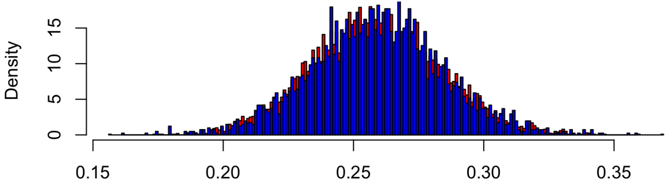

As we can see from the agreement of the red (the fit beta) and the blue (the true data), the $\theta \sim \mathrm{Beta}(79.0, 225.9)$ is a very nice fit for the data!

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Homework Problems

1. What is empirical Bayes estimation?
2. Repeat the exercise of building an empirical Bayes estimator with all players who had more than 600 at bats. What are the α and β MLE estimates?
3. Verify the beta density with the estimates from (2) is a good fit for the historical data.
4. What is the posterior mean?
5. Use your estimates from (2) to estimate the batting average of a hitter with 5 at bats and 3 hits.
6. What is the probability this hitter from (5) has a BA greater than 300?
7. Provide a credible region for the BA from the hitter of (5).
8. What kind of evidence would be required under (2) for a batter to have a BA estimate of 500?

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- Bayesian Machinery for the binomial likelihood model
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