

# ACS234

# Maths and Data Modelling

**Tutorial 5**  
**Wednesday 1pm online**

**<https://github.com/ineskris/ACS234/tree/master/Tutorial5>**

## Done in Lecture (week 5/6)

- Polynomial Regression
- General Linear Models

# Polynomial Regression

Simple Polynomial Model

$$y = a_0 + a_1x + a_2x^2 + \dots + a_mx^m + e$$

Estimation (least squares method)

$$Y = X\hat{a} + e$$

$$\hat{a} = (X'X)^{-1}X'Y$$

The techniques of fitting of the polynomial model in one variable can be extended to the fitting of polynomial models in two or more variables. A second-order polynomial is more used in practice, and its model is specified by :

$$y = a_0 + a_1X_1 + a_2X_2 + a_3X_1^2 + a_4X_1X_2 + a_5X_2^2 + e$$

## Exercise 1

x	0	1	2	3
f(x)	2	7	14	23

Based on the data above, estimate the parameters  $a_0, a_1, a_2$  of the **polynomial regression model**. Calculate the MSE error.

## Exercise 2

X1	0	1	2	3
X2	12	12.3	12.6	12.9
f(x)	2	-3.3	-3.2	2.3

Based on the data above, estimate the parameters  $a_0, a_1, a_2, a_3, a_4, a_5$  of the **general polynomial regression** model. Calculate the MSE error.

## Exercise 1 - bis

Based on each dataset, right down the correct matrix  $X$  for a polynomial model with the degree associated.

How many points (at least) do we need to find the estimator  $a$  ?

a) Degree 2

x	-3	1	7
f(x)	0	-1	12

c) Degree 4

x	1	7	8
f(x)	8	7	1

b) Degree 3

x	-1	1	7	12
f(x)	0	-1	12	6

d) Degree 2

x	0	0.5	1	5	20
f(x)	13	2	76	0	0

## Exercise 1 - solution

$$y = a_0 + a_1x + a_2x^2$$

$$Y = X\hat{a}$$

$$\hat{a} = (X'X)^{-1}X'Y$$

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \quad X'X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} = A$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} 98 \times 14 - 36 \times 36 & 36 \times 14 - 6 \times 98 & 6 \times 36 - 14 \times 14 \\ 14 \times 36 - 6 \times 98 & 4 \times 98 - 14 \times 14 & 6 \times 14 - 4 \times 36 \\ 6 \times 36 - 14 \times 14 & 14 \times 6 - 4 \times 36 & 4 \times 14 - 6 \times 6 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 76 & -84 & 20 \\ -84 & 196 & -60 \\ 20 & -60 & 20 \end{pmatrix}$$

$$\det(A) = 4 \times 14 \times 98 + 6 \times 36 \times 14 + 14 \times 6 \times 36 - 14 \times 14 \times 14 - 36 \times 36 \times 4 - 98 \times 6 \times 6 = 80$$

$$X'Y = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \\ 14 \\ 23 \end{pmatrix} = \begin{pmatrix} 46 \\ 104 \\ 270 \end{pmatrix} \quad \hat{a} = \begin{pmatrix} 0.95 & -1.05 & 0.25 \\ -1.05 & 2.45 & -0.75 \\ 0.25 & -0.75 & 0.25 \end{pmatrix} \begin{pmatrix} 46 \\ 104 \\ 270 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$\mathbf{MSE} = 0$$

## Exercise 2 - solution

$$y = a_0 + a_1X_1 + a_2x_2 + a_3X_1^2 + a_4X_1X_2 + a_5X_2^2 + e$$

X1	0	1	2	3
X2	12	12.3	12.6	12.9
f(x)	2	-3.3	-3.2	2.3

See Lecture 4.2

$$\begin{pmatrix} 1 & x_{1,1} & x_{2,1} & x_{1,1}^2 & x_{1,1}x_{2,1} & x_{2,1}^2 \\ 1 & x_{1,2} & x_{2,2} & x_{1,2}^2 & x_{1,2}x_{2,2} & x_{2,2}^2 \\ 1 & x_{1,3} & x_{2,3} & x_{1,3}^2 & x_{1,3}x_{2,3} & x_{2,3}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_{1,n} & x_{2,n} & x_{1,n}^2 & x_{1,n}x_{2,n} & x_{2,n}^2 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 0 & 12 & 0 & 0 & 144 \\ 1 & 1 & 12.3 & 1 & 12.3 & 151.29 \\ 1 & 2 & 12.6 & 4 & 25.2 & 158.76 \\ 1 & 3 & 12.9 & 9 & 38.7 & 166.41 \end{pmatrix}$$

Use a calculator or Python / Matlab code

$$X'X = \begin{pmatrix} 4 & 6 & 49.8 & 14 & 76.2 & 620.46 \\ 6 & 14 & 76.2 & 36 & 178.8 & 968.04 \\ 49.8 & 76.2 & 620.46 & 178.8 & 968.04 & 7735.932 \\ 14 & 36 & 178.8 & 98 & 461.4 & 2284.02 \\ 76.2 & 178.8 & 968.04 & 461.4 & 2284.02 & 12301.686 \\ 620.46 & 968.04 & 7735.932 & 2284.02 & 12301.686 & 96521.6898 \end{pmatrix}$$

$$\hat{y} = \begin{pmatrix} 0.71875 \\ -4.00123047 \\ -3.32679687 \\ 2.74205078 \end{pmatrix}$$

**MSE = 0.586**

$$\hat{a} = \begin{pmatrix} 0.4375 \\ -8.125 \\ 0.1875 \\ 2.6796875 \\ 0.0625 \\ -0.01367188 \end{pmatrix}$$

## Exercise 2

$$y = a_0 + a_1X_1 + a_2X_2 + a_3X_2^2 + e$$

X1	57	59	49	62
X2	8	10	6	11
f(x)	64	71	53	67

Based on the data above, estimate the parameters  $a_0, a_1, a_2, a_3$  of the **general polynomial regression** model. Calculate the MSE error.

## Exercise 2 - solution

$$y = a_0 + a_1X_1 + a_2X_2 + a_3X_2^2 + e$$

X1	57	59	49	62
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Based on the data above, estimate the parameters  $a_0, a_1, a_2, a_3$  of the **general polynomial regression** model. Calculate the MSE error.

$$X = \begin{pmatrix} 1 & 57 & 8 & 64 \\ 1 & 59 & 10 & 100 \\ 1 & 49 & 6 & 36 \\ 1 & 62 & 11 & 121 \end{pmatrix} \quad X'X = \begin{pmatrix} 4 & 227 & 35 & 321 \\ 227 & 12975 & 2022 & 18814 \\ 35 & 2022 & 321 & 3059 \\ 321 & 18814 & 3059 & 30033 \end{pmatrix} = A \quad \det(A) = 4623.9$$

$$\hat{a} = (X'X)^{-1}X'Y$$

$$\hat{a} = \begin{pmatrix} -19.53 \\ -1.41 \\ 32.97 \\ -1.56 \end{pmatrix}$$



## Exercise 1 bis - solution

a) Degree 2

$$X = \begin{pmatrix} 1 & -3 & 9 \\ 1 & 1 & 1 \\ 1 & 7 & 49 \end{pmatrix}$$

c) Degree 4

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 7 & 49 & 343 & 2401 \\ 1 & 8 & 64 & 512 & 4096 \end{pmatrix}$$

5 unknown parameters - 3 equations

b) Degree 3

$$X = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 7 & 49 & 343 \\ 1 & 12 & 144 & 1728 \end{pmatrix}$$

d) Degree 2

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \\ 1 & 5 & 25 \\ 1 & 20 & 400 \end{pmatrix}$$

3 unknown parameters - 5 equations