ACS234
Maths and Data Modelling

Tutorial 5
Wednesday 1pm online

https://github.com/ineskris/ACS234/tree/master/Tutorial5

Done in Lecture (week 5/6)

- Polynomial Regression
- General Linear Models

Polynomial Regression

Simple Polynomial Model $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m + e$

Estimation (least squares method) $Y = X\hat{a} + e$ $\boxed{\hat{a} = (X'X)^{-1}X'Y}$

General Polynomial Regression (degree 2) $y = a_0 + a_1 X_1 + a_2 x_2 + a_3 X_1^2 + a_4 X_1 X_2 + a_5 X_2^2 + e$

Exercice 1

x	0	1	2	3
f(x)	2	7	14	23

Based on the data above, estimate the parameters a0,a1,a2 of the **polynomial regression model**. Calculate the MSE error.

Exercice 2

X1	0	1	2	3
X2	12	12.3	12.6	12.9
f(x)	2	-3.3	-3.2	2.3

Based on the data above, estimate the parameters a0,a1,a2, a3, a4, a5 of the **general polynomial regression** model. Calculate the MSE error.

Exercice 1 - bis

Based on each dataset, right down the correct matrix X for a polynomial model with the degree associated.

How many points (at least) do we need to find the estimator a ?

a) Degree 2

x	-3	1	7
f(x)	0	-1	12

b) Degree 3

x	-1	1	7	12
f(x)	0	-1	12	6

c) Degree 4

x	1	7	8
f(x)	8	7	1

d) Degree 2

x	0	0.5	1	5	20
f(x)	13	2	76	0	0

Exercice 1 - solution

$$y = a_0 + a_1 x + a_2 x^2 \qquad Y = X\hat{a} \qquad \hat{a} = (X'X)^{-1}X'Y$$

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad X'X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} = A$$

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} = \frac{1}{det(A)} \begin{pmatrix} 98 \times 14 - 36 \times 36 & 36 \times 14 - 6 \times 98 & 6 \times 36 - 14 \times 14 \\ 14 \times 36 - 6 \times 98 & 4 \times 98 - 14 \times 14 & 6 \times 14 - 4 \times 36 \\ 6 \times 36 - 14 \times 14 & 14 \times 6 - 4 \times 36 & 4 \times 14 - 6 \times 6 \end{pmatrix}$$

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} 76 & -84 & 20 \\ -84 & 196 & -60 \\ 20 & -60 & 20 \end{pmatrix}$$

$$det(A) = 4 \times 14 \times 98 + 6 \times 36 \times 14 + 14 \times 6 \times 36 - 14 \times 14 \times 14 - 36 \times 36 \times 4 - 98 \times 6 \times 6 = 80$$

$$X'Y = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \\ 14 \\ 23 \end{pmatrix} = \begin{pmatrix} 46 \\ 104 \\ 270 \end{pmatrix}$$

$$\boxed{\hat{a} = \begin{pmatrix} 0.95 & -1.05 & 0.25 \\ -1.05 & 2.45 & -0.75 \\ 0.25 & -0.75 & 0.25 \end{pmatrix} \begin{pmatrix} 46 \\ 104 \\ 270 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}}$$

Exercice 2 - solution

$$y = a_0 + a_1 X_1 + a_2 x_2 + a_3 X_1^2 + a_4 X_1 X_2 + a_5 X_2^2 + e$$

X1	0	1	2	3
X2	12	12.3	12.6	12.9
f(x)	2	-3.3	-3.2	2.3

See Lecture 4.2

$$\begin{pmatrix} 1 & x_{1,1} & x_{2,1} & x_{1,1}^2 & x_{1,1}x_{2,1} & x_{2,1}^2 \\ 1 & x_{1,2} & x_{2,2} & x_{1,2}^2 & x_{1,2}x_{2,2} & x_{2,2}^2 \\ 1 & x_{1,3} & x_{2,3} & x_{1,3}^2 & x_{1,3}x_{2,3} & x_{2,3}^2 \\ \dots & \dots & \dots & & & \\ 1 & x_{1,n} & x_{2,n} & x_{1,n}^2 & x_{1,n}x_{2,n} & x_{2,n}^2 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 0 & 12 & 0 & 0 & 144 \\ 1 & 1 & 12.3 & 1 & 12.3 & 151.29 \\ 1 & 2 & 12.6 & 4 & 25.2 & 158.76 \\ 1 & 3 & 12.9 & 9 & 38.7 & 166.41 \end{pmatrix}$$

Use a calculator or Python / Matlab code

$$X'X == \begin{pmatrix} 4 & 6 & 49.8 & 14 & 76.2 & 620.46 \\ 6 & 14 & 76.2 & 36 & 178.8 & 968.04 \\ 49.8 & 76.2 & 620.46 & 178.8 & 968.04 & 7735.932 \\ 14 & 36 & 178.8 & 98 & 461.4 & 2284.02 \\ 76.2 & 178.8 & 968.04 & 461.4 & 2284.02 & 12301.686 \\ 620.46 & 968.04 & 7735.932 & 2284.02 & 12301.686 & 96521.6898 \end{pmatrix}$$

$$\boxed{\hat{a} == \begin{pmatrix} 0.4375 \\ -8.125 \\ 0.1875 \\ 2.6796875 \\ 0.0625 \\ -0.01367188 \end{pmatrix}}$$

Exercice 1 bis - solution

a) Degree 2

$$X = \begin{pmatrix} 1 & -3 & 9 \\ 1 & ? & 1 \\ 1 & 7 & ? \end{pmatrix}$$

c) Degree 4

b) Degree 3

$$X = \begin{pmatrix} 1 & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}$$

d) Degree 2

General Linear Models

There are three components to any GLM:

- *Random Component* : noise model or error model. e
- *Systematic Component* - the linear predictor $\eta = X\beta$
- *Link Function*, *η or $g(\mu)$* - specifies the link between random and systematic components. $E[Y] = g^{-1}(X\beta)$

General linear model (GLM) includes multiple linear regression.

Example - logistic regression

$$X\beta = ln(\frac{\mu}{1-\mu}) \qquad \mu = E[Y]$$