# ACS234
# Maths and Data Modelling

**Tutorial**
**Wednesday 1pm LT04**

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**https://github.com/ineskris/ACS234**

# Done in Lecture (week 1 - week 3)

- Polynomials
- Ill-conditioning
- Lagrange
- Newton

# Polynomial Interpolation

$$f(x) = a_1x^n + a_2x^{n-1} + \ldots + a_{n+1}$$

n points -> n-1 = min order of the polynomial

**Exercice 1** - Can you find the polynomial function that goes though these 3 points ?

a)

| x | 0 | 1 | 2 |
|---|---|---|---|
| f(x) | 0 | 4 | 0 |

b)

| x | 0 | 1 | 2 |
|---|---|---|---|
| f(x) | 2 | 1 | 0 |

**Exercice 1bis** - Polynomial function ?

| x | 0 | 10 | 20 | 30 |
|---|---|---|---|---|
| f(x) | -250 | 0 | 50 | -100 |

**Exercice 1 - Solution**

a) $f(x) = -4x^2 + 8x$

b) $f(x) = -x + 2$

1bis) $f(x) = -x^2 + 35x - 250$

# Lagrange Interpolation

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \qquad L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \qquad L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

**Exercice 2** - Construct $P_2(x)$ from the data points

a) (0, –1) ; (1, –1) ; (2, 7).

**Exercice 2bis** - Use the Lagrange Polynomial to find the unique polynomial of degree 3 that agrees with the following data :

| x | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| f(x) | 3 | -4 | 5 | -6 |

## Exercice 2 solution

$$P_2(x) = (-1)\frac{(x-1)(x-2)}{2} + (-1)\frac{(x)(x-2)}{-1} + 7\frac{x(x-1)}{2} = \frac{-1}{2}(x-1)(x-2) + x(x-2) + \frac{7}{2}x(x-1)$$

## Exercice 2bis guided solution

We must have $p_3(-1) = 3 \quad p_3(0) = -4 \quad p_3(2) = -6$

We construct the Lagrange polynomials $\{\mathscr{L}_{3,j}(x)\}_{j=0}^{3}$ $\qquad \mathscr{L}_{n,j}(x) = \prod_{i=0,i\neq j}^{3} \frac{x - x_i}{x_j - x_i}$

This yields

$$\mathscr{L}_{3,0}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} = \frac{x(x^2-3x+2)}{(-1)(-2)(-3)} = \frac{-1}{6}(x^3-3x^2+2x)$$

$$\mathscr{L}_{3,1}(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)} = \frac{(x^2-1)(x-2)}{(1)(-1)(-2)} = \frac{1}{2}(x^3-2x^2-x+2)$$

$$\mathscr{L}_{3,2}(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)} = \frac{x(x^2-x-2)}{(2)(1)(-1)} = \frac{-1}{2}(x^3-x^2-2x)$$

$$\mathscr{L}_{3,3}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)} = \frac{x(x^2-1)}{(3)(2)(1)} = \frac{1}{6}(x^3-x)$$

By substituting $x_i$ for $x$ in each Lagrange polynomial $\mathscr{L}_{3,j}(x)$ , for $j = 0,1,2,3$ , it can be verified that $\mathscr{L}_{3,j}(x_j) = 0$ for $i \neq j$

It follows that the Lagrange interpolating polynomial $p_3(x)$ is given by :

$$p_3(x) = \sum_{j=0}^{3} y_j \mathscr{L}_{3,j}(x) = y_0(x)\mathscr{L}_{3,0}(x) + y_1(x)\mathscr{L}_{3,1}(x) + y_2(x)\mathscr{L}_{3,2}(x) + y_3(x)\mathscr{L}_{3,3}(x) = -6x^3 + 8x^2 + 7x - 4$$

# Newton Interpolation

**2nd order** $$P_2(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2)$$

$$a_1 = y_1 \qquad a_2 = \frac{y_2 - y_1}{x_2 - x_1} \qquad a_3 = \frac{\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}$$

**Exercice 3**- Construct $P_2(x)$ from the data points

a)

| x | 0 | 1 | 2 |
|---|---|---|---|
| f(x) | 0 | 4 | 0 |

b)

| x | 0 | 1 | 2 |
|---|---|---|---|
| f(x) | 2 | 1 | 0 |

## Exercice 3 solution

a)

$$P_2(x) = 4x - 4x(x-1)$$

b) $$P_2(x) = 2 - x$$

**Exercice 3bis**

The data

| | | | | |
|---|---|---|---|---|
| x | 0 | 1 | 3 | 4 |
| f(x) | 1 | 0 | 2 | 1 |

a) Write the cubic interpolating polynomial in the Newton form :
$p_3(x) = a_1 + a_2 x + a_3 x(x-1) + a_4 x(x-1)(x-3)$

b) Use the Lagrange Polynomial to find the unique polynomial of degree 3.

c) Verify that solutions in (a) and (b) are the same.

**Exercice 3bis solution**

a) $a_1 = 1$ $a_2 = -1$ $a_3 = \frac{2}{3}$ $a_4 = \frac{-1}{3}$