

ACS234

Maths and Data Modelling

Tutorial 7
Wednesday 1pm online

<https://github.com/ineskris/ACS234/blob/master/Tutorial7/Tutorial7.ipynb>

Done in Lecture (week 8/9)

- ODEs systems
- Euler's Method

Ordinary Differential Equations

	1st-order ODE system	nth-order ODE
Formula	$\frac{dy}{dt} = f(t, y)$	$\frac{dy^n}{dt} = f(t, y(t), y'(t), \dots, y^{n-1}(t))$
Example	$\frac{dy}{dt} = y \quad y(0) = 1 \quad \boxed{y(t) = \exp(t)}$	$\frac{dy^2}{dt} = -\frac{dy}{dt} + 6y \quad \boxed{y(t) = A \exp(2t) + B \exp(-3t)}$

Euler's Method

Taylor series : Let f be indefinitely differentiable at a point a : $f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \quad y(t_0 + h) \approx y(t_0) + hy'(t_0)$

Euler's method : $\frac{dy}{dt} = f(t, y) \quad y_1 \approx y_0 + hf(t_0, y_0) \quad t_1 = t_0 + h \quad y(t_1) = y_1 \quad y(t_0) = y_0$

$$\frac{dy}{dt} = f(t, y) \quad y(t_0) = y_0$$

$$y_{i+1} = y_i + hf(t_i, y_i)$$

Stability

Solution of an ODE is :

- Stable if solutions resulting from perturbations of initial value remain close to original solution
- Asymptotically stable if solutions resulting from perturbations converge back to original solution
- Unstable if solutions resulting from perturbations diverge away from original solution without bound

Exercise 1

Use Euler's method to integrate these 1st order ODE system with the given step size h , number of steps and the initial condition. Compute for each step the exact analytical solution too.

a) $h = 0.1$	$n_{step} = 2$	b) $h = 0.5$	$n_{step} = 3$	c) $h = 1$	$n_{step} = 2$
$\frac{dy}{dt} = 3y$	$y(0) = 2$	$\frac{dy}{dt} = t^2 + 1$	$y(1) = 4$	$\frac{dy}{dt} = \ln(t)$	$y(1) = 0$

Exercise 2

Find the step size h in question 1c) to have an absolute error inferior to 0.05 between the for the approximate y_2 .

Exercise 3

$$\frac{dy}{dt} = -3y \quad y(0) = 2$$

For the 1st order ODE above, find the condition on the step size h for the difference equation to be stable. See Hand out part 5.

Exercise 1 - solution

- a) Analytical solution : $f(t) = 2 \exp(3t)$ $f(0.5) = 2 \exp(0.1) \approx 2.69$
 $f(0.2) = 2 \exp(0.2) \approx 3.64$
- Euler's method : $t_1 = t_0 + h = 0.1$ $y_1 = y_0 + hf(t_0, y_0) = 2 + 0.1 \times 3 \times y_0 = 2.6$
 $t_2 = t_1 + h = 0.2$ $y_2 = y_1 + hf(t_1, y_1) = 2.6 + 0.1 \times 3 \times 2.6 = 3.38$
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- b) Analytical solution : $f(t) = \frac{t^3}{3} + t + \frac{8}{3}$ $f(1.5) = \frac{1.5^3}{3} + 1.5 + \frac{8}{3} \approx 5.29$
 $f(2) \approx 7.33$
 $f(2.5) = 10.375$
- Euler's method : $t_2 = t_1 + h = 1.5$ $y_2 = y_1 + hf(t_1, y_1) = 4 + 0.5 \times 2 = 5$
 $t_3 = t_2 + h = 2$ $y_3 = y_2 + hf(t_2, y_2) = 5 + 0.5 \times 5 = 7.5$
 $t_4 = t_3 + h = 2.5$ $y_4 = y_3 + hf(t_3, y_3) = 7.5 + 0.5 \times 7.33 = 11.16$
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- c) Analytical solution : $f(t) = t \ln t - t + 1$ $f(2) \approx 0.386$
 $f(3) \approx 1.296$
- Euler's method : $t_2 = t_1 + h = 2$ $y_2 = y_1 + hf(t_1, y_1) = 0 + 1 \times 0 = 0$
 $t_3 = t_2 + h = 3$ $y_3 = y_2 + hf(t_2, y_2) = 0 + 1 \times \ln(2) = 0.693$

Exercise 2 - solution

$$h = 0.7$$

See Jupyter notebook

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Exercise 3 - solution

$$y_{i+1} = y_i - 3hy_i$$

$$y_{i+1} = (1 - 3h)y_i$$

$$|1 - 3h| < 1$$

$$h < 2/3$$

