

ACS234

Maths and Data Modelling

Tutorial 9
Wednesday 1pm online

- High Order ODE
- RK4 method

ODE

1st-order ODE system

nth-order ODE

Formula	$\frac{dy}{dt} = f(t, y) \quad y(t_0) = y_0$	$\frac{dy^n}{dt} = f(t, y(t), y'(t), \dots, y^{n-1}(t), u(t))$
Example	$\frac{dy}{dt} = y \quad y(0) = 1 \quad \boxed{y(t) = \exp(t)}$	$\frac{dy^2}{dt} = -\frac{dy}{dt} + 6y \quad \boxed{y(t) = A \exp(2t) + B \exp(-3t)}$
Euler	$y_{i+1} = y_i + hf(t_i, y_i) \quad t_{i+1} = t_i + h$	$\begin{aligned} x_1(t) &= y(t) & \dot{x}_1(t) &= x_2(t) \\ x_2(t) &= y'(t) & \dot{x}_2(t) &= x_3(t) \\ & \dots & \dots \\ x_n(t) &= y^{n-1}(t) & \dot{x}_{n-1}(t) &= x_n(t) \\ & & \dot{x}_n(t) &= f(t, x_1(t), \dots, x_n(t)) \end{aligned}$
Mid Point	$\begin{aligned} y_{i+\frac{1}{2}} &= y_i + \frac{h}{2} f(t_i, y_i) \\ t_{i+\frac{1}{2}} &= t_i + \frac{h}{2} \end{aligned} \quad y_{i+1} = y_i + hf(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	
Heun	$\begin{aligned} y_{i+1}^p &= y_i + hf(t_i, y_i) & t_{i+1} &= t_i + h \\ y_{i+1} &= y_i + \frac{h}{2} [f(t_i, y_i) + f(t_{i+1}, y_{i+1}^p)] \end{aligned}$	<p>Method :</p> <p>—> state-space model (simultaneous equations of n 1st-order ODEs)</p> <p>--> apply a numerical method (e.g. Euler's, Mid-point, Heun's, RK4) to each of the n 1st-order ODEs</p> <p>>> in each step, the values of the state variables are updated simultaneously in each step of calculation</p> <p>>> when all state variables are updated, move to the next step of calculation</p>
RK4	$\begin{aligned} y_{i+1} &= y_i + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 &= f(t_i, y_i) \\ k_2 &= f(t_i + \frac{h}{2}, y_i + \frac{hk_1}{2}) \\ k_3 &= f(t_i + \frac{h}{2}, y_i + \frac{hk_2}{2}) \\ k_4 &= f(t_i + h, y_i + hk_3) \end{aligned}$	

Exercise 1

Compute the first two steps of the RK4 with the 1st order ODE system below with the given step size h and the initial condition. Compare with the results obtained in Tutorial 8 Exercise 1a.

$$h = 0.1 \quad \frac{dy}{dt} = 3y \quad y(0) = 2$$

Exercise 2

Use classical RK4 method to solve the second-order below (cf hand out) :

$$y''(t) = -y'(t) + \sin(ty) \quad y(0) = 1 \quad y'(0) = 2$$

Exercise 1 - solution

Analytical solution : $f(t) = 2 \exp(3t)$
 $f(0) = 2$

	Analytical	Euler	Mid Point	Heun	RK4
y1	2.699	2.6	2.69	2.69	
y2	3.64	3.38	3.61805	3.61805	

$$k_1 = f(t_0, y_0) = 3 \times 2 = 6$$

$$k_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{hk_1}{2}) = 3 \times 2.3 = 6.9$$

$$k_3 = f(t_0 + \frac{h}{2}, y_0 + \frac{hk_2}{2}) = 3 \times 2.345 = 7.035$$

$$k_4 = f(t_0 + h, y_0 + hk_3) = 3 \times 2.7035 = 8.1105$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.699675$$

$$k_1 = 8.099025$$

$$k_2 = 9.31387875$$

$$k_3 = 9.4961068125$$

$$k_4 = 10.9478570438$$

$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 3.64412255281$$

Exercise 2 - solution

$$x''(t) = -x'(t) + \sin(tx) \quad x(0) = 1 \quad x'(0) = 2$$

$$x_1'(t) = x_2(t) \quad x_1(0) = 1 \quad (\text{a})$$

$$x_2'(t) = -x_2(t) + \sin(tx_1(t)) \quad x_2(0) = 2 \quad (\text{b})$$

Calculating $k_{1,1}$ and $k_{1,2}$: