ACS234
Maths and Data Modelling

Tutorial 7
Wednesday 1pm online

https://github.com/ineskris/ACS234/blob/master/Tutorial7/Tutorial7.ipynb

Done in Lecture (week 8/9)

- ODEs systems
- Euler’s Method

Ordinary Differential Equations

	1st-order ODE system	nth-order ODE
Formula	$\frac{dy}{dt} = f(t, y)$	$\frac{dy^n}{dt} = f(t, y(t), y'(t), \ldots, y^{n-1}(t))$
Example	$\frac{dy}{dt} = y \quad y(0) = 1 \quad \boxed{y(t) = \exp(t)}$	$\frac{dy^2}{dt} = -\frac{dy}{dt} + 6y \quad \boxed{y(t) = A\exp(2t) + B\exp(-3t)}$

Euler's Method

Taylor series : Let f be indefinitely differentiable at a point a: $f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad y(t_0 + h) \approx y(t_0) + hy'(t_0)$

Euler's method : $\frac{dy}{dt} = f(t, y) \quad y_1 \approx y_0 + hf(t_0, y_0) \quad t_1 = t_0 + h \quad y(t_1) = y_1 \quad y(t_0) = y_0$

$$\frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0$$

$$y_{i+1} = y_i + hf(t_i, y_i)$$

Stability

Solution of an ODE is :

- Stable if solutions resulting from perturbations of initial value remain close to original solution
- Asymptotically stable if solutions resulting from perturbations converge back to original solution
- Unstable if solutions resulting from perturbations diverge away from original solution without bound

Exercice 1

Use Euler's method to integrate these 1st order ODE system with the given step size h, number of steps and the initial condition. Compute for each step the exact analytical solution too.

a) $h = 0.1$ $\quad n_{step} = 2$

$\frac{dy}{dt} = 3y$ $\quad y(0) = 2$

b) $h = 0.5$ $\quad n_{step} = 3$

$\frac{dy}{dt} = t^2 + 1$ $\quad y(1) = 4$

c) $h = 1$ $\quad n_{step} = 2$

$\frac{dy}{dt} = ln(t)$ $\quad y(1) = 0$

Exercice 2

Find a step size h in question 1c) to have an absolute error inferior to 0.05 between the analytical solution and the approximate y_2.

Exercice 3

$\frac{dy}{dt} = -3y$ $\quad y(0) = 2$

For the 1st order ODE above, find the condition on the step size h for the difference equation to be stable. See Hand out part 5.

Exercice 1 - solution

a) Analytical solution : $f(t) = 2\exp(3t)$ $f(0.5) = 2\exp(0.1) \approx 2.69$

$$f(0.2) = 2\exp(0.2) \approx 3.64$$

Euler's method :

$$y_1 = y_0 + hf(t_0, y_0) = 2 + 0.1 \times 3 \times y_0 = 2.6 \qquad t_1 = t_0 + h = 0.1$$

$$y_2 = y_1 + hf(t_1, y_1) = 2.6 + 0.1 \times 3 \times 2.6 = 3.38 \qquad t_2 = t_1 + h = 0.2$$

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b) Analytical solution : $f(t) = \frac{t^3}{3} + t + \frac{8}{3}$ $f(1.5) = \frac{1.5^3}{3} + 1.5 + \frac{8}{3} \approx 5.29$

$$f(2) \approx 7.33$$

$$f(2.5) = 10.375$$

Euler's method :

$$y_2 = y_1 + hf(t_1, y_1) = 4 + 0.5 \times 2 = 5 \qquad t_2 = t_1 + h = 1.5$$

$$y_3 = y_2 + hf(t_2, y_2) = 5 + 0.5 \times 5 = 7.5 \qquad t_3 = t_2 + h = 2$$

$$y_4 = y3 + hf(t_3, y_3) = 7.5 + 0.5 \times 7.33 = 11.16$$

Exercice 1 - solution

Integration By Parts

c) Analytical solution :

$$f(t) = \int lnt dt = t \times lnt - \int t \times \frac{1}{t} dt = tlnt - t + C$$

$$f(1) = 1 \times 0 - 1 + C = 0 \qquad C = 1$$

$$f(t) = tlnt - t + 1$$

$$f(2) \approx 0.386$$
$$f(3) \approx 1.296$$

Euler's method :

$$y_2 = y_1 + hf(t_1, y_1) = 0 + 1 \times 0 = 0 \qquad t_2 = t_1 + h = 2$$

$$y_3 = y_2 + hf(t_2, y_2) = 0 + 1 \times ln(2) = 0.693$$

Exercice 2 - solution

$f(t) = tlnt - t + 1$ $\quad f(2) \approx 0.386$

$t_2 = t_1 + h = 1 + h$ $\quad y_2 = y_1 + hf(t_1, y_1) = 0 + h \times ln(1 + h)$

With $h = 0.7$ $\quad h \times ln(1 + h) \approx 0.3714$

$$|h \times ln(1 + h) - 2(ln2 - \frac{1}{2})| < 0.05$$

See Jupyter notebook
https://github.com/ineskris/ACS234/blob/master/Tutorial7/Tutorial7.ipynb

Exercice 3 - solution

$y_{i+1} = y_i - 3hy_i$ $\quad y_{i+1} = (1 - 3h)y_i$ $\quad |1 - 3h| < 1$ $\quad h < 2/3$