

# ACS234

# Maths and Data Modelling

**Tutorial 7**  
**Wednesday 1pm online**

<https://github.com/ineskris/ACS234/blob/master/Tutorial7/Tutorial7.ipynb>

## Done in Lecture (week 8/9)

- ODEs systems
- Euler's Method

# Ordinary Differential Equations

	1st-order ODE system	nth-order ODE
Formula	$\frac{dy}{dt} = f(t, y)$	$\frac{dy^n}{dt} = f(t, y(t), y'(t), \dots, y^{n-1}(t))$
Example	$\frac{dy}{dt} = y \quad y(0) = 1 \quad \boxed{y(t) = \exp(t)}$	$\frac{dy^2}{dt} = -\frac{dy}{dt} + 6y \quad \boxed{y(t) = A \exp(2t) + B \exp(-3t)}$

## Euler's Method

Taylor series : Let  $f$  be indefinitely differentiable at a point  $a$ :  $f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \quad y(t_0 + h) \approx y(t_0) + hy'(t_0)$

Euler's method :  $\frac{dy}{dt} = f(t, y) \quad y_1 \approx y_0 + hf(t_0, y_0) \quad t_1 = t_0 + h \quad y(t_1) = y_1 \quad y(t_0) = y_0$

$$\frac{dy}{dt} = f(t, y) \quad y(t_0) = y_0$$

$$y_{i+1} = y_i + hf(t_i, y_i)$$

## Stability

Solution of an ODE is :

- Stable if solutions resulting from perturbations of initial value remain close to original solution
- Asymptotically stable if solutions resulting from perturbations converge back to original solution
- Unstable if solutions resulting from perturbations diverge away from original solution without bound

## Exercise 1

Use Euler's method to integrate these 1st order ODE system with the given step size  $h$ , number of steps and the initial condition. Compute for each step the exact analytical solution too.

a) $h = 0.1$	$n_{step} = 2$	b) $h = 0.5$	$n_{step} = 3$	c) $h = 1$	$n_{step} = 2$
$\frac{dy}{dt} = 3y$	$y(0) = 2$	$\frac{dy}{dt} = t^2 + 1$	$y(1) = 4$	$\frac{dy}{dt} = \ln(t)$	$y(1) = 0$

## Exercise 2

Find the step size  $h$  in question 1c) to have an absolute error inferior to 0.05 between the analytical solution and the approximate  $y_3$ .

## Exercise 3

$$\frac{dy}{dt} = -3y \quad y(0) = 2$$

For the 1st order ODE above, find the condition on the step size  $h$  for the difference equation to be stable. See Hand out part 5.

## Exercise 1 - solution

a) Analytical solution :  $f(t) = 2 \exp(3t)$   $f(0.5) = 2 \exp(0.1) \approx 2.69$

$$f(0.2) = 2 \exp(0.2) \approx 3.64$$

Euler's method :  $t_1 = t_0 + h = 0.1$   $y_1 = y_0 + hf(t_0, y_0) = 2 + 0.1 \times 3 \times y_0 = 2.6$

$$t_2 = t_1 + h = 0.2 \quad y_2 = y_1 + hf(t_1, y_1) = 2.6 + 0.1 \times 3 \times 2.6 = 3.38$$

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b) Analytical solution :  $f(t) = \frac{t^3}{3} + t + \frac{8}{3}$   $f(1.5) = \frac{1.5^3}{3} + 1.5 + \frac{8}{3} \approx 5.29$

$$f(2) \approx 7.33$$

$$f(2.5) = 10.375$$

Euler's method :  $y_2 = y_1 + hf(t_1, y_1) = 4 + 0.5 \times 2 = 5$   $t_2 = t_1 + h = 1.5$

$$y_3 = y_2 + hf(t_2, y_2) = 5 + 0.5 \times 5 = 7.5 \quad t_3 = t_2 + h = 2$$

$$y_4 = y_3 + hf(t_3, y_3) = 7.5 + 0.5 \times 7.33 = 11.16$$

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c) Analytical solution :  $f(t) = t \ln t - t + 1$   $f(2) \approx 0.386$

$$f(3) \approx 1.296$$

Euler's method :  $y_2 = y_1 + hf(t_1, y_1) = 0 + 1 \times 0 = 0$   $t_2 = t_1 + h = 2$

$$y_3 = y_2 + hf(t_2, y_2) = 0 + 1 \times \ln(2) = 0.693$$

## Exercise 2 - solution

$$h = 0.7$$

See Jupyter notebook

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## Exercise 3 - solution

$$y_{i+1} = y_i - 3hy_i$$

$$y_{i+1} = (1 - 3h)y_i$$

$$|1 - 3h| < 1$$

$$h < 2/3$$