ACS234
Maths and Data Modelling

Tutorial 8
Wednesday 1pm online

https://github.com/ineskris/ACS234/blob/master/Tutorial8/Tutorial8.ipynb

Done in Lecture (week 9/10)

- High Order ODE
- Euler's, Mid Point, Heun's Method

ODE solvers

	1st-order ODE system	nth-order ODE
Formula	$\frac{dy}{dt} = f(t, y)$ $\quad y(t_0) = y_0$	$\frac{dy^n}{dt} = f(t, y(t), y'(t), \ldots, y^{n-1}(t), u(t))$
Example	$\frac{dy}{dt} = y$ $\quad y(0) = 1$ $\quad \boxed{y(t) = \exp(t)}$	$\frac{dy^2}{dt} = -\frac{dy}{dt} + 6y$ $\quad \boxed{y(t) = A\exp(2t) + B\exp(-3t)}$
Euler	$y_{i+1} = y_i + hf(t_i, y_i)$ $\quad t_{i+1} = t_i + h$	
Mid Point	$y_{i+\frac{1}{2}} = y_i + \frac{h}{2}f(t_i, y_i)$ $t_{i+\frac{1}{2}} = t_i + \frac{h}{2}$ $y_{i+1} = y_i + hf(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	
Heun	$y^p_{i+1} = y_i + hf(t_i, y_i)$ $\quad t_{i+1} = t_i + h$ $y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, y^p_{i+1})]$	

$$x_1(t) = y(t) \qquad \dot{x}_1(t) = x_2(t)$$

$$x_2(t) = y'(t) \qquad \dot{x}_2(t) = x_3(t)$$

$$\ldots \qquad \ldots$$

$$x_n(t) = y^{n-1}(t) \qquad \dot{x}_{n-1}(t) = x_n(t)$$

$$\dot{x}_n(t) = f(t, x_1(t), \ldots, x_n(t))$$

Method :
—> state-space model (simultaneous equations of n 1st-order ODEs)
--> apply a numerical method (e.g. Euler's, Mid-point, Heun's, RK4) to each of the n 1st-order ODEs
>> in each step, the values of the state variables are updated simultaneously in each step of calculation
>> when all state variables are updated, move to the next step of calculation

Exercice 1

Compute the first two steps of the Mid Point and the Heun method with these 1st order ODE system with the given step size h and the initial condition. Compare the results with the Euler method obtained in Tutorial 7 Exercice 1.

a) $h = 0.1$

$$\frac{dy}{dt} = 3y \qquad y(0) = 2$$

b) $h = 0.5$

$$\frac{dy}{dt} = t^2 + 1 \qquad y(1) = 4$$

c) $h = 1$

$$\frac{dy}{dt} = ln(t) \qquad y(1) = 0$$

Exercice 2

Solve the ODE below using the Euler Method from t = 0 to t = 0.2 with a step size h = 0.1.

$$y''(t) = -y'(t) + sin(ty) \qquad y(0) = 1 \qquad y'(0) = 2$$

Exercice 1 - solution

a) Analytical solution : $f(t) = 2\exp(3t)$ $\boxed{f(0.1) = 2\exp(3 * 0.1) \approx 2.699}$

$$\boxed{f(0.2) = 2\exp(0.2) \approx 3.64}$$

Euler :

$$h = 0.1$$

$$\boxed{y_1 = y_0 + hf(t_0, y_0) = 2 + 0.1 \times 3 \times y_0 = 2.6} \qquad t_1 = t_0 + h = 0.1$$

$$\boxed{y_2 = y_1 + hf(t_1, y_1) = 2.6 + 0.1 \times 3 \times 2.6 = 3.38} \qquad t_2 = t_1 + h = 0.2$$

Mid Point :

$$y_{\frac{1}{2}} = y_0 + \frac{h}{2} f(t_0, y_0) = 2 + \frac{0.1}{2} \times 3 \times 2 = 2.3 \qquad t_{1+\frac{1}{2}} = t_1 + \frac{h}{2} = 0.05$$

$$\boxed{y_1 = y_0 + hf(t_{\frac{1}{2}}, y_{\frac{1}{2}}) = 2 + 0.1 \times 3 \times 2.3 = 2.69}$$

$$y_{1+\frac{1}{2}} = y_1 + \frac{h}{2} f(t_1, y_1) = 2.69 + \frac{0.1}{2} \times 3 \times 2.69 = 3.0935 \qquad t_1 = t_{1+\frac{1}{2}} + \frac{h}{2} = 0.1$$

$$\boxed{y_2 = y_1 + hf(t_{1+\frac{1}{2}}, y_{1+\frac{1}{2}}) = 2.69 + 0.1 \times 3 \times 3.0935 = 3.61805}$$

Heun :

$$y_1^p = y_0 + hf(t_0, y_0) = 2.6 \qquad t_1 = t_0 + h = 0.1$$

$$\boxed{y_1 = y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, y_1^p)] = 2 + 0.05 \times (6 + 7.8) = 2.69}$$

$$y_2^p = y_1 + hf(t_1, y_1) = 2.69 + 0.1 \times 3 \times 2.69 = 3.4971$$

$$\boxed{y_2 = y_1 + \frac{h}{2}[f(t_1, y_1) + f(t_2, y_2^p)] = 2.69 + 0.05[8.07 + 10.4913] = 3.61805}$$

Exercice 1 - solution

b) Analytical solution : $f(t) = \frac{t^3}{3} + t + \frac{8}{3}$

$$\boxed{f(1.5) = \frac{1.5^3}{3} + 1.5 + \frac{8}{3} \approx 5.291}$$

$$\boxed{f(2) \approx 7.33}$$

Euler :

$h = 0.5$

$$\boxed{y_2 = y_1 + hf(t_1, y_1) = 4 + 0.5 \times 2 = 5}$$

$$\boxed{y_3 = y_2 + hf(t_2, y_2) = 5 + 0.5 \times (1.5^2 + 1) = 6.625}$$

$t_2 = t_1 + h = 1.5$

$t_3 = t_2 + h = 2$

Mid Point :

$$y_{1+\frac{1}{2}} = y_1 + \frac{h}{2} f(t_1, y_1) = 4 + \frac{0.5}{2} \times 2 = 4.5$$

$$\boxed{y_2 = y_1 + hf(t_{1+\frac{1}{2}}, y_{1+\frac{1}{2}}) = 4 + 0.5 \times (1.25^2 + 1) = 5.28125}$$

$$y_{2+\frac{1}{2}} = y_2 + \frac{h}{2} f(t_2, y_2) = 5.28125 + \frac{0.5}{2} \times (1.5^2 + 1) = 6.09375$$

$$\boxed{y_3 = y_2 + hf(1.75, y_{2+\frac{1}{2}}) = 5.28125 + 0.5 \times (1.75^2 + 1) = 7.3125}$$

$t_{1+\frac{1}{2}} = t_1 + \frac{h}{2} = 1.25$

$t_2 = t_{1+\frac{1}{2}} + \frac{h}{2} = 1.5$

$t_{2+\frac{1}{2}} = t_2 + \frac{h}{2} = 1.75$

Heun :

$$y_2^p = y_1 + hf(t_1, y_1) = 4 + 0.5 \times (1 + 1) = 5$$

$$\boxed{y_2 = y_1 + \frac{h}{2}[f(t_1, y_1) + f(t_2, y_2^p)] = 4 + 0.25 \times (2 + 1.5^2 + 1) = 5.3125}$$

$$y_3^p = y_2 + hf(t_2, y_2) = 5.3125 + 0.5 \times (1.5^2 + 1) = 6.9375$$

$$\boxed{y_3 = y_2 + \frac{h}{2}[f(t_2, y_2) + f(t_3, y_3^p)] = 5.3125 + 0.25[3.25 + 5] = 7.375}$$

$t_2 = t_1 + h = 1.5$

$t_3 = t_2 + h = 2$

Exercice 1 - solution

c) Analytical solution : $f(t) = tlnt - t + 1$ $\boxed{f(2) \approx 0.386}$

$\boxed{f(3) \approx 1.296}$

Euler :

$h = 1$

$\boxed{y_2 = y_1 + hf(t_1, y_1) = 0 + 1 \times 0 = 0}$

$\boxed{y_3 = y_2 + hf(t_2, y_2) = 0 + 1 \times ln(2) = 0.693}$

$t_2 = t_1 + h = 2$

Mid Point :

$y_{1+\frac{1}{2}} = y_1 + \frac{h}{2} f(t_1, y_1) = 0 + \frac{1}{2} \times ln(1) = 0$

$\boxed{y_2 = y_1 + hf(t_{1+\frac{1}{2}}, y_{1+\frac{1}{2}}) = 0 + 1 \times (ln(1.5)) \approx 0.405}$

$y_{2+\frac{1}{2}} = y_2 + \frac{h}{2} f(t_2, y_2) = 0.405 + \frac{1}{2} \times ln(2) \approx 0.7515$

$\boxed{y_3 = y_2 + hf(t_{2+\frac{1}{2}}, y_{2+\frac{1}{2}}) = 0.405 + 1 \times (ln(2.5)) \approx 1.32}$

$t_{1+\frac{1}{2}} = t_1 + \frac{h}{2} = 1.5$

$t_2 = t_{1+\frac{1}{2}} + \frac{h}{2} = 2$

$t_{2+\frac{1}{2}} = t_2 + \frac{h}{2} = 2.5$

Heun :

$y_2^p = y_1 + hf(t_1, y_1) = 0 + 1 \times (ln(1)) = 0$

$t_2 = t_1 + h = 2$

$\boxed{y_2 = y_1 + \frac{h}{2}[f(t_1, y_1) + f(t_2, y_2^p)] = 0 + 0.5 \times (0 + ln(2)) \approx 0.3465}$

$y_3^p = y_2 + hf(t_2, y_2) \approx 0.3465 + 1 \times (ln(2)) = 1.0397$

$t_3 = t_2 + h = 3$

$\boxed{y_3 = y_2 + \frac{h}{2}[f(t_2, y_2) + f(t_3, y_3^p)] = 0.3465 + 0.5[ln(2) + ln(3)] \approx 1.242}$

Exercice 2 - solution

$$x''(t) = -x'(t) + sin(tx) \qquad x(0) = 1 \qquad x'(0) = 2$$

$$x_1'(t) = x_2(t) \qquad x_1(0) = 1 \qquad \text{(a)}$$
$$x_2'(t) = -x_2(t) + sin(tx_1(t)) \qquad x_2(0) = 2 \qquad \text{(b)}$$

$$x_2(t_1) = x_2(t_0) + hf(t_0, x_2(t_0)) = 2 + 0.1 \times (-2 + sin(0 \times 1)) = 1.8$$

$$x_1(t_1) = x_1(t_0) + hf(t_0, x_1(t_0)) = 1 + 0.1 \times 2 = 1.2$$

$$x_2(t_2) = x_2(t_1) + hf(t_1, x_2(t_1)) = 1.8 + 0.1 \times (-2 + sin(0.1 \times 1.2))$$

$$x_1(t_2) = x_1(t_1) + hf(t_1, x_1(t_1)) = 1.2 + 0.1 \times 1.8 = 1.38$$