ACS234
Maths and Data Modelling

Tutorial 9
Wednesday 1pm online

- High Order ODE
- RK4 method

ODE

<table>
<tr><th></th><th>1st-order ODE system</th><th>nth-order ODE</th></tr>
<tr><td>Formula</td><td>$\frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0$</td><td>$\frac{dy^n}{dt} = f(t, y(t), y'(t), \ldots, y^{n-1}(t), u(t))$</td></tr>
<tr><td>Example</td><td>$\frac{dy}{dt} = y \qquad y(0) = 1 \qquad \boxed{y(t) = \exp(t)}$</td><td>$\frac{dy^2}{dt} = -\frac{dy}{dt} + 6y \qquad \boxed{y(t) = A\exp(2t) + B\exp(-3t)}$</td></tr>
<tr><td>Euler</td><td>$y_{i+1} = y_i + hf(t_i, y_i) \qquad t_{i+1} = t_i + h$</td><td rowspan="4">$x_1(t) = y(t)$
$x_2(t) = y'(t)$
...
$x_n(t) = y^{n-1}(t)$

$\dot{x}_1(t) = x_2(t)$
$\dot{x}_2(t) = x_3(t)$
...
$\dot{x}_{n-1}(t) = x_n(t)$
$\dot{x}_n(t) = f(t, x_1(t), \ldots, x_n(t))$

Method :
—> state-space model (simultaneous equations of n 1st-order ODEs)
--> apply a numerical method (e.g. Euler's, Mid-point, Heun's, RK4) to each of the n 1st-order ODEs
>> in each step, the values of the state variables are updated simultaneously in each step of calculation
>> when all state variables are updated, move to the next step of calculation</td></tr>
<tr><td>Mid Point</td><td>$y_{i+\frac{1}{2}} = y_i + \frac{h}{2}f(t_i, y_i)$
$t_{i+\frac{1}{2}} = t_i + \frac{h}{2}$
$y_{i+1} = y_i + hf(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$</td></tr>
<tr><td>Heun</td><td>$y^p_{i+1} = y_i + hf(t_i, y_i) \qquad t_{i+1} = t_i + h$
$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, y^p_{i+1})]$</td></tr>
<tr><td>RK4</td><td>$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$k_1 = f(t_i, y_i)$
$k_2 = f(t_i + \frac{h}{2}, y_i + \frac{hk_1}{2})$
$k_3 = f(t_i + \frac{h}{2}, y_i + \frac{hk_2}{2})$
$k_4 = f(t_i + h, y_i + hk_3)$</td></tr>
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Exercice 1

Compute the first two steps of the RK4 with the 1st order ODE system below with the given step size h and the initial condition. Compare with the results obtained in Tutorial 8 Exercice 1a.

$$h = 0.1 \qquad \frac{dy}{dt} = 3y \quad y(0) = 2$$

Exercice 2

Use classical RK4 method to solve the second-order below (cf hand out) :

$$y''(t) = -y'(t) + sin(ty) \qquad y(0) = 1 \qquad y'(0) = 2$$

$$h = 0.1$$

Exercice 1 - solution

Analytical solution : $f(t) = 2\exp(3t)$

$f(0) = 2$

	Analytical	Euler	Mid Point	Heun	RK4
y1	2.699	2.6	2.69	2.69	
y2	3.64	3.38	3.61805	3.61805	

$$k_1 = f(t_0, y_0) = 3 \times 2 = 6$$

$$k_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{hk_1}{2}) = 3 \times 2.3 = 6.9$$

$$k_3 = f(t_0 + \frac{h}{2}, y_0 + \frac{hk_2}{2}) = 3 \times 2.345 = 7.035$$

$$k_4 = f(t_0 + h, y_0 + hk_3) = 3 \times 2.7035 = 8.1105$$

$$\boxed{y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.699675}$$

$$k_1 = 8.099025$$

$$k_2 = 9.31387875$$

$$k_3 = 9.4961068125$$

$$k_4 = 10.9478570438$$

$$\boxed{y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 3.64412255281}$$

Exercice 2 - solution

$$x''(t) = -x'(t) + sin(tx) \qquad x(0) = 1 \qquad x'(0) = 2$$

$$x_1'(t) = x_2(t) \qquad x_1(0) = 1 \qquad \text{(a)}$$
$$x_2'(t) = -x_2(t) + sin(tx_1(t)) \qquad x_2(0) = 2 \qquad \text{(b)}$$

Calculating $k_{1,1}$ and $k_{2,1}$:

$$k_{1,1} = f_1(t_0, x_{1,0}, x_{2,0}) = x_{2,0} = 2$$
$$k_{2,1} = f_2(t_0, x_{1,0}, x_{2,0}) = -x_{2,0} + sin(t_0 x_{1,0}) = -2$$

Calculating $k_{1,2}$ and $k_{2,2}$:

$$k_{1,2} = f_1(t_0 + \tfrac{1}{2}h, x_{1,0} + \tfrac{1}{2}hk_{1,1}, x_{2,0} + \tfrac{1}{2}hk_{2,1}) = x_{2,0} + \tfrac{1}{2}hk_{2,1} = 2 - h = 1.9$$
$$k_{2,1} = f_2(t_0 + \tfrac{1}{2}h, x_{1,0} + \tfrac{1}{2}hk_{1,1}, x_{2,0} + \tfrac{1}{2}hk_{2,1}) = -x_{2,0} - \tfrac{1}{2}hk_{2,1} + sin((t_0 + \tfrac{1}{2})(x_{1,0} + \tfrac{1}{2}hk_{1,1})) = -1.9 + sin((1.1)/2)$$

Calculating $k_{1,3}$ and $k_{2,3}$ …

See page 80 hand-out