Structured compartment models of infection in Python

In this section we list all epidemiological models that are featured in PyRoss in order of complexity. These models can be called as a system of Ordinary Differential Equations (ODEs), Stochastic Jump Processes (SJPs) or a hybrid of the two which switches from SJP to ODE when the population reaches a user defined threshold, at which point it is assumed that random fluctuations are a negligible percentage of the total population. These integration methods build the foundation of PyRoss, upon which investigation into the effects of control such as self isolation or forcasting made from real world data can be performed.

We consider a population aggregated by age into M groups labelled by $i = 1, 2, \dots M$. In what follows, we provide several mathematical models of infection which have been implemented in PyRoss. We consider a structured metapopulation

$$\boldsymbol{n} = (n_1, \dots n_{M\times L}) \tag{1}$$

consisting of M age-compartments and L classes of epidemiological states. The ξ-th reaction can be written down in its most general form as

$$\sum_{i=1}^{M\times L} f_i^{\xi} Y_i \to \sum_{i=1}^{M\times L} g_i^{\xi} Y_i$$

where $\boldsymbol{Y_i}$ stands for an age-compartment of a epidemiological state (of dimension $M \times L$), and $\boldsymbol{f}^{\xi}, \boldsymbol{g}^{\xi}$ keeps track of the number of each participating in the ξ-th reaction.

I. SIR

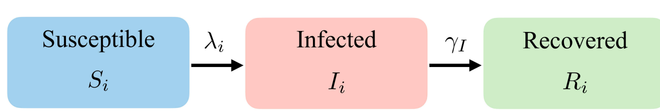
The population within age group i is partitioned into susceptibles S_i, infectives I_i, and removed individuals R_i. The sum of these is the size of the population in age group i, $N_i = S_i + I_i + R_i$ [1–5]. For this model, vital dynamics and the change in age structure on the time scale of the epidemic in this model is ignored. Therefore each N_i and, consequently, the total population size

$$N = \sum_{i=1}^{M} N_i \tag{2}$$

remain constant in time. We assume that the rate of infection of a susceptible individual in age group i is

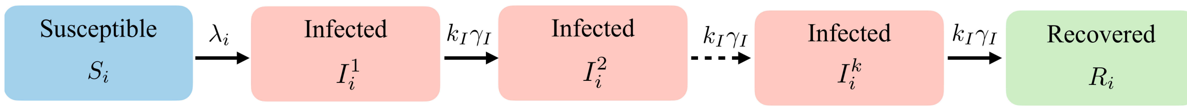
$$\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}(t) \frac{I_j}{N_j} \right), \quad i, j = 1, \dots M \tag{3}$$

where β is the probability of infection on contact (assumed intrinsic to the pathogen) and C_{ij}^a and C_{ij}^s are, respectively, the number of contacts between asymptomatic and symptomatic infectives in age-group j with susceptibles in age-group i (reflecting the structure of social contacts). We take the age-independent recovery rate γ to be identical for



Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I \to 2I$	$(-1, 1, 0)$	$\lambda_I n_S$
$I \to R$	$(0, -1, 1)$	$\gamma_I n_I$

Table I. **Schematic of the SIR model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\beta, \gamma_I)$. The reaction takes the state $\boldsymbol{n} = (n_S, n_I, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. For simplicity we consider $M = 1$, and thus, the sum in λ_i, (6), becomes trivial. we define $\lambda_I = \beta C n_I / N$. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SIR`.



Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I^1 \to 2I^1$	$(-1, 1, \dots 0, 0, 0)$	$\lambda_I n_S$
$\vdots$	$\vdots$	$\vdots$
$I^{k-1} \to I^k$	$(0, 0, \cdots -1, 1, 0)$	$k_I \gamma_I n_{I^{k-1}}$
$I^k \to R$	$(0, 0, \dots 0, -1, 1)$	$k_I \gamma_I n_I$

Table II. **Schematic of the SEIIR with stages (SIkR) model and elementary reaction steps, and their rates rates.** The parameters for this model are: $\boldsymbol{\theta} = (k_I, \beta, \gamma_I)$. The reaction takes the state $\boldsymbol{n} = (n_S, n_{I^1}, \dots, n_{I^{k_I}}, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. For simplicity we consider $M = 1$, and thus, the sum in λ_i, (6), becomes trivial. we define $\lambda_I = \beta C n_I / N$. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SIkR`.

both asymptomatic and symptomatic individuals whose fractions are, respectively, α_i and $\bar{\alpha}_i = 1 - \alpha_i$. The social contact matrix C_{ij} denotes the average number of contacts made per day by an individual in class i with an individual in class j. Clearly, the total number of contacts between group i to group j must equal the total number of contacts from group j to group i, and thus, $N_i C_{ij} = N_j C_{ji}$.

With these assumptions the progress of the epidemic is governed by the age-structured SIR model. Table I gives the elementary reaction steps, and their rates. The Stochastic Jump Processes well-defined deterministic limit. The deterministic limit of the SIR model is given in terms of the following ODEs:

$$\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i, \\
\dot{I}_i &= \lambda_i(t) S_i - \gamma_I I_i, \\
\dot{R}_i &= \gamma_I I_i.
\end{aligned}$$

The rate of infection of a susceptible individual in age group i is

$$\lambda_i(t) = \beta \sum_{j=1}^{M} \sum_{n=1}^{k} C_{ij} \frac{I_j^n}{N_j}, \tag{4}$$

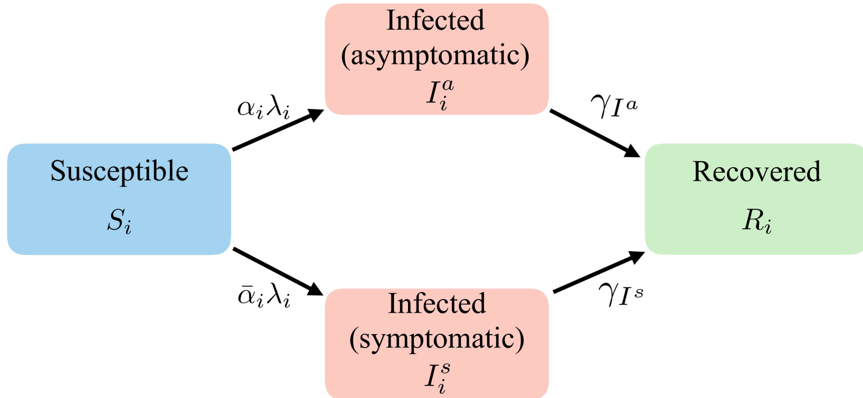
II. SIR WITH STAGES

The I class is the SIR model is now allowed to have k-stages [6]. Table II gives the elementary reaction steps, and their rates. The deterministic limit of the SIkR model, which is of I, is

$$\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i, \\
\dot{I}_i^1 &= \lambda_i(t) S_i - k_I \gamma_I I_i^1, \\
\dot{I}_i^2 &= k \gamma_I I_i^1 - k_I \gamma_I I_i^2, \\
&\vdots \\
\dot{I}_i^k &= k_I \gamma_I I_i^{k-1} - k_I \gamma_I I_i^k, \\
\dot{R}_i &= k_I \gamma_I I_i^k.
\end{aligned} \tag{5}$$

III. SIIR

The population within age group i is partitioned into susceptibles S_i, asymptomatic infectives I_i^a, symptomatic infectives I_i^s and removed individuals R_i. The sum of these is the size of the population in age group i, $N_i =$



Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I^a \to 2I^a$	$(-1, 1, 0, 0)$	$\alpha\lambda_I^a n_S$
$S + I^a \to I^s + I^a$	$(-1, 0, 1, 0)$	$\bar{\alpha}\lambda_I^a n_S$
$S + I^s \to I^a + I^s$	$(-1, 1, 0, 0)$	$\alpha\lambda_I^s n_S$
$S + I^s \to 2I^s$	$(-1, 0, 1, 0)$	$\bar{\alpha}\lambda_I^s n_S$
$I^a \to R$	$(0, -1, 0, 1)$	$\gamma_{I^a} n_{I^a}$
$I^s \to R$	$(0, 0, -1, 1)$	$\gamma_{I^s} n_{I^s}$

Table III. **Schematic of the SIIR model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\alpha_i, \beta, \gamma_{I^a}, \gamma_{I^s})$. The reaction takes the state $\boldsymbol{n} = (n_S, n_{I^a}, n_{I^s}, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. For simplicity we consider $M = 1$, and thus, the sum in λ_i, (6), becomes trivial. we define $\lambda_I^a = \beta C^a n_{I^a}/N$ and $\lambda_I^s = \beta C^s n_{I^s}/N$ such that $\lambda = \lambda_I^a + \lambda_I^s$. This can be easily generalized to the age-structured model. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SIR`.

$S_i + I_i^a + I_i^s + R_i$ [1–4]. We ignore vital dynamics and the change in age structure on the time scale of the epidemic in this model. We assume that the rate of infection of a susceptible individual in age group i is

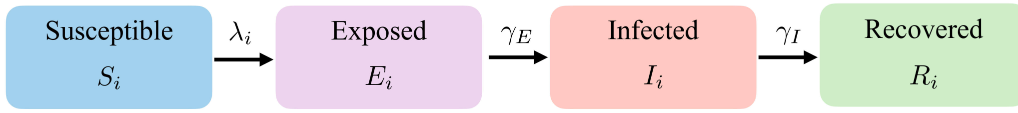
$$\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}^a(t) \frac{I_j^a}{N_j} + C_{ij}^s(t) \frac{I_j^s}{N_j} \right), \quad i, j = 1, \ldots M \tag{6}$$

where β is the probability of infection on contact (assumed intrinsic to the pathogen) and C_{ij}^a and C_{ij}^s are, respectively, the number of contacts between asymptomatic and symptomatic infectives in age-group j with susceptibles in age-group i (reflecting the structure of social contacts). We assume that symptomatic infectives reduce their contacts compared to asymptomatic infectives and set $C_{ij}^s = f^s C_{ij}^a \equiv f^s C_{ij}$, where $0 \leq f^s \leq 1$ is the proportion by which this self-isolation takes place.

With these assumptions the progress of the epidemic is governed by the age-structured SIIR model. Table III gives the elementary reaction steps, and their rates. The deterministic limit is given as,

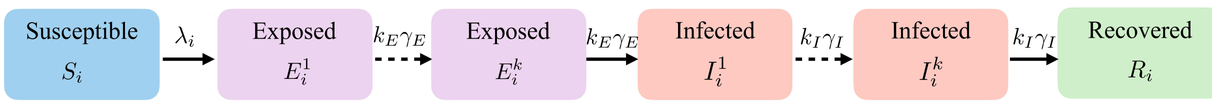
$$\begin{aligned} \dot{S}_i &= -\lambda_i(t) S_i, \\ \dot{I}_i^a &= \alpha_i \lambda_i(t) S_i - \gamma_{I^a} I_i^a, \\ \dot{I}_i^s &= \bar{\alpha}_i \lambda_i(t) S_i - \gamma_{I^s} I_i^s, \\ \dot{R}_i &= \gamma_{I^a} I_i^a + \gamma_{I^s} I_i^s. \end{aligned} \tag{7}$$

Here γ_{I^a} is the recovery rate for asymptomatic infectives, γ_{I^s} is the recovery rate for symptomatic infectives, α_i is the fraction of asymptomatic infectives.



Step	$\boldsymbol{r}_\xi$	w_ξ
$S+I \to E+I$	$(-1,1,0,0)$	$\lambda_I n_S$
$E \to I$	$(0,-1,1,0)$	$\gamma_E n_E$
$I \to R$	$(0,0,-1,1)$	$\gamma_I n_I$

Table IV. **Schematic of the SEIR model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\beta, \gamma_I, \gamma_E)$. The reaction takes the state $\boldsymbol{n} = (n_S, n_E, n_I, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. For simplicity we consider $M = 1$, and thus, the sum in λ_i, (6), becomes trivial. we define $\lambda_I = \beta C n_I/N$. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SEIR`.



Step	$\boldsymbol{r}_\xi$	w_ξ
$S+I^1 \to E^1+I^1$	$(-1,1,0\ldots,0,0,0,\ldots,0,0)$	$\lambda_I n_S$
$E^1 \to E^2$	$(0,-1,1,\ldots,0,0,0,\ldots,0,0)$	$k_E\gamma_E n_E$
$\vdots$	$\vdots$	$\vdots$
$E^{k-1} \to E^k$	$(0,0,0\ldots,-1,1,0,\ldots,0,0)$	$k_E\gamma_E n_E$
$E^k \to I^1$	$(0,0,0\ldots,0,-1,1,\ldots,0,0)$	$k_E\gamma_E n_E$
$\vdots$	$\vdots$	$\vdots$
$I^k \to R$	$(0,0,0\ldots,0,0,0,\ldots,-1,1)$	$k_I\gamma_I n_I$

Table V. **Schematic of the SEIR with stages (SEkIkR) model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (k_I, k_E, \beta, \gamma_I, \gamma_E)$. The reaction takes the state $\boldsymbol{n} = (n_S, n_{E^1}, \ldots, n_{E^{k_E}}, n_{I^1}, \ldots, n_{I^{k_I}}, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. For simplicity we consider $M = 1$, and thus, the sum in λ_i, (6), becomes trivial. we define $\lambda_I = \beta C n_I/N$. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SEkIkR`.

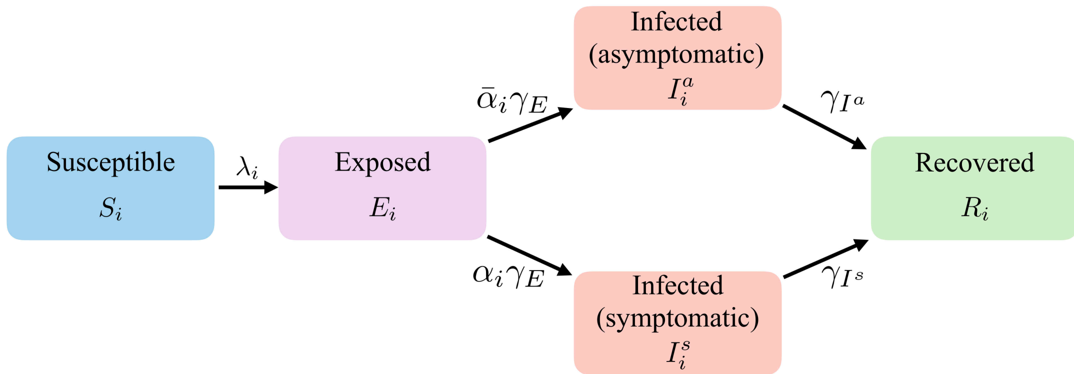
IV. SEIR

Adding a exposed class to the SIR, we obtain the SEIR. Table IV gives the elementary reaction steps, and their rates. The deterministic equation giving its time-evolution is

$$\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i, \\
\dot{E}_i &= \lambda_i(t) S_i - \gamma_E E_i \\
\dot{I}_i &= \gamma_E E_i - \gamma_I I_i, \\
\dot{R}_i &= \gamma_I I_i.
\end{aligned} \tag{8}$$

The rate of infection of a susceptible individual in age group i is

$$\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij} \frac{I_j}{N_j} \right), \tag{9}$$



Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I^a, I^s \to E + I^a, I^s$	$(-1, 1, 0, 0, 0)$	$\lambda_I^a n_S, \lambda_I^s n_S$
$E \to I^a$	$(0, -1, 1, 0, 0)$	$\alpha\gamma_E n_E$
$E \to I^s$	$(0, -1, 0, 1, 0)$	$\bar{\alpha}\gamma_E n_E$
$I^a \to R$	$(0, 0, -1, 0, 1)$	$\gamma_{I^a} n_{I^a}$
$I^s \to R$	$(0, 0, 0, -1, 1)$	$\gamma_{I^s} n_{I^s}$

Table VI. **Schematic of the SEIIR model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\alpha_i, \beta, \gamma_E, \gamma_{I^a}, \gamma_{I^s})$. The reaction takes the state $\boldsymbol{n} = (n_S, n_E, n_{I^a}, n_{I^s}, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. The infection rates λ_I^a and λ_I^s are defined in III. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SIR`.

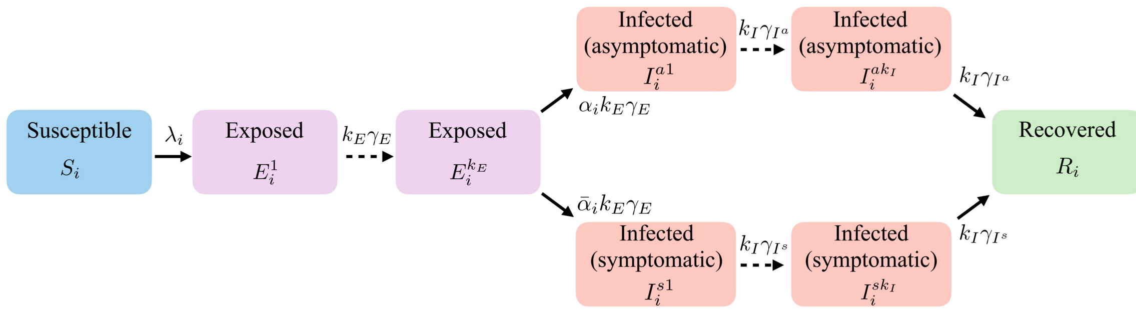
V. SEIR WITH STAGES

The deterministic limit of a SEIR model is now extended to an age-structured $k-$staged SEkIkR model. Table V gives the elementary reaction steps, and their rates. The deterministic limit is:

$$\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i, \\
\dot{E}_i^1 &= \lambda_i(t) S_i - k_E\gamma_E E_i^1 \\
\dot{E}_i^2 &= k_E\gamma_E E_i^1 - k_E\gamma_E E_i^2 \\
&\vdots \\
\dot{E}_i^k &= k_E\gamma_E E_i^{k-1} - k_E\gamma_E E_i^k \\
\dot{I}_i^1 &= k_E\gamma_E E_i^k - k_I\gamma_I I_i^1, \\
\dot{I}_i^2 &= k_I\gamma_I I_i^1 - k_I\gamma_I I_i^2, \\
&\vdots \\
\dot{I}_i^k &= k_I\gamma_I I_i^{(k-1)} - k_I\gamma_I I_i^k, \\
\dot{R}_i &= k_I\gamma_I I_i^k.
\end{aligned} \tag{10}$$

The rate of infection of a susceptible individual in age group i is

$$\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij} \frac{I_j}{N_j} \right), \tag{11}$$



Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I^1 \to E^1 + I^1$	$(-1, 1, 0 \ldots, 0, 0, 0, \ldots, 0, 0, 0, \ldots, 0, 0)$	$\lambda_I n_S$
$E^1 \to E^2$	$(0, -1, 1, \ldots, 0, 0, 0, \ldots, 0, 0, 0, \ldots, 0, 0)$	$k_E \gamma_E n_E$
$\vdots$	$\vdots$	$\vdots$
$E^{k-1} \to E^k$	$(0, 0, 0 \ldots, -1, 1, 0, \ldots, 0, 0, 0, \ldots, 0, 0)$	$k_E \gamma_E n_E$
$E^k \to I^{a1}$	$(0, 0, 0 \ldots, 0, -1, 1, \ldots, 0, 0, 0, \ldots, 0, 0)$	$\alpha k_E \gamma_E n_E$
$\vdots$	$\vdots$	$\vdots$
$E^k \to I^{s1}$	$(0, 0, 0 \ldots, 0, -1, 0, \ldots, 0, 0, 1, \ldots, 0, 0)$	$\bar{\alpha} k_E \gamma_E n_E$
$\vdots$	$\vdots$	$\vdots$

Table VII. **Schematic of the SEIIR with stages (SEkIkIkR) model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (k_I, k_E, \alpha_i, \beta, \gamma_{I^a}, \gamma_{I^s}, \gamma_E)$. The reaction takes the state $\boldsymbol{n} = (n_S, n_{E^1}, \ldots, n_{E^{k_E}}, n_{I^{a1}}, \ldots, n_{I^{ak_I}}, n_{I^{s1}}, \ldots, n_{I^{sk_I}}, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. For simplicity we consider $M = 1$, and thus, the sum in λ_i, (6), becomes trivial. we define $\lambda_I = \beta C n_I / N$. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SEkIkIksR`.

VI. SEIIR

We can add an exposed class, that has caught the infection but is not infectious, to the SIR model to obtain an SEIR model. Table VI gives the elementary reaction steps, and their rates. The deterministic equations are

$$
\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i, \\
\dot{E}_i &= \lambda_i(t) S_i - \gamma_E E_i \\
\dot{I}_i^a &= \alpha_i \gamma_E E_i - \gamma_{I^a} I_i^a, \\
\dot{I}_i^s &= \bar{\alpha}_i \gamma_E E_i - \gamma_{I^a} I_i^s, \\
\dot{R}_i &= \gamma_{I^a} I_i^a + \gamma_{I^s} I_i^s.
\end{aligned}
\tag{12}
$$

The rate of infection of a susceptible individual in age group i is

$$
\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}^a \frac{I_j^a}{N_j} + C_{ij}^s \frac{I_j^s}{N_j} \right), \tag{13}
$$

VII. SEIIR WITH STAGES

We now extend the SEIIR model to have stages in exposed, asymptomatic infectives, and symptomatic infectives classes. Table VII gives the elementary reaction steps, and their rates. The deterministic dynamics is given as

$$
\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i, \\
\dot{E}_i^1 &= \lambda_i(t) S_i - k_E \gamma_E E_i^1 \\
\dot{E}_i^2 &= k_E \gamma_E E_i^1 - k_E \gamma_E E_i^2 \\
&\vdots \\
\dot{E}_i^{k_E} &= k_E \gamma_E E_i^{k_E - 1} - k_E \gamma_E E_i^{k_E} \qquad (14)\\
\dot{I}_i^{a1} &= \alpha_i k_E \gamma_E E_i^k - k_I \gamma_{I^a} I_i^{a1}, \\
\dot{I}_i^{a2} &= k_I \gamma_{I^a} I_i^{a1} - k_I \gamma_{I^a} I_i^{a2}, \\
&\vdots \\
\dot{I}_i^{ak_I} &= k_{I^a} \gamma_{I^a} I_i^{a(k_I - 1)} - k_I \gamma_{I^a} I_i^{ak_I}, \\
\dot{I}_i^{s1} &= \bar{\alpha}_i k_E \gamma_E E_i^{k_E} - k_I \gamma_{I^s} I_i^{a1}, \qquad (15)\\
\dot{I}_i^{s2} &= k_I \gamma_{I^s} I_i^{s1} - k_I \gamma_{I^s} I_i^{s2}, \qquad (16)\\
&\vdots \qquad (17)\\
\dot{I}_i^{sk_I} &= k_I \gamma_{I^s} I_i^{s(k_I - 1)} - k_I \gamma_{I^s} I_i^{sk_I}, \qquad (18)\\
\dot{R}_i &= k_I \gamma_{I^a} I_i^{ak_I} + k_I \gamma_{I^s} I_i^{sk_I}.
\end{aligned}
$$

The rate of infection of a susceptible individual in age group i is

$$
\lambda_i(t) = \beta \sum_{j=1}^{M} \sum_{n=1}^{k_I} \left(C_{ij}^a \frac{I_j^{an}}{N_j} + C_{ij}^s \frac{I_j^{sn}}{N_j} \right), \tag{19}
$$

VIII. SEI5R

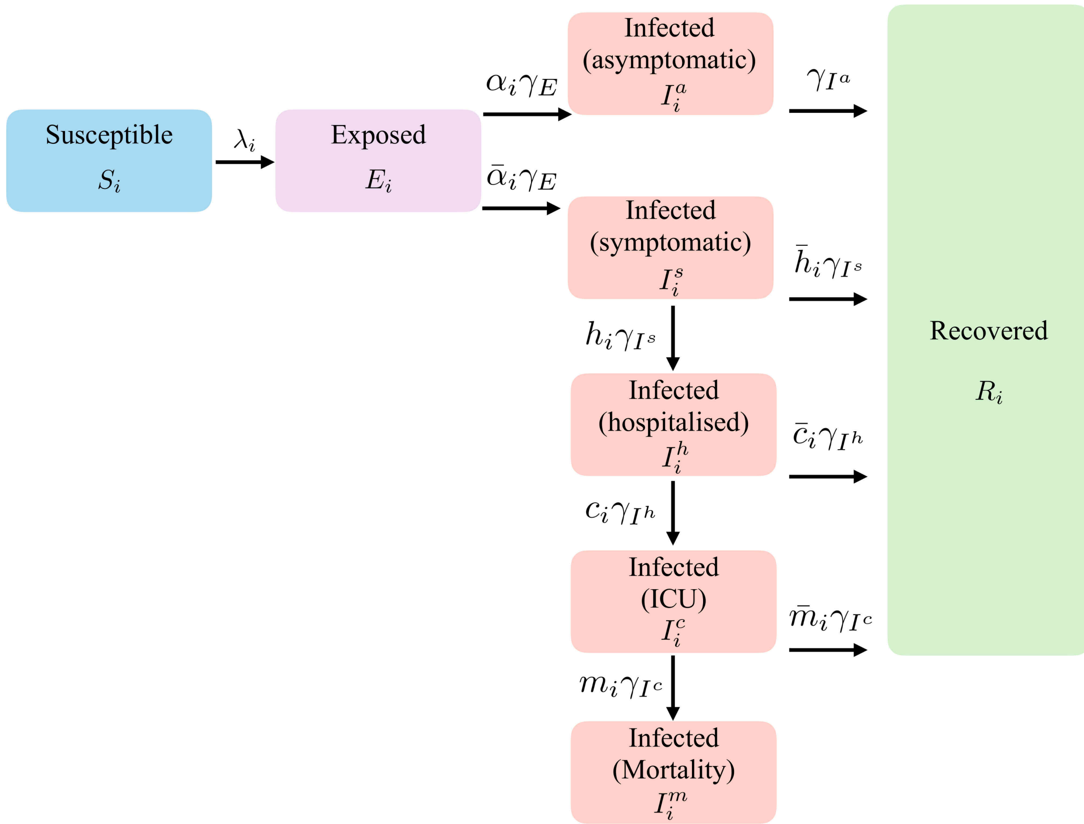
The SEIR model is now extended to include fives types of infectives: I_i^h: infectives who are hospitalized; I_i^c: infectives who are in ICU; and I_i^m: mortality. Table VIII gives the elementary reaction steps, and their rates. The deterministic limit of the dynamics is

$$
\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i + \sigma_i, \\
\dot{E}_i &= \lambda_i(t) S_i - \gamma_E E_i \\
\dot{I}_i^a &= \alpha_i \gamma_E E_i - \gamma_{I^a} I_i^a, \\
\dot{I}_i^s &= \bar{\alpha}_i \gamma_E E_i - \gamma_{I^s} I_i^s, \\
\dot{I}_i^h &= h_i \gamma_{I^s} I_i^s - \gamma_{I^h} I_i^h, \\
\dot{I}_i^c &= c_i \gamma_{I^h} I_i^h - \gamma_{I^c} I_i^c, \\
\dot{I}_i^m &= m_i \gamma_{I^c} I_i^c, \\
\dot{R}_i &= \gamma_{I^a} I_i^a + \bar{h}_i \gamma_{I^s} I_i^s + \bar{c}_i \gamma_{I^h} I_i^h + \bar{m}_i \gamma_{I^c} I_i^c. \\
\dot{N}_i &= \sigma_i - m_i \gamma_{I^c} I_i^c
\end{aligned} \tag{20}
$$

The rate of infection of a susceptible individual in age group i is

$$
\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}^a \frac{I_j^a}{N_j} + C_{ij}^s \frac{I_j^s}{N_j} + C_{ij}^h \frac{I_j^h}{N_j} \right), \tag{21}
$$

Here $\bar{h}_i = 1 - h_i$, $\bar{m}_i = 1 - m_i$, $C_{ij}^s = f^s C_{ij}^a \equiv f^s C_{ij}$ and $C_{ij}^s = f^h C_{ij}^a \equiv f^h C_{ij}$. I^c is the number of ICU cases and I^m is the mortality due to the infection.

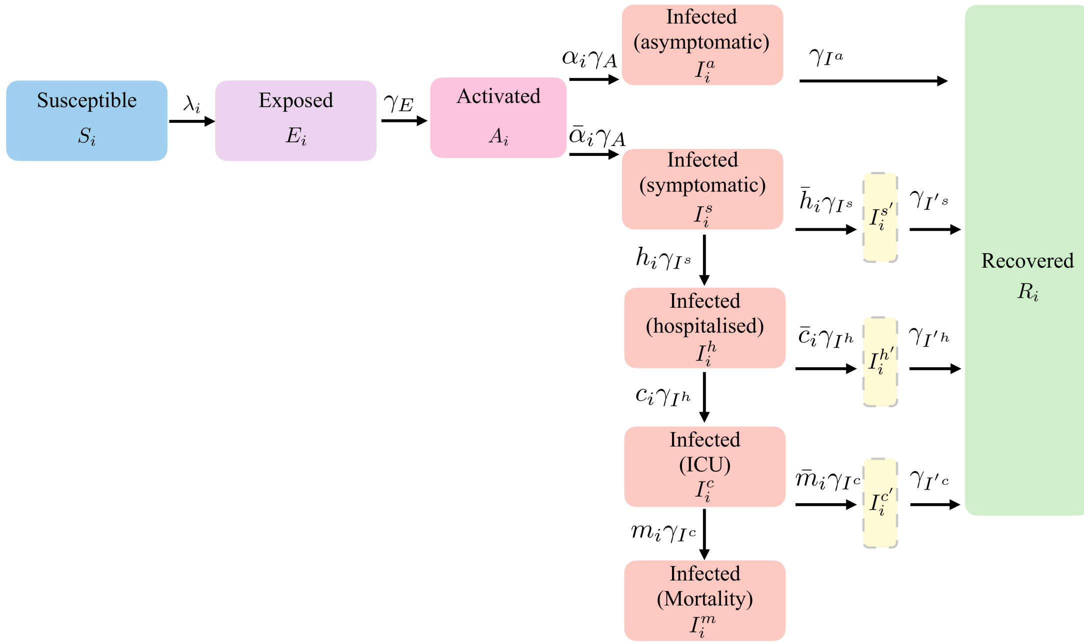


Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I^a, I^s \to E + I^a, I^s$	$(-1, 1, 0, 0, 0, 0, 0, 0)$	$\lambda_I^a n_S, \lambda_I^s n_S$
$E \to I^a$	$(0, -1, 1, 0, 0, 0, 0, 0)$	$\alpha\gamma_E n_E$
$I^a \to R$	$(0, 0, -1, 0, 0, 0, 0, 1)$	$\gamma_{I^a} n_{I^a}$
$E \to I^s$	$(0, -1, 0, 1, 0, 0, 0, 0)$	$\bar{\alpha}\gamma_E n_E$
$I^s \to R$	$(0, 0, 0, -1, 0, 0, 0, 1)$	$\bar{h}\gamma_{I^s} n_{I^s}$
$I^s \to I^h$	$(0, 0, 0, -1, 1, 0, 0, 0)$	$h\gamma_{I^s} n_{I^s}$
$I^h \to R$	$(0, 0, 0, 0, -1, 0, 0, 1)$	$\bar{c}\gamma_{I^h} n_{I^s}$
$I^h \to I^c$	$(0, 0, 0, 0, -1, 1, 0, 0)$	$c\gamma_{I^h} n_{I^s}$
$I^c \to R$	$(0, 0, 0, 0, 0, -1, 0, 1)$	$\bar{m}\gamma_{I^c} n_{I^s}$
$I^c \to I^m$	$(0, 0, 0, 0, 0, -1, 1, 1)$	$m\gamma_{I^c} n_{I^s}$

Table VIII. **Schematic of the SEI5R model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\alpha_i, \beta, \gamma_E, \gamma_{I^a}, \gamma_{I^s}, \gamma_{I^h}, \gamma_{I^c}, h_i, c_i, m_i)$. The reaction takes the state $\boldsymbol{n} = (n_S, n_E, n_{I^a}, n_{I^s}, n_{I^h}, n_{I^c}, n_{I^m}, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. The infection rates λ_I^a and λ_I^s are defined in III . The class SIR can be instantiated in PyRoss using `pyross.deterministic.SEI5R`.

IX. SEI8R

The SEIR model is now extended to include eight types of infectives: I_i^h: infectives who are hospitalized; $I_i^{h'}$, I_i^c: infectives who are in ICU; and $I_i^{c'}$, I_i^m: mortality. Table IX gives the elementary reaction steps, and their rates. The deterministic limit of this case is:



Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I^a, I^s \to E + I^a, I^s$	$(-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$\lambda_I^a n_S, \lambda_I^s n_S$
$E \to I^a$	$(0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$	$\alpha\gamma_A n_E$
$I^a \to R$	$(0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 1)$	$\gamma_{I^a} n_{I^a}$
$A \to I^s$	$(0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0)$	$\bar{\alpha}\gamma_A n_E$
$I^s \to I^{s'}$	$(0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0)$	$\bar{h}\gamma_{I^s} n_{I^s}$
$I^{s'} \to R$	$(0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 1)$	$\gamma_{I^{s'}} n_{I^{s'}}$
$I^s \to I^h$	$(0, 0, 0, 0, -1, 0, 1, 0, 0, 0, 0)$	$h\gamma_{I^s} n_{I^s}$
$I^h \to I^{h'}$	$(0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0)$	$\bar{c}\gamma_{I^h} n_{I^s}$
$I^{h'} \to R$	$(0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 1)$	$\gamma_{I^{h'}} n_{I^{h'}}$
$I^h \to I^c$	$(0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0)$	$c\gamma_{I^h} n_{I^s}$
$I^c \to I^{c'}$	$(0, 0, 0, 0, 0, 0, -1, 0, 1, 0, 0)$	$\gamma_{I^{c'}} n_{I^c}$
$I^{c'} \to R$	$(0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 1)$	$\gamma_{I^{c'}} n_{I^{c'}}$
$I^c \to I^m$	$(0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 1)$	$m\gamma_{I^c} n_{I^s}$

Table IX. **Schematic of the SEI8R model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\alpha_i, \beta, \gamma_E, \gamma_{I^a}, \gamma_{I^s}, \gamma_{I^h}, \gamma_{I^c}, \gamma_{I_{s'}}, \gamma_{I^{h'}}, \gamma_{I_{c'}}, h_i, c_i, m_i)$. Elementary reaction steps and their rates for the SEAI8R model for $M = 1$. The reaction takes the state $\boldsymbol{n} = (n_S, n_E, n_{I^a}, n_{I^s}, n_{I_{s'}}, n_{I^h}, n_{I_{h'}}, n_{I^c}, n_{I_{c'}}, n_{I^m}, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. The infection rates λ_I^a and λ_I^s are defined in . The class SIR can be instantiated in PyRoss using `pyross.deterministic.SEI8R`.

$$
\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i + \sigma_i, \\
\dot{E}_i &= \lambda_i(t) S_i - \gamma_E E_i \\
\dot{I}_i^a &= \alpha_i \gamma_E E_i - \gamma_{I^a} I_i^a, && (22) \\
\dot{I}_i^s &= \bar{\alpha}_i \gamma_E E_i - \gamma_{I^s} I_i^s, \\
\dot{I}_i^{s'} &= \bar{h}_i \gamma_{I^s} I_i^s - \gamma_{I^{s'}} I_i^{s'} && (23) \\
\dot{I}_i^h &= h_i \gamma_{I^s} I_i^s - \gamma_{I^h} I_i^h, \\
\dot{I}_i^{h'} &= \bar{c}_i \gamma_{I^h} I_i^h - \gamma_{I^{h'}} I_i^{h'} && (24) \\
\dot{I}_i^c &= c_i \gamma_{I^h} I_i^h - \gamma_{I^c} I_i^c, \\
\dot{I}_i^{c'} &= \bar{m}_i \gamma_{I^c} I_i^c - \gamma_{I^{c'}} I_i^{c'} && (25) \\
\dot{I}_i^m &= m_i \gamma_{I^c} I_i^c, \\
\dot{R}_i &= \gamma_{I^a} I_i^a + \gamma_{I^{s'}} I_i^{s'} + \gamma_{I^{h'}} I_i^{h'} + \gamma_{I^{c'}} I_i^{c'}. \\
\dot{N}_i &= \sigma_i - m_i \gamma_{I^c} I_i^c
\end{aligned}
$$

The rate of infection of a susceptible individual in age group i is

$$
\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}^a \frac{I_j^a}{N_j} + C_{ij}^s \frac{I_j^s}{N_j} + C_{ij}^h \frac{I_j^h}{N_j} \right), \tag{26}
$$

Here $\bar{h}_i = 1 - h_i$, $\bar{m}_i = 1 - m_i$, $C_{ij}^s = f^s C_{ij}^a \equiv f^s C_{ij}$ and $C_{ij}^s = f^h C_{ij}^a \equiv f^h C_{ij}$. I^c is the number of ICU cases and I^m is the mortality due to the infection.

X. SEAIIR

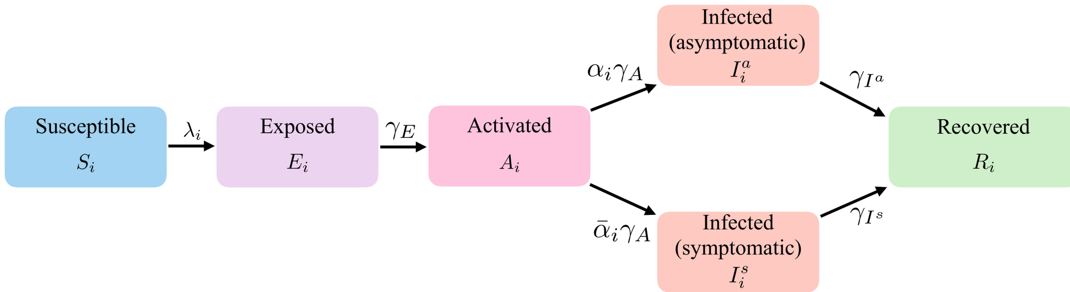
This model is an extension of the SEIR model, introducing the additional class A, which is both asymptomatic and infectious. In other words, this models shows what ensues if *everyone* who gets infected, undergoes a latency period where they are both asymptomatic and infectious. This class is potentially quite important, as there is some evidence that people are infectious before they start showing symptoms. Table X gives the elementary reaction steps, and their rates. The deterministic limit of this case

$$
\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i \\
\dot{E}_i &= \lambda_i(t) S_i - \gamma_E E_i \\
\dot{A}_i &= \gamma_E E_i - \gamma_A A_i \\
\dot{I}_i^a &= \alpha_i \gamma_A A_i - \gamma_{I^a} I_i^a \\
\dot{I}_i^s &= \bar{\alpha}_i \gamma_A A_i - \gamma_{I^s} I_i^s \\
\dot{R}_i &= \gamma_{I^a} I_i^a + \gamma_{I^s} I_i^s
\end{aligned} \tag{27}
$$

The rate of infection of a susceptible individual in age group i is

$$
\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}^a \frac{I_j^a}{N_j} + C_{ij}^a \frac{A_j}{N_j} + C_{ij}^s \frac{I_j^s}{N_j} \right), \tag{28}
$$

The A and I^a classes should behave virtually the same (so their contact matrices should be equal). The two are kept distinct to keep track of the fact that some people remain asymptomatic even in the I stage. Since it's difficult to find data on the ratio of I^s to I^a, it is possible to disregard the distinction and simply use I instead.



Step	$\boldsymbol{r}_\xi$	w_ξ
$S + A, I^a, I^s \to E + A, I^a, I^s$	$(-1, 1, 0, 0, 0)$	$\lambda_A^a n_S, \lambda_I^a n_S, \lambda_I^s n_S$
$E \to A$	$(0, -1, 1, 0, 0)$	$\gamma_E n_E$
$A \to I^a$	$(0, 0, -1, 1, 0)$	$\alpha\gamma_A n_A$
$A \to I^s$	$(0, 0, -1, 0, 1)$	$\bar{\alpha}\gamma_A n_A$
$I^a \to R$	$(0, 0, 0, -1, 0)$	$\gamma_{I^a} n_{I^a}$
$I^s \to R$	$(0, 0, 0, 0, -1)$	$\gamma_{I^s} n_{I^s}$

Table X. **Schematic of the SEAIIR model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\alpha_i, \beta, \gamma_E, \gamma_A, \gamma_{I^a}, \gamma_{I^s})$. Elementary reaction steps and their rates for the SEAIR model. The reaction takes the state $\boldsymbol{n} = (n_S, n_E, n_A, n_{I^a}, n_{I^s})$ to the state $\boldsymbol{n} + \boldsymbol{r}_\alpha$ for $M = 1$. In addition to the infection rates λ_I^a and λ_I^s, we define $\lambda_A^a = \beta C^a n_A/N$. Thus, for this simplified model we can write for the total rate of infection, $\lambda = \lambda_A^a + \lambda_I^a + \lambda_I^s$. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SEAIR`.

XI. SEAI5R

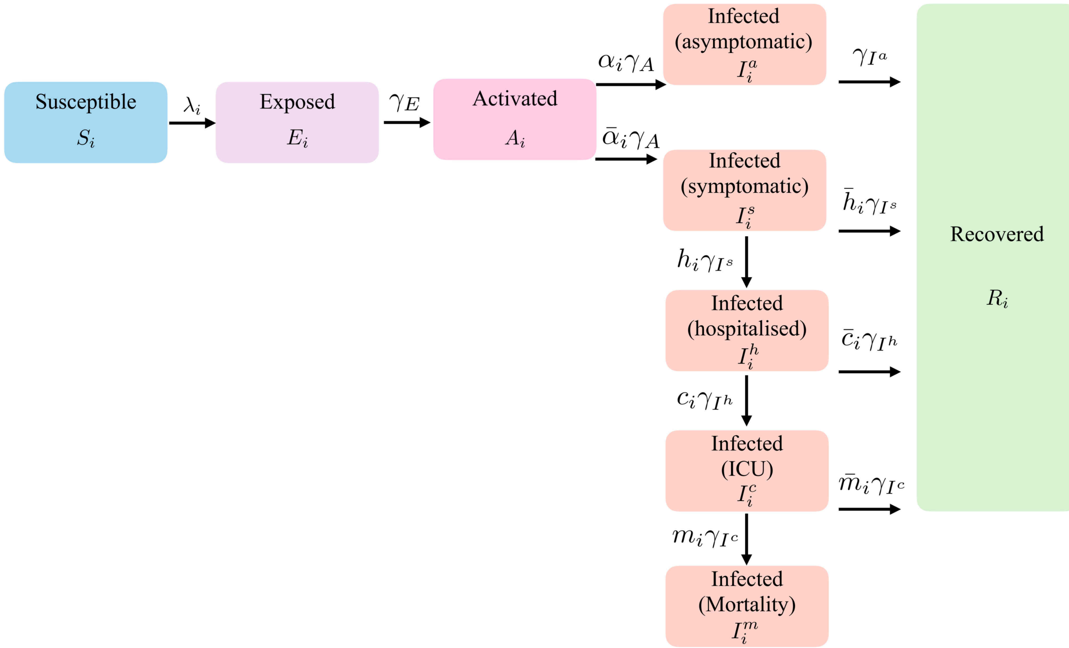
We now extend SEIR model to have fives types of infectives (I_i^h: infectives who are hospitalized, I_i^c: infectives who are in ICU, and I_i^m: mortality). Table XI gives the elementary reaction steps, and their rates. The deterministic limit is:

$$
\begin{aligned}
\dot{S}_i &= -\lambda_i(t) S_i + \sigma_i, \\
\dot{E}_i &= \lambda_i(t) S_i - \gamma_E E_i \\
\dot{A}_i &= \gamma_E E_i - \gamma_A A_i \quad (29)\\
\dot{I}_i^a &= \alpha_i \gamma_A A_i - \gamma_{I^a} I_i^a, \quad (30)\\
\dot{I}_i^s &= \bar{\alpha}_i \gamma_A A_i - \gamma_{I^s} I_i^s, \\
\dot{I}_i^h &= h_i \gamma_{I^s} I_i^s - \gamma_{I^h} I_i^h, \\
\dot{I}_i^c &= c_i \gamma_{I^h} I_i^h - \gamma_{I^c} I_i^c, \\
\dot{I}_i^m &= m_i \gamma_{I^c} I_i^c, \\
\dot{R}_i &= \gamma_{I^a} I_i^a + \bar{h}_i \gamma_{I^s} I_i^s + \bar{c}_i \gamma_{I^h} I_i^h + \bar{m}_i \gamma_{I^c} I_i^c. \\
\dot{N}_i &= \sigma_i - m_i \gamma_{I^c} I_i^m
\end{aligned}
$$

The rate of infection of a susceptible individual in age group i is

$$
\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}^a \frac{I_j^a}{N_j} + C_{ij}^a \frac{A_j}{N_j} + C_{ij}^s \frac{I_j^s}{N_j} + C_{ij}^h \frac{I_j^h}{N_j} \right), \tag{31}
$$

Here $\bar{h}_i = 1 - h_i$, $\bar{m}_i = 1 - m_i$, $C_{ij}^s = f^s C_{ij}^a \equiv f^s C_{ij}$ and $C_{ij}^s = f^h C_{ij}^a \equiv f^h C_{ij}$. I^c is the number of ICU cases and I^m is the mortality.

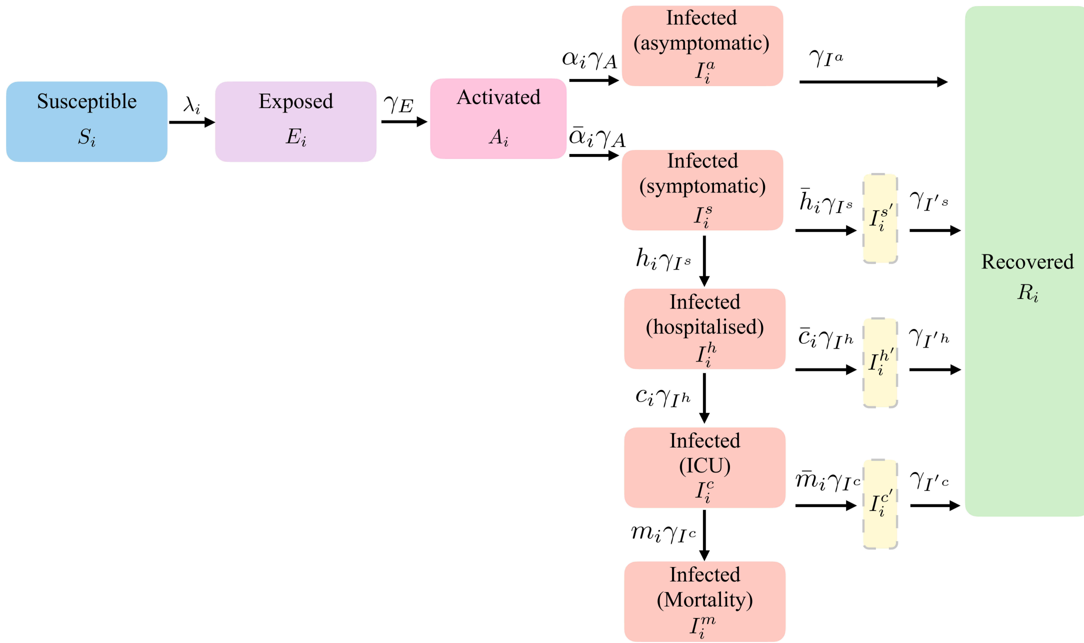


Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I^a, I^s \to E + I^a, I^s$	$(-1, 1, 0, 0, 0, 0, 0, 0, 0)$	$\lambda_I^a n_S, \lambda_I^s n_S$
$E \to A$	$(0, -1, 1, 0, 0, 0, 0, 0)$	$\gamma_E n_E$
$A \to I^a$	$(0, 0, -1, 1, 0, 0, 0, 0, 0)$	$\alpha\gamma_A n_E$
$I^a \to R$	$(0, 0, 0, -1, 0, 0, 0, 0, 1)$	$\gamma_{I^a} n_{I^a}$
$A \to I^s$	$(0, 0, -1, 0, 1, 0, 0, 0, 0)$	$\bar{\alpha}\gamma_A n_E$
$I^s \to R$	$(0, 0, 0, 0, -1, 0, 0, 0, 1)$	$\bar{h}\gamma_{I^s} n_{I^s}$
$I^s \to I^h$	$(0, 0, 0, 0, -1, 1, 0, 0, 0)$	$h\gamma_{I^s} n_{I^s}$
$I^h \to R$	$(0, 0, 0, 0, 0, -1, 0, 0, 1)$	$\bar{c}\gamma_{I^h} n_{I^s}$
$I^h \to I^c$	$(0, 0, 0, 0, 0, -1, 1, 0, 0)$	$c\gamma_{I^h} n_{I^s}$
$I^c \to R$	$(0, 0, 0, 0, 0, 0, -1, 0, 1)$	$m\gamma_{I^c} n_{I^s}$
$I^c \to I^m$	$(0, 0, 0, 0, 0, 0, -1, 1, 1)$	$m\gamma_{I^c} n_{I^s}$

Table XI. **Schematic of the SEAI5R model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\alpha_i, \beta, \gamma_E, \gamma_A, \gamma_{I^a}, \gamma_{I^s}, \gamma_{I^h}, \gamma_{I^c}, h_i, c_i, m_i)$. Elementary reaction steps and their rates for the SEAI5R model for $M = 1$. The reaction takes the state $\boldsymbol{n} = (n_S, n_E, n_A, n_{I^a}, n_{I^s}, n_{I^h}, n_{I^c}, n_{I^m}, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. The infection rates λ_I^a and λ_I^s are defined in . The class SIR can be instantiated in PyRoss using `pyross.deterministic.SEAI5R`.

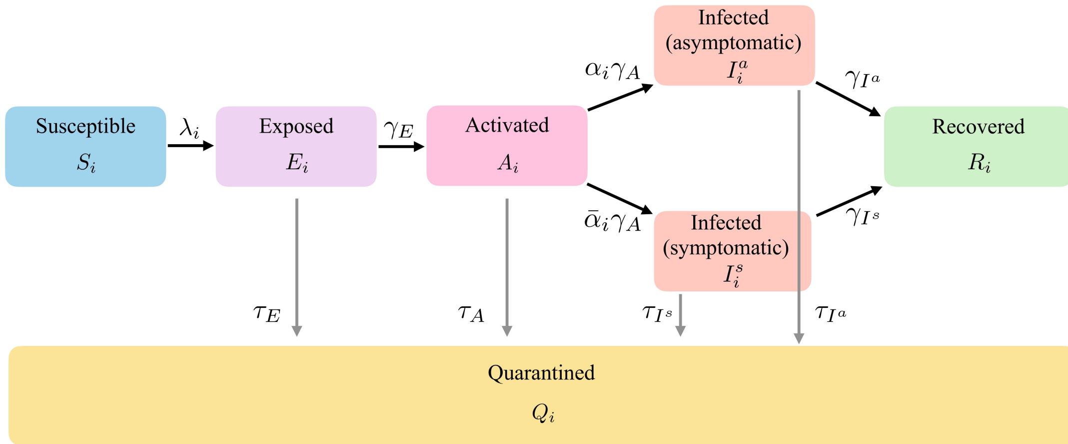
XII. SEAI8R

We now extend SEIR model to have eight types of infectives (I_i^h: infectives who are hospitalized, I_i^c: infectives who are in ICU, and I_i^m: mortality). Table XII gives the elementary reaction steps, and their rates. The deterministic limit is:



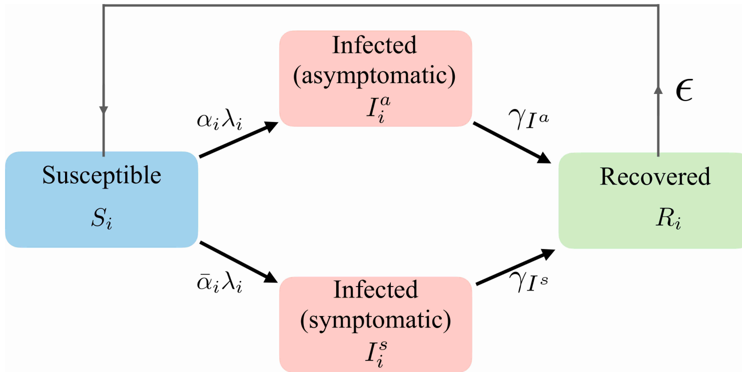
Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I^a, I^s \to E + I^a, I^s$	$(-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$\lambda_I^a n_S, \lambda_I^s n_S$
$E \to A$	$(0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$\gamma_E n_E$
$A \to I^a$	$(0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$	$\alpha\gamma_A n_E$
$I^a \to R$	$(0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 1)$	$\gamma_{I^a} n_{I^a}$
$A \to I^s$	$(0, 0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0)$	$\bar{\alpha}\gamma_A n_E$
$I^s \to I^{s'}$	$(0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0)$	$\bar{h}\gamma_{I^s} n_{I^s}$
$I^{s'} \to R$	$(0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 1)$	$\gamma_{I^{s'}} n_{I^{s'}}$
$I^s \to I^h$	$(0, 0, 0, 0, 0, -1, 0, 1, 0, 0, 0, 0)$	$h\gamma_{I^s} n_{I^s}$
$I^h \to I^{h'}$	$(0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0)$	$\bar{c}\gamma_{I^h} n_{I^s}$
$I^{h'} \to R$	$(0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 1)$	$\gamma_{I^{h'}} n_{I^{h'}}$
$I^h \to I^c$	$(0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0)$	$c\gamma_{I^h} n_{I^s}$
$I^c \to I^{c'}$	$(0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 0, 0)$	$\gamma_{I^{c'}} n_{I^c}$
$I^{c'} \to R$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 1)$	$\gamma_{I^{c'}} n_{I^{c'}}$
$I^c \to I^m$	$(0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 1)$	$m\gamma_{I^c} n_{I^s}$

Table XII. **Schematic of the SEAI8R model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\alpha_i, \beta, \gamma_E, \gamma_A, \gamma_{I^a}, \gamma_{I^s}, \gamma_{I^h}, \gamma_{I^c}, \gamma_{I^{s'}}, \gamma_{I^{h'}}, \gamma_{I^{c'}}, h_i, c_i, m_i)$. Elementary reaction steps and their rates for the SEAI8R model for $M = 1$. The reaction takes the state $\boldsymbol{n} = (n_S, n_E, n_A, n_{I^a}, n_{I^s}, n_{I^{s'}}, n_{I^h}, n_{I^{h'}}, n_{I^c}, n_{I^{c'}}, n_{I^m}, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. The infection rates λ_I^a and λ_I^s are defined in . The class SIR can be instantiated in PyRoss using `pyross.deterministic.SEAI8R`.



Step	$\boldsymbol{r}_\xi$	w_ξ
$S + A, I^a, I^s \to E + A, I^a, I^s$	$(-1, 1, 0, 0, 0, 0, 0)$	$\lambda_A^a n_S, \lambda_I^a n_S, \lambda_I^s n_S$
$S \to Q$	$(-1, 0, 0, 0, 0, 1, 0)$	$\tau_S n_S$
$E \to A$	$(0, -1, 1, 0, 0, 0, 0)$	$\gamma_E n_E$
$E \to Q$	$(0, -1, 0, 0, 0, 1, 0)$	$\tau_E n_E$
$A \to I^a$	$(0, 0, -1, 1, 0, 0, 0)$	$\alpha\gamma_A n_A$
$A \to I^s$	$(0, 0, -1, 0, 1, 0, 0)$	$\bar{\alpha}\gamma_A n_A$
$A \to Q$	$(0, 0, -1, 0, 0, 1, 0)$	$\tau_A n_A$
$I^a \to Q$	$(0, 0, 0, -1, 0, 1, 0)$	$\tau_{I^a} n_{I^a}$
$I^a \to R$	$(0, 0, 0, -1, 0, 0, 1)$	$\gamma_{I^a} n_{I^a}$
$I^s \to Q$	$(0, 0, 0, 0, -1, 1, 0)$	$\tau_{I^s} n_{I^s}$
$I^s \to R$	$(0, 0, 0, 0, -1, 0, 1)$	$\gamma_{I^s} n_{I^s}$

Table XIII. **Schematic of the SEAIRQ model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\alpha_i, \beta, \gamma_E, \gamma_A, \gamma_{I^a}, \gamma_{I^s}, \tau_E, \tau_A, \tau_{I^a}, \tau_{I^s})$. Elementary reaction steps and their rates for the SEAIRQ model. The reaction takes the state $\boldsymbol{n} = (n_S, n_E, n_A, n_{I^a}, n_{I^s}, n_Q, n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$ for $M = 1$. In addition to the infection rates λ_I^a and λ_I^s, see Table III on page 3, we define $\lambda_A^a = \beta C^a n_A/N$. Thus, for this simplified model we can write for the total rate of infection, $\lambda = \lambda_A^a + \lambda_I^a + \lambda_I^s$. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SEAIRQ`.



Step	$\boldsymbol{r}_\xi$	w_ξ
$S + I^a \to 2I^a$	$(-1, 1, 0, 0)$	$\alpha\lambda_I^a n_S$
$S + I^a \to I^s + I^a$	$(-1, 0, 1, 0)$	$\bar{\alpha}\lambda_I^a n_S$
$S + I^s \to I^a + I^s$	$(-1, 1, 0, 0)$	$\alpha\lambda_I^s n_S$
$S + I^s \to 2I^s$	$(-1, 0, 1, 0)$	$\bar{\alpha}\lambda_I^s n_S$
$I^a \to R$	$(0, -1, 0, 1)$	$\gamma_{I^a} n_{I^a}$
$I^s \to R$	$(0, 0, -1, 1)$	$\gamma_{I^s} n_{I^s}$
$R \to S$	$(1, 0, 0, -1)$	ϵn_R

Table XIV. **Schematic of the SIIRS model and elementary reaction steps, and their rates.** The parameters for this model are: $\boldsymbol{\theta} = (\alpha_i, \beta, \gamma_{I^a}, \gamma_{I^s}, \epsilon)$. Elementary reaction steps and their rates for the non-age-structred SIRS model. The reaction takes the state $\boldsymbol{n} = (n_S, n_{I^a}, n_{I^s} n_R)$ to the state $\boldsymbol{n} + \boldsymbol{r}_\xi$. For this simplified model the sum in λ_i, (3), becomes trivial. Thus, let's define $\lambda_I^a = \beta C^a n_{I^a}/N$ and $\lambda_I^s = \beta C^s n_{I^s}/N$ such that $\lambda = \lambda_I^a + \lambda_I^s$. This can be easily generalized to the age-structured model. The class SIR can be instantiated in PyRoss using `pyross.deterministic.SIRS`.

$$\dot{S}_i = -\lambda_i(t)S_i + \sigma_i,$$

$$\dot{E}_i = \lambda_i(t)S_i - \gamma_E E_i$$

$$\dot{A}_i = \gamma_E E_i - \gamma_A A_i \tag{32}$$

$$\dot{I}_i^a = \alpha_i \gamma_A A_i - \gamma_{I^a} I_i^a, \tag{33}$$

$$\dot{I}_i^s = \bar{\alpha}_i \gamma_A A_i - \gamma_{I^s} I_i^s,$$

$$\dot{I}_i^{s'} = \bar{h}_i \gamma_{I^s} I_i^s - \gamma_{I^{s'}} I_i^{s'} \tag{34}$$

$$\dot{I}_i^h = h_i \gamma_{I^s} I_i^s - \gamma_{I^h} I_i^h,$$

$$\dot{I}_i^{h'} = \bar{c}_i \gamma_{I^h} I_i^h - \gamma_{I^{h'}} I_i^{h'} \tag{35}$$

$$\dot{I}_i^c = c_i \gamma_{I^h} I_i^h - \gamma_{I^c} I_i^c,$$

$$\dot{I}_i^{c'} = \bar{m}_i \gamma_{I^c} I_i^c - \gamma_{I^{c'}} I_i^{c'} \tag{36}$$

$$\dot{I}_i^m = m_i \gamma_{I^c} I_i^c,$$

$$\dot{R}_i = \gamma_{I^a} I_i^a + \gamma_{I^{s'}} I_i^{s'} + \gamma_{I^{h'}} I_i^{h'} + \gamma_{I^{c'}} I_i^{c'}.$$

$$\dot{N}_i = \sigma_i - m_i \gamma_{I^c} I_i^c$$

The rate of infection of a susceptible individual in age group i is

$$\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}^a \frac{I_j^a}{N_j} + C_{ij}^a \frac{A_j}{N_j} + C_{ij}^s \frac{I_j^s}{N_j} + C_{ij}^h \frac{I_j^h}{N_j} \right), \tag{37}$$

Here $\bar{h}_i = 1 - h_i$, $\bar{m}_i = 1 - m_i$, $C_{ij}^s = f^s C_{ij}^a \equiv f^s C_{ij}$ and $C_{ij}^s = f^h C_{ij}^a \equiv f^h C_{ij}$. I^c is the number of ICU cases and

I^m is the mortality due to the infection.

XIII. SEAIIRQ

We now introduce the Q-class, which represents people who have been tested and put into quarantine (and can therefore not infect anyone else). This point of Q-class is to model proper contact tracing. Table XIII gives the elementary reaction steps, and their rates. The deterministic dynamics of the SEAIRQ model is given as:

$$\begin{aligned}
\dot{S}_i &= -\lambda_i(t)S_i \\
\dot{E}_i &= \lambda_i(t)S_i - (\gamma_E + \tau_E)E_i \\
\dot{A}_i &= \gamma_E E_i - (\gamma_A + \tau_A)A_i \\
\dot{I}_i^a &= \alpha_i \gamma_A A_i - (\gamma_{I^a} + \tau_{I^a})I_i^a \\
\dot{I}_i^s &= \bar{\alpha}_i \gamma_A A_i - (\gamma_{I^s} + \tau_{I^s})I_i^s \\
\dot{R}_i &= \gamma_{I^a} I_i^a + \gamma_{I^s} I_i^s \\
\dot{Q}_i &= \tau_S S_i + \tau_E E_i + \tau_A A_i + \tau_{I^s} I_i^s + \tau_{I^a} I_i^a
\end{aligned} \tag{38}$$

The rate of infection of a susceptible individual in age group i is

$$\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}^a \frac{I_j^a}{N_j} + C_{ij}^a \frac{A_j}{N_j} + C_{ij}^s \frac{I_j^s}{N_j} \right), \tag{39}$$

Here τ_{E,A,I^s,I^a} is the testing rate in the population, these are in general different for different classes. We have presumed that people in the incubation stage E can also be tested. The τ_S terms model the effects of false-positives, resulting in susceptibles being put into quarantine. Note that this model does not keep track of what happens to people once they're put into Q (which is especially important to do if $\tau_S > 0$). Since Q is a closed system, this can all be done after the initial SEAIR simulation has been completed.

XIV. SIIRS

We now extend the age-structured SIR model to allow for recovered persons to be susceptible and for change in the population of each age group. Table XIV gives the elementary reaction steps, and their rates. The deterministic dynamics of the resulting SIRS model is:

$$\begin{aligned}
\dot{S}_i &= -\lambda_i(t)S_i + \sigma_i + \epsilon(\gamma_{I^a} I_i^a + \gamma_{I^s} I_i^s) \\
\dot{I}_i^a &= \alpha_i \lambda_i(t)S_i - \gamma_{I^a} I_i^a + l_i \\
\dot{I}_i^s &= \bar{\alpha}_i \lambda_i(t)S_i - \gamma_{I^a} I_i^s \\
\dot{R}_i &= \gamma_{I^a} I_i^a + \gamma_{I^s} I_i^s. \\
\dot{N}_i &= \sigma_i + l_i
\end{aligned} \tag{40}$$

Here ϵ is fraction of recovered who is susceptible. σ_i denotes of the arrival of new susceptibles, while l_i are new asymptomatic infectives. This means that N_i is now dynamical. The rate of infection of a susceptible individual in age group i is

$$\lambda_i(t) = \beta \sum_{j=1}^{M} \left(C_{ij}^a \frac{I_j^a}{N_j} + C_{ij}^s \frac{I_j^s}{N_j} \right), \tag{41}$$

```
model_spec = {
    "classes" : ["S", "I", "R"],


  "S" : {
    "linear"    : [],
    "infection" : [ ["I", "-beta"] ]
  },

  "I" : {
    "linear"    : [ ["I", "-gamma"] ],
    "infection" : [ ["I", "beta"] ]
  },

  "R" : {
    "linear"    : [ ["I", "gamma"] ],
    "infection" : []
  }
}
```

Table XV. **Instantiation of the Spp classs** using `pyross.deterministic.Spp`.

XV. EVEN MORE ?

If the plethora of models described in the preceding sections are not enough, then PyRoss provides the additional class `pyross.deterministic.Spp` (pronounced *"S plus plus"*), which has the ability to simulate any generic compartmental model. The model is specified by providing a Python dictionary, and supports age-differentiated parameters. For example the SIR model would be:

Currently, the *Spp* class supports the two types of terms which all the compartmental models above share: linear terms and infection terms. Future incarnations of the class could be extended to simulate any generic model of the form

$$\sum_{i=1}^{M\times L} f_i^{\xi} Y_i \to \sum_{i=1}^{M\times L} g_i^{\xi} Y_i$$

where the rates could be both time and state dependent.

Note that `pyross.deterministic.Spp` is designed with generality rather than optimality in mind. A model implemented using `pyross.deterministic.Spp` will in general perform worse than any of the corresponding hard-coded classes above.

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[5] Rajesh Singh and R Adhikari, "Age-structured impact of social distancing on the covid-19 epidemic in india," arXiv preprint arXiv:2003.12055 (2020).

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