ON THE ZERO FORCING NUMBER OF CORONA AND LEXICOGRAPHIC PRODUCT OF GRAPHS

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ABSTRACT. The zero forcing number of a graph G, denoted by Z(G), is the minimum cardinality of a set S of black vertices (where vertices in $V(G) \setminus S$ are colored white) such that V(G) is turned black after finitely many applications of "the color change rule": a white vertex is turned black if it is the only white neighbor of a black vertex. In this paper, we study the zero forcing number of corona product, $G \odot H$ and lexicographic product, $G \circ H$ of two graphs G and G. It is shown that if G and G are connected graphs of order G and G and G are connected graphs of order G and G and G are connected graphs of order G and G are connected graphs of order G and G are connected graph G of order G and an arbitrary graph G containing G are connected graph G of order G and an arbitrary graph G containing G are connected graph G of order G and an arbitrary graph G containing G are connected graph G of order G and G are G and an arbitrary graph G containing G and G are connected graph G of order G and G are G and G and G are G and G ar

1. Introduction

Let G = (V(G), E(G)) be a simple, undirected, connected graph with $|V(G)| \geq 2$. The number of vertices and edges of G are called the order and the size of G respectively. The degree of a vertex $v \in V$, denoted by $deg_G(v)$, is the number of edges incident to the vertex v in G. If there is no ambiguity, we will use the notation deg(v) instead of $deg_G(v)$. An end vertex is a vertex of degree one. Given $u, v \in V$, $u \sim v$ means that u and v are adjacent vertices and $u \nsim v$ means that u and v are not adjacent. We define the open neighborhood of a vertex v in G, $N_G(v) = \{u \in V(G) : u \sim v\}$ and the closed neighborhood of v, $N_G[v] = N_G(v) \cup \{v\}$. If there is no ambiguity, we will simply write N(v) or N[v]. If $u \in N_G(v)$ then u is said to be a neighbor of v. We denote a path, cycle, complete graph and empty graph on v vertices by v and v are not adjacent.

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trivial unless otherwise stated.

The notion of a zero forcing set, as well as the associated zero forcing number, of a simple graph was introduced in [1] to bound the minimum rank of associated matrices for numerous families of graphs. Let each vertex of a graph G be given one of two colors, "black" and "white" by convention. Let S denote the initial set of black vertices of G. The color-change rule converts the color of a vertex u_2 from white to black if the white vertex u_2 is the only white neighbor of a black vertex u_1 ; we say that u_1 forces u_2 , which we denote by $u_1 \to u_2$. And a sequence, $u_1 \to u_2 \to \cdots \to u_i \to u_{i+1} \to \cdots \to u_t$, obtained through iterative applications of the color-change rule is called a forcing chain. The set S is said to be a zero forcing set of G if all the vertices of G will be turned black after finitely many applications of the color-change rule. The zero forcing number of G, denoted by Z(G), is the minimum of |S| over all zero forcing sets $S \subseteq V(G)$. A zero forcing set of cardinality Z(G) is called a forcing basis for G. For surveys on the zero forcing parameter, see [9, 10]. For more on the zero forcing parameter in graphs, see [2, 3, 5, 6, 7, 12].

If F is a field, $M_n(F)$ denotes the set of all $n \times n$ matrices over F. An nsquare matrix A is said to be a symmetric matrix if $A^T = A$. The set of all real symmetric n-square matrices is denoted by S_n . To a given graph G with vertex set $\{1, 2, \dots, n\}$, we associate a class of real, symmetric matrices as follows:

$$S(G) = \{A = [a_{ij}] | A \in S_n, \text{ for } i \neq j, a_{ij} \neq 0 \Leftrightarrow ij \in E(G)\}.$$

Note that there is no restriction on the value of a_{ii} with $i = 1, 2, \dots, n$ and the adjacency matrix A(G) belongs to S(G), where the adjacency matrix of a graph G is a square (0,1)-matrix of size n, whose (i,j)-th entry is 1 if and only if v_i is adjacent to v_i , since there are no loops in the graph, the diagonal entries of the adjacency matrix are zero. On the other hand, the graph of an n-square symmetric matrix A, denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, 2, \cdots, n\}$ and the edge set

$$\{ij|a_{ij}\neq 0, 1\leq i\neq j\leq n\}.$$

The minimum rank of G is defined to be

$$mr(G) = min\{rank(A)|A \in S(G)\},\$$

while the maximum nullity of G is defined as

$$M(G) = \max\{null(A)|A \in S(G)\}.$$

We have

$$mr(G) + M(G) = |V(G)|.$$

The underlying idea for the zero forcing set of a graph is that a black vertex is associated with a coordinate in a vector that is required to be zero, while a white vertex indicates a coordinate that can be either zero or nonzero. Changing a vertex from white to black is essentially noting that the corresponding coordinate is forced to be zero if the vector is in the kernel of a matrix in S(G) and all black vertices indicate coordinates assumed to be or previously forced to be 0. Hence the use of the term "zero forcing set", see [1].

The support of a vector $x = (x_i)$, denoted by Supp(x), is the set of indices i such that $x_i \neq 0$. Let Z be a zero forcing set of G and $A \in S(G)$. If $x \in null(A)$ and $Supp(x) \cap Z = \phi$, then x = 0, stated in [1, 14]. Also from [1, 14], we have $M(G) \leq Z(G)$ for a graph G.

In this paper, we consider corona product and lexicographic product of graphs in the context of zero forcing number. This paper consists of three sections. Section 1 includes introduction. Sections 2 and 3 include several results related to the zero forcing number of corona and lexicographic product of graphs, respectively.

2. Corona Product of Graphs

Let G and H be two graphs of order n_1 and n_2 respectively. The corona product of G and H is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the i^{th} -copy of H with the i^{th} -vertex of G. We will denote by $V = \{v_1, v_2, \cdots, v_{n_1}\}$, the set of vertices of G and by $H_i = (V_i, E_i)$, the i-th copy of H, where $V_i = \{u_1^i, u_2^i, \cdots, u_{n_2}^i\}$, such that $v_i \sim u_k^i$ for every $u_k^i \in V_i$. Note that the subgraph of $G \odot H$ induced by V_i is H_i and the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. For any integer $k \geq 2$, we define the graph $G \odot^k H$ recursively from $G \odot H$ as $G \odot^k H = (G \odot^{k-1} H) \odot H$. It is also noted that $|G \odot^{k-1} H| = n_1(n_2 + 1)^{k-1}$ and $|G \odot^k H| = |G \odot^{k-1} H| + n_1n_2(n_2 + 1)^{k-1}$.

We call the copies of H in $G \odot H$ as the copies of H in 1st-corona, the newly added copies of H in $G \odot H$ to obtain $G \odot^2 H$ as the copies of H in 2nd-corona and generally the newly added copies of H in $G \odot^{k-1} H$ to obtain $G \odot^k H$ as the copies of H in k^{th} -corona.

In $G \odot^k H$ for any positive integer k, we name the vertices in $G \odot^{k-1} H$ as the root vertices of the copies of H in k^{th} -corona, that are joined to these vertices in $G \odot^k H$.

As one can color the vertices of $G \odot^k H$ in more than one ways, but in this paper for a disconnected graph H(containing isolated vertices) of order at least two, we will consider the zero forcing set of $G \odot^k H$, that contains only the vertices of H but not the vertices of G.

In Figure 1, the graph with grey vertices is $G = P_2$, the copies of H with black vertices are the copies of H in first corona and with white vertices are the copies of H in 2nd corona. The black and grey vertices are the root vertices of the corresponding copies of H with white vertices.

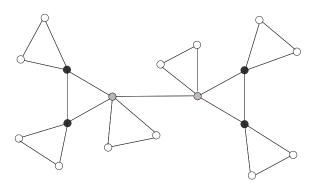


FIGURE 1. $P_2 \odot^2 P_2$

We first recall the useful result obtained in [8].

Proposition 2.1. [8] Let G be a connected graph of order $n \geq 2$. Then

- (a) Z(G) = 1 if and only if $G = P_n$,
- (b) Z(G) = n 1 if and only if $G = K_n$.

Note that for a connected graph G of order n, we have

$$1 \le Z(G) \le n - 1. \tag{1}$$

Lemma 2.2. Let G be a connected graph of order $n \geq 2$ and let H be a graph of order at least two. Let H_i be the subgraph of $G \odot H$ corresponding to the i^{th} -copy of H.

- (i) If S is a zero forcing set of $G \odot H$, then $V_i \cap S \neq \phi$ for every $i \in \{1, ..., n\}$.
- (ii) If H is a connected graph and S is a zero forcing set of $G \odot H$, then for every $i \in \{1, ..., n\}$, $S \cap V_i$ is a zero forcing set of H_i .
- Proof. (i) Suppose $V_i \cap S = \phi$, for some i. Then the vertex $v_i \in V$, initially black or forced to black in the zero forcing process, will not force any vertex in V_i to turn black because it has more than one white neighbors, a contradiction. (ii) Suppose contrary that $S \cap V_i$ is not a zero forcing set for H_i . Then there exists a black vertex say $u_i^i \in V_i$, that has more than one white neighbors in graph H_i , so no forcing situation can occur. Note that the vertex $u_i^i \in V_i$ also has more than one white neighbors in $G \odot H$, a contradiction.

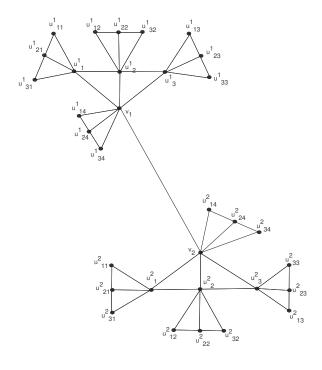


FIGURE 2. Graph $P_2 \odot^2 P_3$

Here we introduce some terminology related to the following theorem. We assume that $B' = \{v_1, v_2, ..., v_t\}$ is a forcing basis for G and by using B' one can color all the other vertices of G by a sequence of forces in the following order: $v_{t+1}, v_{t+2}, ..., v_{n_1}$, with appropriate indexing of vertices. We denote the vertices of i-th copies of H in l-th corona $(1 \le l \le k)$ by $u^i_{j_1j_2...j_l}$, where $1 \le j_1 \le n_2$ and $1 \le j_p \le n_2 + 1$ for each $2 \le p \le l$. Assume that Z(H) = m and a forcing basis for i-th-copy H_i of H in 1st-corona is denoted by $B^i = \{u^i_1, u^i_2, ..., u^i_m\}$ and forcing basis for i-th copies of H in l-th-corona $(2 \le l \le k)$ by $B^i_{j_2...j_l} = \{u^i_{1j_2j_3...j_l}, u^i_{2j_2j_3...j_l}, ..., u^i_{mj_2j_3...j_l}\}$, where $1 \le j_2, ..., j_l \le n_2 + 1$. We denote the collection of forcing basis of all copies of H in first corona by B_1 i.e $\bigcup_{i=1}^{n_1} B^i = B_1$, similarly the collection of forcing basis of all copies of H in l-th corona by B_l i.e $\bigcup_{i=1}^{n_2} (\bigcup_{j_2=1}^{n_2+1} B^i_{j_2...j_l}) = B_l$ and $|B_l| = n_1(n_2+1)^{l-1}Z(H)$. FIGURE 2 helps in understanding the indices as mentioned.

Theorem 2.3. Let G and H be connected graphs of order n_1 and n_2 respectively, then

$$Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1} Z(H).$$

Proof. We prove the result by mathematical induction. For k=1, we have to show that

$$Z(G \odot H) = Z(G) + n_1 Z(H). \tag{2}$$

First, we show that

$$Z(G \odot H) \le Z(G) + n_1 Z(H). \tag{3}$$

We define $B = B' \cup B_1$ and $|B| = Z(G) + n_1 Z(H)$. We claim that B is a zero forcing set of $G \odot H$. To prove the claim, first assume that B is initially colored black and we color all the vertices of H_i with $1 \le i \le t$ which are associated with the vertices of B' using the corresponding sets B^i . Now all the vertices in H_i associated with v_i , $1 \le i \le t$ are colored black. Note that there is a vertex v_i belonging to B' that has only one white neighbor v_{t+1} . Thus $v_i \to v_{t+1}$. Then we color all the vertices in H_{t+1} by using the black vertices in B^{t+1} . Continuing this process, we can color all the vertices of $G \odot H$.

Note that the degree of each vertex of G is increased by n_2 and the degree of each vertex of H is increased by 1 in $G \odot H$. Let $v_i \in B'$ and consider the corresponding copy H_i of H. Note that at least Z(H) + 1 vertices are required as initially colored black to start the zero forcing process in each of these H_i 's, $1 \le i \le t$. Then $v_t \to v_{t+1}$ and to continue the process at least Z(H) more vertices are required in H_{t+1} as initially colored black. Continue the process until all the vertices are turned black. Hence,

$$Z(G \odot H) \ge Z(G) + n_1 Z(H). \tag{4}$$

By (3) and (4), (2) holds.

Suppose that the result is true for k-1, i.e

$$Z(G \odot^{k-1} H) = Z(G \odot^{k-2} H) + n_1(n_2 + 1)^{k-2} Z(H).$$
(5)

Now we have to show that the result is true for k, i.e

$$Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1} Z(H).$$
(6)

We define $B^{\wp} = B' \cup B_1 \cup B_2 \cup \cdots \cup B_{(k-1)}$ and $|B^{\wp}| = Z(G) + n_1 Z(H) + n_1(n_2+1)Z(H) + \cdots + n_1(n_2+1)^{k-2}Z(H) = \alpha$. Then B^{\wp} is a zero forcing set of $G \odot^{k-1} H$ by (5). Therefore, we color all the vertices of $G \odot^{k-1} H$ using B^{\wp} . Now all the vertices from 1st-corona to $(k-1)^{th}$ -corona in $G \odot^k H$ are colored black. It suffices to show that $n_1 n_2 (n_2+1)^{k-1}$ vertices in the copies of H in k^{th} -corona in $G \odot^k H$ will be colored black by taking $n_1(n_2+1)^{k-1}Z(H)$ more vertices as initially colored black.

First, we show that

$$Z(G \odot^k H) \le Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1} Z(H). \tag{7}$$

We define $B = B^{\wp} \cup B_k$ where $|B| = \alpha + n_1(n_2 + 1)^{k-1}Z(H)$. We claim that B is a zero forcing set of $G \odot^k H$. Let B is initially colored black. Note that the degree of each vertex in H_i in k^{th} -corona in $G \odot^k H$ is increased by one. We color all the vertices of copies of H in k^{th} -corona by using $B^i_{j_2\cdots j_k}$ and the corresponding root vertex in $(k-1)^{th}$ -corona. We obtain the derived set of all black vertices in $G \odot^k H$ resulting from repeatedly applying the color-change rule. Hence, (7) holds.

Note that in each copy of H in k^{th} -corona at least Z(H) vertices are required as initially colored black to continue the zero forcing process. Hence,

$$Z(G \odot^k H) \ge Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1} Z(H).$$
(8)

By (7) and (8), (6) holds. Hence, the result is true for any positive integer k.

By using Theorem 2.3 and Proposition 2.1, we have the following immediate corollaries:

Corollary 2.4. Let G and H be connected graphs of order n_1 , $n_2 \geq 2$ respectively. Then $Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}$ if and only if $H \cong P_{n_2}$.

Corollary 2.5. Let G and H be connected graphs of order $n_1, n_2 \geq 2$ respectively. Then $Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}(n_2 - 1)$ if and only if $H \cong K_{n_2}$.

The wheel graph of order n+1 is defined as $W_{1,n}=K_1\odot C_n$, where K_1 is the singleton graph. Any three pairwise adjacent vertices of a wheel form a zero forcing set of $W_{1,n}$.

Remark 2.6. Let $W_{1,n}$, $n \geq 3$, be a wheel graph. Then $Z(W_{1,n}) = 3$.

The fan graph F_{n_1,n_2} is defined as the join graph $\overline{K_{n_1}} + P_{n_2}$. The case $n_1 = 1$ corresponds to the usual fan graph F_{1,n_2} . Note that $F_{1,n_2} = K_1 \odot P_{n_2}$, where K_1 is the singleton graph. Two adjacent vertices of P_{n_2} where one must be an end vertex form a zero forcing set of F_{1,n_2} .

Remark 2.7. Let $F_{1,n}$, $n \geq 2$, be a fan graph. Then $Z(F_{1,n}) = 2$.

Theorem 2.8. Let G be a connected graph of order $n_1 \geq 2$ and let H be a graph of order $n_2 \geq 2$. Then

$$Z(G \odot^k H) \le n_1(n_2+1)^{k-1} Z(K_1 \odot H).$$

Proof. We denote by $K_1 \odot H_i$ the subgraph of $G \odot H$, obtained by joining the vertex $v_i \in V$ with all the vertices of H_i . Let A_i be a forcing basis for $K_1 \odot H_i$ and $A = \bigcup_{i=1}^{n_1} A_i$ with $|A| = n_1 Z(K_1 \odot H)$. We show that A is a zero forcing set of $G \odot H$. Now there are two cases:

Case 1: H contains no isolated vertex. We have that $v_i \in A_i$. So A contains all the vertices of G. Note that any black vertex in H_i has only one white neighbor, so after finite many applications of the color-change rule all the vertices in H_i for each $i = 1, 2, \dots, n_1$ are turned black. Now all the vertices in $G \odot H$ are colored black.

Case 2: H contains isolated vertices, so there exists at least one vertex $x \in H$ such that $x \nsim u$, for all $u \in H$. We have that v_i does not belong to the zero forcing set of minimum cardinality of $K_1 \odot H_i$. So A does not contain any vertex from G.

Subcase 2.1: H has only one isolated vertex. Let x_i is the isolated vertex of H_i also $x_i \in A_i$ and $x_i \nsim u_p^i$ for any $u_p^i \in V_i$. Thus $x_i \to v_i$, $1 \le i \le n_1$. Now all the vertices of G are forced to black. Note that any black vertex in H_i has only one white neighbor, so after finite many applications of the color-change rule all the vertices in H_i for each $i = 1, 2, \dots, n_1$ are turned black. Now all the vertices in $G \odot H$ are colored black.

Subcase 2.2: H has more than one isolated vertices, then all isolated vertices in H_i belong to A_i except one, say y_i does not belong to A_i , then v_i will be forced by any isolated black vertex in H_i , $1 \le i \le n_1$. Now all the vertices of G are forced to black and after finite iterative applications of the color-change rule for connected subgraph of H_i all the vertices in these graphs are turned black for each $i = 1, 2, \dots, n_1$, and then $v_i \to y_i$. Hence $Z(G \odot H) \le n_1 Z(K_1 \odot H)$. Therefore, the result follows.

Corollary 2.9. Let G be a connected graph of order $n_1 \geq 2$ and H be a disconnected graph of order $n_2 \geq 2$. Then

$$Z(G \odot^k H) = n_1(n_2+1)^{k-1} Z(K_1 \odot H) = n_1(n_2+1)^{k-1} (n_2-1)$$

if and only if $H \cong \overline{K_{n_2}}$.

Proof. Suppose $H \cong \overline{K_{n_2}}$. For k=1, we have to show that $Z(G \odot H) = n_1(n_2-1)$. We define $B_i = V_i - \{u_l^i\}$, for any $1 \leq l \leq n_2$, and for each $i=1,2,\cdots,n_1$ and $B=\cup_{i=1}^{n_1}B_i$. We claim that B is a zero forcing set of $G \odot H$ with $|B|=n_1(n_2-1)$. To prove the claim, we first assume that B is initially colored black. Note that every initial black vertex of H_i has single white neighbor v_i , so $u_1^i \to v_i$. Now there is only one white vertex u_l^i in each

 V_i and this vertex is the single white neighbor of the corresponding vertex v_i , so $v_i \to u_i^i$ for each $i = 1, 2, \dots, n_1$.

Consider $v_i \in V$ and the corresponding copy H_i of H for any $i, 1 \leq i \leq n_1$. Note that at least $n_2 - 1$ vertices of H_i are required to start the zero forcing process. Hence, $Z(G \odot H) \geq n_1(n_2 - 1)$. Therefore, the result follows.

On the other hand, $Z(G \odot^k H) = n_1(n_2 - 1)$ implies $H \cong \overline{K_{n_2}}$. Suppose $H \ncong \overline{K_{n_2}}$ and let $u^i_{(n_2-2)} \sim u^i_{n_2}$, for $1 \le i \le n_1$. We define $B_i = \{u^i_1, u^i_2, \cdots, u^i_{(n_2-1)}\}$, $1 \le i \le n_1 - 1$ and $B_{n_1} = \{u^{n_1}_1, u^{n_1}_2, \cdots, u^{n_1}_{(n_2-2)}\}$. Let $B = \bigcup_{i=1}^{n_1-1} B_i \cup B_{n_1}$ with $|B| = n_1(n_2 - 1) - 1$. We show that B is a zero forcing set of $G \odot H$. Assume that B is initially colored black. Note that each isolated vertex in each H_i , $1 \le i \le n_1$, has only one white neighbor $v_i \in V$, so v_i will be forced to black for each $i = 1, 2, \cdots, n_1$. Now all the vertices of G are colored black. Since $u^i_{(n_2-2)}$ has only one white neighbor $u^i_{n_2}$ for each $i = 1, 2, \cdots, n_1$, so $u^i_{(n_2-2)} \to u^i_{n_2}$, $1 \le i \le n_1$. Note that v_{n_1} has only one white neighbor $u^{n_1}_{(n_2-1)}$, so $v_{n_1} \to u^{n_1}_{(n_2-1)}$. Now all the vertices are colored black. So B is a zero forcing set of cardinality $n_1(n_2-1)-1$, a contradiction. Therefore, the result follows.

The following definitions are introduced in [5]. Fix a graph T. A vertex of degree at least three is called a major vertex. An end vertex u is called a terminal vertex of a major vertex v if d(u,v) < d(u,w) for every other major vertex w. The terminal degree of a major vertex v in T, denoted by $ter_T(v)$, is the number of terminal vertices of v. A major vertex v is an exterior major vertex (emv) if it has positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of all major vertices of G and let ex(G) denote the number of emvs of G. We further define an exterior degree two vertex to be a vertex of degree two that lies on a path from a terminal vertex to its major vertex and an interior degree two vertex to be a vertex of degree two such that the shortest path to any terminal vertex includes a major vertex.

Theorem 2.10. [4, 11, 13] If T is a tree that is not a path, then $dim(T) = \sigma(T) - ex(T)$.

Theorem 2.11. [8] For any tree T, we have Z(T) = dim(T) iff T has no interior degree two vertices and each major vertex v of T satisfies $ter_T(v) \ge 2$.

Theorem 2.12. Let T be a tree of order $n \geq 3$, that has no interior degree two vertices and each major vertex v of T satisfies $ter_T(v) \geq 2$, then

$$Z(T \odot^k K_1) = \begin{cases} \sigma(T), & k = 1, \\ 2^{k-2}n, & k \ge 2. \end{cases}$$

Proof. Since $T \odot^k K_1$ is a tree, with no interior degree two vertex and each major vertex v satisfies $ter_T(v) \geq 2$. Now for k = 1, $\sigma(T \odot K_1) = n$ and $ex(T \odot K_1) = n - \sigma(T)$, $dim(T \odot K_1) = \sigma(T) = Z(T \odot K_1)$ by Theorem 2.10 and Theorem 2.11 we obtain the result. Since we have $\sigma(T \odot^2 K_1) = 2n$, $ex(T \odot^2 K_1) = n$, so we obtain the result for k = 2 by Theorem 2.10 and Theorem 2.11, $dim(T \odot^2 K_1) = n = Z(T \odot^2 K_1)$.

Let α be the number of connected components of a graph H. Let us denote the connected components of H by C_l , where $1 \leq l \leq \alpha$.

Theorem 2.13. Let G be a connected graph of order n_1 and H be a graph of order n_2 . Let α be the number of connected components of H of order greater than one and let β be the number of isolated vertices of H. Then

than one and let
$$\beta$$
 be the number of isolated vertices of H . Then
$$Z(G \odot^k H) \leq \begin{cases} n_1(n_2+1)^{k-1} \sum_{l=1}^{\alpha} Z(C_l) + n_1(n_2+1)^{k-1}(\beta-1), & \alpha \geq 1, \beta \geq 2, \\ Z(G \odot^{k-1} H) + n_1(n_2+1)^{k-1} \sum_{l=1}^{\alpha} Z(C_l), & \alpha \geq 1, \beta = 0, \\ n_1(n_2+1)^{k-1} \sum_{l=1}^{\alpha} Z(C_l) + n_1(n_2+1)^{k-1} - 1, & \alpha \geq 1, \beta = 1, \\ n_1(n_2+1)^{k-1}(n_2-1), & \alpha = 0, \beta \geq 2. \end{cases}$$

Proof. We define K_l^i , $1 \le l \le \alpha$, be a forcing basis for connected component C_l^i of H_i , $1 \le i \le n_1$.

We suppose $\alpha \geq 1, \beta \geq 2$. We define P_i to be the set of vertices of $G \odot H$ formed by all but one of the isolated vertices of H_i , $1 \leq i \leq n_1$. Let us show that $B = \bigcup_{i=1}^{n_1} (\bigcup_{l=1}^{\alpha} K_l^i \cup P_i)$ is a zero forcing set of $G \odot H$ with $|B| = n_1 \sum_{l=1}^{\alpha} Z(C_l) + n_1(\beta - 1)$. Let B is initially colored black. Note that $p_1^i \in P_i$ has only one white neighbor $v_i \in V$, so $p_1^i \to v_i$ for each $i, 1 \leq i \leq n_1$. Now all the vertices of G are colored black. We color all the vertices of connected components C_l^i , $1 \leq l \leq \alpha$, of H_i using K_l^i and the corresponding vertex $v_i \in V$, $1 \leq i \leq n_1$. Note that the vertex v_i has only one white neighbor p_{β}^i , the isolated vertex of H_i not belonging to P_i , so $v_i \to p_{\beta}^i$, $1 \leq i \leq n_1$, and we have the derived set of all black vertices in $G \odot H$. As a consequence, $Z(G \odot H) \leq n_1 \sum_{l=1}^{\alpha} Z(C_l) + n_1(\beta - 1)$. Therefore, the result follows.

Now suppose $\alpha \geq 1$, and $\beta = 0$. Let $B = B' \cup_{i=1}^{n_1} (\cup_{l=1}^{\alpha} K_l^i)$, where $B' = \{v_1, v_2, \dots, v_t\}$ is a forcing basis for G and by using B' one can color all the other vertices of G by a sequence of forces in the following order: $v_{t+1}, v_{t+2}, \dots, v_n$, with appropriate indexing of vertices. We show that B is a zero forcing set of $G \odot H$. Consider iterative applications of the color-change

rule with initial black set B. We color all the vertices of H_i with $1 \leq i \leq t$ which are associated with the vertices of B' using the corresponding sets K_l^i , $1 \leq l \leq \alpha$. Now all the vertices in H_i associated with v_i , $1 \leq i \leq t$ are colored black. Note that there is a vertex v_i belonging to B' that has only one white neighbor $v_{t+1} \in V$. Thus $v_i \to v_{t+1}$. Then we color all the vertices in H_{t+1} using the black vertices in K_l^{t+1} , $1 \leq l \leq \alpha$. Continuing this process, we can color all the vertices of $G \odot H$. So $Z(G \odot H) \leq Z(G) + n_1 \sum_{l=1}^{\alpha} Z(C_l)$. Therefore, the result follows.

Now suppose $\alpha \geq 1$, $\beta = 1$. Let R be the set of all isolated vertices in each H_i , $1 \leq i \leq n_1$, except one say r_{n_1} . We define $B = \bigcup_{i=1}^{n_1} (\bigcup_{l=1}^{\alpha} K_l^i) \cup R$. We show that B is a zero forcing set of $G \odot H$. Let B is initially colored black. Note that the vertex $r_i \in R$, $1 \leq i \leq n_1 - 1$, has only one white neighbor $v_i \in V$, so $r_i \to v_i$, $1 \leq i \leq n_1 - 1$. We color all the vertices of H_i , $1 \leq i \leq n_1 - 1$, which are associated with v_i , $1 \leq i \leq n_1 - 1$ using the corresponding sets K_l^i , $1 \leq l \leq \alpha$. Now all the vertices in H_i associated with v_i , $1 \leq i \leq n_1 - 1$ are colored black. Now v_{n_1-1} has only one white neighbor v_{n_1} , so $v_{n_1-1} \to v_{n_1}$. We color $C_l^{n_1}$ in H_{n_1} using $K_l^{n_1}$, $1 \leq l \leq \alpha$. Now v_{n_1} has only one white neighbor v_{n_1} , so $v_{n_1} \to v_{n_1}$ and we have the derived set of all black vertices. So $Z(G \odot H) \leq n_1 \sum_{l=1}^{\alpha} Z(C_l) + n_1 - 1$. Therefore, the result follows.

Now suppose $\alpha = 0, \beta \geq 2$. Here H is an empty graph so the result followed by Corollary 2.9.

3. Lexicographic Product of Graphs

Let G and H be two graphs. The lexicographic product of G and H, denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H) = \{(a,v) \mid a \in V(G) \text{ and } v \in V(H)\}$, where (a,v) is adjacent to (b,w) whenever $ab \in E(G)$ or a=b and $vw \in E(H)$. For any vertex $a \in V(G)$ and $b \in V(H)$, we define the vertex set $H(a) = \{(a,v) \in V(G \circ H) \mid v \in V(H)\}$ and $G(b) = \{(v,b) \in V(G \circ H) \mid v \in V(G)\}$. It is clear that the graph induced by H(a), called a layer H(a), is isomorphic to H and the graph induced by H(a), called a layer H(a), is isomorphic to H(a) and the graph induced by H(a) and H(a) and

Let G be a connected graph and H be a non-trivial graph containing $k \ge 1$ components H_1, H_2, \dots, H_k with $|V(H_j)| \ge 2$ for each $j = 1, 2, \dots, k$. For any vertex $a \in V(G)$ and $1 \le i \le k$, we define the vertex set $H_i(a) = \{(a, v) \in A\}$

 $V(G \circ H) | v \in V(H_i) \}$. Let $|V(H_i)| = m_i$, $1 \le i \le k$. From the definition of $G \circ H$, it is clear that for every $(a, v) \in V(G \circ H)$, $deg_{G \circ H}(a, v) = deg_G(a) \cdot |V(H)| + deg_H(v)$. If G is a disconnected graph having $k \ge 2$ components G_1 , G_2, \ldots, G_k , then $G \circ H$ is also a disconnected graph having k components such that $G \circ H = G_1 \circ H \cup G_2 \circ H \cup \ldots \cup G_k \circ H$ and each component $G_i \circ H$ is the lexicographic product of connected component G_i of G with G0 with G1 therefore throughout this section, we will assume G2 to be connected.

Observation 3.1. For any $a, b \in V(G)$, either $H(a) \sim H(b)$ or $H(a) \not\sim H(b)$ in $G \circ H$.

First, we give a general lower bound on the zero forcing number of lexicographic product of graphs. Note that given any connected graph G, then Z(G) = 1 if and only if $G \cong P_n$, $n \geq 2$. So, if $Z(G \circ H) = 1$ for some graph H, then clearly $G \circ H$ is a path graph, i.e G is the trivial graph K_1 and H is a path or viceversa. So, we have the following result:

Remark 3.2. If G and H are non trivial graphs, then $Z(G \circ H) \geq 2$.

Lemma 3.3. Let G be a connected graph on n vertices. There exists a forcing basis S for $G + K_1$ such that $S \subseteq V(G)$.

Proof. Let $V(G + K_1) = V(G) \cup \{v\}$. If $v \notin S$ we have nothing to prove. Suppose that $v \in S$. Since G is connected and $deg_{G+K_1}(v) = n$ and also v is initially colored black so by equation (1), there exists at least one white vertex $x \in N_{G+K_1}(v)$ such that $(S \setminus \{v\}) \cup \{x\}$ is a forcing basis for $G + K_1$.

Theorem 3.4. Let G be a connected graph and H be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, H_3, \ldots H_k$ and $m_i \geq 2$. Let Z be a zero forcing set of $G \circ H$. For any vertex $a \in V(G)$, if $Z_i(a) = Z \cap H_i(a)$ for every $i \in \{1, 2, \ldots k\}$, then $Z_i(a) \neq \phi$. Moreover, if B_i is a forcing basis for H_i , then $|Z_i(a)| \geq |B_i|$.

Proof. Suppose that for some $i \in \{1, 2, ..., k\}$ there exists a vertex $a \in V(G)$ such that $Z_i(a) = \phi$. Then, by Observation 3.1 any vertex in $H_i(a)$ cannot be forced by any vertex in $H_i(b)$, $i \neq j$ for any $a \neq b \in V(G)$, a contradiction.

Now suppose that $|Z_i(a)| < |B_i|$ and $Z_i(a) = \{(a, z_1), (a, z_2), \dots, (a, z_t)\}$ for some forcing basis B_i of H_i , where $\{z_1, z_2, \dots, z_t\} \subset V(H_i)$. Then, each black vertex in $H_i(a)$ has more than one white neighbors and no vertex of $H_i(a)$ can be forced by any vertex in $H_j(v)$ for any $v \in V(G)$, $i \neq j$. Hence, $|Z_i(a)| \geq |B_i|$.

From above theorem, we have an immediate corollary:

Corollary 3.5. Let G be a connected graph and H be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, H_3, \ldots H_k$ and $m_i \geq 2$. Let $Z(a) = \bigcup_{1 \leq i \leq k} Z_i(a)$ for $a \in V(G)$. Then Z(a) is a zero forcing set of H(a).

Proposition 3.6. Let G be a connected graph and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Let $a \in V(G)$ and Z be a forcing basis for $G \circ H$. If $Z(a) = Z \cap H(a)$ and $\alpha(a) = |Z(a)|$. Then

$$\alpha(a) \le \sum_{i=1}^k m_i.$$

Proof. For any $(b,x) \in V(G \circ H)$, $deg_{G \circ H}(b,x) = \sum_{b \sim u} |H(u)| + deg_H(x)$. Since G is connected so for any $b \in V(G)$, there exist at least one vertex $v \in V(G)$ such that $b \sim v$ and $deg_{G \circ H}(b,x) \geq |H(v)| + deg_H(x)$. To start the zero forcing process at least all the vertices of H(v) along with $deg_H(x)$ vertices are initially colored black. Since Z is a forcing basis for $G \circ H$. Hence, $Z \cap H(v) = H(v)$ and $\alpha(a) \leq \sum_{i=1}^k m_i$ for any $a \in V(G)$.

Corollary 3.7. Let G be a connected graph and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Then there exists at least one vertex $x \in V(G)$ such that $\alpha(x) = \sum_{i=1}^k m_i$.

The projection of $S \subseteq V(G \circ H)$ onto G, denoted by $P_G(S)$, is the set of vertices $a \in V(G)$ for which there exists a vertex $(a, v) \in S$. Similarly, the projection of $S \subseteq V(G \circ H)$ onto H, $P_H(S)$, is the set of vertices $v \in V(H)$ for which there exists a vertex $(a, v) \in S$.

Lemma 3.8. Let G be a connected graph of order n and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Let Z be a forcing basis for $G \circ H$ and $Z_i = Z \cap V(G \circ H_i)$, where $G \circ H_i$ is the induced subgraph of $G \circ H$. Then $P_G(Z_i) = V(G)$.

Proof. Let $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $V(H_i) = \{v_1^i, v_2^i, \cdots, v_{m_i}^i\}$ for $1 \le i \le k$. Suppose $P_G(Z_i) \ne V(G)$, i.e there exists a vertex $u_j \in V(G)$ such that $u_j \notin P_G(Z_i)$. This implies $(u_j, v_p^i) \notin Z$ for any $v_p^i \in V(H_i)$ for $1 \le p \le m_i$. Hence, $H_i(u_j) \cap Z = \phi$, a contradiction by Theorem 3.4.

Lemma 3.9. Let G be a connected graph of order n and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Then

$$Z(G \circ H) \le n(\sum_{i=1}^{k} m_i) - k.$$

Proof. Let $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $V(H_i) = \{v_1^i, v_2^i, \cdots, v_{m_i}^i\}$ for $1 \leq i \leq k$. We define $Z = V(G \circ H) \setminus \{(u_1, v_2^i) | 1 \leq i \leq k\}$ and $|Z| = n(\sum_{i=1}^k m_i) - k$. We claim that Z is a zero forcing set of $G \circ H$. To prove the claim, assume that Z is initially colored black. Since for any $i \neq j$, $v_p^i \not\sim v_q^i$ in H hence $(u_r, v_p^i) \not\sim (u_r, v_q^i)$, $1 \leq r \leq n$, in $G \circ H$. Since for any i, H_i is connected, so there exists at least one vertex v_l^i such that $v_2^i \sim v_l^i$ in H_i and hence $(u_r, v_2^i) \sim (u_r, v_l^i)$, $1 \leq r \leq n$, in $G \circ H$. Therefore, $(u_1, v_l^i) \rightarrow (u_1, v_2^i)$ for $1 \leq r \leq n$. Hence, $Z(G \circ H) \leq n(\sum_{i=1}^k m_i) - k$.

This bound is sharp for $G = K_n$ and $H = K_{m_1}, K_{m_2}, \dots, K_{m_k}$.

Lemma 3.10. Let G be a connected graph of order n and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Then

$$Z(G \circ H) \ge (n-1)k + \sum_{i=1}^{k} m_i.$$

Proof. The result follows from Corollary 3.7 and Lemma 3.8. \Box

This bound is sharp for $G = K_{1,n-1}$ and $H = P_{m_1}, P_{m_2}, \cdots, P_{m_k}$.

Lemma 3.11. Let G be a connected graph of order n and H_1, H_2, \dots, H_k , $k \geq 2$ are singleton components. Then $Z(G \circ H) \leq nk - 2$.

Proof. Let $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H_i) = \{x_i\}$ for $1 \leq i \leq k$. We define, for $u_1 \sim u_2$, $Z = V(G \circ H) \setminus \{(u_1, x_k), (u_2, x_k)\}$ with |Z| = nk - 2. Since $x_1 \not\sim x_k$ so $(u_1, x_1) \to (u_2, x_k)$. Similarly, $(u_2, x_1) \to (u_1, x_k)$. Now all the vertices in $G \circ H$ are colored black. Therefore, the result follows. \square

Now we study the zero forcing number of lexicographic product of graphs for some specific families of graphs and H contains one component only. Note that $K_m \circ K_n \cong K_{mn}$, so $Z(K_M \circ K_n) = mn - 1$. Therefore, from now on we consider the graphs when at most one of the factors of the product is a complete graph.

Lemma 3.12. For any connected graph H of order m, $Z(K_n \circ H) = Z(H) + (n-1)m$.

Proof. Note that for any $a \in V(K_n)$, $H(a) \sim H(b)$ for all $b \in V(K_n) \setminus \{a\}$. Therefore all the vertices in H(b) for all $b \in V(K_n) \setminus \{a\}$ are initially colored black. Now $H(a) \cong H$ so Z(H) vertices are required as initially colored black to complete the zero forcing process.

Since the lexicographic product of graphs is not commutative, i.e $K_n \circ H \ncong H \circ K_n$. Therefore we study the case when the second factor is a complete graph.

Lemma 3.13. For any connected non complete graph G of order m, $m(n-1)+1 \le Z(G \circ K_n) \le nm-2$.

Proof. Suppose $V(G) = \{u_1, u_2, \dots, u_m\}$ and $V(K_n) = \{v_1, v_2, \dots, v_n\}$. It is easy to check that $Z = V(G \circ K_n) \setminus \{(u_1, v_2), (u_n, v_2)\}$ is a zero forcing set of $G \circ K_n$. Hence $Z(G \circ K_n) \leq nm - 2$.

By Corollary 3.7, there exists at least one vertex $a \in V(G)$ such that $\alpha(a) = n$. Since $H(v) \cong K_n$, therefore for m-1 layers H(v) at least n-1 vertices from each layer are required as initially colored black to color all the vertices of $G \circ K_n$. Hence $Z(G \circ K_n) \geq n + (m-1)(n-1) = m(n-1) + 1$.

Now we study the zero forcing number of $P_n \circ H$, for $n \geq 3$ and a connected graph H. Suppose $V(P_n) = \{u_1, u_2, \cdots, u_n\}$. Since $H(u_i) \sim H(u_{i+1})$, $1 \leq i \leq n-1$. Note that to color the vertices of $H(u_1)$, Z(H) vertices in $H(u_1)$ and all the vertices in $H(u_2)$ are required as initially colored black and to color the vertices of $H(v_3)$, Z(H) vertices in $H(v_3)$ and all the vertices in $H(v_4)$ are required as initially colored black. Continue the process untill all the vertices in $P_n \circ H$ are turned black.

Proposition 3.14. For a connected graph H and $n \geq 3$,

$$Z(P_n \circ H) = \begin{cases} \frac{n(Z(H)+m)}{2}, & n \text{ is even} \\ \frac{n(Z(H)+m)+Z(H)-m}{2}, & n \text{ is odd.} \end{cases}$$

Corollary 3.15. For $n, m \geq 3$,

$$Z(P_n \circ K_m) = \begin{cases} \frac{n(2m-1)}{2}, & n \text{ is even} \\ nm - \frac{n+1}{2}, & n \text{ is odd.} \end{cases}$$

Now we study the zero forcing number of $C_n \circ H$, for $n \geq 4$ and a connected graph H. Suppose $V(C_n) = \{u_1, u_2, \dots, u_n\}$. Since $H(u_i) \sim H(u_{i+1})$, $1 \leq i \leq n$ and $u_{n+1} = u_1$. Note that to color the vertices of $H(u_1)$, Z(H) vertices in $H(u_1)$ and all the vertices in $H(u_2)$ and $H(u_n)$ are required as initially colored black and to color the vertices of $H(v_3)$, Z(H) vertices in $H(v_3)$ and all the vertices in $H(v_4)$ are required as initially colored black. Continue the process until all the vertices in $C_n \circ H$ are turned black.

Proposition 3.16. For a connected graph H and $n \geq 4$,

$$Z(C_n \circ H) = \begin{cases} \frac{n(Z(H)+m)}{2}, & n \text{ is even} \\ \frac{n(m+Z(H))+m-Z(H)}{2}, & n \text{ is odd.} \end{cases}$$

Corollary 3.17. For $n \ge 4$ and $m \ge 3$,

$$Z(C_n \circ K_m) = \begin{cases} \frac{n(2m-1)}{2}, & n \text{ is even} \\ \frac{n(2m-1)+1}{2}, & n \text{ is odd.} \end{cases}$$

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