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# On minimum rank and zero forcing sets of a graph

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ABSTRACT

For a graph $G$ on $n$ vertices and a field $F$, the minimum rank of $G$ over $F$, written as $\mathrm{mr}^F(G)$, is the smallest possible rank over all $n \times n$ symmetric matrices over $F$ whose $(i,j)$th entry (for $i \neq j$) is nonzero whenever $ij$ is an edge in $G$ and is zero otherwise. The maximum nullity of $G$ over $F$ is $M^F(G) = n - \mathrm{mr}^F(G)$. The minimum rank problem of a graph $G$ is to determine $\mathrm{mr}^F(G)$ (or equivalently, $M^F(G)$). This problem has received considerable attention over the years. In [F. Barioli, W. Barrett, S. Butler, S.M. Cioabă, D. Cvetković, S.M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K.V. Meulen, A.W. Wehe, AIM Minimum Rank–Special Graphs Work Group, Zero forcing sets and the minimum rank of graphs, Linear Algebra Appl. 428 (2008) 1628–1648], a new graph parameter $Z(G)$, the zero forcing number, was introduced to bound $M^F(G)$ from above. The authors posted an attractive question: What is the class of graphs $G$ for which $Z(G) = M^F(G)$ for some field $F$? This paper focuses on exploring the above question.


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## 1. Introduction and preliminary results

A *graph* $G$ consists of a set $V(G)$ of *vertices* together with a set $E(G)$ of unordered pairs of vertices called *edges*. We often use $uv$ for an edge $\{u, v\}$. Two vertices $u$ and $v$ are *adjacent* to each other if $uv \in E(G)$. In this paper, all graphs are finite and have no loops or multiple edges. Let $|G|$ denote the number of vertices of $G$. For $S \subseteq V(G)$, the *subgraph of $G$ induced by $S$* is the graph $G[S]$ with vertex set $S$ and edge set $\{uv \in E(G) : u, v \in S\}$. Denote by $G - S$ the subgraph of $G$ induced by $V(G) \setminus S$. For a vertex $v$ of $G$, we use $G - v$ for $G - \{v\}$. The *neighborhood* of $v$ is the set $N_G(v) = \{u \in V(G) : uv \in E\}$, and the *closed neighborhood* of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. A *complete graph* is a graph in which every two distinct vertices are adjacent. The complete graph on $n$ vertices is denoted by $K_n$. A complete bipartite graph with partite sets having $p$ and $q$ vertices is denoted by $K_{p,q}$. The *n-path* is the graph $P_n$ with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. The *n-cycle* is the graph $C_n$ with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Throughout this paper, we denote by $I_n$ the $n \times n$ identity matrix and $J_n$ the $n \times n$ matrix with all entries equal to 1. We use $I$ and $J$ for $I_n$ and $J_n$, respectively, when the order $n$ is clear from the context. Let char$(F)$ denote the *characteristic* of a field $F$. Denote by $F^{m\times n}$ the set of all $m \times n$ matrices over $F$. We write $F^m$ for $F^{m\times 1}$ in short.

Denote by $S_n(F)$ the set of $n \times n$ symmetric matrices over $F$. The *graph* of a matrix $A = [a_{ij}]$ in $S_n(F)$, denoted by $\mathcal{G}(A)$, is the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{ij : a_{ij} \neq 0 \text{ and } 1 \leqslant i < j \leqslant n\}$. Note that throughout this paper the vertices of $G$ are implicitly labeled in coordination with the rows (columns) of $A$ by the statement $\mathcal{G}(A) = G$. Denote the set $\{A \in S_{|G|}(F) : \mathcal{G}(A) = G\}$ by $\mathcal{S}^F(G)$. Given a graph $G$ and a field $F$, the *minimum rank* of $G$ over $F$, written as $\mathrm{mr}^F(G)$, is defined to be

$$\mathrm{mr}^F(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}^F(G)\}.$$

The *maximum nullity* (or *maximum corank*) of $G$ over $F$ is defined to be

$$M^F(G) = \max\{\mathrm{nullity}(A) : A \in \mathcal{S}^F(G)\},$$

where nullity$(A)$ is the nullity of $A$. It is well known that $\mathrm{mr}^F(G) + M^F(G) = |G|$. We write $\mathrm{mr}(G)$ for $\mathrm{mr}^{\mathbb{R}}(G)$ and $M(G)$ for $M^{\mathbb{R}}(G)$ in short. For matrix (resp. graph) terminology not defined in this paper, please see [12,17] or [22] (resp. [10,11] or [15]).

The *minimum rank problem* of a graph $G$ is to determine $\mathrm{mr}^F(G)$ (or equivalently, $M^F(G)$). This problem has received considerable attention in the literature (see for example [1,2,4–7,9,13,14,18,19] and references therein). In spite of the many efforts and different approaches the minimum rank/maximum nullity problem remains largely open. This problem has been solved for relatively few classes of graphs (see [1,3,5–8,13,14,19–21] and references therein). Recently, in [1], a graph parameter $Z(G)$, the zero forcing number, has been introduced as a technique to bound $M^F(G)$ from above. To define $Z(G)$, we adopt some notation and terminology from [1,4,19].

**Definition 1**

- *Color-change rule*: Suppose that $G$ is a graph with each vertex colored either white or black. If $u$ is a black vertex in $G$ and exactly one neighbor $v$ of $u$ is white, then change the color of $v$ to black, we say that $u$ *forces* $v$ and write $u \to v$.
- Given a coloring of $G$, the *derived coloring* is the result of applying the color-change rule until no more changes are possible. It was remarked in [1, p. 1633] that the derived coloring (of a specific coloring) is in fact unique. A process of obtaining the derived coloring is called a *zero forcing process* on $G$. If $u_1 \to v_1, u_2 \to v_2, \ldots, u_r \to v_r$ are the forces in the order in which they are performed in a zero forcing process, then $(u_1, u_2, \ldots, u_r)$ is called the *zero forcing sequence* of the zero forcing process with corresponding *color change sequence* $(v_1, v_2, \ldots, v_r)$.
- Given a graph $G$, a subset $S$ of vertices is called a *zero forcing set* for $G$ if it has the property that when initially the vertices in $S$ are colored black and the remaining vertices are colored white, then the derived coloring of $G$ is all black. The smallest size of a zero forcing set for $G$ is denoted by $Z(G)$ and is called the *zero forcing number* of $G$. A zero forcing set for $G$ of size $Z(G)$ is called a *minimum* zero forcing set of $G$.

**Theorem 2** (Proposition 2.4 of [1]). *For any graph $G$ and any field $F$, $M^F(G) \leqslant Z(G)$.*

Using this technique, the authors of [1] successfully determine $M(G)$ (or $M^F(G)$) and establish $Z(G) = M(G)$ (or $Z(G) = M^F(G)$) for many interesting classes of graphs. At the end of the paper [1] the authors posed the following attractive question: What is the class of graphs $G$ for which $Z(G) = M^F(G)$ for some field $F$? Our goal in this paper is to investigate which graphs has the property $Z(G) = M(G)$ (or $Z(G) = M^F(G)$) and to determine $M(G)$ (or $M^F(G)$). In Section 2 we show that if $G$ is a block-clique graph or a unit interval graph, then $Z(G) = M(G)$. The assertion for block-clique graphs generalizes Proposition 3.23 of [1]. In Section 3, we show that for the $d$-dimensional hypercube $Q_d$, $M^F(Q_d)$ is field independent. This result generalizes Theorem 3.1 of [1]. In Section 3, several families of product graphs $G$ are demonstrated that $Z(G) = M^F(G)$ for every field $F$.

## 2. The minimum rank of block-clique graphs and unit interval graphs.

In [1], the authors show that if $G$ is a block-clique graph (defined below) such that no vertex is contained in more than two blocks, then $Z(G) = M(G)$. In Theorem 7 we show that their conclusion is in fact true for any block-clique graph $G$.

A vertex $v$ of a graph is called a *cut-vertex* if deleting $v$ and all edges incident to it increases the number of connected components. A *block* of a graph $G$ is a maximal connected induced subgraph of $G$ that has no cut-vertices. We call a complete subgraph of a graph $G$ a *clique* of $G$. A graph is *block-clique* (also called 1-*chordal*) if every block is a clique. A block $H$ of a block-clique graph $G$ is a *pendent block* of $G$ if $H$ has at most one cut-vertex of $G$. Let $v$ be a cut-vertex of $G$. If $G - v$ consists of two disjoint graphs $W_1$ and $W_2$ and let $G_i (i = 1, 2)$ be the subgraph of $G$ induced by $\{v\} \bigcup V(W_i)$, then $G$ is called the *vertex-sum* at $v$ of the two graphs $G_1$ and $G_2$, and denoted by $G = G_1 \oplus_v G_2$.

**Theorem 3** (Cut-vertex Reduction Theorem [2,24]). *If $G = G_1 \oplus_v G_2$, then* $\mathrm{mr}(G) = \min\{\mathrm{mr}(G_1) + \mathrm{mr}(G_2), \mathrm{mr}(G_1 - v) + \mathrm{mr}(G_2 - v) + 2\}$.

Consequently, we have

**Corollary 4.** $M(G_1 \oplus_v G_2) = \max\{M(G_1) + M(G_2), M(G_1 - v) + M(G_2 - v)\} - 1$.

**Lemma 5.** *The following assertions hold for $G = G_1 \oplus_v G_2$.*

(i) $Z(G) \geqslant Z(G_1) + Z(G_2) - 1$.
(ii) $Z(G) \leqslant \min\{Z(G_1) + Z(G_2 - v), Z(G_1 - v) + Z(G_2)\}$.

**Proof.** Denote by $V_1$ (resp. $V_2$) the vertex set of $G_1$ (resp. $G_2$).

(i) Let $S$ be a minimum zero forcing set of $G$. Consider a zero forcing process $\mathcal{P}$ on $G$ with initial set of black vertices $S$. For the case of $v \notin S$, we may suppose without loss of generality that, in the process $\mathcal{P}$, $v$ is forced by a vertex of $V_1$. In this case, we see that $S \cap V_1$ is a zero forcing set for $G_1$ and $(S \cap V_2) \cup \{v\}$ is a zero forcing set for $G_2$. For the case of $v \in S$, it is easy to see that $S \cap V_i$ is a zero forcing set for $G_i$ for $i = 1, 2$. In either case, $Z(G) + 1 \geqslant |S \cap V_1| + |(S \cap V_2) \cup \{v\}| \geqslant Z(G_1) + Z(G_2)$ and so (i) holds.

(ii) By symmetric, it suffices to show that $Z(G) \leqslant Z(G_1) + Z(G_2 - v)$. Denote by $S_1$ (resp. $S_2$) a minimum zero forcing set for $G_1$ (resp. $G_2 - v$). There is a zero forcing process on $G_1$ (resp. $G_2 - v$) with initial set of black vertices $S_1$ (resp. $S_2$) and zero forcing sequence $(x_1, x_2, \ldots, x_p)$ (resp. $(y_1, y_2, \ldots, y_q)$). If $v \notin \{x_1, x_2, \ldots, x_p\}$, then there is a zero forcing process for $G$ with initial set of black vertices $S_1 \cup S_2$ and zero forcing sequence $(x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$. If $v \in \{x_1, x_2, \ldots, x_p\}$, say $v = x_i$, then there is a zero forcing process for $G$ with initial set of black vertices $S_1 \cup S_2$ and zero forcing sequence $(x_1, x_2, \ldots, x_{i-1}, y_1, y_2, \ldots, y_q, x_i, x_{i+1}, \ldots, x_p)$. In either case, $Z(G) \leqslant |S_1| + |S_2| = Z(G_1) + Z(G_2 - v)$. $\square$

**Lemma 6.** *If $v$ is a vertex in graph $G$, then $Z(G - v) - 1 \leqslant Z(G) \leqslant Z(G - v) + 1$.*

**Proof.** Denote by $S$ (resp. $S_v$) a minimum zero forcing set for $G$ (resp. $G - v$). It is easy to see that $S_v \cup \{v\}$ is a zero forcing set for $G$, and hence $Z(G) \leqslant Z(G-v)+1$. To prove the remaining inequality, notice that if a zero forcing process on $G - v$ is started with the initial set of black vertices $S \setminus \{v\}$, then the derived coloring of the process has a set of black vertices $F$ with $|N_G(v) \setminus F| \leqslant 1$. Since $S \cup (N_G(v) \setminus F)$ is a zero forcing set for $G - v$, we have $Z(G-v) \leqslant |S| + |N_G(v) \setminus F| \leqslant Z(G)+1$. This completes the proof of the lemma. □

**Theorem 7.** *If $G$ is a block-clique graph, then $Z(G) = M(G)$.*

**Proof.** Denote by $b(G)$ the number of blocks in $G$. We shall prove the theorem by induction on $b(G)$. If $b(G)$=1, then $G$ is a complete graph and clearly we have $Z(G) = M(G)$. Assume $b(G) \geqslant 2$ and $Z(H) = M(H)$ for any block-clique graph $H$ with $b(H) < b(G)$. There is a cut vertex $v$ such that all except at most one of the blocks that contain $v$ are pendent blocks; let $t$ denote the number of pendent blocks that contain $v$. We consider two cases.

**Case 1.** One of the $t$ pendent blocks is of size at least 3. In this case, we may assume that $G = G_1 \oplus_v G_2$ where $G_2$ is a clique of size at least 3. By Corollary 4, the induction hypothesis and the fact that $Z(K_n) = n - 1$ for $n \geqslant 2$,

$$M(G) \geqslant M(G_1) + M(G_2) - 1 = Z(G_1) + Z(G_2) - 1 = Z(G_1) + (|G_2| - 1) - 1.$$

By Lemmas 5 (ii), $Z(G) \leqslant Z(G_1) + Z(G_2 - v) = Z(G_1) + |G_2| - 2$.Since $M(G) \leqslant Z(G)$, we then have $M(G) = Z(G) = Z(G_1) + |G_2| - 2$.

**Case 2.** All the $t$ pendent blocks are of size 2. In this case, we may assume that $G = G_1 \oplus_v G_2$ where $G_2$ is a star with center $v$ and $t$ leaves $v_1, v_2, \ldots, v_t$.

For the subcase of $t \geqslant 2$, by Corollary 4, the induction hypothesis and the fact that $Z(tK_1) = t$,

$$M(G) \geqslant M(G_1 - v) + M(G_2 - v) - 1 = Z(G_1 - v) + Z(G_2 - v) - 1 = Z(G_1 - v) + t - 1.$$

By Lemmas 5 (ii) and the fact that $Z(K_{1,t}) = t - 1$ for $t \geqslant 2, Z(G) \leqslant Z(G_1 - v) + Z(G_2) = Z(G_1 - v) + t - 1$.Since $M(G) \leqslant Z(G)$, we then have $M(G) = Z(G) = Z(G_1 - v) + t - 1$.

For the subcase of $t = 1$, by Corollary 4 and the induction hypothesis, we have $M(G) \geqslant M(G_1) + M(G_2) - 1 = Z(G_1) + Z(G_2) - 1 = Z(G_1)$. Next, we show that $Z(G) \leqslant Z(G_1)$. Denote by $S_1$ a minimum zero forcing set for $G_1$. Let $\mathcal{P}$ be a zero forcing process on $G_1$ with initial set of black vertices $S_1$. Let us consider two cases. If $v \in S_1$, then clearly $(S_1 \setminus \{v\}) \cup \{v_1\}$ is a zero forcing set for $G$. We next show that if $v \notin S_1$, then $S_1$ is a zero forcing set for $G$. Since $v$ is not a cut-vertex of $G_1$, $N_{G_1}(v)$ induces a clique in $G$. Therefore $v$ cannot perform a force in the process $\mathcal{P}$. These show that $Z(G) \leqslant Z(G_1)$. Finally, since $M(G) \leqslant Z(G)$, we conclude that $M(G) = Z(G) = Z(G_1)$. □

A *clique covering* of $G$ is a set of cliques of $G$ which together contain each edge of $G$ at least once. The *clique covering number* $\mathrm{cc}(G)$ of $G$ is the smallest cardinality of a clique covering of $G$. It is well known [14] that $\mathrm{mr}(G) \leqslant \mathrm{cc}(G)$ for any graph $G$. In [1], it was also shown that if $G$ is a block-clique graph such that no vertex is contained in more than two blocks, then $\mathrm{mr}(G) = \mathrm{cc}(G)$. Notice that there are infinitely many block-clique graphs $G$ for which $\mathrm{mr}(G) < \mathrm{cc}(G)$. For example, if $q \geqslant 3$ then we have $\mathrm{mr}(K_{1,q}) = 2$ and $\mathrm{cc}(K_{1,q}) = q$.

An *interval graph* is a graph $G$ for which we can associate with each vertex $v$ an interval $I(v)$ in the real line such that two distinct vertices $u$ and $v$ are adjacent if and only if $I(u) \cap I(v) \neq \emptyset$. The set of intervals $\{I(v)\}_{v \in V(G)}$ is called an *interval representation* for $G$. A graph is a *unit interval graph* if it is an interval graph which has an interval representation in which all intervals have equal length.

In Theorem 9 we use the following characterization of unit interval graphs to show that if $G$ is a unit interval graph then $\mathrm{cc}(G) = \mathrm{mr}(G)$ and $Z(G) = M(G)$. We remark that, for $q \geqslant 2$, $K_{1,q}$ is an interval graph with $\mathrm{cc}(K_{1,q}) = q$ and $\mathrm{mr}(K_{1,q}) = 2$.

**Theorem 8** [27]**.** *A graph $G$ is a unit interval graph if and only if there is an order on vertices such that for each vertex $v$, the closed neighborhood of $v$ is a set of consecutive vertices in that order.*

The order defined in Theorem 8 is called a *consecutive order* of $G$. The idea of the proof of Theorem 9 is to show that for any unit interval graph $G$ we have $\mathrm{cc}(G) = |G| - Z(G)$.

**Theorem 9.** *If $G$ is a connected unit interval graph, then $\mathrm{cc}(G) = \mathrm{mr}(G)$ and $Z(G) = M(G)$.*

**Proof.** Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $(v_1, v_2, \ldots, v_n)$ is a consecutive order of $G$. For positive integers $i$ and $j$ with $i \leqslant j$, denote by $[i, j]$ the set $\{i, i+1, i+2, \ldots, j\}$. Let $[v_i, v_j]$ denote the collection of vertices $v_k$ such that $i \leqslant k \leqslant j$. Let $\ell(j) = \min\{k : v_k \in N_G[v_j]\}$. Define

$$S = \{v_1\} \cup \{v_k : k \in [3, n] \text{ and } \ell(k) = \ell(k-1)\}.$$

Consider a zero forcing process $\mathcal{P}$ on $G$ that is started with the initial set of black vertices $S$. Denote by $F$ the set of of black vertices in the derived coloring of the process $\mathcal{P}$.

We shall show that $S$ is a zero forcing set for $G$, that is, to show that $F = V(G)$. Assume to the contrary that $V(G) \setminus F \neq \emptyset$. Let $t = \min\{k : v_k \in V(G) \setminus F\}$. Notice that $t \geqslant 2$. We claim that $v_t$ has a neighbor in $\{v_1, v_2, v_3, \ldots, v_{t-1}\}$. Since $G$ is connected, there exists an edge $v_i v_j$ in $G$ such that $i < t \leqslant j$. Hence, by Theorem 8, it follows that $v_t$ is adjacent to $v_i$. Let $r(t) = \max\{k : v_k \text{ is adjacent to } v_{\ell(t)}\}$. Notice that $v_{\ell(t)} \in F$ and $r(t) \geqslant t$. To get a contradiction we consider two cases of $r(t)$. If $r(t) = t$, then all neighbors of $v_{\ell(t)}$ except $v_t$ lie in $F$. It follows that $v_t$ must be forced by $v_{\ell(t)}$ at some time during the zero forcing process $\mathcal{P}$. Hence $v_t \in F$, a contradiction to the choice of $t$. Next we consider the case when $r(t) > t$. In this case, by Theorem 8 and the choice of $\ell(t)$, we have

$$\ell(t) = \ell(t+1) = \ell(t+2) = \cdots = \ell(r(t)),$$

and hence $\{v_{t+1}, v_{t+2}, v_{t+3}, \ldots, v_{r(t)}\} \subseteq S \subseteq F$. By the choices of $r(t)$ and $t$, all neighbors of $v_{\ell(t)}$ except $v_t$ lie in $F$. Thus, $v_t$ must be forced by $v_{\ell(t)}$ during the zero forcing process $\mathcal{P}$, a contradiction to $v_t \notin F$.

Denote by $\mathcal{C}_j$ the set $\{v_k : \ell(j) \leqslant k \leqslant j\}$. By Theorem 8, it is clear that $\mathcal{C}_j$ is a clique of $G$ for each integer $j$ in $[1, n]$. Next we define

$$\mathcal{C} = \{\mathcal{C}_{k-1} : k \in [3, n] \text{ and } \ell(k) \neq \ell(k-1)\} \cup \{\mathcal{C}_n\},$$

and show that $\mathcal{C}$ is a clique covering of $G$. For an edge $v_i v_j (i < j)$, we denote by $j^*$ the largest integer $t$ such that $\ell(j) = \ell(s)$ for any integer $s \in [j, t]$. If $j^* < n$, then $\ell(j^*) \neq \ell(j^* + 1)$. Thus, by definition of $\mathcal{C}$, it follows that $\mathcal{C}_{j^*} \in \mathcal{C}$ and $\mathcal{C}_{j^*}$ contains the edge $v_i v_j$. If $j^* = n$, then $\ell(j) = \ell(j+1) = \ell(j+2) = \ldots = \ell(n)$ and hence $\mathcal{C}_n$ contains the edge $v_i v_j$. Since $v_i v_j$ is an arbitrary edge, we conclude that $\mathcal{C}$ is a clique covering of $G$.

By the definitions of $S$ and $\mathcal{C}$ together with Theorem 2 and the fact that $\mathrm{cc}(G) \geqslant \mathrm{mr}(G)$,

$$|S| = n - |\mathcal{C}| \leqslant n - \mathrm{cc}(G) \leqslant n - \mathrm{mr}(G) = M(G) \leqslant Z(G) \leqslant |S|,$$

which give the theorem and show that $|S| = Z(G)$ and $|\mathcal{C}| = \mathrm{cc}(G)$. $\square$

## 3. The minimum rank of product graphs

In this section, several families of product graphs $G$ are demonstrated that $M^F(G) = Z(G)$ for every field $F$.

### 3.1. Cartesian products

The *Cartesian product* of two graphs $G$ and $H$ is the graph $G \square H$ with vertex set $V(G) \times V(H)$ and edge set $\{(g, h)(g', h') : gg' \in E(G) \text{ with } h = h', \text{ or } g = g' \text{ with } hh' \in E(H)\}$. Note that the Cartesian product is commutative and associative (see page 29 of [25]). The *d-dimensional hypercube* $Q_d$ is defined recursively: $Q_1 = K_2$ and $Q_{d+1} = Q_d \square K_2$. In [1], the authors showed that $\mathrm{mr}^F(Q_d) = Z(Q_d) = 2^{d-1}$ whenever $\mathrm{char}(F) = 2$ or ($\mathrm{char}(F) \neq 2$ and $\sqrt{2} \in F$). It was shown in [13] that $M^F(Q_d) = Z(Q_d) = 2^{d-1}$ for any field $F$ of order at least 6. In the following, we show that in fact $\mathrm{mr}^F(Q_d) = 2^{d-1}$ for any field $F$. That is, $M^F(Q_d)$ is field independent.

**Theorem 10.** *If $F$ is a field, then $M^F(Q_d) = Z(Q_d) = 2^{d-1}$.*

**Proof.** Since $2^d - \text{mr}^F(Q_d) = M^F(Q_d) \leqslant Z(Q_d) \leqslant 2^{d-1}$, it suffices to prove that $\text{mr}^F(Q_d) \leqslant 2^{d-1}$. First we set two $2 \times 2$ symmetric matrices $H_1$ and $L_1$ over $F$ as

$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad L_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We then define inductively two sequences of symmetric matrices $\{H_d\}_{d=1}^{\infty}$ and $\{L_d\}_{d=1}^{\infty}$ as follows: Given $H_{d-1}$ and $L_{d-1}$, define

$$H_d = \begin{bmatrix} H_{d-1} & I \\ I & -H_{d-1} \end{bmatrix} \quad \text{and} \quad L_d = \begin{bmatrix} L_{d-1} & I \\ I & -L_{d-1} \end{bmatrix}.$$

By a simple induction argument on $d$ it can be shown that $H_d^2 = (d+1)I$, $L_d^2 = dI$ and $\mathcal{G}(H_d) = \mathcal{G}(L_d) = Q_d$. If $\text{char}(F)$ is not a factor of $d$, then define

$$B_d = \begin{bmatrix} d^{-1}H_{d-1} & I \\ I & H_{d-1} \end{bmatrix}.$$

Since $\mathcal{G}(B_d) = Q_d$ and

$$\begin{bmatrix} I & \mathbf{0} \\ -H_{d-1} & I \end{bmatrix} B_d = \begin{bmatrix} d^{-1}H_{d-1} & I \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

we have $\text{mr}^F(Q_d) \leqslant \text{rank}(B_d) = 2^{d-1}$. If $\text{char}(F)$ is a factor of $d$, since $\mathcal{G}(L_d) = Q_d$ and

$$\begin{bmatrix} I & \mathbf{0} \\ L_{d-1} & I \end{bmatrix} L_d = \begin{bmatrix} L_{d-1} & I \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

we have $\text{mr}^F(Q_d) \leqslant \text{rank}(L_d) = 2^{d-1}$. In any case, we have $\text{mr}^F(Q_d) \leqslant 2^{d-1}$. This proves the theorem. □

**Theorem 11.** *If $F$ is a field and $n \geqslant 2$, then $M^F(K_2 \square K_{1,n}) = Z(K_2 \square K_{1,n}) = n$.*

**Proof.** Denote by $\{v, w_1, \dots, w_n\}$ (resp. $\{u_1, u_2\}$) the vertex set of $K_{1,n}$ (resp. $K_2$), where $v$ is the vertex that has maximum degree in $K_{1,n}$. Clearly, the set $S = \{(u_1, v), (u_1, w_1), (u_1, w_2), \dots, (u_1, w_{n-1})\}$ is a zero forcing set for $K_2 \square K_{1,n}$, and hence $M^F(K_2 \square K_{1,n}) \leqslant Z(K_2 \square K_{1,n}) \leqslant |S| = n$. Consider the following two $(n+1) \times (n+1)$ matrices over $F$:

$$B = \begin{bmatrix} 1 & \mathbf{1}^T \\ \mathbf{1} & I_n \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} n-3 & \mathbf{1}^T \\ \mathbf{1} & I_n \end{bmatrix},$$

where $\mathbf{1} = [1, 1, \dots, 1]^T$ is a column vector in $\mathbb{R}^n$. To show that $n$ is a lower bound for $M^F(K_2 \square K_{1,n})$, we consider the following $2(n+1) \times 2(n+1)$ symmetric matrix over $F$:

$$A = \begin{bmatrix} B & I \\ I & C \end{bmatrix}.$$

Note that

$$\begin{bmatrix} I & -B \\ \mathbf{0} & I \end{bmatrix} A = \begin{bmatrix} \mathbf{0} & I - BC \\ I & C \end{bmatrix}.$$

It is easy to check that $\text{rank}(I - BC) = 1$, and hence $\text{rank}(A) = n + 2$. Since $\mathcal{G}(A) = K_2 \square K_{1,n}$, we get $M^F(K_2 \square K_{1,n}) = 2(n+1) - \text{mr}^F(K_2 \square K_{1,n}) \geqslant 2(n+1) - \text{rank}(A) = n$. This completes the proof of the theorem. □

It was shown in Theorem 3.6 of [1] that if $G$ is a graph with $|V(G)| \leqslant n$, then $Z(G \square P_n) = M(G \square P_n) = |G|$. We note that the proof of Theorem 3.6 in [1] in fact gives the sharper result that if there is a matrix $A \in \mathcal{S}^{\mathbb{R}}(G)$ such that $A$ has at most $n$ distinct eigenvalues, then $Z(G \square P_n) = M(G \square P_n) = |G|$.

Let $G = (G_1\Box\cdots\Box G_r)\Box(H_1\Box\cdots\Box H_s)\Box(Q_1\Box\cdots\Box Q_t)$ such that $G_k$'s, $H_k$'s, and $Q_k$'s are complete bipartite graphs, complete graphs and paths, respectively. Note that $\frac{1}{\sqrt{mn}}A(K_{m,n}) + I_{m+n}$ has spectrum $\{2, 1^{(m+n-2)}, 0\}$, $\frac{1}{n}A(K_n) + \frac{1}{n}I_n$ has spectrum $\{1, 0^{(n-1)}\}$, and there exists $B \in \mathcal{S}^{\mathbb{R}}(P_n)$ with spectrum $\{0, 1, \ldots, n-1\}$ (see Theorem 2 of [16]). Using well-known properties on eigenvalues of Kronecker product of graphs (see page 207 of [15]), it can be seen that there is a matrix $A \in \mathcal{S}^{\mathbb{R}}(G)$ such that the spectrum of $A$ is contained in $\{0, 1, 2, \ldots, \ell_G\}$, where $\ell_G = 2r + s - t + \sum_{k=1}^{t} |Q_k|$. What we have just proved can be summarized in the following theorem:

**Theorem 12.** *For a graph $G$, let $\sigma_G = \min_{A\in\mathcal{S}^{\mathbb{R}}(G)} |\mathrm{spec}(A)|$, where $|\mathrm{spec(A)}|$ denotes the number of distinct eigenvalues of $A$.*

(a) *If $\sigma_G \leqslant n$, then $Z(G\Box P_n) = M(G\Box P_n) = |G|$.*
(b) *If $G = (G_1\Box\cdots\Box G_r)\Box(H_1\Box\cdots\Box H_s)\Box(Q_1\Box\cdots\Box Q_t)$ such that $G_k$'s, $H_k$'s, and $Q_k$'s are complete bipartite graphs, complete graphs and paths, respectively. Then $\sigma_G \leqslant 2r + s - t + \sum_{k=1}^{t} |Q_k| + 1$,*

We remark that the idea of Theorem 12(a) is implicit in the proof technique of Theorem 3.10 of [1]. Theorem 12 also contains Proposition 3.3 of [1] as a special case, where the authors use the Colin de Verdière-type parameter to show that $Z(K_s\Box P_n) = M(K_s\Box P_n) = s$.

The following upper bound for the parameter $Z$ for any Cartesian product is useful in the proof of Theorem 14.

**Lemma 13** (Proposition 2.5 of [1])**.** *For any two graphs $G$ and $H$, $Z(G\Box H) \leqslant \min\{Z(G)|H|, Z(H)|G|\}$.*

In Example 3.4 of [1], an exhaustive search was used to show that $M^{\mathbb{Z}_2}(K_3\Box K_2) = 2$, and hence $M^F(K_3\Box K_2)$ depends on the field $F$. In the following theorem, we show that $M^F(K_s\Box K_2)$ can be determined effectively for any field $F$ and for any $s \geqslant 2$.

**Theorem 14.** *Suppose $F$ is a field and $s \geqslant 2$.*

(a) *If $F \neq \mathbb{Z}_2$ then $M^F(K_s\Box K_2) = Z(K_s\Box K_2) = s$.*
(b) *If $s$ is even, then $M^{\mathbb{Z}_2}(K_s\Box K_2) = s$; otherwise $M^{\mathbb{Z}_2}(K_s\Box K_2) = s - 1$.*

**Proof.** (a) By Theorem 2 and Lemma 13, we get $M^F(K_s\Box K_2) \leqslant Z(K_s\Box K_2) \leqslant s$ for any field $F$. To prove the required lower bound for $M^F(K_s\Box K_2)$, we divide the proof into three cases. In these cases, we denote by $A$ a $2s \times 2s$ symmetric matrix over $F$.

**Case 1.** $\mathrm{char}(F)$ divides $s$. In this case, define $A$ to be the following matrix:

$$A = \begin{bmatrix} I+J & I \\ I & I-J \end{bmatrix}.$$

Clearly, we have $A \in \mathcal{S}^F(K_s\Box K_2)$. Since $\mathrm{char}(F)$ divides $s$ and

$$\begin{bmatrix} I & \mathbf{0} \\ J-I & I \end{bmatrix} A = \begin{bmatrix} I+J & I \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

we get $s = \mathrm{nullity}(A) \leqslant M^F(K_s\Box K_2)$, as required.

**Case 2.** $\mathrm{char}(F) = 2$ and $s$ is odd. Since $F \neq \mathbb{Z}_2$, we can pick $a \in F$ such that $a \neq 0$ and $a \neq 1$. Let $b = a(a+1)^{-1}$. Clearly, we have $b \neq 0$. Define $A$ to be the following matrix:

$$A = \begin{bmatrix} I+aJ & I \\ I & I+bJ \end{bmatrix}.$$

Clearly, we have $A \in \mathcal{S}^F(K_s\Box K_2)$. Since $\mathrm{char}(F) = 2$ and $s$ is odd, it can be seen that $J^2 = J$ and

$$\begin{bmatrix} I & \mathbf{0} \\ -bJ-I & I \end{bmatrix} A = \begin{bmatrix} I+aJ & I \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

It follows that $s = \mathrm{nullity}(A) \leqslant M^F(K_s\Box K_2)$, as required.

**Case 3.** $\mathrm{char}(F) \neq 2$ and $\mathrm{char}(F)$ does not divide $s$. Let $c = 2s^{-1}$. Define $A$ to be the following matrix:

$$A = \begin{bmatrix} cJ - I & I \\ I & cJ - I \end{bmatrix}.$$

Since $A \in \mathcal{S}^F(K_s \square K_2)$ and

$$\begin{bmatrix} I & \mathbf{0} \\ -cJ + I & I \end{bmatrix} A = \begin{bmatrix} cJ - I & I \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

we have $s = \mathrm{nullity}(A) \leqslant M^F(K_s \square K_2)$, as required.

(b) Construct a $2s \times 2s$ symmetric matrix $A$ over $\mathbb{Z}_2$ as follows:

$$A = \begin{bmatrix} I + J & I \\ I & I + J \end{bmatrix}.$$

Clearly, we have $A \in \mathcal{S}^{\mathbb{Z}_2}(K_s \square K_2)$ and

$$\begin{bmatrix} I & \mathbf{0} \\ J - I & I \end{bmatrix} A = \begin{bmatrix} I + J & I \\ sJ & \mathbf{0} \end{bmatrix}.$$

If $s$ is even, then we have $s = \mathrm{nullity}(A) \leqslant M^{\mathbb{Z}_2}(K_s \square K_2) \leqslant Z(K_s \square K_2) \leqslant s$, where the last inequality follows from Lemma 13.

It remains to consider the case when $s$ is odd. In this case, by the above matrix equation, it can be seen that $\mathrm{mr}^F(K_s \square K_2) \leqslant \mathrm{rank}(A) = s + 1$. We shall show that $s + 1$ is also a lower bound for $\mathrm{mr}^F(K_s \square K_2)$. To this end, let us consider an arbitrary matrix $B$ in $\mathcal{S}^{\mathbb{Z}_2}(K_s \square K_2)$. We note that $B$ has the following form

$$\begin{bmatrix} J + D_1 & I \\ I & J + D_2 \end{bmatrix},$$

where $D_1$ and $D_2$ are diagonal matrices. Denote by $Q$ the matrix $J + JD_1 + D_2J + D_2D_1 + I$. It can readily be checked that all the diagonal entries of $Q$ are zero if and only if both $D_1$ and $D_2$ are zero matrices. Since

$$\begin{bmatrix} I & \mathbf{0} \\ J + D_2 & I \end{bmatrix} B = \begin{bmatrix} J + D_1 & I \\ Q & \mathbf{0} \end{bmatrix},$$

it can be seen that $Q$ is not a zero matrix, and hence $\mathrm{rank}(B) \geqslant s + 1$. We conclude that $\mathrm{mr}^F(K_s \square K_2) \geqslant s + 1$. This completes the proof of the theorem. $\square$

### *3.2. Direct and strong products*

The *direct product* of two graphs $G$ and $H$ is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and edge set $\{(g, h)(g', h') : gg' \in E(G) \text{ and } hh' \in E(H)\}$. The *strong product* of two graphs $G$ and $H$ is the graph $G \boxtimes H$ with vertex set $V(G) \times V(H)$ and edge set $E(G \square H) \cup E(G \times H)$. Note that $G \boxtimes H = G \square H \cup G \times H$ and that the direct and strong products are associative and commutative (see page 163 and page 148 of [25]).

**Theorem 15.** *If $n \geqslant 2$, then $M(P_{2k+1} \times K_n) = Z(P_{2k+1} \times K_n) = (2k + 1)n - 4k$.*

**Proof.** Let $V(P_{2k+1}) = \{x_1, x_2, \ldots, x_{2k+1}\}$, $E(P_{2k+1}) = \{x_1x_2, x_2x_3, \ldots, x_{2k}x_{2k+1}\}$ and $V(K_n) = \{y_1, y_2, \ldots, y_n\}$. Denote by $\overline{S}$ the vertex subset $\{(x_i, y_j) : 2 \leqslant i \leqslant 2k + 1 \text{and } 1 \leqslant j \leqslant 2\}$ of $P_{2k+1} \times K_n$. Consider a zero forcing process $\mathcal{P}$ on $P_{2k+1} \times K_n$ with initial set of black vertices $V(P_{2k+1} \times K_n) \setminus \overline{S}$. Using $((x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), \ldots, (x_{2k}, y_1), (x_{2k}, y_2))$ as the zero forcing sequence of $\mathcal{P}$, it is easy to see that $V(P_{2k+1} \times K_n) \setminus \overline{S}$ is a zero forcing set for $P_{2k+1} \times K_n$, and so $Z(P_{2k+1} \times K_n) \leqslant (2k + 1)n - 4k$.

For every integer $n \geqslant 2$, we construct a real $n \times n$ matrix $A_n$ as follows:

$$A_n = [C_1 - C_2, 2C_1 - C_2, 3C_1 - C_2, \ldots, nC_1 - C_2],$$

where $C_1 = [1, 1, \dots, 1]^T$ and $C_2 = [1, 2, \dots, n]^T$ are two column vectors in $\mathbb{R}^n$. Note that $A_n$ has zero diagonal entries and nonzero off-diagonal entries. Moreover, $\operatorname{rank}(A_n) = 2$. Next, using $A_n$ as a building block to construct a $(2k+1)n \times (2k+1)n$ symmetric matrix $B_{2k+1,n}$ as follows:

$$B_{2k+1,n} = \begin{bmatrix} \mathbf{0} & A_n & \mathbf{0} & \cdots & & & & \mathbf{0} \\ A_n^T & \mathbf{0} & A_n^T & \mathbf{0} & \cdots & & & \mathbf{0} \\ \mathbf{0} & A_n & \mathbf{0} & A_n & \mathbf{0} & \cdots & & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_n^T & \mathbf{0} & A_n^T & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & A_n & \mathbf{0} & A_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & & \cdots & \mathbf{0} & A_n^T & \mathbf{0} & A_n^T \\ \mathbf{0} & \mathbf{0} & & & \cdots & \mathbf{0} & A_n & \mathbf{0} \end{bmatrix}.$$

It can be seen that $B_{2k+1,n} \in \mathcal{S}^{\mathbb{R}}(P_{2k+1} \times K_n)$ and $\operatorname{rank}(B_{2k+1,n}) = 4k$. By what we have proved above and Theorem 2, $(2k+1)n - 4k = \operatorname{nullity}(B_{2k+1,n}) \leqslant M(P_{2k+1} \times K_n) \leqslant Z(P_{2k+1} \times K_n) \leqslant (2k+1)n - 4k$. This completes the proof of the theorem. $\square$

In Section 3.1 of [1], the authors used techniques involving Kronecker product to study the maximum nullity/zero forcing number of the Cartesian product of two graphs. In the following, we use ideas involving the celebrated property of Kronecker product (see for example [23, Theorem 4.2.15])

$$\operatorname{rank}(A \otimes B) = \operatorname{rank}(A)\operatorname{rank}(B) \tag{1}$$

to study the maximum nullity/zero forcing number of the direct product and the strong product of two graphs.

Let $A = [a_{ij}] \in F^{m \times n}$ and $B \in F^{p \times q}$, where $F$ is a field. The *Kronecker product* of $A$ and $B$, denoted by $A \otimes B$, is the $mp \times nq$ matrix over $F$ with the block structure

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}.$$

**Theorem 16.** *If $m \geqslant 3$ and $n \geqslant 2$, then $M(K_m \times K_n) = Z(K_m \times K_n) = mn - 4$.*

**Proof.** Let $V(K_m \times K_n) = \{(x_i, y_j) : 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n\}$ and $S = V(K_m \times K_n) \setminus \{(x_2, y_2), (x_3, y_1), (x_1, y_2), (x_2, y_1)\}$. Then $S$ is a zero-forcing set for $K_m \times K_n$ with the zero forcing sequence $((x_1, y_1), (x_2, y_2), (x_3, y_1), (x_1, y_2))$, and so $Z(K_m \times K_n) \leqslant |S| = mn - 4$.

Next, we want to show that $mn - 4$ is a lower bound for $M(K_m \times K_n)$. Using the notation $A_n$ defined in the proof of Theorem 15 we consider an $mn \times mn$ matrix $A_m \otimes A_n$. Note that $A_n$ is skew-symmetric. The elementary fact about Kronecker product $(A_m \otimes A_n)^T = A_m^T \otimes A_n^T$ (see, for example, [26, page 8]) shows that $A_m \otimes A_n$ is symmetric. Moreover, it can readily be seen that $A_m \otimes A_n \in \mathcal{S}^{\mathbb{R}}(K_m \times K_n)$. Then by equation (1), $\operatorname{rank}(A_m \otimes A_n) = \operatorname{rank}(A_m)\operatorname{rank}(A_n) = 4$, and hence $mn - 4 = \operatorname{nullity}(A_m \otimes A_n) \leqslant M(K_m \times K_n) \leqslant Z(K_m \times K_n) \leqslant mn - 4$. $\square$

Using Lemma 19 below we will exhibit a large new class of product graphs $G$ for which $Z(G) = M^F(G)$ for any field $F$, that is, $G$ has a field independent minimum rank (see [13] for much more about field independence of the minimum rank of a graph). To achieve the proof of Lemma 19, we need some notation and facts.

Using the same idea used in [1] to show that $Z(P_s \boxtimes P_t) \leqslant s + t - 1$, we have the following upper bound estimates for $Z(G \boxtimes H)$ and $Z(G \times H)$. Lemma 17 is useful in the proof of Lemma 19 and is also independently interesting from a combinatorial point of view.

**Lemma 17.** *For graphs $G$ and $H$, $Z(G \boxtimes H) \leqslant |G|Z(H) + Z(G)|H| - Z(G)Z(H)$ and $Z(G \times H) \leqslant |G|Z(H) + Z(G)|H| - Z(G)Z(H)$.*

**Proof.** Denote by $S_G$ (resp. $S_H$) a minimum zero forcing set of $G$ (resp. $H$). Let $s = |V(G) \setminus S_G|$ and $t = |V(G) \setminus S_H|$. Denote by $\mathcal{P}_G$ (resp. $\mathcal{P}_H$) a zero forcing process on $G$ (resp. $H$) with zero forcing sequence $(g_1, g_2, \ldots, g_s)$ (resp. $(h_1, h_2, \ldots, h_t)$) and its corresponding color change sequence $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ (resp. $(\beta_1, \beta_2, \ldots, \beta_t)$). Consider a zero forcing process $\mathcal{P}$ on $G \boxtimes H$started from the initial set of black vertices $S = \{(g, h) : g \in S_G \text{ or } h \in S_H\}$ with the following zero forcing sequence:

$$\Phi = \begin{array}{lllll} (\,(g_1,h_1), & (g_1,h_2), & (g_1,h_3), & \cdots & (g_1,h_t), \\ \ (g_2,h_1), & (g_2,h_2), & (g_2,h_3), & \cdots & (g_2,h_t), \\ \ \vdots & \vdots & \vdots & & \vdots \\ \ (g_s,h_1), & (g_s,h_2), & (g_s,h_3), & \cdots & (g_s,h_t)\,), \end{array}$$

where the time steps are equipped with the lexicographical order $\prec$ such that for two time steps $(i,j)$ and $(i',j')$ we have $(i,j) \prec (i',j')$ if and only if $i < i'$ or ($i = i'$ and $j < j'$). We want to show that the zero forcing process $\mathcal{P}$ will eventually reach the situation in which all vertices of $G \boxtimes H$ are black.

Let $Q(i,j)$ be the statement: Vertex $(\alpha_i, \beta_j)$ is forced by vertex $(g_i, h_j)$ at time step $(i,j)$ in the process $\mathcal{P}$. We shall prove by induction on $(i,j)$ that $Q(i,j)$ holds at each time step $(i,j)$. For the induction basis, we consider the case $(i,j) = (1,1)$. Since $N_G(g_1) \setminus \{\alpha_1\} \subseteq S_G$ and $N_H(h_1) \setminus \{\beta_1\} \subseteq S_H$, by the definition of $S$ we see that $(g_1, h_1)$ has exactly one white neighbor $(\alpha_1, \beta_1)$ in $G \boxtimes H$ at time step $(1,1)$, and hence $(\alpha_1, \beta_1)$ is forced by $(g_1, h_1)$ at time step $(1,1)$ in the process $\mathcal{P}$.

Let $(i,j) \succ (1,1)$ be a given time step and assume that $Q(i',j')$ holds for all time steps $(i',j') \prec (i,j)$. We shall show that $Q(i,j)$ holds. First we claim that $(g_i, h_j)$ is a black vertex at time step $(i,j)$. Indeed, if it is not, then $g_i \notin S_G$ and $h_j \notin S_H$. It follows that $g_i$ (resp. $h_j$) must be forced by some vertex $g_{i'} \in V(G)$ with $i' < i$ (resp. $h_{j'} \in V(H)$ with $j' < j$) in the process $\mathcal{P}_G$ (resp. $\mathcal{P}_H$). Since $(i',j') \prec (i,j)$, by the induction hypothesis, $(g_i, h_j)$ is forced by $(g_{i'}, h_{j'})$ at time step $(i',j')$ in the process $\mathcal{P}$, a contradiction.

Next, we want to show that vertex $(g_i, h_j)$ has exactly one white neighbor $(\alpha_i, \beta_j)$ at time step $(i,j)$. Denote by $(g,h)$ a white neighbor of $(g_i, h_j)$ in $G \boxtimes H$ at time step $(i,j)$. Since $(g,h) \notin S$, $(g,h) = (\alpha_k, \beta_\ell)$ for some integers $k$ and $\ell$. It follows that $\{\alpha_i, \alpha_k\} \subseteq N_G(g_i)$ and $\{\beta_j, \beta_\ell\} \subseteq N_H(h_j)$ with $\{\alpha_i, \alpha_k\} \cap S_G = \emptyset$ and $\{\beta_j, \beta_\ell\} \cap S_H = \emptyset$. Since $g_i$ (resp. $h_j$) has exactly one white neighbor $\alpha_i$ (resp. $\beta_j$) at time $i$ (resp. $j$) in the zero forcing process $\mathcal{P}_G$ (resp. $\mathcal{P}_H$) on $G$ (resp. $H$), it can be seen that $k \leqslant i$ (resp. $\ell \leqslant j$). If $k < i$ then, by the fact that $(k,\ell) \prec (i,j)$ and the induction hypothesis, we see that $Q(k,\ell)$ holds. It follows that $(\alpha_k, \beta_\ell)$ is a black vertex at time step $(i,j)$, a contradiction. Thus it must be $k = i$. If $\ell < j$ then, by using the fact that $(k,\ell) \prec (i,j)$ and induction hypothesis again, we see that $Q(k,\ell)$ holds and hence $(\alpha_k, \beta_\ell)$ is a black vertex at time step $(i,j)$. That is a contradiction. Thus it must be $\ell = j$. From what we have already proved, we conclude that $(g,h) = (\alpha_i, \beta_j)$ and $Q(i,j)$ holds. This completes the inductive step.

Thus $Q(i,j)$ holds at each time step $(i,j)$ in the zero-forcing process $\mathcal{P}$. Therefore, $S$ is a zero forcing set of $G \boxtimes H$, and hence $Z(G \boxtimes H) \leqslant |S| = |G|Z(H) + Z(G)|H| - Z(G)Z(H)$.

To prove $Z(G \times H) \leqslant |G|Z(H) + Z(G)|H| - Z(G)Z(H)$, we just replace $G \boxtimes H$ by $G \times H$ in the above proof. This completes the proof of the lemma. $\square$

Let $\mathcal{S}_1^F(G)$ (resp. $\mathcal{S}_0^F(G)$) denote the set of all matrices $A$ in $\mathcal{S}^F(G)$ such that $A$ has non-zero (resp. zero) diagonal entries. We have the following observation, whose proof is straightforward and is omitted.

**Observation 18.** Suppose $F$ is a field, and $G$ and $H$ are graphs.

(a) If $A \in \mathcal{S}_1^F(G)$ and $B \in \mathcal{S}_1^F(H)$, then $A \otimes B \in \mathcal{S}_1^F(G \boxtimes H)$.
(b) If $A \in \mathcal{S}_0^F(G)$ and $B \in \mathcal{S}_0^F(H)$, then $A \otimes B \in \mathcal{S}_0^F(G \times H)$.

For $i \in \{0, 1\}$, define $\mathrm{mr}_i^F(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}_i^F(G)\}$ and $M_i^F(G) = \max\{\mathrm{nullity}(A) : A \in \mathcal{S}_i^F(G)\}$. Clearly we have $\mathrm{mr}_i^F(G) + M_i^F(G) = |G|$ for $i = 0, 1$. Denote by $\mathcal{A}^F$ (resp. $\mathcal{A}_0^F$, resp. $\mathcal{A}_1^F$) the collection of graphs $G$ for which $Z(G) = M^F(G)$ (resp. $Z(G) = M_0^F(G)$, resp. $Z(G) = M_1^F(G)$) holds. By Theorem 2 it can be seen that $\mathcal{A}_0^F \subseteq \mathcal{A}^F$ and $\mathcal{A}_1^F \subseteq \mathcal{A}^F$.

**Lemma 19.** *Suppose $F$ is a field and $s = |G|Z(H) + Z(G)|H| - Z(G)Z(H)$.*

(a) *If $G \in \mathcal{A}_1^F$ and $H \in \mathcal{A}_1^F$, then $G \boxtimes H \in \mathcal{A}_1^F$ and $Z(G \boxtimes H) = s$.*
(b) *If $G \in \mathcal{A}_0^F$ and $H \in \mathcal{A}_0^F$, then $G \times H \in \mathcal{A}_0^F$ and $Z(G \times H) = s$.*

**Proof.** (a) Denote by $A$ (resp. $B$) a matrix in $\mathcal{S}_1^F(G)$ (resp. $\mathcal{S}_1^F(H)$) with $\operatorname{rank}(A) = \operatorname{mr}_1^F(G)$ (resp. $\operatorname{rank}(B) = \operatorname{mr}_1^F(H)$).

With these notations, we have the following result:

$$\begin{aligned} |G||H| - s = & \ (|G| - Z(G))(|H| - Z(H)) = (|G| - M_1^F(G))(|H| - M_1^F(H)) \\ = & \ \operatorname{mr}_1^F(G)\operatorname{mr}_1^F(H) = \operatorname{rank}(A)\operatorname{rank}(B) = \operatorname{rank}(A \otimes B) \\ \geqslant & \ \operatorname{mr}_1^F(G \boxtimes H) (\text{by Observation 18(a)}) \\ = & \ |G||H| - M_1^F(G \boxtimes H) \geqslant |G||H| - M^F(G \boxtimes H) \\ \geqslant & \ |G||H| - Z(G \boxtimes H) (\text{by Theorem 2}) \\ \geqslant & \ |G||H| - s (\text{by Lemma 17}). \end{aligned}$$

Consequently, $M_1^F(G \boxtimes H) = M^F(G \boxtimes H) = Z(G \boxtimes H) = s$.

(b) This part follows exactly the same lines as in the proof of (a) and is thus omitted. □

From what we have already proved and results in [1,13] we have the following easy observations, whose proofs we omit because they are not difficult.

1. For any field $F$, $\{C_{4n}, P_{2n+1}, K_{m,n} : n \geqslant 1, m \geqslant 2\} \subseteq \mathcal{A}_0^F$.
2. For any field $F$, $\{C_{3n}, P_{3n-1}, K_{n+1} : n \geqslant 1\} \cup \{P\} \subseteq \mathcal{A}_1^F$, where $P$ is the Petersen graph.
3. $\{Q_d, P_{2k+1} \times K_n, K_r \times K_s : d \geqslant 2, k \geqslant 1, n \geqslant 2, r \geqslant 3, s \geqslant 2\} \subseteq \mathcal{A}_0^{\mathbb{R}}$.
4. $\{C_n, P_k : n \geqslant 5, k \geqslant 2\} \subseteq \mathcal{A}_1^{\mathbb{R}}$.

Let us define $\mathcal{A}$ as follows: $\mathcal{A} = \bigcap_F \mathcal{A}^F$, where the intersection is taken over all fields $F$. Notice that if $G$ is a graph in $\mathcal{A}$, then $M^F(G)$ is field independent. Starting from the above observations, with Lemma 19 at hand, we can exhibit a large new class of product graphs $G$ for which $Z(G) = M^F(G)$ for any field $F$. As a result, we have

$$C_{40} \times P_{25} \times K_{37,51} \times C_{84} \times K_{106,17} \in \mathcal{A} \text{ and } C_{63} \boxtimes P_{32} \boxtimes K_{2009} \boxtimes P \in \mathcal{A}.$$

### *3.3. Strongly regular graphs*

Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $\overline{G}$ be the complement graph of $G$ with $V(\overline{G}) = \{v_1', v_2', \ldots, v_n'\}$ and $E(\overline{G}) = \{v_i'v_j' : v_iv_j \notin E(G) \text{ and } i \neq j\}$. In the following, we define a graph product between $G$ and $\overline{G}$. Denote by $G \ominus \overline{G}$ the graph with vertex set $V(G) \cup V(\overline{G})$ and edge set $E(G) \cup E(\overline{G}) \cup \{v_iv_i' : 1 \leqslant i \leqslant n\}$. A *strongly regular graph* $G$ with parameters $(n, k, a, c)$ is a $k$-regular graph on $n$ vertices that is neither complete nor empty, where the number of common neighbors of every two adjacent (resp. distinct non-adjacent) vertices is $a$ (resp. $c$). A strongly regular graph $G$ is called *primitive* if both $G$ and $\overline{G}$ are connected. The following results about an $(n, k, a, c)$ strongly regular graph $G$ are well known (see, for example, Chapter 5 of [12] or Chapter 10 of [15]). The adjacency matrix $A$ of $G$ has the equation $A^2 = kI + aA + c(J - A - I)$, and its complement $\overline{G}$ is also a strongly regular graph with parameter $(n, n-k-1, n-2k+c-2, n-2k+a)$.

**Theorem 20.** *If $G$ is a strongly regular graph, then $Z(G \ominus \overline{G}) = M(G \ominus \overline{G})$. In particular, if $G$ is primitive then $Z(G \ominus \overline{G}) = |G|$; otherwise $Z(G \ominus \overline{G}) = |G| - 1$.*

**Proof.** Let $G$ be a strongly regular graph on the parameter $(n, k, a, c)$. We denote by $A$ and $B$ the adjacency matrices of $G$ and its complement $\overline{G}$, respectively. Since $B = J - A - I$, it is straightforward to see that $BA = (k - c)B + (k - a - 1)A$. To shorten notation, we let $r = k - a - 1$ and $s = k - c$.

First we consider the case that $G$ is primitive. In this case, by Lemma 10.1.1(c) of [15], we see that $r > 0$ and $s > 0$. Let us define $H$ to be the following $2n \times 2n$ symmetric matrix:

$$H = \begin{bmatrix} A - sI & -rsI \\ -rsI & rs(B - rI) \end{bmatrix}.$$

Since $H \in \mathcal{S}^{\mathbb{R}}(G \ominus \overline{G})$,

$$\begin{bmatrix} I & \mathbf{0} \\ B - rI & I \end{bmatrix} H = \begin{bmatrix} A - sI & -rsI \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and $V(G)$ is a zero forcing set for $G \ominus \overline{G}$, we get $n = \text{nullity}(H) \leqslant M(G \ominus \overline{G}) \leqslant Z(G \ominus \overline{G}) \leqslant n$.

It remains to consider the case that $G$ is connected and $\overline{G}$ is disconnected, say $\overline{G}$ has components $G_1, \ldots, G_t$. Let $V_1 = \{v \in V(G) : v \text{ is adjacent to some vertex of } G_1\}$. For any vertex $v$ of $V_1$, it can readily be checked that $(V_1 \setminus \{v\}) \cup V(G_2) \cup V(G_3) \cup \cdots \cup V(G_t)$ is a zero forcing set for $G \ominus \overline{G}$, and hence $Z(G \ominus \overline{G}) \leqslant n - 1$. Since $G$ is connected and $\overline{G}$ is disconnected, by Lemma 10.1.1(c) of [15], we have $r > 0$ and $s = 0$, and hence $BA = rA$. Let us define matrix $P$ as follows:

$$P = \begin{bmatrix} -rB & rI \\ rI & A + rI \end{bmatrix}.$$

Since $P \in \mathcal{S}^{\mathbb{R}}(G \ominus \overline{G})$ and

$$\begin{bmatrix} I & B \\ \mathbf{0} & I \end{bmatrix} P = \begin{bmatrix} \mathbf{0} & rJ \\ rI & A + rI \end{bmatrix},$$

we have $n - 1 = \text{nullity}(P) \leqslant M(G \ominus \overline{G}) \leqslant Z(G \ominus \overline{G}) \leqslant n - 1$. This completes the proof of the theorem. □

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