Grundy dominating sequences and zero forcing sets

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A B S T R A C T

If G is a graph, then a sequence $v_1, \ldots, v_m$ of vertices is a closed neighborhood sequence if for all $2 \leq i \leq m$ we have $N[v_i] \not\subseteq \cup_{j=1}^{i-1} N[v_j]$, and it is an open neighborhood sequence if for all $2 \leq i \leq m$ we have $N(v_i) \not\subseteq \cup_{j=1}^{i-1} N(v_j)$. The length of a longest closed (open) neighborhood sequence is the Grundy (Grundy total) domination number of G. In this paper we introduce two similar concepts in which the requirement on the neighborhoods is changed to $N(v_i) \not\subseteq \cup_{j=1}^{i-1} N[v_j]$ or $N[v_i] \not\subseteq \cup_{j=1}^{i-1} N(v_j)$. In the former case we establish a strong connection to the zero forcing number of a graph, while we determine the complexity of the decision problem in the latter case. We also study the relationships among the four concepts, and discuss their computational complexities.

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1. Introduction

In [1] the Grundy domination number $\gamma_{gr}(G)$ of a graph G was introduced as the length of a longest dominating sequence, that is, a sequence of vertices, such that each vertex of the sequence dominates at least one new vertex. While the problem is NP-complete in general, it was demonstrated in the paper that γ_{gr} can be obtained efficiently within the classes of trees, split graphs, and cographs. Interval graphs were added to this list in [2] where in addition a closed formula for the Grundy domination number of Sierpiński graph was proved. The investigation of the Grundy domination number on standard graph products, mostly

products of paths and cycles, was done in [3]. It was also observed in [3] that the edge clique cover number is an upper bound for the Grundy domination number.

It is quite common that along with a domination concept its total counterpart is studied. And indeed, the Grundy total domination number $\gamma_{gr}^t(G)$ of a graph G was studied in [4]. This concept is closely related to the hypergraph of open neighborhoods of a graph. Using a connection with edge covering sequences in hypergraphs the authors proved that also the total Grundy domination number decision problem is NP-complete. Among several additional results proved we extract that $\gamma_{gr}^t(G) \leq 2\gamma_{gr}(G)$ holds for any graph without isolated vertices.

Motivated by the above two concepts, we introduce in this paper the closely related concept of the Z-Grundy domination number $\gamma_{gr}^Z(G)$ of a graph G, which is the length of a longest sequence of vertices such that each vertex in the sequence has at least one neighbor that is not dominated by the previous vertices. We are going to prove in Section 2 that to determine $\gamma_{gr}^Z(G)$ is equivalent to compute $Z(G)$, where $Z(G)$ is the extensively studied zero forcing number [5] which is in turn closely related to the concept of power domination [6,7]. The complexity of the decision problem whether $Z(G)$ is at least some constant was shown to be NP-complete by [8] and the zero forcing number has been determined for many classes of graphs [5,9–11]. This connection on the one hand enables us to deduce $\gamma_{gr}^Z(G)$ for many classes of graphs, and to determine the zero forcing number of Sierpiński graphs and lexicographic products of paths and cycles on the other hand. To make the picture complete we introduce the L-Grundy domination number, which is the maximum length of a sequence $v_1, \ldots, v_r$ of (pairwise different) vertices such that for each vertex v_i the closed neighborhood $N[v_i]$ contains at least one vertex not totally dominated by the previous vertices in the sequence. In Section 3 we give the formal definition of the L-Grundy number and prove relations between the four types of the Grundy domination number. In Section 4 we prove that the decision problem for the L-Grundy domination number is NP-complete even when restricted to bipartite graphs. To prove this result hypergraphs again turn out to be useful.

In the rest of the introduction we recall the formal definitions of the Grundy domination number and the Grundy total domination number, and list some related terminology.

Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of a graph G. The corresponding set $\{v_1, \ldots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. The initial segment $(v_1, \ldots, v_i)$ of S will be denoted by S_i. A sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a *(legal) closed neighborhood sequence* if, for each i

$$N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset. \tag{1}$$

We also say that v_i is a *legal choice for Grundy domination*, when the above inequality holds. If for a closed neighborhood sequence S, the set $\widehat{S}$ is a dominating set of G, then S is called a *dominating sequence* of G. We will use a suggestive term by saying that v_i *footprints* the vertices from $N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$, and that v_i is the *footprinter* of any $u \in N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$. For a dominating sequence S any vertex in $V(G)$ has a unique footprinter in $\widehat{S}$. Clearly the length k of a dominating sequence $S = (v_1, \ldots, v_k)$ is bounded from below by the domination number $\gamma(G)$ of a graph G. We call the maximum length of a legal dominating sequence in G the *Grundy domination number* of a graph G and denote it by $\gamma_{gr}(G)$. The corresponding sequence is called a *Grundy dominating sequence* of G or γ_{gr}-sequence of G.

In a similar way total dominating sequences were introduced in [4], for graphs without isolated vertices. Using the same notation as in the previous paragraph, we say that a sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a *(legal) open neighborhood sequence* if, for each i

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset. \tag{2}$$

If the above is true, each v_i is said to be a *legal choice for Grundy total domination*, and we speak of *total footprinters* of which the meaning should be clear. It is easy to see that an open neighborhood sequence S in G of maximum length yields $\widehat{S}$ to be a total dominating set; the sequence S is then called a *Grundy total dominating sequence* or γ_{gr}^t*-sequence*, and the corresponding invariant the *Grundy total domination number* of G, denoted $\gamma_{gr}^t(G)$. Any legal open neigborhood sequence S, where $\widehat{S}$ is a total dominating set is called a *total dominating sequence*.

2. Z-Grundy domination and zero forcing

In view of (1) and (2) the following definition is natural. Let G be a graph without isolated vertices. A sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a *(legal) Z-sequence* if, for each i

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset. \tag{3}$$

The *Z-Grundy domination number* $\gamma_{gr}^Z(G)$ of the graph G is the length of a longest Z-sequence. Note that such a sequence is a legal closed neighborhood sequence and hence $\gamma_{gr}^Z(G) \leq \gamma_{gr}(G)$. Given a Z-sequence S, the corresponding set $\widehat{S}$ of vertices will be called a *Z-set*.

Remark 2.1. For any graph G without isolated vertices the equality $\gamma_{gr}^Z(G) = \gamma_{gr}(G)$ holds if and only if there exists a Grundy dominating sequence for G each vertex of which footprints some of its neighbors.

In this section we connect the Z-Grundy domination number with the well-established zero forcing number that is defined below. (This is also the reason for using "Z" in the name of the concept.)

Let G be a simple graph with vertex set $V(G) = \{1, \ldots, n\}$. The *minimum rank* $\mathrm{mr}(G)$ of G is the smallest possible rank over all symmetric real matrices whose (i,j)th entry, for $i \neq j$, is nonzero whenever vertices i and j are adjacent in G and is zero otherwise. The *maximum nullity* $\mathrm{M}(G)$ of G is the biggest possible nullity over all the above matrices. Clearly $\mathrm{M}(G) + \mathrm{mr}(G) = |V(G)|$. Next, we present a concept derived from the minimum rank of a graph, cf. [5]. Suppose that the vertices of a graph G are colored white and blue. If a given blue vertex has exactly one white neighbor w, then by the *color change rule* the color of w is changed to blue. A *zero forcing set* for G is a subset B of its vertices such that if initially vertices from B are colored blue and the remaining vertices are colored white, then by a repeated application of the color change rule all the vertices of G are turned to blue. The *zero forcing number* $Z(G)$ of a graph G is the size of a minimum zero forcing set. The zero forcing number is closely related to minimum rank (maximum nullity) because $|V(G)| - \mathrm{mr}(G) \leq Z(G)$, see [5].

The main result of this section reads as follows.

Theorem 2.2. *If G is a graph without isolated vertices, then*

$$\gamma_{gr}^Z(G) + Z(G) = |V(G)|.$$

Moreover, the complement of a (minimum) zero forcing set of G is a (maximum) Z-set of G and vice versa.

Proof. Without loss of generality we may assume that G is connected. Let $n = |V(G)|$.

Let B be a zero forcing set, i.e. the set of vertices initially colored blue, and let $k = n - |B|$. The following two sequences appear in the color change process in which eventually all vertices become blue. These are the sequences $(b_1, \ldots, b_k)$ and $(w_1, \ldots, w_k)$, where b_i is the blue vertex selected at the ith step of the color change process, and w_i is its unique white neighbor at that moment. We now claim that $(w_k, \ldots, w_1)$ is a Z-sequence.

Note that $b_k \neq w_k$ and hence w_k footprints b_k. Let now $i < k$. Then $N(b_i) \subseteq V(G) \setminus \{w_{i+1}, \ldots, w_k\}$, because otherwise at the ith step of the color change process the blue vertex b_i would be adjacent to the white vertex w_i and at least one more white vertex from the set $\{w_{i+1}, \ldots, w_k\}$, which contradicts the color change rule. It follows that w_i footprints b_i. In particular, if B is a minimum zero forcing set, that is, $|B| = n - k = Z(G)$, then $k \leq \gamma_{gr}^Z(G)$ which in turn implies that $\gamma_{gr}^Z(G) \geq n - Z(G)$.

Conversely, let $S = (u_1, \ldots, u_k)$ be a Z-sequence for G. We claim that $X = V(G) - \widehat{S}$ is a zero forcing set. Let $(a_1, \ldots, a_k)$ be a sequence, where a_i is an arbitrary vertex selected from the neighbors of u_i that are footprinted by u_i. At the beginning we color all vertices from X blue and the remaining vertices white. Note that a_k is colored blue because it was footprinted by the last vertex of S. Since u_k is the only white neighbor of a_k, we can color u_k at the first step of the color change process blue. Let now i be an arbitrary index smaller than k and assume that the vertices u_j, where $i < j \leq k$, are already colored blue. Since $a_i \in X \cup \{u_{i+1}, \ldots, u_k\}$, we infer that at this moment a_i is colored blue. Now, by the same argument as we used for u_k, we color u_i blue. By this procedure we end up with all vertices colored blue, which implies that X is a zero forcing set. In particular, if S is a maximum Z-sequence, that is $|\widehat{S}| = \gamma_{gr}^Z(G) = k$, then since $Z(G) \leq n - k$ we get $\gamma_{gr}^Z(G) \leq n - Z(G)$. $\square$

Combining Theorem 2.2 with the inequality $\gamma_{gr}^Z(G) \leq \gamma_{gr}(G)$ we get the following.

Corollary 2.3. *If G is a graph without isolated vertices, then*

$$\gamma_{gr}(G) + Z(G) \geq |V(G)|.$$

In particular, if G has a Grundy dominating sequence that is also a Z-sequence, then

$$\gamma_{gr}(G) + Z(G) = |V(G)|.$$

Proof. The first assertion follows because $\gamma_{gr}^Z(G) \leq \gamma_{gr}(G)$ holds for any graph G. If in addition S is a Grundy dominating sequence that is also a Z-sequence, then $\gamma_{gr}(G) = |\widehat{S}| \leq \gamma_{gr}^Z(G)$ and hence $\gamma_{gr}(G) = \gamma_{gr}^Z(G)$. $\square$

There are many examples where equality holds in Corollary 2.3. For instance, $\gamma_{gr}(P_n) = n - 1 = \gamma_{gr}^Z(P_n)$ and $Z(P_n) = 1$. To see that $\gamma_{gr}(G) + Z(G)$ can be arbitrary larger than the order of G, consider the stars $K_{1,n}$. Note that $\gamma_{gr}(K_{1,n}) = n$ and that $Z(K_{1,n}) = n - 1$, so that $\gamma_{gr}(K_{1,n}) + Z(K_{1,n}) = 2n - 1$. On the other hand, $\gamma_{gr}^Z(K_{1,n}) = 2$ in accordance with Theorem 2.2.

The above results are two-fold useful. If $Z(G)$ is known, then Theorem 2.2 gives us $\gamma_{gr}^Z(G)$ and a lower on $\gamma_{gr}(G)$. On the other hand, since the upper bound on the zero forcing number is usually obtained by construction, there are more problems with lower bounds. But if $\gamma_{gr}(G)$ is known for a graph G, then $|V(G)| - \gamma_{gr}(G)$ is a lower bound for $Z(G)$. Moreover, if there exists a Grundy dominating sequence that is a Z-sequence, then the value of $Z(G)$ immediately follows by Corollary 2.3. We continue with some applications of these two approaches.

The Cartesian product $G \square H$, the strong product $G \boxtimes H$, and the lexicographic product $G \circ H$ of graphs G and H all have the vertex set $V(G) \times V(G)$. In $G \square H$ vertices (g, h) and (g', h') are adjacent when $(gg' \in E(G)$ and $h = h')$ or $(g = g'$ and $hh' \in E(H))$. The strong product $G \boxtimes H$ is obtained from $G \square H$ by adding all the edges of the form $(g, h)(g', h')$, where $gg' \in E(G)$ and $hh' \in E(H)$; cf. [12]. In the lexicographic product $G \circ H$, vertices (g, h) and (g', h') are adjacent if either $gg' \in E(G)$ or $(g = g'$ and $hh' \in E(H))$.

The formulas for the zero forcing numbers of some graph products derived in [5] (for the products in items (i)–(vii) below), and in [6] (for item (viii)), together with Theorem 2.2, imply the following formulas for their Z-Grundy domination numbers. As usual, Q_n denotes the hypercube of order 2^n, while P_t, C_s, K_ℓ stand for the path of order t, the cycle of order s, and for the complete graph of order ℓ, respectively.

Corollary 2.4.

  (i) *If $n \geq 1$, then $\gamma_{gr}^{Z}(Q_n) = 2^{n-1}$.*

  (ii) *If $s, t \geq 2$, then $\gamma_{gr}^{Z}(K_s \square P_t) = s(t-1)$.*

  (iii) *If $2 \leq |V(G)| \leq t$, then $\gamma_{gr}^{Z}(G \square P_t) = |V(G)|(t-1)$.*

  (iv) *If $2 \leq s \leq t$, then $\gamma_{gr}^{Z}(P_s \square P_t) = s(t-1)$.*

  (v) *If $s \geq 3$ and $t \geq 2$, then $\gamma_{gr}^{Z}(C_s \square P_t) = st - \min\{s, 2t\}$.*

  (vi) *If $s \geq 4$ and $t \geq 2$, then $\gamma_{gr}^{Z}(C_s \square K_t) = t(s-2)$.*

  (vii) *If $s, t \geq 2$, then $\gamma_{gr}^{Z}(P_s \boxtimes P_t) = (s-1)(t-1)$.*

  (viii) *If $3 \leq s \leq t$ and $(s,t) \neq (2r+1, 2r+1)$ for some $r \geq 1$, then $\gamma_{gr}^{Z}(C_s \square C_t) = st - 2s$. Moreover,*
      $\gamma_{gr}^{Z}(C_s \square C_s) = s^2 - 2s + 1$ if s is odd.

Formulas in items (iv), (v), (vii), and (viii) were independently established in [3]. More precisely, these formulas were obtained for the Grundy domination number, yet the sequences applied were also Z-sequences. Hence the results follow by Remark 2.1.

Next, we add the following results on the zero forcing number of some product graphs for which the Z-Grundy domination number was established earlier.

Proposition 2.5.

  (i) *If $s \geq 3$ and $t \geq 2$, then $Z(C_s \boxtimes P_t) = 2t + s - 2$.*

  (ii) *If $s, t > 2$, then*

$$Z(P_s \circ P_t) = \begin{cases} \dfrac{s}{2} \cdot (t+1) - 1, & s \text{ is even} \\ st - \left\lceil \dfrac{s}{2} \right\rceil \cdot (t-1), & s \text{ is odd.} \end{cases}$$

  (iii) *If $s, t > 2$, then*

$$Z(P_s \circ C_t) = \begin{cases} \dfrac{s}{2} \cdot (t+2) - 1, & s \text{ is even} \\ st - \left\lceil \dfrac{s}{2} \right\rceil \cdot (t-2), & s \text{ is odd.} \end{cases}$$

  (iv) *If $s, t > 3$, then*

$$Z(C_s \circ C_t) = \begin{cases} \dfrac{s}{2} \cdot (t+2), & s \text{ is even} \\ st - \left\lfloor \dfrac{s}{2} \right\rfloor \cdot (t-2) - 1, & s \text{ is odd.} \end{cases}$$

Proof. These follow by a similar argument to the above. Grundy domination number of $C_s \boxtimes P_t$, $P_s \circ P_t$, $P_s \circ C_t$, and $C_s \circ C_t$ were determined in [3, Corollary 28, Corollary 11, Corollary 12, Corollary 14] and all lower bounds use Grundy sequences that are Z-Grundy sequences as well. Therefore Corollary 2.3 finishes the proof in all cases. □

For another application consider the Sierpiński graphs introduced in [13] and extensively studied by now, cf. the recent survey [14]. They are defined as follows. Set $[n] = \{1, 2, \ldots, n\}$ and $[n]_0 = \{0, 1, \ldots, n-1\}$. The Sierpiński graph S_p^n $(n, p \geq 1)$ is defined on the vertex set $[p]_0^n$, two different vertices $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ being adjacent if and only if there exists an $h \in [n]$ such that

  1. $u_t = v_t$, for $t = 1, 2, \ldots, h - 1$;

  2. $u_h \neq v_h$; and

  3. $u_t = v_h$ and $u_h = v_t$ for $t = h + 1, h + 2, \ldots, n$;
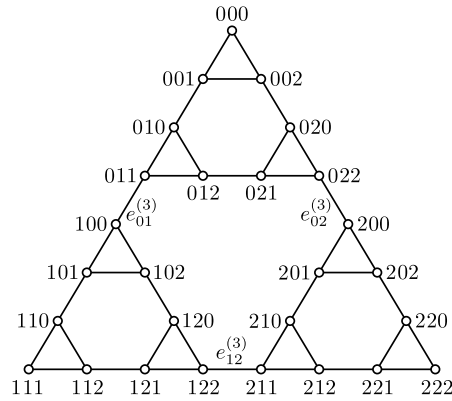
Fig. 1. The Sierpiński graph S_3^3.

It quickly follows from the definition that S_p^n can be constructed from p copies of S_p^{n-1} as follows. For each $i \in [p]_0$ concatenate i to the left of the vertices in a copy of S_p^{n-1} and denote the obtained graph with iS_p^{n-1}. Then for each $i \neq j$ join copies iS_p^{n-1} and jS_p^{n-1} by the single edge $e_{ij}^{(n)} = \{ij^{n-1}, ji^{n-1}\}$. The graph S_3^3 is depicted in Fig. 1.

Proposition 2.6. *If $n \geq 1$ and $p \geq 2$, then*

$$Z(S_p^n) = \frac{p}{2}\left(p^{n-2}(p-2)+1\right).$$

Proof. It was proved in [2, Theorem 3.1] that

$$\gamma_{gr}(S_p^n) = p^{n-1} + \frac{p(p^{n-1}-1)}{2}.$$

In the proof of this theorem it is shown that every vertex added to the appropriate Grundy dominating sequence footprints one of its neighbors. Consequently, the constructed Grundy sequence is also a Z-sequence. Therefore by Corollary 2.3, $\gamma_{gr}(S_p^n) = \gamma_{gr}^Z(S_p^n)$, and hence

$$Z(S_p^n) = |V(S_p^n)| - \gamma_{gr}(S_p^n) = p^n - \left(p^{n-1} + \frac{p(p^{n-1}-1)}{2}\right)$$

$$= \frac{p}{2}\left(p^{n-2}(p-2)+1\right). \qquad \square$$

3. L-Grundy domination number and relations between the four concepts

Observing the defining conditions in (1)–(3) it appears natural to consider the remaining, fourth related concept. It gives the longest sequences among all four versions, and we call it L-Grundy domination. Given a graph G, a sequence $S = (v_1, \dots, v_k)$ of **distinct** vertices from G is called a *(legal) L-sequence* if, for each i

$$N[v_i] \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset. \qquad (4)$$

Then the *L-Grundy domination number*, $\gamma_{gr}^L(G)$, of the graph G is the length of a longest L-sequence. Given an L-sequence S, the corresponding set $\widehat{S}$ of vertices will be called an *L-set* (the requirement that all vertices
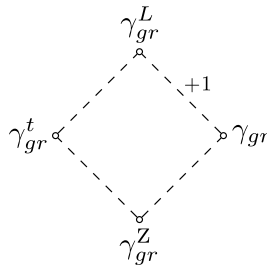
Fig. 2. Relations between Grundy domination numbers. In the Hasse diagram an invariant is below another invariant if and only if this is true for the respective Grundy domination numbers for any graph for which both invariants are well-defined. The label **+1** describes that $\gamma_{gr}^L(G) \geq \gamma_{gr}(G) + 1$ for any graph G with no isolated vertices.

in S are distinct prevents the creation of an infinite sequence by repetition of one and the same vertex). Note that it is possible that some v_i in an L-sequence L-footprints only itself, but at the same time it does not totally dominate any vertex. For instance, consider the star $K_{1,n}$, $n \geq 2$, with the center w and leaves $v_1, \ldots, v_n$. Then $(v_1, \ldots, v_n, w)$ is an L-sequence in which each of the vertices $v_2, \ldots, v_n$ L-footprints only itself. Note also that if a vertex footprints itself, then it will be footprinted later by some other vertex.

The basic inequalities about the four concepts are collected in the next result.

Proposition 3.1. *If G is a graph with no isolated vertices, then*

(i) $\gamma_{gr}^Z(G) \leq \gamma_{gr}(G) \leq \gamma_{gr}^L(G) - 1$,
(ii) $\gamma_{gr}^Z(G) \leq \gamma_{gr}^t(G) \leq \gamma_{gr}^L(G)$,

and all the bounds are sharp.

Proof. The bounds $\gamma_{gr}^Z(G) \leq \gamma_{gr}(G)$, $\gamma_{gr}^Z(G) \leq \gamma_{gr}^t(G)$, and $\gamma_{gr}^t(G) \leq \gamma_{gr}^L(G)$ follow directly from definitions. To prove that $\gamma_{gr}(G) \leq \gamma_{gr}^L(G) - 1$ holds for any graph G with isolated vertices, consider an arbitrary Grundy dominating sequence S of a graph G, and let v_k be the last vertex in the sequence. Then v_k or some of its neighbors has not yet been dominated in previous steps; let us denote the vertex not dominated before the last step by v_r, and let S' be the sequence obtained from S by deleting the last vertex v_k from it. Note that $S' \oplus (v_r, v_t)$, where v_t is any neighbor of v_r, is an L-sequence of length $k + 1$. This readily implies the bound $\gamma_{gr}^L(G) \geq \gamma_{gr}(G) + 1$.

The sharpness of $\gamma_{gr}^Z(G) \leq \gamma_{gr}(G)$ is demonstrated by Sierpiński graphs S_p^n. To see that $\gamma_{gr}^Z(G) \leq \gamma_{gr}^t(G)$ is sharp consider for instance stars $K_{1,n}$ with $n \geq 2$. To establish the sharpness of $\gamma_{gr}^t(G) \leq \gamma_{gr}^L(G)$, use [4, Theorem 4.2], where the graphs G for which $\gamma_{gr}^t(G) = |V(G)|$ holds are characterized. (A simple family in this class of graphs are paths of even order.) Finally, to show the sharpness of $\gamma_{gr}(G) \leq \gamma_{gr}^L(G) - 1$, the class of graphs G for which $\gamma_{gr}(G) = |V(G)| - 1$, considered briefly in [1], presents a family with sharpness of this bound (examples are stars $K_{1,n}$ and paths P_n). □

Proposition 3.1 can be summarized as the partial order from Fig. 2.

As an application of the left inequality in Proposition 3.1(ii), we get another lower bound for the zero forcing number of an arbitrary graph with no isolated vertices. Let $\beta(G)$ stand for the *vertex cover number* of a graph G, and $\alpha(G)$ be the *independence number* of G.

Proposition 3.2. *If G is a graph with no isolated vertices, then $Z(G) \geq \alpha(G) - \beta(G)$.*

Proof. Note that Proposition 3.1(ii) implies that $Z(G) \geq |V(G)| - \gamma_{gr}^t(G)$ for any graph G with no isolated vertices. By [15, Proposition 3.2], $\gamma_{gr}^t(G) \leq 2\beta(G)$, which in turn implies that $Z(G) \geq |V(G)| - 2\beta(G)$. Finally, by the well-known formula $\alpha(G) = |V(G)| - \beta(G)$, we conclude that $Z(G) \geq \alpha(G) - \beta(G)$. $\square$

It is easy to see that the bound $Z(G) \geq \alpha(G) - \beta(G)$ is sharp in stars $K_{1,n}$ and in odd paths P_{2k+1}. It would be interesting to characterize the extremal graphs for this bound.

It was proved in [4] that $\gamma_{gr}^t(G) \leq 2\gamma_{gr}(G)$. We can improve this result as follows.

Proposition 3.3. *If G is a graph without isolated vertices, then $\gamma_{gr}^t(G) \leq 2\gamma_{gr}^Z(G)$ and the bound is sharp.*

Proof. Let $S = (s_1, \ldots, s_k)$ be a Grundy total dominating sequence and let A_i be the set of vertices totally footprinted by s_i. We will prove that at most half of the vertices can be removed from S in such a way that the resulting sequence S' is a legal Z-sequence. If $s_i \in \widehat{S}$ prevents S from being a legal Z-sequence, then $N(s_i) \setminus \bigcup_{j=1}^{i-1} N[s_j] = \emptyset$. Since S is a total dominating sequence this implies $A_i \subseteq \{s_1, \ldots, s_{i-1}\}$. Let $A = \{s_i \in \widehat{S} : A_i \subseteq \{s_1, \ldots, s_{i-1}\}\}$.

We will first prove that for any $s_i, s_j \in A$, with $i < j$, $A_i \cap A_j = \emptyset$. Suppose that $s_t \in A_i \cap A_j$. Then $t < i < j$. Since $s_t \in A_i$, s_t is totally footprinted by s_i and thus it cannot be totally footprinted by s_j, a contradiction.

Finally suppose that for some vertex $s_j \in A$, $s_i \in A \cap A_j$. Then $i < j$. As $s_i \in A$, there exists $t < i$, such that s_i totally footprints s_t. Therefore, when s_j is added to the sequence, s_i is already totally footprinted by s_t, a contradiction with $s_i \in A_j$.

Thus for any $s_i \in A$ there exists $s_t \in (\widehat{S} \setminus A) \cap A_i$ that is not contained in $\bigcup_{s_j \in A \setminus \{s_i\}} A_j$. Therefore $|A| \leq \frac{k}{2}$ and if we define S' to be the sequence obtained from S by removing all its elements that belong to A, then S' is a legal Z-sequence, $\gamma_{gr}^Z(G) \geq k - |A| \geq k - \frac{k}{2} = \frac{1}{2}\gamma_{gr}^t(G)$.

The bound in Proposition 3.3 is sharp. For example $\gamma_{gr}^t(K_n) = 2$ and $\gamma_{gr}^Z(K_n) = 1$ when $n \geq 2$. Another example is a graph G obtained by gluing together two copies of K_n at one vertex. If $n \geq 3$, then $\gamma_{gr}^t(G) = 4 = 2\gamma_{gr}^Z(G)$. $\square$

A similar inequality holds between the L-Grundy domination number and the Grundy domination number. Its proof also proceeds along the same lines as the proof of Proposition 3.3.

Proposition 3.4. *If G is a graph, then $\gamma_{gr}^L(G) \leq 2\gamma_{gr}(G)$ and the bound is sharp.*

Proof. Let $S = (s_1, \ldots, s_k)$ be a legal L-sequence with $\gamma_{gr}^L(G) = k$ and let A_i be the set of vertices L-footprinted by s_i. We will prove that at most half of the vertices can be removed from S in such a way that the resulting sequence S' is a legal closed neighborhood sequence. If $s_i \in \widehat{S}$ prevents S from being a legal closed neighborhood sequence, then $N[s_i] \setminus \bigcup_{j=1}^{i-1} N[s_j] = \emptyset$. Since S is an L-sequence this implies that $A_i \subseteq \{s_1, \ldots, s_{i-1}\}$. Let $A = \{s_i \in \widehat{S} : A_i \subseteq \{s_1, \ldots, s_{i-1}\}\}$.

We will first prove that for any $s_i, s_j \in A$, with $i < j$, $A_i \cap A_j = \emptyset$. Suppose that $s_t \in A_i \cap A_j$. Then $t < i < j$. Since $s_t \in A_i$, s_t is L-footprinted by s_i and thus it cannot be L-footprinted by s_j, a contradiction.

Finally suppose that for some vertex $s_j \in A$, $s_i \in A \cap A_j$. Then $i < j$. As $s_i \in A$, there exists $t < i$, such that s_i L-footprints s_t. Therefore, when s_j is added to the sequence, s_i is already L-footprinted by s_t, a contradiction with $s_i \in A_j$.

Thus for any $s_i \in A$ there exists $s_t \in (\widehat{S} \setminus A) \cap A_i$ that is not contained in $\bigcup_{s_j \in A \setminus \{s_i\}} A_j$. Therefore $|A| \leq \frac{k}{2}$ and if we define S' to be the sequence obtained from S by removing all its elements that belong to A, then S' is a legal closed neighborhood sequence in G, $\gamma_{gr}(G) \geq k - |A| \geq k - \frac{k}{2} = \frac{1}{2}\gamma_{gr}^L(G)$. Again, the bound from Proposition 3.4 is sharp, as $\gamma_{gr}^L(K_n) = 2$ and $\gamma_{gr}(K_n) = 1$ for every $n \geq 2$. Another example is a graph G obtained by gluing together two copies of K_n at one vertex. If $n \geq 3$, then $\gamma_{gr}^L(G) = 4 = 2\gamma_{gr}(G)$. $\square$

Similar relations between the Grundy domination number and the Z-Grundy domination number, as well as between the L-Grundy domination number and the Grundy total domination number, do not hold. That is, there exist graphs G with $\gamma_{gr}(G) > C\gamma_{gr}^{Z}(G)$ and graphs G with $\gamma_{gr}^{L}(G) > C\gamma_{gr}^{t}(G)$, where C is an arbitrarily chosen positive constant. For example, if G is the star $K_{1,n}$, $n \geq 3$, then $\gamma_{gr}(K_{1,n}) = n$, $\gamma_{gr}^{L}(K_{1,n}) = n+1$, and $\gamma_{gr}^{t}(K_{1,n}) = \gamma_{gr}^{Z}(K_{1,n}) = 2$. It was noticed in [4] that $\gamma_{gr}(G)$ can be arbitrarily bigger than $\gamma_{gr}^{t}(G)$, the stars again form a simple family demonstrating this fact.

4. On the computational complexity of L-Grundy domination

Recall that NP-completeness results for decision versions of γ_{gr} and γ_{gr}^{t} have been known, see [1,4], respectively. Furthermore, it was established in [8] that the zero forcing number yields an NP-complete problem, which combined with the formula $\gamma_{gr}^{Z}(G) + Z(G) = |V(G)|$ from Theorem 2.2 implies that the decision version of γ_{gr}^{Z} is also NP-complete. Hence it remains to consider the remaining invariant, γ_{gr}^{L}, and the corresponding computational complexity problem:

> L-GRUNDY DOMINATION NUMBER
> *Input:* A graph $G = (V, E)$, and an integer k.
> *Question:* Is $\gamma_{gr}^{L}(G) \geq k$?

In the study of the above problem we involve covering sequences in hypergraphs, in a similar way as in [1], where the connection between dominating sequences in graphs and covering sequences in hypergraphs have been established, see also [4] for another similar application of covering sequences in hypergraphs.

An *edge cover* of a hypergraph $\mathcal{H} = (X, \mathcal{E})$ with no isolated vertices, is a set of hyperedges from $\mathcal{E}$ that cover all vertices of X. That is, the union of the hyperedges from an edge cover is the ground set X. The minimum number of hyperedges in an edge cover of $\mathcal{H}$ is called the *(edge) covering number* of $\mathcal{H}$ and is denoted by $\rho(\mathcal{H})$. A sequence $\mathcal{C} = (C_1, \ldots, C_r)$, where $C_i \in \mathcal{E}$, is called a *(legal) hyperedge sequence* of $\mathcal{H}$, if for any i, $1 \leq i \leq r$, C_i is picked in such a way that it covers some vertex not captured by previous steps; that is, $C_i \setminus (\cup_{j<i} C_j) \neq \emptyset$. In this case $\mathcal{C}$ is called an *edge covering sequence*, and the maximum length r of an edge covering sequence of $\mathcal{H}$ is called the *Grundy covering number* of $\mathcal{H}$, and is denoted by $\rho_{gr}(\mathcal{H})$. It was shown in [1] that determining whether $\rho_{gr}(\mathcal{H})$ is bounded from below by a given constant is an NP-complete problem.

Theorem 4.1. L-GRUNDY DOMINATION NUMBER *is NP-complete, even when restricted to bipartite graphs.*

Proof. Let G be an arbitrary bipartite graph, with (A, B) being the bipartition of $V(G)$. Let G^* be the bipartite graph obtained from G by adding the set I of $|B|$ independent vertices to the set A, and connecting each vertex of I to each vertex of B by an edge. Clearly, the bipartition of $V(G^*)$ is $(A \cup I, B)$, and $|A \cup I| = |A| + |B|$.

From the (bipartite) graph G we derive the following hypergraph (which is one of the components of the open neighborhood hypergraph of G):

$$\mathcal{H}(G, B) = (A, \{N_G(b) \mid b \in B\}).$$

In an analogous way $\mathcal{H}(G^*, B)$ is defined, and clearly $\rho_{gr}(\mathcal{H}(G, B)) = \rho_{gr}(\mathcal{H}(G^*, B))$. The following claim is the crucial step in the proof.

Claim. $\gamma_{gr}^{L}(G^*) = |A| + |B| + \rho_{gr}(\mathcal{H}(G, B))$.

Proof (of the claim). It is easy to see that $\gamma_{gr}^L(G^*) \geq |A| + |B| + \rho_{gr}(\mathcal{H}(G, B))$. Indeed, consider the sequence, which first takes all vertices of $A \cup I$, and then emulates the Grundy covering sequence $(C_1, \ldots, C_k)$ of $\mathcal{H}(G^*, B)$ (or equivalently, of $\mathcal{H}(G, B)$), by choosing b_i as the ith term, where $C_i = N_G(b_i)$. It is clear that the resulting sequence is an L-sequence of G^* with length $|A| + |B| + \rho_{gr}(\mathcal{H}(G, B))$.

For the reversed inequality, let S be an L-sequence in G^*. First note that whenever a vertex $b \in B$ appears in S, then at most one vertex from I can appear in S after b. Now, consider two possibilities. Suppose that the first vertex b from B that appears in S appears in S before the first vertex from I. If all vertices from B appear in S before any vertex from I appeared in S, then the length of S is at most $|A| + |B| + 1$, which is clearly not bigger than $|A| + |B| + \rho_{gr}(\mathcal{H}(G, B))$. Otherwise, the sequence S starts with some vertices from $A \cup B$, then a vertex from I appears, and after that some vertices from $A \cup B$ come in S. In this case we also get $|\widehat{S}| \leq |A| + |B| + 1$ as desired.

Second possibility is that the first vertex from I that appears in S appears in S before the first vertex from B. Then all vertices from B are totally dominated at the time the first vertex from B is added to S. This implies that $|B \cap \widehat{S}| \leq \rho_{gr}(\mathcal{H}(G, B))$, which in turn implies that $|\widehat{S}| \leq |A| + |B| + \rho_{gr}(\mathcal{H}(G, B))$, which finally implies $\gamma_{gr}^L(G^*) \leq |A| + |B| + \rho_{gr}(\mathcal{H}(G, B))$. $\square$

Since determining the Grundy covering number of a hypergraph (with no isolated vertices) is NP-hard, and $\mathcal{H}(G, B)$ can represent an arbitrary hypergraph, we infer from the claim that determining $\gamma_{gr}^L(G^*)$ is also NP-hard for any bipartite graph G. $\square$

5. Concluding remarks

1. Chang et al. introduced the concept of k-power domination [16] as a natural generalization of power domination. An analogous concept of the so-called k-forcing was introduced by Amos et al. [17]; the latter concept generalizes the zero forcing number. (For a connection between the two concepts, see [18].) Given a graph G, the *k-forcing set* is a subset $S \subseteq V(G)$ (S are the vertices initially colored blue) such that in the color change process all neighbors of a blue vertex with at most k white neighbors become blue, and at the end of this propagation procedure all vertices are blue. Following our approach from Section 2, we present the concept of *k–Z-Grundy domination number* of a graph G, defined as follows.

   Let G be a graph with $\delta(G) \geq k$. A sequence $S = (v_1, \ldots, v_\ell)$, where $v_i \in V(G)$, is called a *(legal) k–Z-sequence* if, for each i there exists a vertex $u_i \in N(v_i)$, such that $u_i \in N[v_j]$ holds for less than k vertices v_j, $j \in \{1, \ldots, i - 1\}$. (For $k = 1$ this definition coincides with that of the (legal) Z-sequence.) Note that if S is a maximal k–Z-sequence, then $\widehat{S}$ is a k-dominating set, i.e., $\widehat{S}$ has the property that each vertex outside $\widehat{S}$ has at least k neighbors in $\widehat{S}$. The corresponding invariant, the k-domination number $\gamma_k(G)$ of a graph G, was introduced in [19], and studied in several papers, see e.g. [20–22]. The *k-Z-Grundy domination number* $\gamma_{gr}^{Z,k}(G)$ of the graph G is the length of a longest k–Z-sequence. By the above connection with k-domination, we infer that $\gamma_{gr}^{Z,k}(G) \geq \gamma_k(G)$.

2. Propagation time for zero forcing was introduced in [23] as the minimum number of propagation steps over all smallest zero-forcing sets. Being in a given state, a propagation step consists of coloring all vertices blue that may be colored blue in the state. Analogous concept for the power domination was introduced in [8,24] and independently in [25,26].

   The interpretation of these concepts in terms of the Z-Grundy domination is the following. White vertices that will turn blue at the same propagation step have the property that the set of the corresponding blue neighbors partitions into private neighbors of each of the white neighbors, where private neighbors are considered with respect to the set of white vertices, before the propagation step. In terms of the Z-Grundy dominating sequence S that is built in the reversed process to zero forcing propagation, this implies that a subsequence of the corresponding (white) vertices can be permuted in

an arbitrary order such that the resulting sequence is still a Z-Grundy dominating sequence. Hence, given a zero forcing set, the number of propagation steps coincides with the number of uniquely determined consecutive subsequences of the Z-sequence with the above permutation property, when permutability is observed in the backwards order.

More precisely, given a Z-Grundy sequence S, with $S = (v_1, \ldots, v_r)$, we look for the minimum index i_1 such that $(v_{i_1}, \ldots, v_r)$ is permutable in S, then consider the minimum index i_2 such that $(v_{i_2}, \ldots, v_{i_1-1})$ is permutable in S, and so on, until we reach the beginning of S. This reversed order permutability gives a uniquely determined set of indices $i_1, \ldots, i_t$, and t coincides with the number of propagation steps in a given zero-forcing procedure. This leads to the following result.

Proposition 5.1. *Given a graph G, the zero forcing propagation time of G coincides with the minimum number of consecutive permutable subsequences obtained in the reversed order over all longest Z-sequences in G.*

3. There are many equality cases, for which it would be interesting to characterize the extremal graphs. We list some of them:
   - Characterization of graphs G with $|V(G)| = \gamma_{gr}^L(G)$.
   - Characterization of graphs G with $\gamma_{gr}^L(G) = \gamma_{gr}^Z(G) + 1$.
   - Characterization of graphs G with $\gamma_{gr}^L(G) = 2\gamma_{gr}(G)$.
   - Characterization of graphs G with $\gamma_{gr}^Z(G) = \gamma_{gr}(G)$.

   A similar problem about characterizing graphs for which $\gamma_{gr}^t(G) = 2\gamma_{gr}(G)$ holds were posed in [4]. All the above questions can be asked in some specific, interesting families of graphs, such as trees, k-regular graphs, and graph products. In particular, in our next project, we plan to consider different Grundy domination numbers in grid-like and toroidal graphs with respect to various graphs products.

4. Recall that the zero forcing number is closely related to minimum rank (and maximum nullity) via the formula: $|V(G)| - \mathrm{mr}(G) \le Z(G)$. It would be interesting to establish a (Grundy)-domination-type concept that would be directly connected to the minimum rank of a graph.

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