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Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph

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ABSTRACT

The minimum rank of a simple graph G is defined to be the smallest possible rank over all symmetric real matrices whose ijth entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise; maximum nullity is taken over the same set of matrices. The zero forcing number is the minimum size of a zero forcing set of vertices and bounds the maximum nullity from above. The spread of a graph parameter at a vertex v or edge e of G is the difference between the value of the parameter on G and on $G - v$ or $G - e$. Rank spread (at a vertex) was introduced in [4]. This paper introduces vertex spread of the zero forcing number and edge spreads for minimum rank/maximum nullity and zero forcing number. Properties of the spreads are established and used to determine values of the minimum rank/maximum nullity and zero forcing number for various types of grids with a vertex or edge deleted.

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1. Introduction

All matrices discussed are real and symmetric; the set of $n \times n$ real symmetric matrices will be denoted by $S_n(\mathbb{R})$. A *graph* $G = (V_G, E_G)$ means a simple undirected graph (no loops, no multiple edges) with a finite nonempty set of vertices V_G and edge set E_G (an edge is a two-element subset of vertices). For $A \in S_n(\mathbb{R})$, the *graph* of A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leqslant i < j \leqslant n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$.

Let G be a graph. The *set of symmetric matrices described by* G is

$$\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}.$$

The *maximum nullity of* G is

$$\mathrm{M}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\},$$

and the *minimum rank* of G is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}.$$

Clearly $\mathrm{mr}(G) + \mathrm{M}(G) = |G|$, where the *order* $|G|$ is the number of vertices of G. See [10] for a survey of results and discussion of the motivation for the minimum rank/maximum nullity problem. The rank spread (at a vertex), i.e., the difference between the minimum rank of a graph and the minimum rank after deleting a vertex, was introduced in [4]. Here we introduce and study the (vertex) spread of the zero forcing number, and the edge spread for minimum rank and zero forcing number. Definitions and general properties of the spreads are given in Section 2. In Section 3 these properties are applied to determine spreads (and thus minimum rank, maximum nullity, and zero forcing number after deletion of a vertex or edge) of various families of graphs having a grid structure.

The zero forcing number was introduced in [1] and the associated terminology was extended in [2,3,11,12]. Let G be a graph with each vertex colored either white or black. Vertices change color according to the *color-change rule*: If u is a black vertex and exactly one neighbor w of u is white, then change the color of w to black. When the color-change rule is applied to u to change the color of w, we say u *forces* w and write $u \rightarrow w$. Given a coloring of G, the *derived set* is the set of black vertices obtained by applying the color-change rule until no more changes are possible. A *zero forcing set* for G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, then the derived set is all the vertices of G. The *zero forcing number* $\mathrm{Z}(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$. The zero forcing number is now widely used in the computation of minimum rank and maximum nullity.

Theorem 1.1 [1, Proposition 2.4]**.** *For any graph* G, $\mathrm{M}(G) \leqslant \mathrm{Z}(G)$.

Families of graphs G for which $\mathrm{Z}(G) = \mathrm{M}(G)$ are studied in [1,12]. Of particular use in this paper is the following (shortened) result (definitions of the graphs T_m, $P_m \,\Box\, P_n$, and $P_m \boxtimes P_n$ are given in Sections 3.1, 3.2, and 3.4 below).

Theorem 1.2 [1, Theorem 4.10]**.** *For each of the following families of graphs,* $\mathrm{Z}(G) = \mathrm{M}(G)$.

1. *Any graph G such that $|G| \leqslant 6$.*
2. *Any complete graph K_n, path P_n, or cycle C_n.*
3. *Any tree T.*
4. *Any supertriangle T_m.*
5. *Any grid graph $P_m \,\Box\, P_n$.*
6. *Any king grid $P_m \boxtimes P_n$.*

In [1, Example 4.1] it is shown that for the pentasun H_5 (shown in Fig. 1 below), $\mathrm{Z}(H_5) > \mathrm{M}(H_5)$. An *optimal zero forcing set* is a zero forcing set having the minimum number of elements. For a given zero forcing set, construct the derived set, listing the forces in the order in which they were performed.

This list is a *chronological list of forces*, and an *optimal chronological list of forces* is a chronological list of forces of an optimal zero forcing set. A *forcing chain* (for a particular chronological list of forces) is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k-1$, $v_i \to v_{i+1}$. A *maximal forcing chain* is a forcing chain that is not a proper subsequence of another zero forcing chain. The following result will be used.

Theorem 1.3 [2]. *For any connected graph G of order more than one, no vertex is in every optimal zero forcing set of G.*

Definition 1.4. Let G be a graph, let Z be a zero forcing set, and let $\mathcal{F}$ be a chronological list of forces of Z. The *chain set* of $\mathcal{F}$ is the set of maximal forcing chains of $\mathcal{F}$. If a chain set $\mathcal{C}$ contains a chain (v) consisting of only one vertex, then we say that $\mathcal{C}$ contains v as a *singleton*. An *optimal chain set* is a chain set from a chronological list of forces of an optimal zero forcing set.

The *path cover number* $\mathrm{P}(G)$ of G is the smallest positive integer m such that there are m vertex-disjoint induced paths in G such that every vertex of G is a vertex of one of the paths. Path cover number was first used in the study of minimum rank and maximum eigenvalue multiplicity in [13] (since the matrices in $\mathcal{S}(G)$ are symmetric, algebraic and geometric multiplicities of eigenvalues are the same, and since the diagonal is free, maximum eigenvalue multiplicity is the same as maximum nullity). In [13] it was shown that for a tree T, $\mathrm{P}(T) = \mathrm{M}(T)$; however, in [4] it was shown that $\mathrm{P}(G)$ and $\mathrm{M}(G)$ are not comparable for graphs unless some restriction is imposed on the type of graph. Recently Sinkovic established a relationship for outerplanar graphs (a graph is *outerplanar* if it has a drawing in the plane without crossing edges such that one face contains all vertices).

Theorem 1.5 [15]. *If G is an outerplanar graph, then* $\mathrm{P}(G) \geqslant \mathrm{M}(G)$.

The vertices in a forcing chain induce a path in G because the forces in a forcing chain occur chronologically in the order of the chain (since only a black vertex can force). The maximal forcing chains in an optimal chain set are disjoint, together contain all the vertices of G, and the elements of the set Z are the initial vertices of these chains. Thus we have the following result.

Proposition 1.6 [11]. *For any graph G,* $\mathrm{P}(G) \leqslant \mathrm{Z}(G)$.

Let $G = (V_G, E_G)$ be a graph and $W \subseteq V_G$. The *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{v, w\} \in E_G : v, w \in W\}$. The subgraph induced by $\overline{W} = V_G \setminus W$ is usually denoted by $G - W$, or in the case W is a single vertex $\{v\}$, by $G - v$. If e is an edge of $G = (V_G, E_G)$, the subgraph $(V_G, E_G \setminus \{e\})$ is denoted by $G - e$. The *contraction* of edge $e = \{u, v\}$ of G is obtained by identifying the vertices u and v to a single vertex, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. A *minor* of G arises by performing a sequence of deletions of edges, deletions of isolated vertices, and/or contractions of edges. A graph parameter β is *minor monotone* if for any minor H of G, $\beta(H) \leqslant \beta(G)$.

Definition 1.7. Let G be a graph, let $e = \{v, w\}$ be an edge in G, let Z be a zero forcing set of G, and let $\mathcal{F}$ be a chronological list of forces of Z where $v \to w$ exists in $\mathcal{F}$. Then $\mathcal{F} - e$ denotes the removal of $v \to w$ from $\mathcal{F}$. If $\mathcal{C}$ is the chain set of $\mathcal{F}$, then $\mathcal{C} - e$ denotes the chain set for $\mathcal{F} - e$.

Note that with the notation of Definition 1.7, $\mathcal{F} - e$ is a chronological list of forces for the zero forcing set $Z' = Z \cup \{w\}$ of $G - e$.

Colin de Verdière introduced several minor monotone graph parameters equal to the maximum nullity among all matrices satisfying several conditions including the Strong Arnold Hypothesis (defined below). Parameters satisfying the Strong Arnold Hypothesis are now called *Colin de Verdière type parameters* and have proved useful in the study of minimum rank/maximum nullity. A real symmetric matrix A is said to satisfy the *Strong Arnold Hypothesis* (*SAH*) provided there does not exist a nonzero real symmetric matrix X satisfying $AX = 0$, $A \circ X = 0$, and $I \circ X = 0$, where $\circ$ denotes the Hadamard (entrywise) product and I is the identity matrix. In [7], Colin de Verdière introduced the parameter $\nu(G)$ that is defined to be the maximum nullity among matrices A satisfying the three conditions:

- $\mathcal{G}(A) = G$;
- A satisfies the Strong Arnold Hypothesis;
- A is positive semidefinite.

The parameter $\xi(G)$ was introduced in [5] as a Colin de Verdière type parameter intended for use in computing maximum nullity and minimum rank, by removing any unnecessary restrictions while preserving minor monotonicity. Define $\xi(G)$ to be the maximum nullity among real symmetric matrices that satisfy the two conditions:

- $\mathcal{G}(A) = G$.
- A satisfies the Strong Arnold Hypothesis.

Clearly, for every graph G, $\nu(G) \leqslant \xi(G) \leqslant \mathrm{M}(G)$. Both ν [7] and ξ [5] have been shown to be minor monotone.

A *clique* is subgraph that is a complete graph. The *union* of $G_i = (V_i, E_i)$ is $\cup_{i=1}^{h} G_i = (\cup_{i=1}^{h} V_i, \cup_{i=1}^{h} E_i)$; a disjoint union is denoted $\dot\cup_{i=1}^{h} G_i$. The following observation is useful when bounding minimum rank of a graph from above by exhibiting a low rank matrix, often by expressing the graph as a union of cliques.

Observation 1.8 [10]. If $G = \cup_{i=1}^{h} G_i$, then $\mathrm{mr}(G) \leqslant \sum_{i=1}^{h} \mathrm{mr}(G_i)$.

2. Zero spread, null spread, and rank spread

2.1. Vertex spread

The *rank spread* of v, defined in [4], is

$$\mathrm{r}_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v),$$

and it is known [14] that

$$0 \leqslant \mathrm{r}_v(G) \leqslant 2.$$

In analogy with the rank spread, we can define the null spread and the zero spread.

Definition 2.1. G be a graph and v be a vertex in G.

1. The *null spread* of v is $\mathrm{n}_v(G) = \mathrm{M}(G) - \mathrm{M}(G - v)$.
2. The *zero spread* of v is $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G - v)$.

Observation 2.2. For any graph G and vertex v of G,

$$\mathrm{r}_v(G) + \mathrm{n}_v(G) = 1,$$

and thus

$$-1 \leqslant \mathrm{n}_v(G) \leqslant 1.$$

The following bound on zero spread has also been obtained independently [12].

Theorem 2.3. *For every graph G and vertex v of G,*

$$-1 \leqslant \mathrm{z}_v(G) \leqslant 1.$$

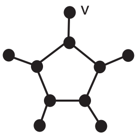

Fig. 1. The pentasun H_5.

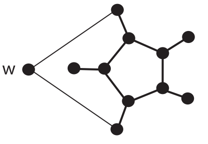

Fig. 2. The graph G for Example 2.5.

Proof. If Z is a optimal zero forcing set for $G - v$, then $Z \cup \{v\}$ is a zero forcing set of G. Thus $\mathrm{Z}(G) \leqslant \mathrm{Z}(G - v) + 1$ and $\mathrm{z}_v(G) \leqslant 1$. Now let Z be an optimal zero forcing set for G. Construct a particular chronological list of forces $\mathcal{F}$. If a force $v \to u$ appears in $\mathcal{F}$ for some vertex u, then $Z \cup \{u\}$ is a zero forcing set with chronological list of forces obtained from $\mathcal{F}$ by deleting $v \to u$; otherwise, Z is a zero forcing set with chronological list of forces $\mathcal{F}$. Thus $\mathrm{Z}(G - v) \leqslant \mathrm{Z}(G) + 1$ and $\mathrm{z}_v(G) \geqslant -1$. $\square$

As might be expected from the loose relationship between zero forcing number and maximum nullity, the parameters $\mathrm{n}_v(G)$ and $\mathrm{z}_v(G)$ are not comparable.

Example 2.4. Let v be a leaf (degree one vertex) of the pentasun H_5 shown in Fig. 1; $\mathrm{M}(H_5) = 2$ [4], $\mathrm{Z}(H_5) = 3$ [1], and $\mathrm{Z}(H_5 - v) = 2$ (a set of two consecutive leaves, one of which was consecutive with v in H_5, is a zero forcing set). Then $\mathrm{M}(H_5 - v) = 2$ since $\mathrm{M}(H_5 - v) \leqslant \mathrm{Z}(H_5 - v)$ and $H_5 - v$ is not a path. Therefore $\mathrm{n}_v(H_5) = 0 < 1 = \mathrm{z}_v(H_5)$.

Example 2.5. Construct a graph G from the pentasun H_5 by adding a new vertex w that is adjacent to two nonconsecutive leaves as shown in Fig. 2. Then $\mathrm{M}(G) = \mathrm{Z}(G) = 3$ (both can be computed by the software [8]). Since $G - w = H_5$, $\mathrm{M}(G - w) = 2$ and $\mathrm{Z}(G - w) = 3$. Therefore $\mathrm{z}_w(G) = 0 < 1 = \mathrm{n}_w(G)$.

However, under certain circumstances we can use one spread to determine the other.

Observation 2.6. Let G be a graph such that $\mathrm{M}(G) = \mathrm{Z}(G)$ and let v be a vertex of G.

1. $\mathrm{n}_v(G) \geqslant \mathrm{z}_v(G)$.
2. [6] If $\mathrm{z}_v(G) = 1$, then $\mathrm{n}_v(G) = 1$.
3. If $\mathrm{n}_v(G) = -1$, then $\mathrm{z}_v(G) = -1$.

Theorem 2.7. *Let $G = (V, E)$ be a graph and $v \in V$. Then there exists an optimal chain set of G that contains v as a singleton if and only if $\mathrm{z}_v(G) = 1$.*

Proof. Let G be a graph, v be a vertex in G, Z be an optimal zero forcing set of G where there exists an optimal chain set of Z with a singleton containing v. Clearly $Z \backslash \{v\}$ is a zero forcing set for $G - v$. Therefore $\mathrm{Z}(G - v) \leqslant \mathrm{Z}(G) - 1$, so $\mathrm{z}_v(G) \geqslant 1$. But $\mathrm{z}_v(G) \leqslant 1$ by Theorem 2.3, so $\mathrm{z}_v(G) = 1$.

Let G be a graph and v be a vertex in G such that $\mathrm{z}_v(G) = 1$. Let Z be an optimal zero forcing set for $G - v$ and define $Z' = Z \cup \{v\}$. Clearly Z' is a zero forcing set for G with the same chronological list of forces $\mathcal{F}$ as for Z in $G - v$. Since $\mathrm{z}_v(G) = 1$, Z' is an optimal zero forcing set. Clearly v is a singleton in the chain set of $\mathcal{F}$. $\square$

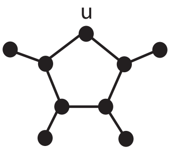

Fig. 3. A counterexample to the converse of Theorem 2.8.

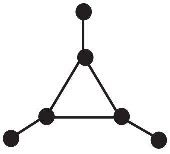

Fig. 4. The graph H_3 in Example 2.11.

Theorem 2.8. *Let $G = (V, E)$ be a graph and $v \in V$. If $\mathrm{z}_v(G) = -1$, then $v \notin Z$ for all optimal zero forcing sets Z of G. Equivalently, if $v \in Z$ for some optimal zero forcing set Z of G, then $\mathrm{z}_v(G) \geqslant 0$.*

Proof. We prove the second statement. Let Z be an optimal zero forcing set of G where $v \in Z$. Construct a chronological list of forces $\mathcal{F}$. If $v \rightarrow w$ appears in $\mathcal{F}$, then let $Z' = Z \backslash \{v\} \cup \{w\}$; if not, let $Z' = Z \backslash \{v\}$. Clearly Z' is a zero forcing set for $G - v$ and $|Z'| \leqslant |Z|$, so $\mathrm{z}_v(G) \geqslant 0$. $\square$

Since Theorem 2.7 is an equivalence, it is natural to ask whether the same is true for Theorem 2.8. That is, if v is never in an optimal zero forcing set of G, then does $\mathrm{z}_v(G) = -1$? The next example provides a negative answer.

Example 2.9. For the graph G shown in Fig. 3, the vertex u is never in an optimal zero forcing set, yet $\mathrm{z}_u(G) = 0$, because $\mathrm{Z}(G) = \mathrm{Z}(G - u) = 2$.

Graphs for which all vertices have constant rank spread have been studied; those having constant rank spread 0 (respectively, 1, 2) are called *rank null* (*rank weak*, *rank strong*). Examples of rank null and rank weak graphs are easy to find, but it is not known whether a rank strong graph exists. Equivalently, it is not known whether there exists a graph having null spread -1 for all vertices. Known examples of rank null and rank weak graphs provide examples of graphs where every vertex has zero spread 1 or 0.

Example 2.10. For the n-cycle, $\mathrm{z}_v(C_n) = 1$ for all vertices v, because $\mathrm{Z}(C_n) = 2$ and $\mathrm{Z}(C_n - v) = \mathrm{Z}(P_{n-1}) = 1$.

Example 2.11. For the graph H_3 shown in Fig. 4, $\mathrm{z}_v(H_3) = 0$ for all vertices v, because $\mathrm{Z}(H_3) = 2 = \mathrm{Z}(H_3 - v)$.

Theorem 2.12. *There does not exist a graph such that every vertex has zero spread -1.*

Proof. Suppose there exists a graph $G = (V_G, E_G)$ such that $\mathrm{z}_v(G) = -1$ for all $v \in V_G$. By Theorem 2.8, $\mathrm{z}_v(G) = -1$ implies v is not in any optimal zero forcing set of G, for all $v \in V_G$; this is a contradiction. $\square$

Remark 2.13. By Observation 2.6 and Theorem 2.12, if a rank strong graph G exists, then $\mathrm{Z}(G) > \mathrm{M}(G)$.

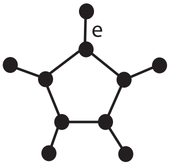

Fig. 5. The pentasun H_5 for Example 2.18.

2.2. Edge spread

In analogy with the rank, null, and zero spreads for vertex deletion, we can define spreads for edge deletion.

Definition 2.14. Let G be a graph and e be an edge in G.

1. The *rank edge spread* of e is $\mathrm{r}_e(G) = \mathrm{mr}(G) - \mathrm{mr}(G - e)$.
2. The *null edge spread* of e is $\mathrm{n}_e(G) = \mathrm{M}(G) - \mathrm{M}(G - e)$.
3. The *zero edge spread* of e is $\mathrm{z}_e(G) = \mathrm{Z}(G) - \mathrm{Z}(G - e)$.

Observation 2.15. For any graph G and edge e of G, $\mathrm{r}_e(G) + \mathrm{n}_e(G) = 0$.

Observation 2.16 [14]**.** For any graph G and edge e of G, $-1 \leqslant \mathrm{r}_e(G) \leqslant 1$ and thus $-1 \leqslant \mathrm{n}_e(G) \leqslant 1$.

Theorem 2.17. *For every graph G and every edge e of G,*

$$-1 \leqslant \mathrm{z}_e(G) \leqslant 1.$$

Proof. Let G be a graph and $e = \{v, w\}$ be an edge in G. First, let Z be an optimal zero forcing set of $G - e$. If both v and w are in Z, then Z is a zero forcing set for G. Otherwise, without loss of generality assume v is black when w is forced. Then $Z \cup \{w\}$ is a zero forcing set of G. In either case, $\mathrm{Z}(G) \leqslant \mathrm{Z}(G - e) + 1$ and $\mathrm{z}_e(G) \leqslant 1$.

Now let Z be a optimal zero forcing set for G. Construct a particular chronological list of forces $\mathcal{F}$. Without loss of generality, assume $v \in Z$ or v is forced before w is forced. If the force $v \to w$ appears in $\mathcal{F}$ then $Z' = Z \cup \{w\}$ is a zero forcing set with chronological list of forces obtained from $\mathcal{F}$ by deleting $v \to w$. If the force $v \to w$ does not appear in $\mathcal{F}$ then $Z' = Z$ is a zero forcing set with chronological list of forces $\mathcal{F}$. Thus $\mathrm{Z}(G - e) \leqslant \mathrm{Z}(G) + 1$ and $\mathrm{z}_e(G) \geqslant -1$. $\square$

We note that the bounds on the zero edge spread are the same as the bounds on the null edge spread. However, they are not comparable.

Example 2.18. For the pentasun H_5 and e an edge incident with a degree one vertex (as shown in Fig. 5), $\mathrm{Z}(H_5) = 3 = \mathrm{Z}(H_5 - e)$, $\mathrm{M}(H_5) = 2$, and $\mathrm{M}(H_5 - e) = 3$. Therefore $\mathrm{n}_e(H_5) = -1 < 0 = \mathrm{z}_e(H_5)$.

Example 2.19. For the graph G constructed from the pentasun by adding an edge e between two consecutive leaves (shown in Fig. 6), $\mathrm{Z}(G) = 3 = \mathrm{Z}(G - e)$, $\mathrm{M}(G) = 3$, and $\mathrm{M}(G - e) = 2$. Therefore $\mathrm{n}_e(G) = 1 > 0 = \mathrm{z}_e(G)$.

As with vertex spread, under certain circumstances we can use one spread to determine the other.

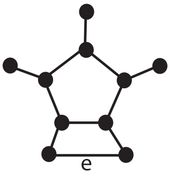

Fig. 6. Graph G for Example 2.19.

Observation 2.20. Let G be a graph such that $\mathrm{M}(G) = \mathrm{Z}(G)$ and let e be an edge of G.

1. $\mathrm{n}_e(G) \geqslant \mathrm{z}_e(G)$.
2. If $\mathrm{z}_e(G) = 1$, then $\mathrm{n}_e(G) = 1$.
3. If $\mathrm{n}_e(G) = -1$, then $\mathrm{z}_e(G) = -1$.

Recall that no vertex is in every optimal zero forcing set. The situation is somewhat different for edges.

Theorem 2.21. *Let $G = (V, E)$ be a graph and $e \in E$. If $\mathrm{z}_e(G) = -1$, then for every optimal zero forcing chain set of G, e is an edge in a chain. Equivalently, if there is an optimal zero forcing chain set of G such that e is not an edge in any chain, then $\mathrm{z}_e(G) \geqslant 0$.*

Proof. We will prove the second statement. Let Z be an optimal zero forcing set such that for some chronological list of forces, e is not in the optimal chain set $\mathcal{C}$. Then Z is a zero forcing set for $G - e$ with the same chronological list of forces. Thus $\mathrm{Z}(G - e) \leqslant \mathrm{Z}(G)$ and $\mathrm{z}_e(G) \geqslant 0$. $\square$

Question 2.22. Is the converse of Theorem 2.21 true? That is, if G is a graph, e is an edge of G, and $\mathrm{z}_e(G) \geqslant 0$, does this imply that there is an optimal zero forcing chain set of G such that e is not an edge in any chain?

The next result provides a partial converse.

Theorem 2.23. *Let G be a graph and e be an edge in G. If $\mathrm{z}_e(G) = 1$, then there exists an optimal chain set such that e is not an edge in any chain.*

Proof. Let G be a graph and $e = \{v, w\}$ be an edge in G such that $\mathrm{z}_e(G) = 1$. Choose an optimal zero forcing set Z of $G - e$ with $w \notin Z$ (such an optimal zero forcing set exists by Theorem 1.3). The vertex v requires all but one of its neighbors to be colored black for v to force, and in $G - e$, this requirement was already filled by previous conditions. Thus $Z' = Z \cup \{w\}$ is a zero forcing set of G, using the same chronological list of forces as for Z in $G - e$, and Z' is an optimal zero forcing set of G by our assumption that $\mathrm{z}_e(G) = 1$. The optimal chain set for Z' derived from a chronological list of forces for Z and $G - e$ does not contain e since w was in the zero forcing set and v was colored by some previous force from Z. $\square$

As suggested by describing Theorem 2.23 as a partial converse to Theorem 2.21, it is also possible to have $\mathrm{z}_e(G) = 0$ and have an optimal chain set such that e is not an edge in any chain.

Example 2.24. As shown in Example 2.19, $\mathrm{z}_e(G) = 0$ for the graph G shown in Fig. 6 (the pentasun with edge between two leaves). The set Z of leaves of G is an optimal zero forcing set, and there is an optimal chain set for Z that does not include e.

The idea of transmission of zero forcing across a boundary can be used to bound the zero forcing number. This bound can be used to compute the zero forcing number of graphs obtained by the deletion of an edge from grid graphs (see Section 3.2 below). For a graph $G = (V_G, E_G)$ and subset $W \subset V$, $\partial(W)$ equals the number of edges in E_G with one endpoint in W and one endpoint outside W.

Theorem 2.25. *For any graph $G = (V_G, E_G)$ and $W \subseteq V_G$,*

$$\mathrm{Z}(G) \geqslant \mathrm{Z}(G[W]) + \mathrm{Z}(G[\overline{W}]) - \partial(W).$$

Proof. Let Z be an optimal zero forcing set with the chronological list of forces $\mathcal{F}$. For each edge $e = \{w, v\}$ such that $w \in W, v \in \overline{W}$, if $w \to v$ or $v \to w$ appears in $\mathcal{F}$, then remove e from $\mathcal{C}$ (adjoining v or w to Z each time), to obtain $\mathcal{C}'$. Observe that $\mathcal{C}' = \mathcal{C}_W \cup \mathcal{C}_{\overline{W}}$, where $\mathcal{C}_W$ is a chain set for $G[W]$ for zero forcing set Z_W consisting of the first vertices of the chains in $\mathcal{C}_W$, and similarly for $\overline{W}$. Note that the maximum number of such edges removed is $\partial(W)$. Thus

$$\mathrm{Z}(G) + \partial(W) = |Z| + \partial(W) \geqslant |Z_W| + |Z_{\overline{W}}| \geqslant \mathrm{Z}(G[W]) + \mathrm{Z}(G[\overline{W}]). \qquad \square$$

As with vertex spread, we can look for examples of graphs having constant edge spread, and in some cases can adapt well known examples.

Example 2.26. $\mathrm{z}_e(C_n) = 1$ for all edges e, because $\mathrm{Z}(C_n) = 2$ and $\mathrm{Z}(C_n - e) = \mathrm{Z}(P_n) = 1$.

Example 2.27. $\mathrm{z}_e(P_n) = -1$ for all edges e, because $\mathrm{Z}(P_n) = 1$ and $\mathrm{Z}(P_n - e) = 2$ because $P_n - e$ is the union of two disjoint paths.

Theorem 3.19 in Section 3.3 below shows that every edge in a square triangular grid graph has edge spread 0.

3. Supertriangles, grid graphs, triangular grids, and king grids

In this section, we establish the zero spread, null spread and rank spread of most vertices and edges of supertriangles, triangular grids, king grids and (rectangular) grid graphs, all defined below. This includes establishing the zero forcing number, minimum rank, and maximum nullity of the triangular grids (these parameters were known previously for the other graphs).

3.1. Supertriangles

The mth *supertriangle*, T_m, is an equilateral triangular grid with m vertices on each side; T_5 is shown in Fig. 7. When diagrammed as in Fig. 7, the edges in a supertriangle T_m form three sets of *lines*, namely *horizontal* edges, *diagonal* edges (running upper left to lower right), and *counterdiagonal* edges (running upper right to lower left). If each of these types of lines is numbered from the point to the base, then vertex v is described by a triple $v = (v_h, v_d, v_c)$, where v_h is the horizontal line index, v_d is the diagonal line index, and v_c is the counterdiagonal line index. For any vertex $v = (v_h, v_d, v_c)$, the three line indices are related by the formula

$$v_h + v_d + v_c = 2m + 1.$$

A vertex will be denoted by its triple of line indices. For the supertriangle T_m, $\mathrm{M}(T_m) = \mathrm{Z}(T_m) = m$ and $\mathrm{mr}(T_m) = \frac{1}{2}m(m-1)$ [1].

Theorem 3.1. *For every edge e and vertex v in T_m, $\mathrm{z}_v(T_m) = \mathrm{n}_v(T_m) = \mathrm{n}_e(T_m) = \mathrm{z}_e(T_m) = 1$. Equivalently, if G is obtained from T_m by deleting exactly one edge, or G is obtained from T_m by deleting exactly one vertex, then $\mathrm{M}(G) = \mathrm{Z}(G) = m - 1$.*

Proof. Without loss of generality, assume

$$e = \{(s, k, 2m + 1 - k - s), (s, k + 1, 2m - k - s)\}.$$

(note the missing edge e in the middle of Fig. 8).

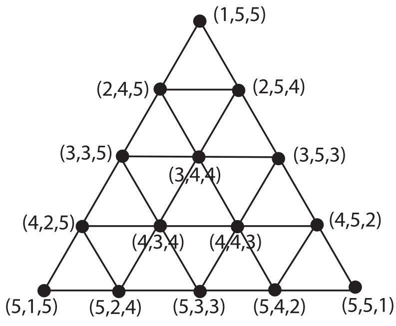

Fig. 7. The supertriangle T_5.

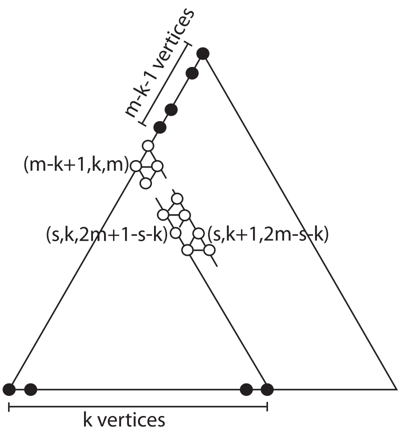

Fig. 8. Diagram for Theorem 3.1.

In $T_m - e$, let Z be the vertices having line indices $(m, 1, m), \ldots, (m, k, m-k+1), (1, m, m), \ldots, (m-k-1, k+2, m)$ (the $m-1$ black vertices shown in the diagram in Fig. 8). We show that Z is a zero forcing set. The black vertices $(m, 1, m), \ldots, (m, k, m-k+1)$ force the triangle of vertices w having $w_d \leqslant k$ to turn black. Then

$$\begin{aligned}
(s, k, 2m-k-s+1) &\rightarrow (s-1, k+1, 2m-k-s+1),\\
(s-1, k, 2m-k-s+2) &\rightarrow (s-2, k+1, 2m-k-s+2),\\
&\vdots\\
(m-k+1, k, m) &\rightarrow (m-k, k+1, m).
\end{aligned}$$

Thus the entire left edge (vertices v with $v_c = m$) is black. This is a zero forcing set for T_m using only diagonal edges, so Z is a zero forcing set for $T_m - e$. Thus $\mathrm{z}_e(T_m) \geqslant 1$, and so by Theorem 2.17, $\mathrm{z}_e(T_m) = 1$. In [1], it was shown that $\mathrm{M}(T_m) = \mathrm{Z}(T_m)$, so by Observation 2.20 $\mathrm{n}_e(T_m) = 1$.

If $v = (s, k+1, 2m-s-k)$ is the right vertex in e, then the same set Z with cardinality $m-1$ is a zero forcing set. Thus $\mathrm{z}_v(T_m) = 1$ and $\mathrm{n}_v(T_m) = 1$. $\square$

3.2. Grid graphs

The Cartesian product $P_m \,\square\, P_n$ is a called a *grid graph*. Since the product is commutative, we assume $m \leqslant n$. It is convenient to label the vertices of the grid graph $P_m \,\square\, P_n$ with m rows and n columns as

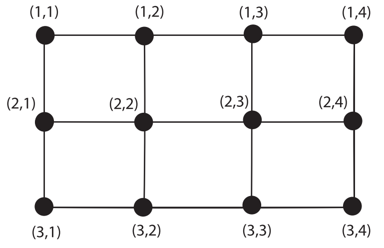

Fig. 9. The 3×4 grid graph $P_3 \,\square\, P_4$.

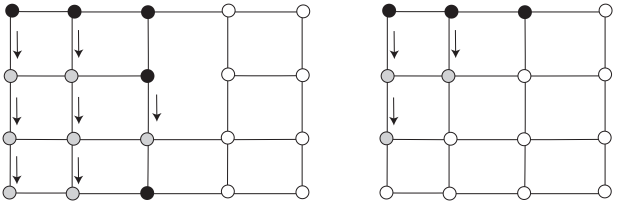

Fig. 10. Diagrams for Observations 3.6 and 3.7.

ordered pairs (i, j), where i is the row coordinate and j is the column coordinate, starting from the upper left corner, as shown in Fig. 9. In [1] it was shown that $\mathrm{Z}(P_m \,\square\, P_n) = \mathrm{M}(P_m \,\square\, P_n) = m$ (and $\mathrm{mr}(P_m \,\square\, P_n) = mn - m$).

Let $P_m \,\square\, P_n$ be a grid graph. We can view the vertices as in four quadrants. If a result is established for the upper left quadrant $i \leqslant \lceil \frac{m}{2} \rceil$ and $j \leqslant \lceil \frac{n}{2} \rceil$, then symmetry provides analogous results for vertices in the other three quadrants.

For certain vertices v of a grid graph, the minimum rank, maximum nullity, and zero forcing number of the graph obtained by deleting v are established in [6]. Equivalently, the values of $\mathrm{r}_v(P_m \,\square\, P_n)$, $\mathrm{n}_v(P_m \,\square\, P_n)$, and $\mathrm{z}_v(P_m \,\square\, P_n)$ have been established for certain vertices v. For completeness, we list these results below without proof. Theorem 3.2 below, also taken from [6], establishes the value of Colin de Verdière type parameter ξ for square grids. This theorem plays an important role in our results in edge spread for nonsquare grid graphs and the vertex and edge spread of triangular grids (Section 3.3), in addition to its use in the proof of Theorem 3.5 below.

Theorem 3.2 [6]**.** *For a square grid graph,* $\xi(P_m \,\square\, P_m) = m$*.*

Theorem 3.3 [6]**.** *If* v *is any vertex of the square grid graph* $P_m \,\square\, P_m$*, then* $\mathrm{Z}(P_m \,\square\, P_m - v) = \mathrm{M}(P_m \,\square\, P_m - v) = m - 1$*. Equivalently,* $\mathrm{z}_v(P_m \,\square\, P_m) = \mathrm{n}_v(P_m \,\square\, P_m) = 1$*.*

Theorem 3.4 [6]**.** *Let* $P_m \,\square\, P_n$ *be a grid graph with* $m < n$*. If* $\ell \leqslant \lceil \frac{m}{2} \rceil$ *and* $k \leqslant \lceil \frac{n}{2} \rceil$ *and* $n - m \leqslant k - \ell$*, then* $\mathrm{M}(P_m \,\square\, P_n - (\ell, k)) = \mathrm{Z}(P_m \,\square\, P_n - (\ell, k)) = m - 1$*. Equivalently,* $\mathrm{n}_{(\ell,k)}(P_m \,\square\, P_n) = \mathrm{z}_{(\ell,k)}(P_m \,\square\, P_n) = 1$*. Results in the other three quadrants are obtained by symmetry.*

Theorem 3.5 [6]**.** *Let* $P_m \,\square\, P_n$ *be a grid graph with* $n \geqslant 2m + 1$*. If* $m + 1 \leqslant k \leqslant n - m$*, then* $\mathrm{Z}(P_m \,\square\, P_n - (i, k)) = \mathrm{M}(P_m \,\square\, P_n - (i, k)) = m + 1$*. Equivalently,* $\mathrm{z}_{(i,k)}(P_m \,\square\, P_n) = \mathrm{n}_{(i,k)}(P_m \,\square\, P_n) = -1$*.*

We now examine the edge spread of a square grid and need some technical observations. In a grid graph, the set of vertices $(1, 1), \ldots, (1, t), (2, t), \ldots, (s, t)$ is called an *ell*. In a grid graph minus edge $\{(\ell, t), (\ell, t + 1)\}$, the set of vertices $(1, 1), \ldots, (1, t), (2, t), \ldots, (\ell, t), (\ell + 2, t), \ldots, (s, t)$ is called a *modified ell*. A modified ell of black vertices with $t = 3, \ell = 2, s = 4$ is illustrated in Fig. 10. The gray vertices can be forced by the modified ell, as indicated in the next observation. A configuration obtained from a (modified) ell under rotation or reflection of the ell is also called a (modified) ell.

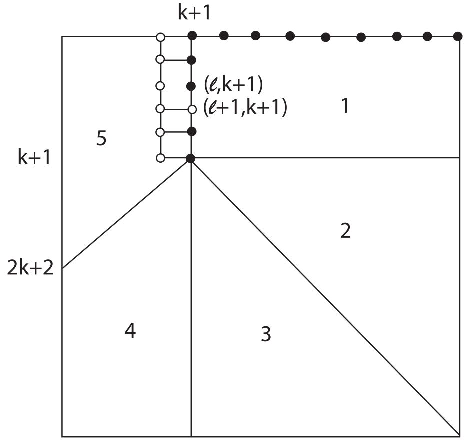

Fig. 11. Diagram of Theorem 3.8.

Observation 3.6. In $P_m \,\square\, P_n$, the black ell $(1, 1), \ldots, (1, t), (2, t), \ldots, (s, t)$ can force all the vertices (i, j), $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t$ to turn black. In $P_m \,\square\, P_n - \{(\ell, t), (\ell, t + 1)\}$, the black modified ell $(1, 1), \ldots, (1, t), (2, t), \ldots, (\ell, t), (\ell + 2, t), \ldots, (s, t)$ can force all the vertices (i, j), $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t$ to turn black. Analogous results are obtained by symmetry.

In a grid graph, the set of vertices $(s - 1, 1), \ldots, (s - 1, t), (s, 1), \ldots, (s, t)$ (where $(s - 1, 1), \ldots, (s - 1, t)$ are omitted if $s = 1$) is called a *wall*. A wall with $s = 1$, $t = 3$ is illustrated in Fig. 10. The gray vertices can be forced by the wall, as indicated in the next observation. A configuration obtained from a wall under rotation or reflection of the wall is also called wall.

Observation 3.7. In $P_m \,\square\, P_n$, the black wall $(s - 1, 1), \ldots, (s - 1, t), (s, 1), \ldots, (s, t)$ (where $(s - 1, 1), \ldots, (s - 1, t)$ are omitted if $s = 1$) can force all the vertices (i, j), $s \leqslant i$ and $i + j \leqslant s + t$ to turn black. Analogous results are obtained by symmetry.

Theorem 3.8 (Bull's-eye theorem). *Let $P_m \,\square\, P_m$ be a square grid graph and let $e = \{(\ell, k), (\ell, k + 1)\}$ with $\ell \leqslant \frac{m}{2}$ and $\ell \leqslant k \leqslant m - \ell$. Then $\mathrm{n}_e(P_m \,\square\, P_m) = \mathrm{z}_e(P_m \,\square\, P_m) = 1$. Equivalently, $\mathrm{M}(P_m \,\square\, P_m - e) = \mathrm{Z}(P_m \,\square\, P_m - e) = m - 1$. Additional results are obtained by symmetry* (see Fig. 12).

Proof. Without loss of generality, $k \leqslant \frac{m}{2}$. We show that $Z = \{(1, k + 1), \ldots, (1, m)\} \cup \{(1, k + 1), \ldots, (k + 1, k + 1)\} \setminus \{(\ell + 1, k + 1)\}$ is a zero forcing set for $G - e$; note that $|Z| = m - 1$. The set Z is a modified ell, so by Observation 3.6 all the vertices (i, j) such that $1 \leqslant i \leqslant k + 1$, $k + 1 \leqslant j \leqslant m$ (region 1 of Fig. 11) are forced to turn black. The set of black vertices now includes the wall $(k, k + 1), \ldots, (k, m), (k + 1, k + 1), \ldots, (k + 1, m)$, so by Observation 3.7 all the vertices (i, j) such that $k + 1 \leqslant i \leqslant j \leqslant m$ (region 2 of Fig. 11) are forced to turn black. The set of black vertices now includes the ell $(k + 1, k + 1), \ldots, (k + 1, m), (k + 2, m), \ldots, (m, m)$, so by Observation 3.6 all the vertices (i, j) such that $k + 1 \leqslant i \leqslant m$, $k + 1 \leqslant j \leqslant m$ (region 3 of Fig. 11) are forced to turn black. The set of black vertices now includes the wall $(k + 1, k + 1), \ldots, (m, k + 1), (k + 1, k + 2), \ldots, (m, k + 2)$, so by Observation 3.7 all the vertices (i, j) such that $1 \leqslant j \leqslant k + 1$, $2k + 2 - j \leqslant i \leqslant m$ (region 4 of Fig. 11) are forced to turn black. The set of black vertices now includes the ell $(1, k + 1), \ldots, (m, k + 1), (m, k), \ldots, (m, 1)$, so by Observation 3.6, all the remaining vertices (region 5 of Fig. 11) are forced to turn black. Thus Z is a zero forcing set of size $m - 1$, and by Theorem 2.17, Z is optimal. So $\mathrm{Z}(P_m \,\square\, P_m - e) = m - 1$. By Observation 2.20, $\mathrm{M}(P_m \,\square\, P_m - e) = m - 1$. $\square$

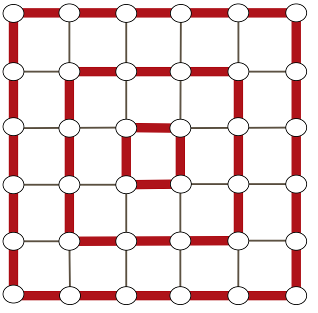

Fig. 12. Diagram of bull's-eye theorem and conjecture: The thick lines (concentric squares) have edge spread 1. The thin lines have edge spread 0 for the grid graph $P_6 \,\Box\, P_6$ that is shown, and are conjectured to have edge spread 0 for all square grid graphs.

By symmetry, every edge e of $P_m \,\Box\, P_m$ can be omitted from some optimal zero forcing set, so $\mathrm{z}_e(P_m \,\Box\, P_m) \geqslant 0$ by Theorem 2.21.

Conjecture 3.9 (*Bull's-eye conjecture*)**.** If e is an edge of $P_m \,\Box\, P_m$ that is not covered by Theorem 3.8, then $\mathrm{z}_e(P_m \,\Box\, P_m) = 0$.

Conjecture 3.9 has been established for $m \leqslant 7$ by use of software [8]. The results established and conjectured above for edge spread in a square grid are summarized in Fig. 12.

We now turn our attention to the edge spread in nonsquare grids.

Theorem 3.10. *Let $P_m \,\Box\, P_n$ be a grid graph with $n > m$ and let $e = \{(\ell, k), (\ell + 1, k)\}$. Then $\mathrm{M}(P_m \,\Box\, P_n - e) = \mathrm{Z}(P_m \,\Box\, P_n - e) = m$, or equivalently, $\mathrm{n}_e(P_m \,\Box\, P_n) = \mathrm{z}_e(P_m \,\Box\, P_n) = 0$.*

Proof. The edge e does not appear in the obvious chronological list of forces (forcing along the horizontal edges) for the zero forcing set of left end vertices $Z = \{(i, 1) : 1 \leqslant i \leqslant m\}$. Thus by Theorem 2.21, $\mathrm{z}_e(P_m \,\Box\, P_n) \geqslant 0$ and so $\mathrm{Z}(P_m \,\Box\, P_n - e) \leqslant m$. By contracting all the edges $\{(i, k), (i, k + 1)\}$, $i = 1, \ldots, m$ (or the edges $\{(i, n - 1), (i, n)\}$, $i = 1, \ldots, m$ if $k = n$), we see that $P_m \,\Box\, P_{n-1}$ is a minor of $P_m \,\Box\, P_n - e$, and $P_m \,\Box\, P_m$ is a subgraph of $P_m \,\Box\, P_{n-1}$. Therefore,

$$m \leqslant \xi(P_m \,\Box\, P_n - e) \leqslant \mathrm{M}(P_m \,\Box\, P_n - e) \leqslant \mathrm{Z}(P_m \,\Box\, P_n - e) \leqslant m. \quad \Box$$

The method of proof in Theorem 3.8 can be used to establish the following theorem.

Theorem 3.11. *Let $P_m \,\Box\, P_n$ be a grid graph with $m < n$. If $\ell \leqslant \lceil \frac{m}{2} \rceil, n - m + \ell \leqslant k \leqslant m - \ell$ and $e = \{(\ell, k), (\ell, k + 1)\}$, then $\mathrm{n}_e(P_m \,\Box\, P_n) = \mathrm{z}_e(P_m \,\Box\, P_n) = 1$. Equivalently, $\mathrm{M}(P_m \,\Box\, P_n - e) = \mathrm{Z}(P_m \,\Box\, P_n - e) = m - 1$. Results in the lower half are obtained by symmetry.*

Theorem 3.12. *Let $P_m \,\Box\, P_n$ be a grid graph with $n \geqslant 2m$ and let $e = \{(\ell, k), (\ell, k + 1)\}$ with $m \leqslant k \leqslant n - m$. Then $\mathrm{Z}(P_m \,\Box\, P_n - e) = m + 1$, or equivalently, $\mathrm{z}_e(P_m \,\Box\, P_n) = -1$.*

Proof. Let $G = P_m \,\Box\, P_n - e$ and $W = \{(i, j) : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant k\}$. Then $G[W] = P_m \,\Box\, P_k$ and $G[\overline{W}] = P_m \,\Box\, P_{n-k}$. Since $m \leqslant k \leqslant n - m$, $\mathrm{Z}(G[W]) = m = \mathrm{Z}(G[\overline{W}])$. Since there are only $m - 1$ edges joining $G[W]$ and $G[\overline{W}]$, $\partial(W) = m - 1$. Then by Theorem 2.25,

$$\mathrm{Z}(G) \geqslant \mathrm{Z}(G[W]) + \mathrm{Z}(G[\overline{W}]) - \partial(W) = m + m - (m - 1) = m + 1.$$

Since $\mathrm{z}_e(P_m \,\Box\, P_n) \geqslant -1$, $\mathrm{z}_e(P_m \,\Box\, P_n) = -1$. $\Box$

Theorem 3.13. *Let $P_m \,\Box\, P_n$ be a grid graph with $n \geqslant 2m$ and let $e = \{(\ell, k), (\ell, k + 1)\}$ with $m \leqslant k \leqslant n - m$. Then $\mathrm{n}_e(P_m \,\Box\, P_n) = -1$. Equivalently, $\mathrm{M}(P_m \,\Box\, P_n - e) = m + 1$.*

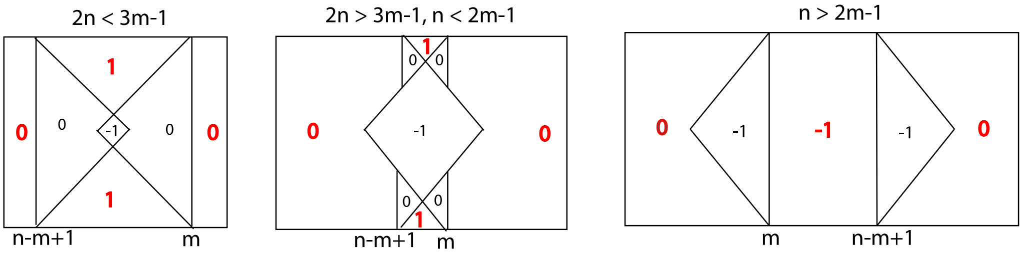

Fig. 13. Schematic diagram of zero edge spreads for horizontal edges (parallel to the long side) in nonsquare grid graphs. Results established are in large type and those conjectured are in small type.

Proof. Let $G_0 = P_m \,\square\, P_n$. Let $e_i = \{(\ell - i + 1, k), (\ell - i + 1, k + 1)\}, i = 1, \ldots, \ell, e_i = \{(i, k), (i, k + 1)\}, i = \ell + 1, \ldots, n,$ and $G_i = G_{i-1} - e_i$. Notice that $G_m = (P_m \,\square\, P_k) \dot\cup (P_m \,\square\, P_{n-k})$. Since $m \leqslant k \leqslant n - m, P_m \,\square\, P_m$ is a subgraph of $P_m \,\square\, P_k$ and also of $P_m \,\square\, P_{n-k}$. By Theorem 3.2, $\mathrm{M}(P_m \,\square\, P_k) \geqslant m$ and $\mathrm{M}(P_m \,\square\, P_{n-k}) \geqslant m$. Therefore, $\mathrm{M}(G_m) = \mathrm{M}(P_m \,\square\, P_k) + \mathrm{M}(P_m \,\square\, P_{n-k}) = 2m$. Thus

$$-m = \mathrm{M}(G) - \mathrm{M}(G_m) = \sum_{i=1}^{m} \mathrm{n}_{e_i}(G_{i-1}) \geqslant \sum_{i=1}^{m} (-1) = -m.$$

Thus the null spread of each vertex must be -1, so $\mathrm{n}_e(P_m \,\square\, P_n) = -1$. □

Since $\mathrm{M}(P_m \,\square\, P_n) = \mathrm{Z}(P_m \,\square\, P_n)$, Theorem 3.12 is a corollary to Theorem 3.13. But whereas Theorem 3.13 relies on analytical results of manifold theory through its use of the parameter ξ, the proofs of Theorem 3.12, and of Theorem 2.25 on which it relies, are combinatorial.

Lemma 3.14. *Let $P_m \,\square\, P_n$ be a grid graph with $m < n, \ell \leqslant \lceil \frac{m}{2} \rceil, \ell + k \leqslant m$, and $e = \{(\ell, k), (\ell, k + 1)\}$. Then $\mathrm{z}_e(P_m \,\square\, P_n) \geqslant 0$, or equivalently, $\mathrm{Z}(P_m \,\square\, P_n - e) \leqslant m$. Analogous results are obtained by symmetry.*

Proof. We show that set of vertices $Z = \{(1, 1), \ldots, (1, m)\}$ is a zero forcing set for $P_m \,\square\, P_n$ that does not use edge e, so by Theorem 2.21, $\mathrm{z}_e(P_m \,\square\, P_n) \geqslant 0$. Z is a wall so by Observation 3.7, Z can force the triangle $1 \leqslant i \leqslant m + 1 - j$; note that e was not used. Then by Observation 3.6, the ell $(1, 1), \ldots, (1, m), (2, 1), \ldots, (m, 1)$ can force all the remaining vertices in the rectangle $1 \leqslant i, j \leqslant m$ to turn black. Clearly all the rest of the vertices are then forced to turn black. □

Lemma 3.15. *Let $P_m \,\square\, P_n$ be a grid graph with $m < n$. If $e = \{(\ell, k), (\ell, k + 1)\}$ with $k \leqslant n - m$ or $k \geqslant m$, then $\mathrm{Z}(P_m \,\square\, P_n - e) \geqslant \mathrm{M}(P_m \,\square\, P_n - e) \geqslant m$. Equivalently, $\mathrm{z}_e(P_m \,\square\, P_n) \leqslant \mathrm{n}_e(P_m \,\square\, P_n) \leqslant 0$.*

Proof. $P_m \,\square\, P_m$ is a subgraph of $P_m \,\square\, P_n - e$ and $\xi(P_m \,\square\, P_m) = m$. □

Corollary 3.16. *Let $P_m \,\square\, P_n$ be a grid graph with $m < n, \ell \leqslant \lceil \frac{m}{2} \rceil, \ell + k \leqslant m, k \leqslant n - m$, and $e = \{(\ell, k), (\ell, k + 1)\}$. Then $\mathrm{n}_e(P_m \,\square\, P_n) = \mathrm{z}_e(P_m \,\square\, P_n) = 0$. Equivalently, $\mathrm{M}(P_m \,\square\, P_n - e) = \mathrm{Z}(P_m \,\square\, P_n - e) = m$. Additional results are obtained by symmetry.*

Fig. 13 summarizes the results in Theorem 3.11, Theorem 3.12, and Corollary 3.16, and Conjecture 3.17 below as to zero edge spread for edges parallel to the long side.

Conjecture 3.17. Let $P_m \,\square\, P_n$ be a grid graph with $m < n, \ell \leqslant \lceil \frac{m}{2} \rceil, k \leqslant \lceil \frac{n}{2} \rceil$, and $e = \{(\ell, k), (\ell, k + 1)\}$. If $m - \ell < k < m$, then $\mathrm{z}_e(P_m \,\square\, P_n) = -1$. If e is not covered by the previous statement nor by any of Theorem 3.11, Theorem 3.12, or Corollary 3.16, then $\mathrm{z}_e(P_m \,\square\, P_n) = 0$. Additional results are conjectured by symmetry.

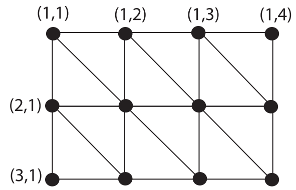

Fig. 14. The 3×4 triangular grid graph.

3.3. Triangular grids

To create the *triangular grid graph* $P_m \boxbslash P_n$ from the grid graph $P_m \square P_n$, we add diagonal edges from (i, j) to $(i+1, j+1)$ where $i = 1, \dots, m-1$ and $j = 1, \dots, n-1$. The triangular grid graph $P_3 \boxbslash P_4$ can be seen in Fig. 14. We first establish the values of zero forcing number and maximum nullity of triangular grids (without removing any edges or vertices).

Theorem 3.18. *If $m \leqslant n$, then $\mathrm{Z}(P_m \boxbslash P_n) = \mathrm{M}(P_m \boxbslash P_n) = m$.*

Proof. The set of left end vertices $Z = \{(1,1), \dots, (m,1)\}$ is a zero forcing set of $P_m \boxbslash P_n$, because $(m,1) \to (m,2)$, $(m-1,1) \to (m-1,2)$, $\dots$, $(1,1) \to (1,2)$, etc. Thus $\mathrm{Z}(P_m \boxbslash P_n) \leqslant m$. Since T_m is a subgraph of $P_m \boxbslash P_n$, $\nu(T_m) = m$ [7], and ν is minor monotone,

$$m = \nu(T_m) \leqslant \nu(P_m \boxbslash P_n) \leqslant \mathrm{M}(P_m \boxbslash P_n) \leqslant \mathrm{Z}(P_m \boxbslash P_n) \leqslant m.$$

Thus $\mathrm{Z}(P_m \boxbslash P_n) = \mathrm{M}(P_m \boxbslash P_n) = m$. $\square$

Next we determine the edge spread for a triangular grid graph.

Theorem 3.19. *For every edge e in $P_m \boxbslash P_m$, $\mathrm{Z}(P_m \boxbslash P_m - e) = \mathrm{M}(P_m \boxbslash P_m - e) = m$. Equivalently, for every edge e in $P_m \boxbslash P_m$, $\mathrm{z}_e(P_m \boxbslash P_m) = \mathrm{n}_e(P_m \boxbslash P_m) = 0$.*

Proof. Let e be an edge in $P_m \boxbslash P_m$. Then either $e = \{(\ell, \ell), (\ell+1, \ell+1)\}$ is a *diagonal edge*, or e is not a diagonal edge, in which case without loss of generality we may assume $e = \{(\ell, k), (\ell+1, k)\}$. Then, regardless of the type of edge of e, as in the proof of Theorem 3.18, the set of left end vertices is a zero forcing set of $P_m \boxbslash P_m - e$ and $\mathrm{Z}(P_m \boxbslash P_m - e) \leqslant m$.

If e is not a diagonal edge, then T_m is a subgraph of $P_m \boxbslash P_m - e$, so $m = \nu(T_m) \leqslant \nu(P_m \boxbslash P_m - e) \leqslant \xi(P_m \boxbslash P_m - e)$. If e is a diagonal edge, then $P_m \square P_m$ is a subgraph of $P_m \boxbslash P_m - e$, so by Theorem 3.2, $m = \xi(P_m \square P_m) \leqslant \xi(P_m \boxbslash P_m - e)$. In either case,

$$m \leqslant \xi(P_m \boxbslash P_m - e) \leqslant \mathrm{M}(P_m \boxbslash P_m - e) \leqslant \mathrm{Z}(P_m \boxbslash P_m - e) \leqslant m.$$

Thus, $\mathrm{Z}(P_m \boxbslash P_m - e) = \mathrm{M}(P_m \boxbslash P_m - e) = m$. $\square$

The proof of the next theorem is similar to that of Theorem 3.19.

Theorem 3.20. *Let e be an edge in $P_m \boxbslash P_n$ where $n > m$ and e is not parallel to a side of size n. Then $\mathrm{Z}(P_m \boxbslash P_n - e) = \mathrm{M}(P_m \boxbslash P_n - e) = m$, or equivalently, $\mathrm{z}_e(P_m \boxbslash P_n) = \mathrm{n}_e(P_m \boxbslash P_n) = 0$.*

Theorem 3.21. *Let $e = \{(\ell, k), (\ell, k+1)\}$ be an edge of $P_m \boxbslash P_n$ where $n > m$. If $k \leqslant \ell - 1$ or $k \geqslant \ell + n - m$, then*

$$\mathrm{M}(P_m \boxbslash P_n - e) = \mathrm{Z}(P_m \boxbslash P_n - e) = m \quad \textit{and} \quad \mathrm{z}_e(P_m \boxbslash P_n) = \mathrm{n}_e(P_m \boxbslash P_n) = 0;$$

otherwise

$$\mathrm{M}(P_m \boxbslash P_n - e) = \mathrm{Z}(P_m \boxbslash P_n - e) = m + 1 \quad \textit{and} \quad \mathrm{z}_e(P_m \boxbslash P_n) = \mathrm{n}_e(P_m \boxbslash P_n) = -1.$$

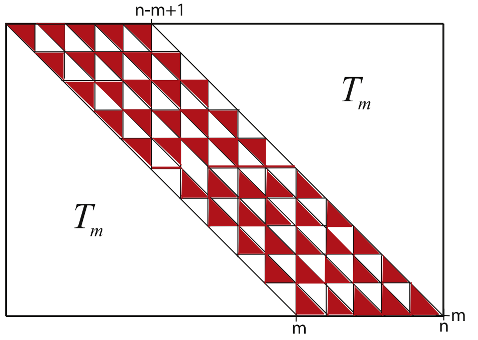

Fig. 15. A covering of the edges of the subgraph $H - e$ in a triangular grid graph, assuming the two large supertriangles are covered.

Proof. If $k \leqslant \ell - 1$ or $k \geqslant \ell + n - m$, without loss of generality assume $k \leqslant \ell - 1$. The graph $P_m \boxslash P_n - e$ contains T_m as a subgraph, and the subset of vertices $Z = \{(1, n), \ldots, (m, n)\}$ is a zero forcing set of $P_m \boxslash P_n - e$ of size m. Thus, $\mathrm{M}(P_m \boxslash P_n - e) = \mathrm{Z}(P_m \boxslash P_n - e) = m$.

Now assume $\ell \leqslant k \leqslant \ell + n - m - 1$. We construct a matrix of rank $nm - m - 1$ in $\mathcal{S}(P_m \boxslash P_n - e)$ by covering $P_m \boxslash P_n - e$ by $nm - m - 1$ cliques. The graph $P_m \boxslash P_n$ is the union of two supertriangles T_m at the ends and a middle section that is a new triangular grid graph $H = P_m \boxslash P_{n-m+1}$ (see Fig. 15). Note that e is an edge of H that is not in either T_m. We construct a matrix of rank $mn - m - 1$ having graph $P_m \boxslash P_n - e$ by covering $P_m \boxslash P_n - e$ by subgraphs. It is well known that T_m can be covered by $\frac{1}{2}m(m-1)$ copies of K_3. Note that two parallel sides of $H - e$ are covered by the coverings of the two copies of T_m, so the inclusion of these edges in the covering of $H - e$ is optional. The graph $H - e$ (not including all the optional edges) can be covered by $(m-1)(n-m)$ copies of K_3 and $n - m - 1$ copies of K_2. The entire graph $P_m \boxslash P_n - e$ now has a covering consisting of $mn - m - 1$ cliques, each of which has minimum rank equal to one. Thus, $\mathrm{mr}(P_m \boxslash P_n - e) \leqslant mn - m - 1$ and $\mathrm{M}(P_m \boxslash P_n - e) \geqslant m + 1$. Since $\mathrm{M}(P_m \boxslash P_n) = m$, $\mathrm{n}_e(P_m \boxslash P_n) \leqslant -1$. By the bounds of the null spread, $\mathrm{n}_e(P_m \boxslash P_n) = -1$. Since $\mathrm{M}(P_m \boxslash P_n) = \mathrm{Z}(P_m \boxslash P_n)$, $\mathrm{z}_e(P_m \boxslash P_n) = -1$. $\square$

Finally we determine the vertex spread for a triangular grid graph.

Theorem 3.22. *For vertex $v = (\ell, k)$ of $P_m \boxslash P_m$,*

$$\mathrm{Z}(P_m \boxslash P_m - v) = \mathrm{M}(P_m \boxslash P_m - v) = \begin{cases} m - 1 & \text{if } \ell = k \\ m & \text{if } \ell \neq k \end{cases},$$

or equivalently,

$$\mathrm{z}_v(P_m \boxslash P_m) = \mathrm{n}_v(P_m \boxslash P_m) = \begin{cases} 1 & \text{if } \ell = k \\ 0 & \text{if } \ell \neq k \end{cases}.$$

Proof. For $\ell = k$, by symmetry we may assume $k \leqslant \lceil \frac{m}{2} \rceil$. We show that $Z = \{(1, 2), \ldots, (1, m)\}$ is a zero forcing set. In the graph $P_m \boxslash P_m$, row 1 of $m - 1$ black vertices can force the $m - 2$ vertices $(2, 3), \ldots, (2, m)$, etc. until at row k the vertices $(k, k + 1), \ldots, (k, m)$ are black. Since vertex (k, k) is deleted, the black vertices in row k can force $(k + 1, k + 1), \ldots, (k + 1, m)$ Forcing of the triangle $\{(i, j) : j \geqslant i \geqslant k + 1\}$ now continues until the corner vertex (m, m) is forced. The vertices (i, i) can be forced for $1 \leqslant i < k$. The rest of the graph $P_m \boxslash P_m - v$ can now be forced, wrapping in both directions around the hole created by the missing vertex (see Fig. 16). Thus Z is a zero forcing set of $P_m \boxslash P_m - v$ with $|Z| = m - 1$, so by Theorem 2.3, $\mathrm{z}_v(P_m \boxslash P_m) = 1$ and $\mathrm{Z}(P_m \boxslash P_m - v) = m - 1$. Since $\mathrm{M}(P_m \boxslash P_m) = \mathrm{Z}(P_m \boxslash P_m)$, $\mathrm{n}_v(P_m \boxslash P_m) \geqslant \mathrm{z}_v(P_m \boxslash P_m)$, and thus $\mathrm{n}_v(P_m \boxslash P_m) = 1$.

For $\ell \neq k$, by symmetry, we may assume $\ell < k$. Then the set $Z = \{(1, 1), \ldots, (m, 1)\}$ is a zero forcing set of $P_m \boxslash P_m - v$ (the argument is similar to the previous case) and thus $\mathrm{Z}(P_m \boxslash P_m - v) \leqslant m$. Since $\ell \neq k$, $P_m \boxslash P_m - v$ has a T_m as a subgraph. Therefore

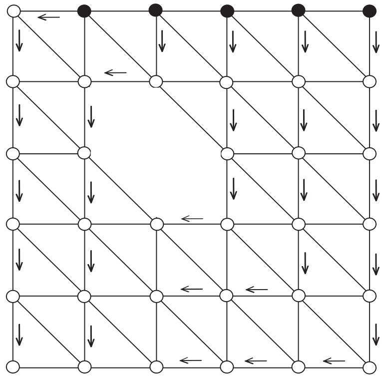

Fig. 16. Optimal zero forcing set on a triangular grid graph minus a diagonal vertex.

$$m \leqslant \xi(T_m) \leqslant \xi(P_m \boxbslash P_m - v) \leqslant \mathrm{M}(P_m \boxbslash P_m - v) \leqslant \mathrm{Z}(P_m \boxbslash P_m - v) \leqslant m.$$

Therefore, $\mathrm{Z}(P_m \boxbslash P_m - v) = \mathrm{M}(P_m \boxbslash P_m - v) = m$. □

Theorem 3.23. *For vertex $v = (\ell, k)$ of $P_m \boxbslash P_n$ with $n > m$,*

$$\mathrm{Z}(P_m \boxbslash P_n - v) = \mathrm{M}(P_m \boxbslash P_n - v) = \begin{cases} m & \text{if } \ell \geqslant k \text{ or } n - k \leqslant m - \ell \\ m+1 & \text{otherwise} \end{cases},$$

or equivalently,

$$\mathrm{z}_v(P_m \boxbslash P_n) = \mathrm{n}_v(P_m \boxbslash P_n) = \begin{cases} 0 & \text{if } \ell \geqslant k \text{ or } n - k \leqslant m - \ell \\ -1 & \text{otherwise} \end{cases}.$$

Proof. If $\ell \geqslant k$ or $n - k \leqslant m - \ell$, without loss of generality assume $\ell \geqslant k$. Let $Z = \{(1, n), \ldots, (m, n)\}$ be a subset of vertices of $P_m \boxbslash P_n - (\ell, k)$. It follows from the same method in the proof of Theorem 3.22 that Z is a zero forcing set of $G - v$, and $|Z| = m$. The graph $P_m \boxbslash P_n - (\ell, k)$ contains a T_m as a subgraph. Thus

$$m \leqslant \nu(T_m) \leqslant \nu(P_m \boxbslash P_n - (\ell, k)) \leqslant \mathrm{M}(P_m \boxbslash P_n - (\ell, k)) \leqslant \mathrm{Z}(P_m \boxbslash P_n - (\ell, k)) \leqslant m$$

Thus $\mathrm{M}(P_m \boxbslash P_n - (\ell, k)) = \mathrm{Z}(P_m \boxbslash P_n - (\ell, k)) = m$ and $\mathrm{z}_v(P_m \boxbslash P_n - (\ell, k)) = \mathrm{n}_v(P_m \boxbslash P_n - (\ell, k)) = 0$.

Now assume $\ell < k$ and $n - k > m - \ell$. Let $G_0 = P_m \boxbslash P_n$ and let $G_i = G_{i-1} - (i, i + k - \ell)$. Notice that G_m is the disjoint union of two graphs H_1 and H_2, each of which contains T_m as an induced subgraph. Therefore,

$$\mathrm{M}(G_m) = \mathrm{M}(H_1) + \mathrm{M}(H_2) \geqslant \nu(H_1) + \nu(H_2) \geqslant 2\nu(T_m) = 2m.$$

We now have

$$-m \geqslant \mathrm{M}(P_m \boxbslash P_n) - \mathrm{M}(G_m) = \sum_{i=1}^{m} \mathrm{n}_{(i,i+k-\ell)}(G_{i-1}) \geqslant \sum_{i=1}^{m}(-1) = -m$$

Thus the null spread of each vertex must be -1. Notice that no matter what order or how many of these m vertices are removed, the null spread of each one will be -1. Thus $\mathrm{n}_v(P_m \boxbslash P_n) = -1$ and since $\mathrm{M}(P_m \boxbslash P_n - (\ell, k)) = \mathrm{Z}(P_m \boxbslash P_n - (\ell, k))$, $\mathrm{n}_v(P_m \boxbslash P_n) \geqslant \mathrm{z}_v(P_m \boxbslash P_n)$, and $\mathrm{z}_v(P_m \boxbslash P_n) = -1$. □

3.4. Strong products of paths (king grids)

The *king grid* $P_m \boxtimes P_n$ is the strong product of P_m with P_n; $P_3 \boxtimes P_4$ is shown in Fig. 17. In [1] it was shown that $\mathrm{Z}(P_m \boxtimes P_n) = \mathrm{M}(P_m \boxtimes P_n) = m + n - 1$. Note that unlike grid graphs and triangular grids,

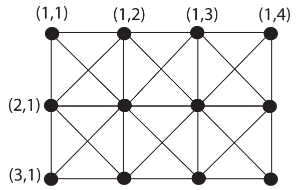

Fig. 17. The 3×4 strong product.

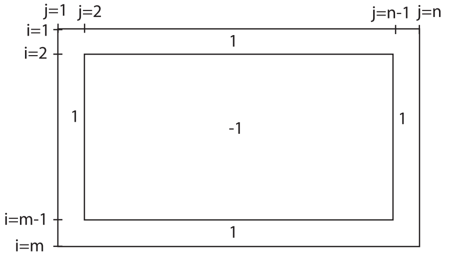

Fig. 18. Schematic diagram of zero and null spread for all vertices in $P_m \boxtimes P_n$, as established in Theorem 3.24. The vertices on the boundaries have $\mathrm{n}_v(P_m \boxtimes P_n) = \mathrm{z}_v(P_m \boxtimes P_n) = 1$.

when establishing the vertex or edge spread of a king grid, there is no need to distinguish $P_m \boxtimes P_m$ and $P_m \boxtimes P_n$, because the zero forcing number depends on both m and n.

Theorem 3.24. *Let $v = (\ell, k)$ be a vertex in $P_m \boxtimes P_n$. If $3 \leqslant \ell \leqslant m-2$ and $3 \leqslant k \leqslant n-2$, then $\mathrm{M}(P_m \boxtimes P_n - v) = \mathrm{Z}(P_m \boxtimes P_n - v) = m + n$; equivalently, $\mathrm{n}_v(P_m \boxtimes P_n) = \mathrm{z}_v(P_m \boxtimes P_n) = -1$. Otherwise, $\mathrm{M}(P_m \boxtimes P_n - v) = \mathrm{Z}(P_m \boxtimes P_n - v) = m + n - 2$ equivalently, $\mathrm{n}_v(P_m \boxtimes P_n) = \mathrm{z}_v(P_m \boxtimes P_n) = 1$. See Fig. 18.*

Proof. Assume $3 \leqslant \ell \leqslant m-2$ and $3 \leqslant k \leqslant n-2$. Recall that $\mathrm{mr}(P_m \boxtimes P_n) = mn - m - n + 1$ and this is realized by covering $P_m \boxtimes P_n$ by $(m-1)(n-1)$ copies of K_4. The graph $P_m \boxtimes P_n - v$ can be completely covered by four fewer copies of K_4 and one C_4 (this is illustrated in Fig. 19, where the thick lines are the C_4 that has replaced four K_4's where v is deleted). Since $\mathrm{mr}(C_4) = 2$, $\mathrm{mr}(P_m \boxtimes P_n - v) \leqslant mn - m - n + 1 - 4 + 2 = (mn-1) - (m+n) = |P_m \boxtimes P_n - v| - (m+n)$. Hence $\mathrm{M}(P_m \boxtimes P_n - v) \geqslant m + n = \mathrm{M}(P_m \boxtimes P_n) + 1$. Since for any graph G, $\mathrm{M}(G - v) \leqslant \mathrm{M}(G) + 1$, $\mathrm{M}(P_m \boxtimes P_n - v) = m + n$ and $\mathrm{n}_v(P_m \boxtimes P_n - v) = -1$. By Observation 2.6(3), $\mathrm{z}_v(P_m \boxtimes P_n - v) = -1$.

If $\ell \leqslant 2$ or $m - 1 \leqslant \ell$ or $k \leqslant 2$ or $n - 1 \leqslant k$, then without loss of generality, assume v is $(\ell, 1)$ or $(\ell, 2)$. We show that the set $Z = \{(1, n), \ldots, (\ell - 1, n), (\ell + 1, n), \ldots, (m - 1, n), (m, n), \ldots, (1, n)\}$ of the $n + m - 2$ vertices on the at the bottom and right border of the graph $P_m \boxtimes P_n - (\ell, k)$ excluding the vertex in row ℓ is a zero forcing set for $P_m \boxtimes P_n - (\ell, k)$. First all vertices (i, j) with $\ell \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ are forced to be colored black. Then the vertices (i, j) with $1 \leqslant i \leqslant m$, $k + 1 \leqslant j \leqslant n$ are forced to be colored black. Then the remaining vertices can be forced and the resulting derived coloring is all black. Thus $\mathrm{Z}(P_m \boxtimes P_n - (\ell, k)) \leqslant m + n - 2$. Since $\mathrm{z}_{(\ell,k)} \leqslant 1$, $\mathrm{z}_{(\ell,k)} = 1$ and $\mathrm{Z}(P_m \boxtimes P_n - (\ell, k)) = m + n - 2$. Since $\mathrm{M}(P_m \boxtimes P_n) = \mathrm{Z}(P_m \boxtimes P_n)$, $\mathrm{n}_{(\ell,k)} = 1$ and $\mathrm{M}(P_m \boxtimes P_n - (\ell, k)) = m + n - 2$. $\square$

Note that $\mathrm{z}_v(P_m \boxtimes P_n) = -1$ occurs only for $m, n \geqslant 5$. Next we consider edge spread for king grids.

Observation 3.25. For every $e \in P_m \boxtimes P_n$, $\mathrm{z}_e(P_m \boxtimes P_n) \geqslant 0$ by Theorem 2.21 because any edge can be omitted from some optimal zero forcing set. Since $\mathrm{M}(P_m \boxtimes P_n) = \mathrm{Z}(P_m \boxtimes P_n)$, $\mathrm{n}_e(P_m \boxtimes P_n) \geqslant 0$.

Lemma 3.26. *For the graph $G_1 = P_2 \boxtimes P_3 - \{(1, 2), (2, 2)\}$ shown on the left in Fig. 20, $\mathrm{mr}(G_1) = 2$. For the graph $G_2 = P_2 \boxtimes P_4 - \{1, 8\}$ shown on the right in Fig. 20, $\mathrm{mr}(G_2) = 3$.*

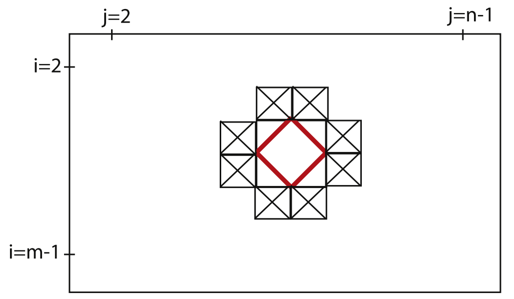

Fig. 19. Covering of king grid for zero spread −1.

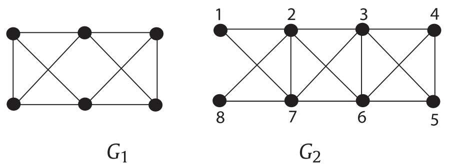

Fig. 20. The graphs G_1 and G_2 for Lemma 3.26.

Proof. It is clear that $\text{mr}(G_1) = 2$ because G_1 can be covered by two 4-cliques where the overlapping edges cancel, and $\text{mr}(G_1) \neq 1$. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

If the vertices are numbered as in Fig. 20, then $\mathcal{G}(A) = G_2$. Since rank $A = 3$, $\text{mr}(G_2) \leqslant 3$, and $\text{mr}(G_2) \geqslant 3$ because G_2 has an induced path P_4. $\square$

Theorem 3.27. *Let $4 \leqslant m \leqslant n$. If e is one of the four edges $\{(2,1),(1,2)\}, \{(m-1,1),(m,2)\}, \{(1,n-1),(2,n)\}, \{(m,n-1),(m-1,n)\}$ of $P_m \boxtimes P_n$, then $\text{n}_e(P_m \boxtimes P_n) = \text{z}_e(P_m \boxtimes P_n) = 1$; if e is any other edge of $P_m \boxtimes P_n$, then $\text{n}_e(P_m \boxtimes P_n) = \text{z}_e(P_m \boxtimes P_n) = 0$. That is, as shown in Fig. 21,*

$$\begin{aligned} \text{M}(P_m \boxtimes P_n - e) &= \text{Z}(P_m \boxtimes P_n - e) \\ &= \begin{cases} m+n-2 & \textit{if } e = \{(2,1),(1,2)\} \textit{ or } e = \{(m-1,1),(m,2)\} \textit{ or} \\ & e = \{(1,n-1),(2,n)\} \textit{ or } e = \{(m,n-1),(m-1,n)\}; \\ m+n-1 & \textit{otherwise}. \end{cases} \end{aligned}$$

Proof. If e is one of the four edges $\{(2,1),(1,2)\}, \{(m-1,1),(m,2)\}, \{(1,n-1),(2,n)\}, \{(m,n-1),(m-1,n)\}$, without loss of generality, let $e = \{(2,1),(1,2)\}$. The set of vertices $Z = \{(m,j) : 1 \leqslant j \leqslant n\} \cup \{(i,1) : 2 \leqslant i \leqslant m-1\}$ is a zero forcing set of size $m-n-2$. Applying the forces in the standard manner for a king grid, the bottom subgraph $P_{m-1} \boxtimes P_n$ (obtained by deleting all the vertices in the first row) will be colored black. Then at the top left corner where e is removed, $(2,1)$ forces $(1,1)$. Then the remaining vertices are forced as usual.

Now assume e is not one of the four edges $\{(2,1),(1,2)\}, \{(m-1,1),(m,2)\}, \{(1,n-1),(2,n)\}, \{(m,n-1),(m-1,n)\}$. If e is a diagonal edge that is not one of $\{(2,1),(1,2)\}, \{(m-1,1),(m,2)\}, \{(1,n-1),(2,n)\}, \{(m,n-1),(m-1,n)\}$, then $P_m \boxtimes P_n - e$ can be covered by $(m-1)(n-1)-1$

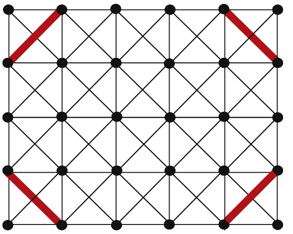

Fig. 21. Zero edge spreads in the king grid $P_m \boxtimes P_n$, $4 \leqslant m \leqslant n$. The four thick edges have $\mathrm{n}_e(P_m \boxtimes P_n) = \mathrm{z}_e(P_m \boxtimes P_n) = 1$ and all other edges have $\mathrm{n}_e(P_m \boxtimes P_n) = \mathrm{z}_e(P_m \boxtimes P_n) = 0$.

copies of K_4 and one copy of K_3, so $\mathrm{mr}(P_m \boxtimes P_n - e) \leqslant (m-1)(n-1) - 1 + 1 = (m-1)(n-1)$. If e is a vertical or horizontal edge not on the border, then $P_m \boxtimes P_n - e$ can be covered by $mn - (m+n-1) - 2$ copies of K_4 and one G_1, so by Lemma 3.26, $\mathrm{mr}(P_m \boxtimes P_n - e) \leqslant \mathrm{mr}(P_m \boxtimes P_n) = (m-1)(n-1) - 2 + 2 = mn - (m+n-1)$. If e is a border edge, then $P_m \boxtimes P_n - e$ can be covered by $mn - (m+n-1) - 3$ copies of K_4 and one G_2, so by Lemma 3.26, $\mathrm{mr}(P_m \boxtimes P_n - e) \leqslant (m-1)(n-1) - 3 + 3 = mn - (m+n-1)$. In all three cases, $\mathrm{Z}(P_m \boxtimes P_n - e) \geqslant \mathrm{M}(P_m \boxtimes P_n - e) \geqslant m + n - 1$, and by Observation 3.25, $\mathrm{Z}(P_m \boxtimes P_n - e) \leqslant \mathrm{Z}(P_m \boxtimes P_n) = m + n - 1$, so $\mathrm{M}(P_m \boxtimes P_n - e) = \mathrm{Z}(P_m \boxtimes P_n - e) = m + n - 1$. □

Now only the spreads of edges for $P_2 \boxtimes P_n, n \geqslant 2$ and $P_3 \boxtimes P_n, n \geqslant 3$ remain to be established. This is done by exhibiting zero forcing sets or construction of matrices realizing minimum rank via graph unions. The spreads for $G = P_2 \boxtimes P_2, P_2 \boxtimes P_3$ also follow from known results about minimum rank of small graphs [9].

Theorem 3.28. *For the following graphs G and edges e,* $\mathrm{n}_e(G) = \mathrm{z}_e(G) = 1$:

1. $G = P_2 \boxtimes P_2$, *all edges* e.
2. $G = P_2 \boxtimes P_3$, *every edge except* $e = \{(1,2),(2,2)\}$.
3. $G = P_2 \boxtimes P_n, n \geqslant 4$ *every edge except* $e = \{(1,k),(2,k)\}, k = 1,\ldots,n$.

For the following graphs G and edges e, $\mathrm{n}_e(G) = \mathrm{z}_e(G) = 0$:

4. $G = P_2 \boxtimes P_3, e = \{(1,2),(2,2)\}$.
5. $G = P_2 \boxtimes P_n, n \geqslant 4$ *and* $e = \{(1,k),(2,k)\}, k = 1,\ldots,n$.

Proof. For all the graphs and edges listed in (1), (2), and (3), without loss of generality it may be assumed that $e = \{(1,1),(2,1)\}$ and $G = P_2 \boxtimes P_3$, or $e = \{(1,k),(2,k+1)\}$. For $e = \{(1,1),(2,1)\}$ and $G = P_2 \boxtimes P_3$, the set of 3 vertices in the second row is a zero forcing set. For $e = \{(1,k),(2,k+1)\}, Z = \{(1,1),(2,1),\ldots,(2,k),(2,k+2),\ldots,(2,n)\}$ is a zero forcing set of n vertices, so $\mathrm{n}_e = \mathrm{z}_e(G) = 1$.

For all the graphs and edges listed in (4) and (5), construct a matrix of rank $n-1$ by covering $G - e$ with one G_1 or G_2 and copies of K_4 as needed. □

The proof of Theorem 3.29 is similar to the proofs of Theorem 3.27 and 3.28, and is omitted.

Theorem 3.29. *For the following graphs G and edges e,* $\mathrm{n}_e(G) = \mathrm{z}_e(G) = 1$:

1. $G = P_3 \boxtimes P_3$, *every edge not having* $(2,2)$ *as an endpoint.*
2. $G = P_3 \boxtimes P_n, n \geqslant 4$ *and* e *any of the edges* $\{(2,1),(1,2)\}, \{(2,1),(3,2)\}, \{(1,n-1),(2,n)\}, \{(3,n-1),(2,n)\}, \{(1,k),(1,k+1)\}, \{(3,k),(3,k+1)\},\ k = 1,\ldots,n-1$.

For the following graphs G and edges e, $\mathrm{n}_e(G) = \mathrm{z}_e(G) = 0$*:*

3. $G = P_3 \boxtimes P_3$, *every edge having* $(2, 2)$ *as an endpoint*.
4. $G = P_3 \boxtimes P_n, n \geqslant 4$ *and* e *not one of the edges* $\{(2,1),(1,2)\}, \{(2,1),(3,2)\}, \{(1,n-1),(2,n)\}, \{(3,n-1),(2,n)\}, \{(1,k),(1,k+1)\}, \{(3,k),(3,k+1)\}, k = 1,\ldots,n-1$.

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