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A technique for computing the zero forcing number of a graph with a cut-vertex

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ABSTRACT

The zero forcing number of a graph is the minimum size of a zero forcing set. This parameter is useful in the minimum rank/maximum nullity problem, as it gives an upper bound to the maximum nullity. Results for determining graphs with extreme zero forcing numbers, for determining the zero forcing number of graphs with a cut-vertex, and for determining the zero forcing number of unicyclic graphs are presented.

1. Introduction

A *graph* $G = (V_G, E_G)$ will mean a simple (no loops, no multiple edges) undirected graph. The vertex set V_G will be assumed finite and nonempty. The edge set E_G consists of two-element subsets of vertices. When $\{x, y\} \in E_G$, we say x and y are *neighbors* or x and y are *adjacent*, and write $x \sim y$. The order of G, denoted $|G|$, refers to the number of vertices $|V_G|$. We denote by K_n, C_n, and P_n the complete graph, the cycle, and the path, respectively, on n vertices. The term *path length* will be used to refer to the number of edges in the path.

The zero forcing number of a graph was introduced in [1] and the related terminology was extended in [2,3,10]. Independently, physicists have studied this parameter, referring to it as the graph infection number, in conjunction with control of quantum systems [5,6,13]. Let G be a graph with each vertex

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initially colored either black or white. From the initial coloring, vertices change color according to the *color-change rule*: if u is a black vertex and exactly one neighbor v of u is white, then change the color of v to black. When the color-change rule is applied to u to change the color of v, we say u *forces* v and write $u \to v$. Given an initial coloring of G, the *derived set* is the set of vertices colored black after the color-change rule is applied until no more changes are possible. In an initial black-white coloring of a graph G, if the set of black vertices Z has derived set that is all the vertices of G, we say Z is a *zero forcing set* for G. A zero forcing set with the minimum number of vertices is called an *optimal zero forcing set*, and this minimum size of a zero forcing set for a graph G is the *zero forcing number* of the graph, denoted $\mathrm{Z}(G)$.

In this paper, we prove results for computing the zero forcing number for certain families of graphs. In Section 2, characterizations are given for graphs having either very high or very low zero forcing numbers. In Section 3, a theorem is given which allows the zero forcing number of a graph with a cut-vertex to be calculated by using the zero forcing numbers of the connected components of the graph after deleting the cut-vertex. Section 4 contains results related to unicyclic graphs. In particular it is shown that the zero forcing number of any unicyclic graph has the same value as another graph parameter for which an algorithm exists for its computation. Section 5 summarizes the main results and proposes some questions for further study. The remainder of Section 1 presents more definitions, notations, and known results that will be used in the subsequent sections.

For a given zero forcing set Z, a *chronological list of forces* is a listing of the forces used to construct the derived set in the order they are performed. A *forcing chain* for a chronological list of forces is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k-1$, $v_i \to v_{i+1}$, and a *maximal forcing chain* is a forcing chain that is not a proper subsequence of any other forcing chain. The collection of maximal forcing chains for a chronological list of forces is called the *chain set* of the chronological list of forces, and an *optimal chain set* is a chain set from a chronological list of forces of an optimal zero forcing set. When a chain set contains a chain consisting of a single vertex, we say that the chain set contains the vertex as a *singleton*. For a zero forcing set Z, a *reversal* of Z is the set of vertices which are last in the forcing chains in the chain set of some chronological list of forces [2]. If Z is a zero forcing set of G then so is any reversal of Z [2]. Since the size of a reversal of a zero forcing set is the same as the size of the zero forcing set, a reversal of an optimal zero forcing set is an optimal zero forcing set. For any connected graph of order more than one, no vertex is in every optimal zero forcing set for the graph [2].

The *union* of $G_i = (V_i, E_i)$ is $\cup_{i=1}^k G_i = (\cup_{i=1}^k V_i, \cup_{i=1}^k E_i)$; a disjoint union is denoted $\dot\cup_{i=1}^k G_i$. Clearly, $\mathrm{Z}(\dot\cup_{i=1}^k G_i) = \sum_{i=1}^k \mathrm{Z}(G_i)$. For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{v, w\} \in E_G : v, w \in W\}$. The subgraph induced by $\overline{W} = V_G \setminus W$ will be denoted by $G - W$, or in the case W is a single vertex $\{v\}$, by $G - v$. For a graph G and a vertex $v \in V_G$, the *zero spread* of v in G is $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G - v)$ [8]. Bounds on the zero spread of a vertex are known. For any graph G and vertex v of G, $-1 \le \mathrm{z}_v(G) \le 1$ [8,11]. Here the definition of zero spread is extended to vertex subsets of size greater than one and bounds are proved.

Definition 1.1. Let G be a graph and $W \subseteq V_G$. The *zero spread* of W in G is $\mathrm{z}_W(G) = \mathrm{Z}(G) - \mathrm{Z}(G - W)$.

Corollary 1.2. *For every graph G and every subset $W \subseteq V_G$, $-|W| \le \mathrm{z}_W(G) \le |W|$.*

Proof. Let $W = \{v_1, \ldots, v_k\}$. Set $G_0 = G$ and define $G_i = G_{i-1} - v_i$ for $i = 1, \ldots, k$. Then $G_k = G - W$. The bounds on the zero spread of a vertex give that for any graph H and any vertex $v \in V_H$, $|\,\mathrm{Z}(H) - \mathrm{Z}(H - v)| \le 1$. Therefore,

$$|\,\mathrm{z}_W(G)| = |\,\mathrm{Z}(G) - \mathrm{Z}(G - W)| = |\sum_{i=0}^{k-1}(\mathrm{Z}(G_i) - \mathrm{Z}(G_{i+1}))| \le \sum_{i=0}^{k-1} |\,\mathrm{Z}(G_i) - \mathrm{Z}(G_{i+1})| \le \sum_{i=0}^{k-1} 1$$

$$= k = |W|. \quad \square$$

The *path cover number* $\mathrm{P}(G)$ of G is the smallest positive integer m such that there are m vertex-disjoint induced paths in G such that every vertex of G is a vertex of one of the paths. For any graph G, $\mathrm{P}(G) \le \mathrm{Z}(G)$ [10].

A primary reason to study the zero forcing number of a graph is its relationship to the maximum nullity of the graph, which is defined here. An association between graphs and matrices is made in the following way. Denote by $S_n(\mathbb{R})$ the set of $n \times n$ real symmetric matrices. The *graph* of $A \in S_n(\mathbb{R})$, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \le i < j \le n\}$. For a graph G, the *set of symmetric matrices described by* G is $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$ and the *maximum nullity of* G is $\mathrm{M}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\}$. For any graph G, $\mathrm{M}(G) \le \mathrm{Z}(G)$ [1]. A graph has maximum nullity of one if and only if the graph is a path [9].

2. Graphs with extreme zero forcing numbers

In this section we consider graphs that have very low or very high zero forcing numbers.

Observation 2.1. Let $G = (V_G, E_G)$ be a graph. Then $\mathrm{Z}(G) = 1$ if and only if $G = P_n$ for some $n \ge 1$.

Proposition 2.2. *Let $G = (V_G, E_G)$ be a connected graph with $|G| \ge 2$. Then $\mathrm{Z}(G) = |G| - 1$ if and only if $G = K_{|G|}$.*

Proof. It is clear that if $G = K_{|G|}$ with $|G| \ge 2$ then $\mathrm{Z}(G) = |G| - 1$.

Let $G = (V_G, E_G)$ be a connected graph with $|G| \ge 2$ and $G \neq K_{|G|}$. Then there exist $x, y \in V_G$ with $x \not\sim y$. Since G is connected, there exists $u \in V_G$ such that $u \sim x$. Let $Z = V_G \setminus \{u, y\}$. Color the vertices in Z black, and the vertices in $\{u, y\}$ white. Now x can force u. Then any vertex adjacent to y can force y. Hence Z is a zero forcing set for G and $\mathrm{Z}(G) \le |Z| = |G| - 2$. $\square$

A definition and a known result will be used in the proof of the next characterization theorem. A graph G is a *graph of two parallel paths* if there exist two independent induced paths of G that cover all the vertices of G and such that the graph can be drawn in the plane in such a way that the paths are parallel and edges (drawn as segments, not curves) between the two paths do not cross [12]. A simple path is not considered to be such a graph. A graph that consists of two connected components, each of which is a path, is considered to be such a graph. It is known that the only graphs with maximum nullity of two are graphs of two parallel paths and those of the types shown in Fig. 1 [12].

Theorem 2.3. *Let $G = (V_G, E_G)$ be a graph. Then $\mathrm{Z}(G) = 2$ if and only if G is a graph of two parallel paths.*

Proof. Let G be a graph of two parallel paths. Consider a drawing of G oriented in the plane so that the two independent induced paths which cover all the vertices of G are each horizontal and no edges

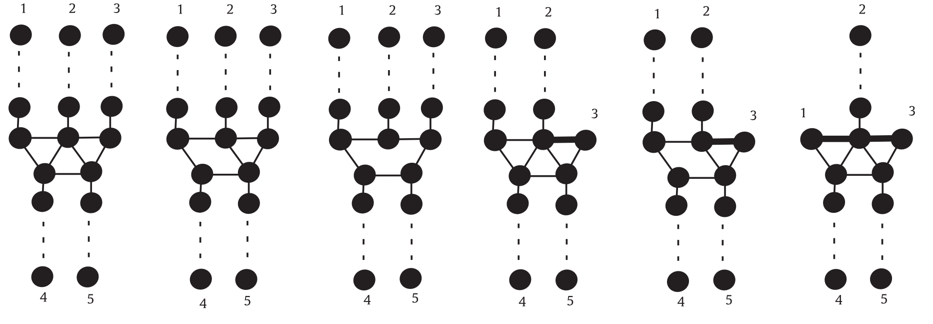

Fig. 1. Graphs which have maximum nullity 2 but are not graphs of two parallel paths. The bold lines indicate a path of length at least one. The dotted lines indicate (possibly nonexistent) paths of arbitrary length.

between the paths cross. Let Z consist of the left-most vertex of each path. Forces can be performed along the top path until a vertex that has a white neighbor w in the bottom path gets forced black. Because the edges between the two paths do not cross, forces can take place along the bottom path until w is forced black. Continuing in this manner, it is clear that Z is a zero forcing set with $|Z| = 2$, so $\mathrm{Z}(G) \le 2$. But G is not a path so by Observation 2.1, $\mathrm{Z}(G) \ge 2$. Hence $\mathrm{Z}(G) = 2$.

Let G be a graph with $\mathrm{Z}(G) = 2$. Since $\mathrm{M}(G) \le \mathrm{Z}(G)$ for any graph G, $\mathrm{M}(G) \le 2$. If $\mathrm{M}(G) = 1$, then G is a path, so by Observation 2.1, $\mathrm{Z}(G) = 1$, a contradiction. Thus $\mathrm{M}(G) = 2$, so G is a graph of two parallel paths or G is one of the types shown in Fig. 1. Note that any vertex which has degree one must be an endpoint of any induced path which contains it. Also for graphs represented in Fig. 1, any induced path which contains a degree two vertex v that is an endpoint of a bold line must have an endpoint either at v or at one of the interior vertices of the path represented by the bold line. By inspection, each graph in the figure has at least five vertices which must be endpoints of induced paths used as a path cover. Therefore, for each graph G represented in Fig. 1, $\mathrm{P}(G) \ge 3$. Since $\mathrm{P}(G) \le \mathrm{Z}(G)$ for any graph G, $\mathrm{Z}(G) \ge 3$, a contradiction. Hence if $\mathrm{Z}(G) = 2$, then G must be a graph of two parallel paths. $\square$

It is also known that a graph G satisfies $\mathrm{Z}(G) \ge |G| - 2$ if and only if G does not contain any of $P_2\dot{\cup}P_2\dot{\cup}P_2$, $P_3\dot{\cup}P_2$, P_4, $\ltimes$, or dart as an induced subgraph [1]. A figure containing these graphs can be found in [1] along with the proof which is linear algebraic. A graph theoretic proof is also possible using only zero forcing techniques, but is omitted here in the interest of brevity.

3. Results for graphs with a cut-vertex

Algorithms to compute the zero forcing number of a graph are implements in the software [7]. However, the run time depends on the number of vertices in the graph and on the zero forcing number. On a standard laptop computer, the software has failed to compute the zero forcing number for a graph with 50 vertices and zero forcing number 35. The zero forcing number of a graph G with a cut-vertex v can be calculated by finding the zero forcing numbers of the connected components of $G - v$ and calculating $\mathrm{z}_v(G)$. Theorem 3.8 gives a formula for $\mathrm{z}_v(G)$ when v is a cut-vertex. For a connected graph G with a cut-vertex v, if $G - v$ has k connected components then the formula from the theorem requires finding the zero forcing numbers of $2k$ graphs. If $G - v$ has a connected component that is large relative to G, the theorem may not be of benefit. However, if each component is reasonably reduced in size relative to G, the formula may be of benefit. Example 3.9 shows how applying the results of this section can result in a substantial reduction in CPU time, which may make computing the zero forcing number of a graph with a cut-vertex practical where it would otherwise be impractical. We begin this section with some preliminary results which lead to the main theorem of the section which gives the zero spread of a cut-vertex.

Lemma 3.1. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Then $\mathrm{Z}(G) \ge \sum_{i=1}^{k} \mathrm{Z}(G_i) - k + 1$.*

Proof. Let Z be an optimal zero forcing set of G with $v \notin Z$. Then there is a vertex u such that $u \to v$. Without loss of generality, let $u \in G_1$. Now $Z \cap V_{G_1}$ is a zero forcing set of G_1 so $\mathrm{Z}(G_1) \le |Z \cap V_{G_1}|$. Also, for $i = 2, \ldots, k$, $(Z \cap V_{G_i}) \cup \{v\}$ is a zero forcing set of G_i so $\mathrm{Z}(G_i) \le |Z \cap V_{G_i}| + 1$. Therefore,

$$\sum_{i=1}^{k} \mathrm{Z}(G_i) \le |Z \cap V_{G_1}| + \sum_{i=2}^{k} (|Z \cap V_{G_i}| + 1) = \sum_{i=1}^{k} |Z \cap V_{G_i}| + k - 1$$
$$= |Z| + k - 1 = \mathrm{Z}(G) + k - 1. \quad \square$$

Corollary 3.2. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Then $\mathrm{z}_v(G) \ge \sum_{i=1}^{k} \mathrm{z}_v(G_i) - k + 1$.*

Proof. By Lemma 3.1, $Z(G) \geq \sum_{i=1}^{k} Z(G_i) - k + 1$. Since v is a cut-vertex, $Z(G - v) = \sum_{i=1}^{k} Z(G_i - v)$. Subtracting gives $z_v(G) \geq \sum_{i=1}^{k} z_v(G_i) - k + 1$. $\square$

Lemma 3.3. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Then $Z(G) \leq \min_{1 \leq j \leq k}\{Z(G_j) + \sum_{i=1, i\neq j}^{k} Z(G_i - v)\}$.*

Proof. Fix j with $1 \leq j \leq k$. Let Z_j be an optimal zero forcing set for G_j. For $i \neq j$, let Z_i be an optimal zero forcing set for $G_i - v$. Set $Z = \cup_{i=1}^{k} Z_i$. Clearly, $Z \cap V_{G_j}$ is a zero forcing set for G_j and for $i \neq j$, $(Z \cap V_{G_i}) \cup \{v\}$ is a zero forcing set for G_i with v not needing to perform a force. Let z be colored black if and only if $z \in Z$. Starting in G_j, perform forces (if necessary) until v is colored black. Now in each $G_i - v$, $i \neq j$, forces can be performed to color all of $G_i - v$ black. (If necessary) return to G_j and perform the remaining forces. Thus Z is a zero forcing set of G. Since j was arbitrary, $Z(G) \leq \min_{1 \leq j \leq k}\{Z(G_j) + \sum_{i=1, i\neq j}^{k} Z(G_i - v)\}$. $\square$

Corollary 3.4. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Then $z_v(G) \leq \min_{1 \leq j \leq k}\{z_v(G_j)\}$.*

Proof. By Lemma 3.3, $Z(G) \leq \min_{1 \leq j \leq k}\{Z(G_j) + \sum_{i=1, i\neq j}^{k} Z(G_i - v)\}$. Since v is a cut-vertex, $Z(G - v) = \sum_{i=1}^{k} Z(G_i - v)$. Subtracting gives $z_v(G) \leq \min_{1 \leq j \leq k}\{z_v(G_j)\}$. $\square$

The following lemma provides information about the distribution of an optimal zero forcing set amongst components having certain properties.

Lemma 3.5. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Let Z be a zero forcing set for G. If $z_v(G_j) = 1$, or if $z_v(G_j) = 0$ and v is not in any optimal zero forcing set for G_j, then $|Z \cap V_{G_j - v}| \geq Z(G_j - v)$.*

Proof. Let Z be a zero forcing set for G. Clearly $(Z \cap (G_j - v)) \cup \{v\}$ must be a zero forcing set of G_j. Suppose $z_v(G_j) = 1$. Then $Z(G_j) \leq |Z \cap V_{G_j - v}| + 1$, so $|Z \cap V_{G_j - v}| \geq Z(G_j) - 1 = Z(G_j - v)$. Suppose $z_v(G_j) = 0$ and v is not in an optimal zero forcing set for G_j. Then $Z(G_j) \leq |Z \cap V_{G_j - v} \cup \{v\}|$ with equality only if v is in an optimal zero forcing set. Hence $Z(G_j) < |Z \cap V_{G_j - v} \cup \{v\}|$, so $|Z \cap V_{G_j - v}| \geq Z(G_j) = Z(G_j - v)$. $\square$

The definition and characterization which follow will be used in the main theorem of the section which gives a formula for the zero spread of a cut-vertex. We will use the fact that $z_v(G) = 1$ if and only if there exists an optimal chain set of G that contains v as a singleton [8].

Definition 3.6. Let G be a graph and $v \in V_G$. The graph $G - v$ is called *optimal chain set extendible* to v if there exists an optimal chain set of G which differs from an optimal chain set of $G - v$ only in that one chain of G is a chain of $G - v$ with v at the end.

Lemma 3.7. *Let G be a graph and $v \in V_G$. The graph $G - v$ is optimal chain set extendible to v if and only if $z_v(G) = 0$ and v is in an optimal zero forcing set for G.*

Proof. Suppose $G - v$ is optimal chain set extendible to v. Then there are optimal chain sets of G and of $G - v$ which are the same size. Since the size of an optimal chain set of a graph is the zero forcing number, $z_v(G) = Z(G) - Z(G - v) = 0$. Also, v is in an optimal zero forcing set which is a reversal in G of the optimal zero forcing set used to construct the chains for $G - v$.

Suppose $z_v(G) = 0$ and v is in an optimal zero forcing set Z for G. Construct an optimal chain set for G from Z. Now v must perform a force, otherwise it is a singleton so $z_v(G) = 1$, a contradiction. By considering each forcing chain in reverse order, it is clear that $G - v$ is optimal chain set extendible to v. □

With the above preliminary results, we are now ready to give a formula for the zero spread of a cut-vertex.

Theorem 3.8. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \leq j \leq k}\{z_v(G_j)\}$, and t denote the number of connected components of $G - v$ which are optimal chain set extendible to v. Then*

$$z_v(G) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } m = 0 \text{ and } t \leq 1 \\ -1 & \text{if } m = 0 \text{ and } t \geq 2, \text{ or if } m = -1 \end{cases}$$

Proof. The proof will be completed by considering each of the cases.

Case 1: Suppose $m = 1$. The bounds on the zero spread of a vertex gives $z_v(G) \leq 1$ and Corollary 3.2 gives $z_v(G) \geq 1$.

Case 2: Suppose $m = -1$. Corollary 3.4 gives $z_v(G) \leq -1$ and the bounds on the zero spread of a vertex gives $z_v(G) \geq -1$.

Case 3: Suppose $m = 0$ and $t \geq 2$. Without loss of generality, let $G_1 - v$ and $G_2 - v$ be optimal chain set extendible to v. Now v is in an optimal zero forcing set Z_1 of G_1. Also, v must perform a force, for if not, then v is a singleton in an optimal zero forcing set of G_1 so $z_v(G_1) = 1$, a contradiction. There exists another optimal zero forcing set Z_1' of G_1 found by reversing the maximal forcing chains. Since $v \in Z_1$ and v performs a force, $v \notin Z_1'$ and there is a chain set such that v does not perform a force. Let Z_2 be an optimal zero forcing set of G_2 with $v \in Z_2$ and for $i = 3, \ldots, k$, let Z_i be an optimal zero forcing set of $G_i - v$. Let $Z = Z_1' \cup (Z_2 \setminus \{v\}) \cup_{i=3}^{k} Z_i$. Now $Z \cap V_{G_1}$ can force all of G_1 with v not used to force. Then for $i = 3, \ldots, k$, $Z \cap V_{G_i}$ can force all of $G_i - v$ with v not used to force. Then $(Z \cap V_{G_2}) \cup \{v\}$ can force all of $G_2 - v$. Thus Z is a zero forcing set of G, so $Z(G) \leq |Z| = \sum_{i=1}^{k} |Z_i| - 1 = \sum_{i=1}^{k} Z(G_i - v) - 1$. Since v is a cut-vertex, this gives $-1 \geq Z(G) - \sum_{i=1}^{k} Z(G_i - v) = Z(G) - Z(G - v) = z_v(G)$. By the bounds on the zero spread of a vertex, $z_v(G) \geq -1$. Hence $z_v(G) = -1$.

Case 4: Suppose $m = 0$ and $t \leq 1$. Corollary 3.4 gives $z_v(G) \leq 0$, so the lower bound remains to be shown. Let Z be an optimal zero forcing set of G with $v \notin Z$. Note that $Z(G) = |Z| = \sum_{i=1}^{k} |Z \cap V_{G_i - v}|$ and $Z(G - v) = \sum_{i=1}^{k} Z(G_i - v)$, so it suffices to show

$$\sum_{i=1}^{k} |Z \cap V_{G_i - v}| \geq \sum_{i=1}^{k} Z(G_i - v) \tag{1}$$

Now there is at most one i, $1 \leq i \leq k$, such that $z_v(G_i) = 0$ and v is in an optimal zero forcing set for G_i, so without loss of generality suppose for $i = 2, \ldots, k$, either $z_v(G_i) = 1$ or that $z_v(G_i) = 0$ but v is not in any optimal zero forcing set for G_i. By Lemma 3.5, $|Z \cap V_{G_i - v}| \geq Z(G_i - v)$ for $i = 2, \ldots, k$. If $|Z \cap V_{G_1 - v}| \geq Z(G_1 - v)$, then (1) is clearly satisfied.

Suppose $|Z \cap V_{G_1 - v}| \leq Z(G_1 - v) - 1$. Then $Z \cap V_{G_1 - v}$ is not a zero forcing set of $G_1 - v$. However, $(Z \cap V_{G_1 - v}) \cup \{v\}$ must be a zero forcing set of G_1. Therefore, since $z_v(G_1) \geq 0$,

$$Z(G_1 - v) \leq Z(G_1) \leq |(Z \cap V_{G_1 - v}) \cup \{v\}| = |Z \cap V_{G_1 - v}| + 1 \leq Z(G_1 - v)$$

so $|Z \cap V_{G_1 - v}| = Z(G_1 - v) - 1$. Also, there must be $j \neq 1$, $u \in V_{G_j - v}$, and $w \in V_{G_1 - v}$ such that $u \rightarrow v \rightarrow w$. Then u is at the end of a forcing chain in $G_j - v$. Since $G_j - v$ is not

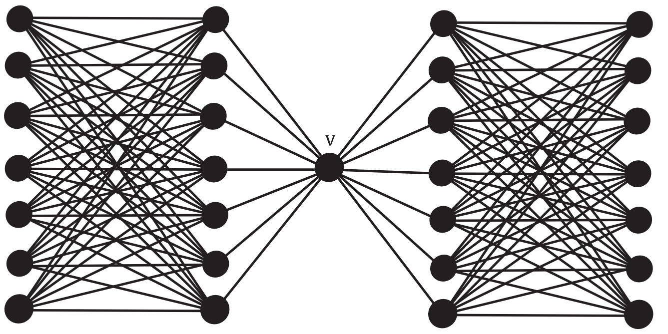

Fig. 2. A graph with a cut-vertex.

optimal chain set extendible to v, $|Z \cap V_{G_j - v}| \geq Z(G_j - v) + 1$. Hence $|Z \cap V_{G_1 - v}| + |Z \cap V_{G_j - v}| \geq \mathrm{Z}(G_1 - v) - 1 + Z(G_j - v) + 1 = \mathrm{Z}(G_1 - v) + Z(G_j - v)$. Applying Lemma 3.5 for $i \neq 1, j$, (1) is satisfied. $\square$

Example 3.9. For the graph G in Fig. 2, v is a cut-vertex, G_1 and G_2 are each the complete bipartite graph $K_{7,8}$, and $G_1 - v$ and $G_2 - v$ are each the complete bipartite graph $K_{7,7}$. The best current program for computing zero forcing number [7] takes 25 seconds (on a 2007 MacBookPro) to find $\mathrm{Z}(G) = 25$. To find $\mathrm{Z}(G_1) = 13$, $\mathrm{Z}(G_2) = 13$, $\mathrm{Z}(G_1 - v) = 12$, and $\mathrm{Z}(G_2 - v) = 12$, the same program on the same machine took .05 seconds, 500 times faster than computing for G. Theorem 3.8 can be used to find $\mathrm{Z}(G)$ from $\mathrm{Z}(G_1)$, $\mathrm{Z}(G_2)$, $\mathrm{Z}(G_1 - v)$, and $\mathrm{Z}(G_2 - v)$.

Remark 3.10. The software [7] currently returns the zero forcing number of a graph and an optimal zero forcing set. Theorem 3.8 could be used in Example 3.9 since $\mathrm{z}_v(G_1) = \mathrm{z}_v(G_2) = 1$. If $\min_{1 \leq j \leq k}\{z_v(G_j)\} = 0$, $\mathrm{z}_v(G_j) = 0$ for at least two j's, and for no more than one j does $\mathrm{z}_v(G_j) = 0$ and v appear in the returned optimal zero forcing set, then $\mathrm{z}_v(G)$ cannot be determined using the theorem. The software is being revised to have the possibility of favoring a specified vertex for inclusion in the optimal zero forcing set that gets returned, thereby covering this one ambiguous case.

4. Zero forcing number for unicyclic graphs

In [9], an algorithm is given which computes $\mathrm{P}(G)$ for any tree or unicyclic graph G. Additionally, in [10], it was proven that for any tree T, $\mathrm{P}(T) = \mathrm{Z}(T)$. Because of this result, the algorithm computes the zero forcing number for any tree. In this section, we prove that for any unicyclic graph G, $\mathrm{P}(G) = \mathrm{Z}(G)$ so the algorithm noted above can be used to compute the zero forcing number for any unicyclic graph.

Let C_n be an n-cycle and let $U \subseteq V_{C_n}$. The graph H obtained from C_n by appending a leaf to each vertex in U is called a *partial n-sun*. The term *segment* of H will refer to any maximal subset of consecutive vertices in U. The segments of H will be denoted $U_1, \ldots, U_t$. For a partial n-sun, H, with segments $U_1, \ldots, U_t$, $\mathrm{P}(H) = \max\left\{2, \sum_{i=1}^{t}\left\lceil \frac{|U_i|}{2}\right\rceil\right\}$ [4]. We prove that for a partial n-sun, the zero forcing number equals the path cover number.

Theorem 4.1. *Let H be a partial n-sun with segments $U_1, \ldots, U_t$. Then*

$$\mathrm{Z}(H) = \max\left\{2, \sum_{i=1}^{t}\left\lceil \frac{|U_i|}{2}\right\rceil\right\}.$$

Proof. Let H be a partial n-sun with segments $U_1, \ldots, U_t$. Since $\mathrm{P}(G) \leq \mathrm{Z}(G)$ for any graph G, $\mathrm{Z}(H) \geq \max\left\{2, \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil\right\}$. Displaying a zero forcing set of size $\max\left\{2, \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil\right\}$ will provide the upper bound. First a few special cases are considered.

If $t = 0$, then H is a cycle and any two consecutive vertices make a zero forcing set.

If $t = 1$, and $|U_1| = 1$, then the degree 1 vertex and either other vertex adjacent to the degree 3 vertex make a zero forcing set.

If $t = 1$, and $|U_1| = 2$, then the two vertices of degree 1 make a zero forcing set.

Now assume that there is at least one segment and if there is only one, it has size at least 3. Note that this implies $\sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil \geq 2$. Suppose each segment is of even order. Let the segments be numbered in the clockwise direction. Let Z' denote the set of vertices obtained as follows: for each segment, select every other leaf vertex starting with the second. Now $|Z'| = \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil$. Construct Z from Z' by removing the last leaf vertex of H_t from Z' and replacing it with the first leaf vertex of H_1. Then Z is a zero forcing set with size $\sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil$.

Now assume G has k odd sized segments for some $1 \leq k \leq t$. Create an induced subgraph G' from G by deleting the first leaf vertex from each odd sized segment. For each odd component U_i in G, let U_i' denote the resulting even component in G'. Now $|G| - |G'| = k$, so $Z(G) \leq Z(G') + k$ by Corollary 1.2. Also G' has no segments of odd size, so by the above argument, $\mathrm{Z}(G') = \sum_{i=1;|U_i|\text{even}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil + \sum_{i=1;|U_i|\text{odd}}^{t} \left\lceil \frac{|U_i'|}{2} \right\rceil$. Hence $\mathrm{Z}(G) \leq \sum_{i=1;|U_i|\text{even}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil + \sum_{i=1;|U_i|\text{odd}}^{t} \left\lceil \frac{|U_i'|}{2} \right\rceil + k = \sum_{i=1;|U_i|\text{even}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil + \sum_{i=1;|U_i|\text{odd}}^{t} \left\lceil \frac{|U_i-1|}{2} \right\rceil + k = \sum_{i=1;|U_i|\text{even}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil + \sum_{i=1;|U_i|\text{odd}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil - k + k = \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil$. $\square$

If there are at least two components of the graph $G - v$ which are paths, each joined to v in G at only one endpoint, then vertex v is called *appropriate*. A vertex v is called a *peripheral leaf* if v is adjacent to only one other vertex u, and u is adjacent to no more than two vertices. The *trimmed form* of a graph G is an induced subgraph obtained by a sequence of deletions of appropriate vertices, isolated paths, and peripheral leaves until no more such deletions are possible. The trimmed form of a graph is unique [4]. The following theorems and remarks describe the consequences on the zero forcing number after applying a "trimming" operation. These consequences will be compared to other consequences of the operations, particularly related to unicyclic graphs, to conclude the main result of the section.

Remark 4.2. If v is an appropriate vertex, then v is a cut-vertex and the case of Theorem 3.8 with $m = 0$ and $t \geq 2$, or the case $m = -1$ applies so $\mathrm{Z}(G - v) = \mathrm{Z}(G) + 1$.

Remark 4.3. If P is an isolated path in G, then Observation 2.1 gives $\mathrm{Z}(G - V_P) = \mathrm{Z}(G) - 1$.

Remark 4.4. If v is a peripheral leaf then v and its neighbor must be in the same maximal forcing chain for any optimal chain set, so $\mathrm{Z}(G - v) = \mathrm{Z}(G)$.

Theorem 4.5. *If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then* $\mathrm{Z}(G) = \mathrm{Z}(\breve{G}) + n_2 - n_1$.

Proof. The proof follows from the uniqueness of the trimmed form and Remarks 4.2–4.4. $\square$

An example will be given at the end of this section which illustrates the use of the above theorem. Here we will continue to progress toward the main result of this section. If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then $\mathrm{P}(G) = \mathrm{P}(\breve{G}) + n_2 - n_1$ [4]. The trimmed form of a unicyclic graph G is either the empty graph or a partial n-sun [4].

Theorem 4.6. *Let $G = (V_G, E_G)$ be a unicyclic graph. Then* $\mathrm{Z}(G) = \mathrm{P}(G)$.

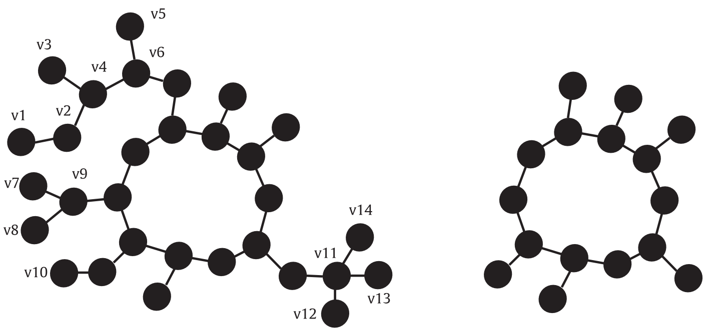

Fig. 3. A unicyclic graph and its trimmed form.

Proof. Let $\breve{G}$ be the unique trimmed form of the unicyclic graph G resulting from a sequence consisting of n_1 appropriate vertex deletions, n_2 isolated path deletions, and n_3 peripheral leaf deletions. Then $\mathrm{Z}(G) = \mathrm{Z}(\breve{G}) + n_2 - n_1 = \mathrm{P}(\breve{G}) + n_2 - n_1 = \mathrm{P}(G)$. $\square$

Example 4.7. In Fig. 3, there is unicylic graph G and its trimmed form $\breve{G}$. A possible order of trimming operations is as follows: Delete the peripheral leaf v_1. Delete appropriate vertex v_4 then the two isolated paths of size one, v_2 and v_3. Delete peripheral leaf v_5. Delete peripheral leaf v_6. Delete appropriate vertex v_9 then the two isolated paths of size one, v_7 and v_8. Delete peripheral leaf v_{10}. Delete appropriate vertex v_{11} then the three isolated paths of size one, v_{12}, v_{13}, and v_{14}. The trimmed form $\breve{G}$ (see graph on right in Fig. 3) was obtained from G (see graph on left in Fig. 3) by deleting $n_1 = 3$ appropriate vertices, $n_2 = 7$ isolated paths, and $n_3 = 4$ peripheral leaves. The trimmed form $\breve{G}$ is a partial n-sun with segments of sizes 1, 2, and 3, so by Theorem 4.1, $\mathrm{Z}(\breve{G}) = 4$. Theorem 4.5 gives $\mathrm{Z}(G) = \mathrm{Z}(\breve{G}) + n_2 - n_1 = 4 + 7 - 3 = 8$.

5. Conclusions and open questions

We have characterizations for graphs G with zero forcing number 1, 2, $|G| - 1$, and $|G| - 2$.

Question 5.1. Can either the linear algebraic or graph theoretic proof techniques used for proving the characterizations listed above be used to characterize graphs G with zero forcing number 3 or $|G| - 3$?

We have proved a formula for the zero spread of a cut-vertex, which allows the zero forcing number of a graph G with a cut-vertex v to be calculated in terms of the zero forcing numbers of the connected components of $G - v$.

Question 5.2. Can the cut-vertex result be generalized to cut sets of size two to be used for computing zero forcing number of 2-connected graphs?

We know that for any graph G, $\mathrm{Z}(G) \geq \mathrm{P}(G)$, and for trees and unicyclic graphs, $\mathrm{Z}(G) = \mathrm{P}(G)$.

Question 5.3. For what other families of graphs does $\mathrm{Z}(G) = \mathrm{P}(G)$?

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