MA-473 Date: 28/9/24

Btp discuss

MA-423 Date: 29/9/24

"non-singular" only reserved for sq matrices

1) $\underline{A_{n\times m} \text{ full rank}} \quad \text{rank}(A) = m$

$$\begin{bmatrix} R_1 \\ 0 \end{bmatrix} x_0 = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}_{n\times n} c$$

$$\begin{bmatrix} R_1 x_0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_m C_1 + 0_\times C_2 \\ 0_\times C_1 + 0_\times C_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \begin{matrix} \}m \\ \}n-m \end{matrix}$$

$$R_1 x_0 = C_1$$

we can't make any assumption on rank(A) possi.

Rank revealing QR decomposition

$$R = \begin{array}{cc|c} & r & m-r \\ & \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} & \begin{matrix} r \\ n-r \end{matrix} \end{array}$$

$\text{rank}(A) = r \ (\leq m)$

$A_{n\times m}$

$R_2 \to$ upper triangular & non singular

$$Q_r \cdots Q_1 A P_1 \cdots P_r = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}$$

$$Q^T A P = R$$

$$AP = QR$$

$\text{rank}(A) \ \cancel{r}\ ? \quad \begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix}$ & R_3 has very small abs values

we can zero out R_3 st rank(A) = r (numerically rank of A)

Case 2

$\text{rank}(A) = r < m$

$(AP)e_1 \quad (AP)e_r$

linear ind why?

$$AP = QR = \begin{array}{cc} r & m-r \\ [Q_1, & Q_2] \end{array} \begin{array}{c} \begin{array}{cc} r & m-r \end{array} \\ \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \end{array} r$$

$$= \left[\begin{array}{c|c} Q_1 R_1 & Q_1 R_2 \\ 0 & 0 \end{array} \right]$$

$Q_1 R_1 \Rightarrow$ will preserve rank of R_1 $(\because Q_1$ is orthogonal$)$

$$[(AP)_1 \ldots (AP)_r] = \begin{bmatrix} Q_1 R_1 \\ 0 \end{bmatrix}$$

orthogonal matrix preserves rank of a matrix on operat

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Kaliyuga : chaos/lawless ness (in our context chaos in existing hierarchical structure)

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for non singular matrix A,

$Ax = b$

$x = A^{-1}b \rightarrow$ costly

$\{x^{(n)}\} \rightarrow \tilde{x}$

$n \rightarrow \infty \qquad x^{(n)} \simeq \tilde{x}$

MA-423

$Q = [Q_1 \; Q_2]$ $\qquad$ $Ax_0 = Q_1 Q_1^T b$

$Q^T = \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$ $\qquad$ $= Q_1 [e_1 \; \; e_r][e_1 \; \; e_r]^T Q_1^T b$

$b = b_1 + b_2$ $\qquad$ $b_1 = Q_1 Q_1^T b$ $\qquad$ $I - Q_1 Q_1^T = Q_2 Q_2^T$

$b_2 = (I - Q_1 Q_1^T) b$

$= Q_2 Q_2^T b = r$

$Q Q^T = I_n$

$Q_1 Q_1^T + Q_2 Q_2^T = I_n$

$\|r\| = \|Q_2 Q_2^T b\|_2 = \|Q_2^T b\|_2 = \|c_2\|$

$c = Q^T b = [Q_1 \; Q_2]^T b = \begin{bmatrix} Q_1^T b \\ Q_2^T b \end{bmatrix} \begin{matrix} \}c_1 \\ \}c_2 \end{matrix}$

Matrix value Decomposition:

* Singular value Decompsi L is implicitly rank revealing decomp. too.

$A = U \Sigma V = [u_1 \; \ldots \; u_m] \begin{bmatrix} \sigma_1 & \\ & \ddots \end{bmatrix} \begin{bmatrix} v_1^* \\ v_m^* \end{bmatrix}$

$V = [v_1 \; \ldots \; v_m]$

$$A v_i = A V e_i \qquad 1 \le i \le m$$

U, V orthogonal matrix.

$$= U \Sigma V^* V e_i$$

$$= U \Sigma e_i$$

$$= U \begin{bmatrix} 0 \\ a_i \\ 0 \end{bmatrix} = \begin{bmatrix} U_1 \dots U_i \dots U_n & U_n \end{bmatrix} \begin{bmatrix} 0 \\ a_i \\ 0 \end{bmatrix}$$

$$= a_i U_i$$

$$A^* = (U \Sigma V)^* = V^* \Sigma^* U^* \alpha$$

Thm: Every matrix has a singular value decomp.

Properties of SVD

i) $A = \sum_{i=1} a_i u_i v_i^*$

ii) $A = V \Sigma^T U^*$

iii) $R(A) = \text{span}(U_1 \dots U_r)$ $\qquad N(A) = \text{span}\{V_{r+1} \dots V_m\}$

$R(A^*) = \text{span}(V_1 \dots V_r)$ $\qquad N(A^*) = \text{span}\{U_{r+1} \dots U_n\}$

$A v_i = a_i u_i \qquad i = 1:r$

$a_i \neq 0 \Rightarrow u_i$ spans colspace (A)

iv) $\|A\|_2 = a_1$

$$\|A\|_2 = \|U \Sigma V^*\|_2$$
$$= \|\Sigma V\|_2$$
$$= \|(\Sigma V)^*\|_2$$
$$= \|V^* \Sigma^*\|_2$$
$$= \|\Sigma^*\|_2 = a_1$$

$\sigma_1 = \|A\|_2 = \max \frac{\|Ax\|_2}{\|x\|_2} = \text{maxmag}(A)$

~~Supp A is sq or tall~~

Supp A is sq or tall & $\text{rank}(A) < n \Rightarrow \text{Null}(A) \neq 0$

$\text{minmag}(A) \Leftarrow$, ~~Null(A) ≠ 0~~

$= 0$

Supp A is broad, $n < m$

$\text{rank}(A) \leq n < m$

$= r$

$\Rightarrow \text{Null}(A) \neq 0 \Rightarrow \text{minmag}(A) = 0$

for $k(A) \rightarrow$ consider $k(A^T)$ ~~for~~ if A is broad.

- A is sq & non singular.

$$F = \begin{bmatrix} 0 & & 1 \\ & \cdot & \\ 1 & & 0 \end{bmatrix} \rightarrow \text{flip matrix}.$$

$$A^{-1} = (U \Sigma V^*)^{-1}$$

$$= (V^*)^{-1} \Sigma^{-1} U^{-1}$$

$$= V \Sigma^{-1} U^*$$

$$F^2 = I$$

$$= VFF \Sigma^{-1} FFU^* = (VF^2) \Sigma^{-1} (UF)^*$$

$$\|A^{-1}\|_2 = 1/\sigma_n \qquad k(A^{-1}) = \frac{\sigma_1}{\sigma_n}$$

$\sigma_1 \gg \sigma_n$

$\Rightarrow$ singular matrices are those

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$\|u_i v_i^*\|_2 =$ $A v_i v_i^*$ ~~$\|A v_i v_i^*\|_2$~~

$\|u_i v_i^*\|_2 = \|u_i\|_2 \|v_i\|_2 = 2$

A

$A v_i = v_i^* v_i = u_i$

$A v = u_i v_i^* v = 0 \quad \forall v \perp v_i$

$\langle v_1 \quad v_i, v_{i n} \quad v_n \rangle \in$ ONB of $\mathbb{F}^r$

$x = \sum_{i=1}^{m} v_i \qquad \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = ?$

$v_i^* v_j = 0 \quad i \neq j$

$1 \quad i = j$

Full rank, Full row rank $\quad A_{n \times m} \quad \text{rank}(A) = n$

$A v_i = a_i u_i$

$A^* A v_i = a_i A^* u_i = a_i^2 v_i$

$A = U \Sigma V^*$ $\qquad A^* A = V \Sigma^* U^* U \Sigma V^*$

$= V \Sigma^* \Sigma V^*$

Every singular matrix is surrounded by non singular matrics

hence it is numerically difficult to find rank-defficient matrix

& vice versa

$$A = U \begin{bmatrix} a_1 & & & & \\ & a_r & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} V^* \qquad r < n \qquad \varepsilon > 0$$

Consider $B = U \begin{bmatrix} a_1 & & & \\ & a_r & & \\ & & \varepsilon/2 & \\ & & & \ddots \ \varepsilon/2 \end{bmatrix} V^*$ rank $B = n$

$$\|A - B\|_2 = \left\| U \left(\begin{bmatrix} a_1 & & & \\ & a_r & & \\ & & 0 & \\ & & & \ddots \ 0 \end{bmatrix} - \begin{bmatrix} a_1 & & & \\ & a_r & & \\ & & \varepsilon/2 & \\ & & & \varepsilon/2 \end{bmatrix} \right) V^* \right\|_2$$

$$= \left\| \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & -\varepsilon/2 & \\ & & & & -\varepsilon/2 \end{bmatrix} V^* \right\|_2$$

$$= \left\| \begin{bmatrix} 0 & \cdots & 0 & -\varepsilon/2 & -\varepsilon/2 \end{bmatrix} \right\|_2 = \max_{1 \le i \le n} (d_{ii})$$

$$= \varepsilon/2$$

Condensed SVD

$$A = U \Sigma V = [u_1 \quad u_n] \begin{bmatrix} a_1 & & & \\ & a_r & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} v_1^* \\ \\ v_m^* \end{bmatrix}$$

$$= [u_1 \quad u_n] \begin{bmatrix} a_1 v_1^* & & & \\ & a_r v_r^* & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \begin{matrix} \} r \\ \} m-r \end{matrix}$$

$$= \begin{bmatrix} a_1 u_1 v_1^* & & & \\ & a_r u_r v_r^* & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$= \sum_{i=1}^{r} a_i u_i v_i^*$$

$$= [u_1 \cdots u_r] \begin{bmatrix} a_1 & \\ & a_r \end{bmatrix} \begin{bmatrix} v_1^* \\ v_r^* \end{bmatrix}$$

$(AA^+)^2 = AA^+$

$P \qquad P^2 = P \qquad A^+AA^+ = A^+ \Rightarrow AA^+AA^+ = AA^+ \Rightarrow P^2 = P$

$B = [1 \; 2]$

$B \begin{bmatrix} x \\ y \end{bmatrix} = 1 \Rightarrow x + 2y = 1$

$B^+ [1] = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} \rightarrow$ closest to origin

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Transformation to upper Hessenberg form :

~~recall~~ recall midsem in exam

$x, y \to n \times 1$ non real vectors

$\|x\|_2 = \|y\|_2$ & $\underline{x^* y \in \mathbb{R}}$

st $x \neq y$

$\langle x, y \rangle = (x_1 - iy_2)(y_1 + i y_2)$
$= x_1 y_1 + i x_1 y_2$
$- i y_1 x_2 + -y_2$

$\langle x+y, x-y \rangle$

$= \|x\|^2 - \|y\|^2 - \underbrace{y^* x + x^* y}$

$x^* y = y^* x$ ($\because x^* y \in \mathbb{R}$)

$= \|x\|^2 - \|y\|^2 = 0$

show Q is unique.

$Q = I - \frac{2}{\|x-y\|_2^2} (x-y)(x-y)^*$

$\hat{Q} = I - \frac{2}{\|\hat{u}\|_2^2} \hat{u} \hat{u}^*$ is another reflector

Supp ,

st $\hat{Q} x = y$

$\Rightarrow x - \frac{2}{\|\hat{u}\|_2^2} \hat{u}(\hat{u}^* x) = y$

$\Rightarrow \frac{2 \hat{u}(\hat{u}^* x)}{\|\hat{u}\|_2^2} = x - y$ or $\underbrace{\left[\frac{2(\hat{u}^* x)}{\|u\|_2^2} \right]}_{\alpha} \hat{u} = x - y$

$\alpha \hat{u} = x - y$

$x \neq y \Rightarrow \hat{u}^* x \neq 0$

$Q = I - \frac{2 (x-y)(x-y)^*}{\|x-y\|_2^2} = I - \frac{2 \hat{u} \hat{u}^*}{\|u\|_2^2}$

$= \hat{Q}$

norm(Q*R-V)   norm(Q'*Q-eye(m))

≈

mgs → bilb(13)
↓
quality of orth. } run 2 times

2nd time will give better orthonormal

i/p 2nd = o/p of 1st

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$A = U\Sigma V^* \in \mathbb{F}^{n\times m}$, $\text{rank}(A)=r$, $A_k = U\Sigma_k V^*$ where

$\Sigma_k = \text{diag}(\sigma_1, \ldots \sigma_k, 0 \ldots 0)$ $\quad \|A - A_k\|_2 = \min\{\|A-B\|_2 :$

$B\in \mathbb{F}^{n\times m}$
$\text{rank}(B)\le k\}$

$= \sigma_{k+1}$

$\dim(V)$ (V: vector space) $:=$ max # linearly indep vectors in S.

rank $A_k = k$

$\|A - A_k\|_2 = \sigma_{k+1}$,

$\therefore \sigma_{k+1} \ge \min\{\|A-B\|_2 : B\in \mathbb{F}^{n\times m}, \text{rank } B \le k\}$. Let $B\in \mathbb{F}^{n\times m}$

rank $B \le k$. Then dim null $B \ge n-k$. Let $S = \text{span}\{v_1, \ldots v_{k+1}\}$

then $\underbrace{\dim(\text{null } B\cap S)\ge 1}$.

$\Downarrow$

$\exists x_0 \in \text{null } B\cap S$ st $x_0 \in \mathbb{R}^m$, $x_0 \ne 0$

$(\because x_0 \in \text{null } B)$

$\therefore (A-B)x_0 = Ax_0$

Now, $x_0 \in S \Rightarrow x_0 = \sum_{j=1}^{k+1} a_j v_j$ & $\|x_0\|_2^2 = \sum_{j=1}^{k+1} \|\alpha_j v_j\|_2^2$

$= \sum_{j=1}^{k+1} |\alpha_j|^2 \underbrace{\|v_j\|_2^2}_{=1}$

$= \sum_{j=1}^{k+1} |\alpha_j|^2$

$\therefore \|x_0\|_2^2 = \sum_{j=1}^{k+1} |\alpha_j|^2$

$\langle x_0, v_i \rangle = \langle \sum_{j=1}^{k+1} \alpha_j v_j , v_i \rangle$
$(1 \leq i \leq k+1)$

$= \sum_{j=1}^{k+1} \alpha_j \langle v_j, v_i \rangle$

$\langle x_0, v_i \rangle = \alpha_i$ (if $j \leq i$)

$$\|(A-B)x_0\|_2^2 = \|Ax_0\|_2^2 = \|A(\sum_{j=1}^{k+1} \alpha_j v_j)\|_2^2 .$$

$$= \|\sum_{j=1}^{k+1} \alpha_j A v_j \|_2^2$$

$$= \|\sum_{j=1}^{k+1} \alpha_j a_j u_j \|_2^2$$

$$= \sum_{j=1}^{k+1} a_j^2 |\alpha_j|^2$$

$$\geq a_{k+1}^2 \underbrace{\sum_{j=1}^{k+1} |\alpha_j|^2}$$

$$= a_{k+1}^2 \|x_0\|_2^2$$

$\therefore \|(A-B)x_0\|_2^2 \geq a_{k+1}^2 \|x_0\|_2^2$ $\therefore$

$\Rightarrow \dfrac{\|(A-B)x_0\|_2}{\|x_0\|_2} \geq a_{k+1}$, $\therefore \|A-B\|_2 = \max_{x \neq 0} \dfrac{\|(A-B)x\|}{\|x\|}$

$$\geq \max_{x \neq 0} \frac{\|(A-B)x_0\|}{\|x_0\|_2}$$

$$= a_{k+1}$$

$\therefore \{ \min \|A-B\|_2 : B \in F^{n\times m},$ rank $B \leq k \} \geq a_{k+1}$ $\forall B$, rank $B \leq k$

Collorary: $A \in \mathbb{F}^{n\times n}$ be non-singular. Let $A = U\Sigma V^*$ be an SVD of A then,

$\sigma_n = \min \{ \|A-B\|_2 : B \in \mathbb{F}^{n\times n}$ is singular $\}$

dist of A from the closest sing. matrix.

$\sigma_n = \min \{ \|\Delta A\| : A+\Delta A$ is sing $\}$

Collorary $\frac{1}{k_2(A)} = \frac{\sigma_n}{\sigma_1}$

$\sigma_1 = \|A\|_2$,

$\|\Delta A\| \leq \frac{1}{k(A)}$

ΔA st $\Delta A + A$ is sing

$A + \Delta A$ is non-singular.

$\Rightarrow A = \begin{bmatrix} \sigma_1 & & \\ & \sigma_{n-1} & \\ & & 0 \end{bmatrix}$

$A = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} V^*$

$A_{n-1} = U \begin{pmatrix} \sigma_1 & & \\ & \sigma_{n-1} & \\ & & 0 \end{pmatrix} V^*$

$A_{n-1} = A - \underbrace{U \begin{pmatrix} 0 & & \\ & 0 & \\ & & \sigma_n \end{pmatrix} V^*}_{\Delta A}$

$\|\Delta A\|_2 = \sigma_n \leftarrow$ nearest sing matrix to A in $\|\ \|_2$

$\Rightarrow \|\Delta A\| \leq \frac{1}{k(A)}$

$\Delta A = -\sigma_n u_n v_n^*$

numerical rank determination via SVD

* Eckart-Young's thm.

A's rank r will be numerically close to rank of

* Hibert matrix are +ve definite, Prove it (ex)

→ point of rank(A) is not revealing exact rank of A but the closest rank around A's neighbour.

MA-473 Lab

$$\frac{y_m^{n+1} - y^n}{k} = \frac{y_{m+1}^{n+1} - 2y_m^{n+1} + y_{m-1}^{n+1}}{h^2}$$

$$y_m^{n+1} - y^n = \lambda y_{m+1}^{n+1} - 2\lambda y_m^{n+1} + \lambda y_{m-1}^{n+1}$$

$$\lambda = \frac{k}{h^2}$$

$$-\lambda y_{m+1}^{n+1} + (1+2\lambda) y_m^{n+1} - \lambda y_{m-1}^{n+1} = y^n$$

$$\begin{bmatrix} (1+2\lambda) & -\lambda & & \\ -\lambda & (1+2\lambda) & & \\ & & (1+2\lambda) & -\lambda \\ & & -\lambda & (1+2\lambda) \end{bmatrix} y^{n+1} = y^n$$

~~MA~~ ~~HSS~~-217

n-8

010 · 011

n=1 · n=2 · 1Ω

* Remember to answer the quesn & don't gives ans that doesn't ans the quesn

8167211360 – term paper mentor.

Numerical method for Jam diffussion

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Stability & Sensitivity of LSP problems.

$$\frac{\|\delta A\|_2}{\|A\|_2} \le k_1(v)\left[\,n u + O(u^2)\right]$$

$$\frac{\|E\|_2}{\|A\|_2} \le n^{\beta} u + O(u^2) = O(u)$$

n is st $O(n^{\beta} u) \simeq O(u)$

$fl(QA) = Q(A+E)$

Q = unitary/orthogonal

bckwd stable

$k_2(A)=1$

moreover we are assuming a very very pessimistic upper bdd.

In reality we don't reach n^{β} unless we're considering very very large systems.

Collorary

$fl(AQ) = fl\left[(AQ)^T\right]^T$

$= fl(Q^T A^T)^T \cong fl(QB)$

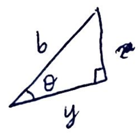
$R = Q_p \cdots Q_1 A$

$fl(R) = fl[Q_p \cdots Q_1 A] = Q_p \cdots Q_1 (A+E)$ where

$\frac{\|E\|_2}{\|A\|_2} \sim O(u)$ $A + E = \underbrace{(Q_1 \cdots Q_p)}_{=Q} fl(R)$

<u>For Normal Eqns</u>

we cannot push back the error alway like we did for QR algo.



; $r = b - Ax_0$

$Ax_0 = b_1$

$\cos\theta = \frac{|\langle b, y\rangle|}{\|b\|_2 \|y\|_2}$

$b = y \pm r$ where y & r are orthogonal.

$\cos\theta = \frac{|\langle y+r, y\rangle|}{\|b\|_2 \|y\|_2}$

$= \frac{\|y\|_2^2 + \langle r, y\rangle}{\|b\|_2 \|y\|_2}$

$= \frac{\|y\|}{\|b\|}$

$\sin\theta = \frac{|\langle b, r\rangle|}{\|b\|_2 \|r\|_2}$

[illegible]

[illegible]

[illegible]

[illegible]

Matrix Eigenvalue Problem

[illegible]

[illegible]

$= 0$

$\boxed{[illegible]}$ [illegible]

[illegible] $\Rightarrow$ [illegible] $= 0$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] $= J$ is called the Jordan [illegible]

Date - 7/9/29

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qr algo ~~vers~~ version taught in class will be diff from other study materials.

Matrix Eigenvalue Problem

if D is diagonal matrix A is diagonalizable

$A v_i = \lambda v_i$

$i = 1, \ldots, n$

$$A[v_1 \quad v_n] = \underbrace{[v_1 \quad v_n]}_{= S} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{= D}$$

if $\boxed{\lambda_i \neq \lambda_j \text{ for } i \neq j}$ → ~~sufficient~~ ~~but~~ ~~not~~ necessary

$\therefore AS = SD \Rightarrow S^{-1}AS = D$

S → invertible

Ex: Prove $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

Jordan Canonical Form: Given any matrix $A_{n \times n}$ & block diagonal matrix $J = \mathrm{diag}(J_{\lambda_1}, J_{\lambda_n})$ & an inv matrix X

with $J_{\lambda_i} = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$ $i = 1 \ldots p$

$A = XJX^{-1}$, J is called the Jordan Canonical form of A.

Schur's theorem will reveal all eigen values of A through unitary transformation.

Schur's Thm: Given any $A \in \mathbb{C}^{n\times n}$ $\exists$ an upp triang matrix T & unitary matrix Q s.t. $Q^* A Q = T$

if A is Hermitian $(A^* = A)$, $T^* = T$,

if A is skew Hermitian $(A^* = -A)$, $T^* = -T$

if $AA^* = A^*A$, A is called normal matrix

$$AA^* = A^*A \iff TT^* = T^*T$$

unitary mat $Q^{-1} = Q^*$

$Q, T \rightarrow$ not unique,

Schur's theorem proof:

$A \in \mathbb{C}^{n\times n}$, let $\lambda \in \mathbb{C}$ & $v (\neq 0) \in \mathbb{C}^n$ & $Av = \lambda v$

if $n = 1$, then $A = [a]$ & the theorem is trivially true

Suppose that the statement holds for sizes $< n$

Let $q_1 = v / \|v\|_2$. Then $\|q_1\|_2 = 1$ & $Aq_1 = \lambda q_1$.

Consider let $\{q_1, w_2, \ldots w_n\}$ be an li set in $\mathbb{C}^n$.

Let $S = [q_1\ w_2 \ldots w_n] = \tilde{Q} R$ (be a QR decom of S)

$\tilde{Q} = [q_1\ q_2 \ldots q_n]$. Now

$\underbrace{\qquad\qquad\qquad}$

orthonormal basis of $\mathbb{C}^n$

$$\tilde{Q}^* A \tilde{Q} = \begin{bmatrix} q_1^* \\ \vdots \\ q_n^* \end{bmatrix} A [q_1 \ldots q_n]$$

$$= \begin{bmatrix} q_1^* \\ \vdots \\ q_n^* \end{bmatrix} \begin{bmatrix} Aq_1 & \ldots & Aq_n \end{bmatrix}$$

$$= \left[\begin{array}{c|ccc} q_1^* A q_1 & q_1^* A q_2 & \ldots & q_1^* A q_n \\ \hline \vdots & & \tilde{A} & \\ q_n^* A q_1 & & & q_n^* A q_n \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda q_1^* q_1 & * \\ \hline \vdots & \tilde{A} \\ \lambda q_n^* q_1 & \end{array}\right]$$

$$\left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \tilde{A} \end{array}\right]$$

By induct^n hypothesis $\exists$ a unitary $\tilde{Q}_1$ & an upper triag. $\tilde{T}$ st

$$\tilde{Q}_1^* \tilde{A} \tilde{Q}_1 = \tilde{T}$$

$$\tilde{A} = \tilde{Q}_1 \tilde{T} \tilde{Q}_1^*$$

$$\tilde{Q}^* A \tilde{Q} = \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \tilde{A} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda & \\ \hline & \tilde{Q}_1 \tilde{T} \tilde{Q}_1^* \end{array}\right] = \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{Q} \end{array}\right] \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \tilde{T} \end{array}\right] \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{Q}^* \end{array}\right]$$

$$\therefore \underbrace{\left[\begin{array}{c|c} 1 & \\ \hline & \tilde{Q}^* \end{array}\right] \tilde{Q}^*}_{Q^*} A \underbrace{\tilde{Q} \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{Q} \end{array}\right]}_{Q} = T \quad (\text{upper triag})$$

$Q^* A Q = T$ for n-size

Collorary:

- λ value taken initally will decide Q T
- if A is Hermetian is $\mathrm{Diag}(A)_i \in \mathbb{R}$
- if T is real & Diagonal $\rightarrow T$ is Hermetian $\Rightarrow A$ is Hermetian

$\Rightarrow$ Only complex Hermian matrices can be turned into real diagonal matrices via unitary transform $\exists$

$\Rightarrow$ ~~Only~~ un

$Av = \lambda v$ $\quad v^* A v = \lambda v^* v = \lambda \|v\|_2^2$

$\overline{v^* A v} = \bar{\lambda} \|v_2\|^2$

$v^* A^* v = \bar{\lambda} \|v\|_2^2$

$v^* A v = \bar{\lambda} \|v\|_2^2 = \lambda \|v\|_2^2 \Rightarrow \bar{\lambda} = \lambda$

$\Rightarrow \lambda \in \mathbb{R}$

Hermitian matrix $\Leftrightarrow$ Real eigenvalue

$\Rightarrow$ eigenvectors are also real

FD FEM

MA-473

$r = (B - \Delta T(1-\theta) A) \omega^{(n)}$

$C = B + \Delta T \theta A$

$\forall v \geq g, \quad (v-\omega)^T (C\omega - r) \geq 0 \quad \omega \geq g.$

$\Updownarrow$

Replica

$A\omega - b \geq 0, \quad \omega \geq g$

FDM : $A = C, \quad b = r$

$(A\omega - b)^T (\omega - g) = 0$

$C\omega - r \geq 0, \quad \omega \geq g$

$(C\omega - r)^T (\omega - g) = 0$

trapezoidal

$$\phi_i = \begin{cases} \dfrac{x_{i+1} - x_i}{x_{i+1} - x_i} & x_i \leq x \leq x_{i+1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \leq x \leq x_i \\ 0 & \text{o.w} \end{cases}$$

rank(A) should be 5

rank(A) ~~3~~

o/p: 5

svd(A)

ans: $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

6. 0(u)

7. 0(u)

~~rank~~ threshold

eps * 9 * ans(1)

: 1.6639e^-19

rank(A, 10^(-16)) changing ε

ans: 7

Ex 4.2.27

svd(A) → able to detect correctly numerical rank(A) ~~efficiency~~

qr(A) → fails to detect num. rank(A)

svd(A) → able to detect ~~as~~ more clearly the proximity of numerical rank deficiency of A.

MA-423 Date: 8/10/24

from prev thm there already exists an unitary matrix Q (possibly complex) st $Q^*AQ = D \to$ real diagonal mat

$A \in \mathbb{R}^{n \times n}$ is symm $\iff \exists$ a real orthogonal matrix Q & a real diag matrix D st $Q^TAQ = D$

Q: build similar to Schur's thm

A → real & symm $n \times n$. Let $\lambda \in \mathbb{R}$ & $v \in \mathbb{R}^n$ st $Av = \lambda v$. Let $q_1 = v/\|v\|_2$. Then $Aq_1 = \lambda q_1$

$\|q_1\|_2 = 2$

① Build an orth. matrix $\tilde{Q} \in \mathbb{R}^{n \times n}$ st $\tilde{Q} = [q_1 \dots q_n]$

② $\tilde{Q}^T A \tilde{Q} = \left[\begin{array}{c|ccc} \lambda_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \tilde{A} & \\ 0 & & & \end{array}\right]$ $\left[\because A^T = A \quad (\tilde{Q}^T A \tilde{Q})^T = \tilde{Q}^T A \tilde{Q}\right]$

③ use the ind^n hypothesis for $\tilde{A}$ when

$$\boxed{\tilde{A}^T = \tilde{A}}$$

Normal Matrix: $Q^* A Q = D \Rightarrow AQ = QD \Rightarrow Aq_i = \lambda_i q_i \quad 1 \le i \le n$

where $Q = [q_1 \cdots q_n]$

$D = \begin{bmatrix} \lambda_1 & & \\ & & \lambda_n \end{bmatrix}$

$\{q_1 \; q_n\} \to$ orthonormal basis of eigen vectors of A.

- if A has complex conj eigen values Q, T need not be real.

Quasi-upper-triag:

$$A = \left[\begin{array}{c|c|c|c} A_1 & \times & \times & \times \\ \hline 0 & A_2 & & \\ & 0 & & \\ \hline 0 & 0 & & A_p \end{array}\right]$$

$\det(A - \lambda I) = \prod_{i=1}^{p} \det(A_i - \lambda I_{(\text{size } A_i)})$

~~if A_i 2x2 matrix~~

if $\text{size}(A_i) \le 2\times2 \quad \forall \; i = 1 .. p$

then A is quasi-upper triangular

$$A = \left[\begin{array}{cc|c} 2 & 2 & 1 \\ 3 & 1 & -1 \\ \hline 0 & a & 5 \end{array}\right]$$

if $a = 0$, quasi-upper triangular

$a \neq 0$, not quasi-upp triang.

when $A \in \mathbb{R}^{n \times n}$ & has complex eigen values T will be an quasi upper triang matrix

→ Finding eigen values of matrices is equ to finding roots of polynomial.

Power Method & its Variants

$c_1 \neq 0$

$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

$A v_i = \lambda_i v_i$, $1 \le i \le n$, & $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$

$A x = c_1 \lambda_1 v_1 + \dots + c_n \lambda_n v_n$

$A^2 x = c_1 \lambda_1^2 v_1 + \dots + c_n \lambda_n^2 v_n$

$A^j x = A^j \left(\sum_{i=1}^{n} c_i v_i \right) = \sum_{i=1}^{n} c_i \lambda_i^j v_i$

$$\frac{A^j x}{\lambda_1^j} = \sum_{i=1}^{n} c_i \left(\frac{\lambda_i}{\lambda_1}\right)^j v_i = c_1 v_1 + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^j c_i v_i$$

$$\left\| \frac{A^j x}{\lambda_1^j} - c_1 v_1 \right\| = \left\| \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^j c_i v_i \right\|$$

$$\leq \sum_{i=2}^{n} \left|\frac{\lambda_i}{\lambda_1}\right|^j \|c_i\| \|v_i\|$$

$$\lim_{j \to \infty} \left\| \frac{A^j x}{\lambda_1^j} - c_1 v_1 \right\| \leq \lim_{j \to \infty} \sum_{i=2}^{n} \left|\frac{\lambda_i}{\lambda_1}\right|^j \|c_i\| \|v_i\|$$

$\left|\frac{\lambda_i}{\lambda_1}\right| < 1 \qquad (|\lambda_1| > |\lambda_2| \ldots)$

$$\Rightarrow \lim_{j \to 0} \left\| \frac{A^j x}{\lambda_1^j} - c_1 v_1 \right\| = 0$$

we don't know λ_1

HSS-217 Date: 9/10/24

- In the West, target consumers for sewing machines were women.
- Sales of sewing machines were done from door to door
- By attaching sewing machine to women, it assumed the gender
- Early customers in India for sewing machines were Europeans / Anglo-Indians
- Till 1880s sewing machines were only sold to Europeans & Anglo-Indians
- Patell was oppose because of his new strategies to advertise the sales of sewing machines to Indians

MA-423 Date: 15/10/29

* Power method needs dominant eigen vector

Shift & Invert Method → (can find any eigen value of the matrix by choosing $\rho \simeq \lambda_i$)

ρ must not be an eigen value.

$(A-\rho I)^{-1}$ → eigen value $\left(\cancel{\tfrac{1}{\lambda_1}-\rho}, \tfrac{1}{\lambda_n-\rho}\right)$ $\left(\tfrac{1}{\lambda_1-\rho} \quad \tfrac{1}{\lambda_n-\rho}\right)$

$\frac{1}{(\lambda_n-\rho)}$ → dominant eV $\quad vAv = \lambda v$

$A v_i = \lambda_i v_i \quad i = 1 \ldots n$

$(A-\rho I)^{-1} v_i = \left(\frac{1}{\lambda_i-\rho}\right) v_i$

$v_i = \left(\frac{1}{\lambda_i-\rho}\right)(A-\rho I)^{-1} v_i$

$(A-\rho I)^{-1} v_i = \frac{1}{(\lambda_i-\rho)} v_i$ → use power method to find the dominant e-v of $(A-\rho I)^{-1}$ say β. invert β & add ρ to get the eig value corresponding to A $\left(\frac{1}{\beta}+\rho\right) = \lambda_i$

- Invert & add the shift back after convergence to get eigen value of A

If q is not an eigen vector then there does not exist any μ that solves the eqn $\mu q = Aq$, therefore there will be an overdetermined sys. ~~it will be an estimate~~ sys

$\underbrace{q}_{A}\mu = \underbrace{Aq}_{b}$ → ①

$\boxed{\mu_0 = \frac{q^* A q}{q^* q}}$ → LSP of ①

→ μ_0 is Rayleigh quotient associated with A & q.

$Av = \lambda v \quad \|v\|_2 = 1 \quad , \rho = q^* A q \quad , \|q\|_2 = 1$

$$|\lambda - \rho| = |v^* A v - q^* A q|$$

$$= |v^* A v - v^* A q + v^* A q - q^* A q|$$

$$= |v^* A(v-q) + (v-q)^* A q|$$

$$\leq |v^* A(v-q)| + |(v-q)^* A q|$$

$$= |v^* A(v-q)| + |\overline{(v-q)^* A q}|$$

$$= |v^* A(v-q)| + |q^* A^* (v-q)|$$

(C.S inq) $\leq \|v\|_2 \, \|A(v-q)\|_2 + \|q\|_2 \|A^*(v-q)\|_2$

$(|x^* y| \leq \|x\|_2 \|y\|_2)$

$$= \|A(v-q)\|_2 + \|A^*(v-q)\|_2$$

$$\leq \|A\|_2 \|v-q\|_2 + \|A^*\| \, \|v-q\|_2$$

$$= 2 \|A\|_2 \|v-q\|_2$$

~~can't~~ ~~apply A~~

Non-convergent iterati using Hessenberg matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = q_0$$

$\rho_0 = x^* A x = 0$

soln $(A - \rho_0 I)\hat{q}_1 = q_0 \Rightarrow \hat{q}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = q_1$

$\rho_1 = q_1^* A q_1 = 0 ,$

Soln $(A - \rho_1 I)\hat{q}_2 = q_1$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{q}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

keeps alt. b/w e_1 & e_2

reason : shift $= 0$ mid value b/w ev 1 & -1

let $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = q_0$ $\quad \rho_0 = \frac{x^* A x}{2} = 1$ (e.v)

v : pm vector.

quad conv g. $\lim_{j \to \infty} \frac{\| q_{j+1} - v \|_2}{\| q_j - v \|_2^2} = C (\neq 0)$

MA 473 Lab.

$$(\omega^T A^T - b^T)(\omega - g) = \omega^T A \omega - \omega^T A g - b^T \omega -$$

PSOR $(A, \hat{b}, x)$:

$\hat{b} = b - Ag$, $x = v - g$

$$\frac{\lambda}{2} \omega_{i-1, v} + (1 - \lambda)\omega_{i, v} + \frac{\lambda}{2} \omega_{i+1, v}$$

HSS-217 Date: 17/10/24

- Patel went door to door to sell advertise/sell sewing machines for his company.

- His strategies were seen as unprofessional & were not accepted by his colleagues.

- Bishandas Basil made the ~~first~~ 1st India made sewing machine USHA.

Bicycles in Modern India

→ cycle made it ~~available~~ possible for ~~owners~~ owners to be automous while travelling & gave ~~much~~ more mobility.

- It was cheaper & more convienient than other means of transport like motors, horse carriages & trains.

(automous in the sense that it didn't depend on other means of transport for travelling)

Date: 18/10/23 MA-423 [1st pt written some pages ~~back~~ back when Date-17/10/24 is cut i.e (~~17/10/24~~)]

base

$$\begin{bmatrix} a_{11} \\ \\ a_{1n} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ a_{12}/a_{11} \\ \\ a_{1n}/a_{11} \end{bmatrix}$$

if $a_{11} = 0$ skip

$\vdots\ \underbrace{}_{m}$

$$Q_x = \begin{bmatrix} \pm \|x\|_2 e^{i\theta} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad a_{11} = (a_{11}) e^{i\theta}$$

$\longrightarrow$

y

$A = Q_{nm} R_{mm}$

if $A_{n \times m}$ $\quad n \geq m$ $\quad A = Q_{mn} R_{n \times m}$ $\quad R(i,i) > 0 \ \forall i = 1 \ldots n$

Show R is uniq

$A^* A \in$ +ve definite $\quad R^* R$, $\quad R$ is cholesky fach so unique

$\downarrow$

R has +ve diag entries

supp $A = Q_1 R_1 = Q_2 R_2$, $\quad Q_2^* Q_1 = R_2 R_1^{-1}$

$(Q_2 Q_1)^* (Q_2^* Q_1) = Q_1 Q_2 Q_2^* Q_1$

$= I$

$$A_{1/2} Q_1 = \left[(A_{1/2} Q_1)^T \right]^T = \left[Q_1^T A_{1/2}^T \right]^T = \left(\begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}_1 \end{bmatrix} \begin{bmatrix} a_{11} & \pm\|b\|_2 \ 0 \ldots 0 \\ c & \hat{A}^T \hat{Q}_1 \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} a_{11} & \pm\|b\|_2 \ 0 \ldots 0 \\ \hat{Q}_1 c & \hat{Q}_1 \hat{A}^T \hat{Q}_1 \end{bmatrix}^T$$

$\hat{Q}_1 \left[c \quad \hat{A}^T \hat{Q}_1 \right]_{(n-1 \times n)}$

$(n-1 \times n-1)$

flop count: $4n \times (n-1)$

$$-\gamma^2 v(v^+ \hat{A} v)\,\theta^T = -\cancel{\gamma(v^T \hat{A} v)}\ -\gamma v(\underbrace{-\gamma \hat{A} v}_{\theta})\,\hat{v}^T$$

$$= \cancel{-\gamma v v \hat{v}}$$

$$= \gamma v v^T \theta \hat{v}$$

Date: MA-473

recap:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$t_n \to t_{n+1/2} \to t_{n+1}$

$$\delta_x^2 u_{ij}^n = u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n$$

$$\delta_y^2 u_{ij}^n = u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n$$

$$\frac{u_{lm}^{n+1/2} - u_{lm}^n}{(\delta\tau/2)} = \frac{1}{h^2}\left[\delta_x^2 u_{lm}^{n+1/2} + \delta_y^2 u_{lm}^n\right] \to (5)$$

$$\frac{u_{lm}^{n+1} - u_{lm}^{n+1/2}}{\delta\tau/2\ldots} = \frac{1}{h^2}\left[\delta_x^2 u_{lm}^{n+1/2} + \delta_y^2 u_{lm}^{n+1}\right] \to (6)$$

$\delta \quad u_{lm}$

HSS-217

Bicycles, Race & Gender

- Bicycles were mainly used a sense of 'social modern' by Europeans until many Indians starting owing them.

- cycles were also ~~gener~~ genderised in association of it with men.

effeminate → making someone feel feminine

MA-423 Date: 21/10/24

$$\hat{A} + vu^T + uv^T + \underbrace{2\alpha uu^T}_{\alpha uu^T + \alpha uu^T}$$

det $w = v + \alpha u$

$$= \hat{A} + (v + \alpha u)u^T + u(v + \alpha u)^T$$

$$= \hat{A} + wu^T + uw^T$$

$$4[n-1 + \ldots + 1] = 2n(n-1)$$

Cost of $\hat{A}_1$ = $= 2n^2 + O(n)$ flops.

Total cost in calculating $H = 4 \sum_{k=0}^{n} (n-k)^2 + O(n^2)$

$= \frac{4}{3} n^3 + O(n^2)$

$v \in$ eigen vector of H

$Qv = 0_{n-2} \quad 0, v \rightarrow$ e. vec of A.

wh

Correctn of Householder reflectors for $\mathbb{C}^n$

$x \in \mathbb{C} = \begin{bmatrix} x_1 \\ x_n \end{bmatrix} \quad Qx = \begin{bmatrix} \pm \|x\|_2 \\ 0 \\ 0 \end{bmatrix}$

$= \underbrace{\begin{bmatrix} -\tau e^{i\theta} \\ \\ 0 \end{bmatrix}}_{y}$ where $\tau \neq \pm \|x\|_2$

$\tau = \|x\|_2$

$u = x - y$

$\theta = x_1 / |x_1|$

$u(x) \begin{bmatrix} \cancel{x_1 + \tau} \\ x_2 \\ x_3 \\ x_n \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ x_2 / (x_1 + \tau) \\ \\ x_n / (x_1 + \tau) \end{bmatrix}$

~~sign~~

$= \text{sign}(x_1$

$x_1 +$

Constn ensure $x^* y$ is real.

QR Algorithm

Under suitable $cond^n$, $\{A_j\}$ will converge to the Schur's form of A.

Reduce A to upper Hessenberg form before iteration $O(n^2)$

to ~~reduce~~ make flop count for each iter

$[Q, H] = hess(A) \rightarrow O(n^3)$

$A_0 = H$

Step 1 : (a) Find reflectors Q_1 to Q_{n-1} st $Q_{n-1} \dots Q_1 H = R$ — $O(n^2)$

(b) Set $A_1 = R Q_1 \dots Q_{n-1}$

$\downarrow$ A_1 upp Hessenberg?

H → upper Hesenberg $\not\Rightarrow$ $Q^{-1} H Q$ is upp Hessenberg for unitary orthogonal Q arising from QR decomp of H.

$$A_0 = \begin{bmatrix} a_{11} & & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ & a_{n,n-1} & a_{nn} \end{bmatrix} \qquad Q_1 = \begin{bmatrix} \hat{Q}_1 & \\ & I_{n-2} \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & & \\ & \hat{Q}_2 & \\ & & I_{n-3} \end{bmatrix} \begin{matrix} 1 \\ 2 \\ n-3 \end{matrix}$$

where $\hat{Q}_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} -\tau \\ 0 \end{bmatrix}$

$$R\theta_1 = \begin{bmatrix} x_{11} & & \\ & \ddots & \\ & & x_{nn} \end{bmatrix} \left[\begin{array}{c|c} \theta_1 & \\ \hline & I_{n-2} \end{array} \right]$$

Applying this tech. on the itn we set ~~$O(n^2)$~~ A_i as upp Hess ~~Q~~ so ~~A_1~~ A_1 as upp Hess. matrix on every itn

MA-473 If we sample the underlying asset S_t at discrete time intervals with equidist time interval. Then we we obtain $S_{t_1}, \ldots, S_{t_{n-1}}, S_{t_n}$. we can consider avg as arithematic mean

$\forall m \in \mathbb{N}$

$$\frac{1}{n}\sum_{i=1}^{n} S_{t_i} = \frac{h}{T}\sum_{i=1}^{n} S_{t_i} = A \to 1$$

or if the time interval is cts then ~~we can have~~ the avg will be

$\boxed{h = T/n}$

$$\hat{S} = \frac{1}{T}\int_0^T S_\theta d\theta \to 2$$

Instead of the arithematic mean we can consider the geo. mean for both discrete & cts.

Discrete:

$$\left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n} = \exp\left(\frac{1}{n}\log\prod_{i=1}^{n} S_{t_i}\right)$$

$$= \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log S_{t_i}\right) \to 3$$

Cts:

$$\hat{S} = \exp\left(\frac{1}{T}\int_0^T \log S_t\, dt\right) \to 4$$

Substituting in original eqⁿ

$$\boxed{\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} R^2 \frac{\partial^2 H}{\partial R^2} + (1 - \lambda R) \frac{\partial H}{\partial R} = 0}$$

For the bdd'y condⁿ $R \to \infty$ we obtain from the payoff $H(R,T) = \left(1 - \frac{1}{T} R_T\right)^+$

$$\lim_{R_T \to \infty} H(R_T, T) = 0$$

$R_t = \frac{1}{t} \int_0^t S_\theta \, d\theta$, the integral R_t is bdd'd

As $S \to \infty$, $R \to \infty$

This is the European call as $S \to 0$ we cannot exercise the European call $\Rightarrow H(R,t) = 0$, $R \to \infty$

Date : 25/10/24

$A_0 = A$, $A_0 = Q_0 R_0$ $(R_0(i,i) > 0 \ \forall i = 1, \ldots n)$ $A_1 := R_0 Q_0$

$= Q_0^T A_0 Q_0$

$A_1 = Q_1 R_1$ $(R_1(i,i) > 0)$ to

$A_2 = R_1 Q_1$

$= Q_1^T A_1 Q_1$

$A_1, A_2 \xrightarrow{\text{converge}}$ Schur form of A

$A_0 = \tilde{Q}_0 \tilde{R}_0$ $\qquad \tilde{A}_1 = \hat{Q}_0^* A_0 \tilde{Q}$

$A_1 = \tilde{Q}_1 \tilde{R}_1$ $\qquad \tilde{A}_2 = \tilde{Q}_1^* A_1 \tilde{Q}_1$

$\vdots$

$\tilde{A}_1 , \tilde{A}_2$

$A = Q_1 R_1 = Q_2 R_2$ $\qquad A_1 := Q_1^* A Q_1 , \quad A_2 := Q_2^* A Q_2 ,$

Then $A_1 = D^* A_2 D$ where

$D = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$

If all the 1st subdiagonal entries of upp. Hessenberg is non zero it's called proper or irreducible upp Hessenberg matrix else it's called improper or reducible upp. Hessenberg matrix.

partition

Eg $A = \left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & 0 & a_{33} \end{array}\right]$ $\qquad \left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{array}\right]$

- QR algo tries to hasten the deflation by zeroing out the ~~sub~~ sub diag entries & dividing the matrix processing into sub-Hessenberg matrices.

Note (impt)

- If $|a_{n,n-1}| \to 0$ then $|a_n| \to \lambda_n$ as $n \to \infty$

$\rho_i = \frac{e_n^* a_n e_n}{e_n e_n}$

HSS-217

- recording takes out sound out of its original context to listen to it anytime & anywhere we need.

Date: 29/10/24

QR algo assumes non sing A so that Q,R decomp is unique in each iter

- This ~~nec~~ is not a necessary condn as long as we're converging to Schur's form

QR iters with singular A,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \\ & a_{n,n-1} & a_{n,n} \end{bmatrix} = QR \text{ -(*)} \qquad a_{i+1,i} \neq 0 \ \forall\ i = 1, \ldots, n-1, \quad A \text{ is singular.}$$

$$A_1 = RQ = \begin{bmatrix} a_{11}^{(1)} & & a_{1n}^{(1)} \\ a_{21}^{(1)} & & \\ & a_{n-1,n-1}^{(1)} & a_{n-1,n}^{(1)} \\ 0 & \cdots & 0 \end{bmatrix}$$

Hessenberg & ~~not~~ form

1^{st} $n-1$ cols of A $\Rightarrow$ proper are l.i set $\Rightarrow$ rank A = n-1.

rank (R) = n-1.

$QR = Q[R_1 \ \ldots \ R_{n-1} \ R_n] = [A_1 \ \ldots \ A_{n-1} \ A_n]$

~~$[QR_1 \ \ldots \ QR_{n-1} \ QR_n] = [A_1 \ \ldots \ A_{n-1} \ A_n]$~~

$\Rightarrow Q[\underbrace{R_1 \ \ldots \ R_{n-1}}_{l.i}] = [\underbrace{A_1 \ \ldots \ A_{n-1}}_{l.i}]$

$\Leftarrow$ l.i

eig values of R will lie in its diag.

$$Q[R_1 \quad R_{n-1}] = Q \begin{bmatrix} r_{11} & & r_{1n-1} \\ & & r_{n-1,n-1} \\ 0 & & 0 \end{bmatrix} \Big\} \text{rank} = \text{rank}([R_1 \quad R_{n-1}])$$

rank

$\Rightarrow r_{i,i} \neq 0 \ \forall \ i = 1, \ldots, n-1 \quad \Rightarrow r_{n,n} = 0$

$\Rightarrow$ entire last row of R is 0.

QR

<u>Double-shift</u> : To extract complex eig values of A while staying in real domain.

$$A - \rho I = Q_\rho R_\rho, \quad R_\rho Q_\rho + \rho I = \hat{A} \Rightarrow \hat{A} = Q_\rho^* A Q_\rho \quad ③$$

$$A - zI = Q_z R_z, \quad R_z Q_z + zI = \tilde{A} \Rightarrow \tilde{A} = Q_z^* A Q_z \quad ④$$

②

①

③ & ④, $\tilde{A} = Q_z^* \hat{A} Q_z = Q_z^* Q_\rho^* A Q_\rho Q_z$

$= Q^* A Q$.

$Q = Q_\rho Q_z$

$$QR = Q_\rho Q_\tau R_\tau R_\rho = Q_\rho (\hat{A} - \rho I) R_\rho \rightarrow \text{by } 2$$

$$= Q_\rho [Q_\rho^* A Q_\rho - \tau Q_\rho^* Q_\rho] R_\rho \rightarrow \text{by } 3$$

$$= \underbrace{Q_\rho Q_\rho^*}_{I} [A - \tau I] Q_\rho R_\rho$$

$$= [A - \tau I] Q_\rho R_\rho$$

$$= (A - \tau I)(A - \rho I) \rightarrow \text{by } 1$$

if $\tau = \bar{\rho}$ then $(A - \rho I)(A - \tau I)$

$= (A - \rho I)(A - \bar{\rho} I) = A^2 - 2 \operatorname{Re} \rho A + |\rho|^2 I \in \mathbb{R}^{n \times n}$

det

~~trace (A) = $\sum$ ... A_{ii} = $\sum \lambda_i$~~

Supp, $(A - \rho I)(A - \tau I) = Q_1 R_1$

$Q_1 = QD$ for $D = \begin{bmatrix} e^{i\theta_1} & \\ & e^{i\theta_2} \end{bmatrix}$ if

$(A - \rho I)(A - \tau I)$ is nonsing

Pf ~~Schur~~

Friend QR-dcn

$\to Q, R$;

ours $\to Q_1, R_1$

$$Q_1^* A Q_1 = (QD)^* A (QD)$$

$$= D^* Q^* A Q D$$

Modulus remains same for jth entries.

$\to$ leads to ~~same~~ same Schur's form.