Finite Precision Systems

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Therefore, in a normalized representation,

$$x = \pm \underbrace{(b_0.b_1 b_2 \cdots)_\beta}_{=s} \times \beta^E,$$

where $1 \leq b_0 \leq \beta - 1, 0 \leq b_j \leq \beta - 1$, for $j \geq 1$.

Finite precision system

Consider a finite precision system $(\beta, p, E_{\text{min}}, E_{\text{max}})$ indicating that numbers stored are 0 and all other normalized numbers in base β, with s stored upto p places and E_{min} and E_{max} being the smallest and largest values of the exponent.

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$$p \rightarrow \textit{precision of the system}$$

The nonzero numbers in the system are

$$x = \pm(b_0.b_1 \cdots b_{p-1})_\beta \times \beta^E,\ 1 \leq b_0 \leq \beta - 1, E_{\min} \leq E \leq E_{\max}.$$

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Any other number that is not a floating point number has to be rounded by a floating point number by some rule of rounding.

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The options for rounding x by a floating point number in $(\beta, p, E_{\min}, E_{\max})$ are

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Therefore for $N_{\min} \leq |x| \leq N_{\max}$,

$$x_- = \begin{cases} (b_0.b_1 \cdots b_{p-1})_\beta \times \beta^E & \text{if } x > 0, \\ -[(b_0.b_1 \cdots b_{p-1})_\beta + \beta^{1-p}] \times \beta^E & \text{if } x < 0, \end{cases}$$

$$x_+ = \begin{cases} [(b_0.b_1 \cdots b_{p-1})_\beta + \beta^{1-p}] \times \beta^E & \text{if } x > 0, \\ -(b_0.b_1 \cdots b_{p-1})_\beta \times \beta^E & \text{if } x < 0 \end{cases}$$

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Example: For $x = 99.95$ which is not a floating point number in the system $(10, 3, -1, 2)$ after normalization,

$$x_- = 9.99 \times 10 = 99.9 \text{ and } x_+ = 10^2 = 100$$

whereas for $y = -x$,

$$y_- = -10^2 \text{ and } y_+ = -9.99 \times 10 = -99.9.$$

Finite precision systems

For $|x| > N_{\max}$,

$$x_- = \begin{cases} N_{\max} & \text{if } x > 0, \\ -\infty & \text{if } x < 0, \end{cases} \qquad x_+ = \begin{cases} \infty & \text{if } x > 0, \\ -N_{\max} & \text{if } x < 0, \end{cases}$$

and for $|x| < N_{\min}$,

$$x_- = \begin{cases} 0 & \text{if } x > 0, \\ -N_{\min} & \text{if } x < 0, \end{cases} \qquad x_+ = \begin{cases} N_{\min} & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

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Rounding modes:

$fl(x) \rightarrow$ rounded value of x.
Depending on the rule of rounding, $fl(x) = x_+$ or $fl(x) = x_-$.

Rounding in finite precision

Rounding Mode	**Rule**
Round down/Round toward $-\infty$	$fl(x) = x_-$
Round up/Round towards ∞	$fl(x) = x_+$
Round towards zero	$fl(x) = \begin{cases} x_- & \text{if } x \geq 0; \\ x_+ & \text{if } x \leq 0 \end{cases}$
Round to nearest	$fl(x)$ is either x_- or x_+ whichever is nearer to x except that $fl(x) = \infty$ if $x > N_{\max}$ and $fl(x) = -\infty$ if $x < -N_{\max}$. If x is the mid point of (x_-, x_+), then the one among x_- and x_+ with the lesser value of b_{p-1} is chosen.

Rounding in finite precision

Example: In the finite precision system $(10, 3, -1, 2)$,

$$fl(99.95) = \begin{cases} 99.9 & \text{in } \textit{round down} \text{ and } \textit{round towards zero}; \\ 100 & \text{in } \textit{round to nearest}; \end{cases}$$

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For any x such that $N_{\min} \leq |x| \leq N_{\max}$,

$$\frac{|fl(x) - x|}{|x|} \leq \frac{\beta^{1-p}}{2} := u$$

in the *round to nearest* rounding mode.

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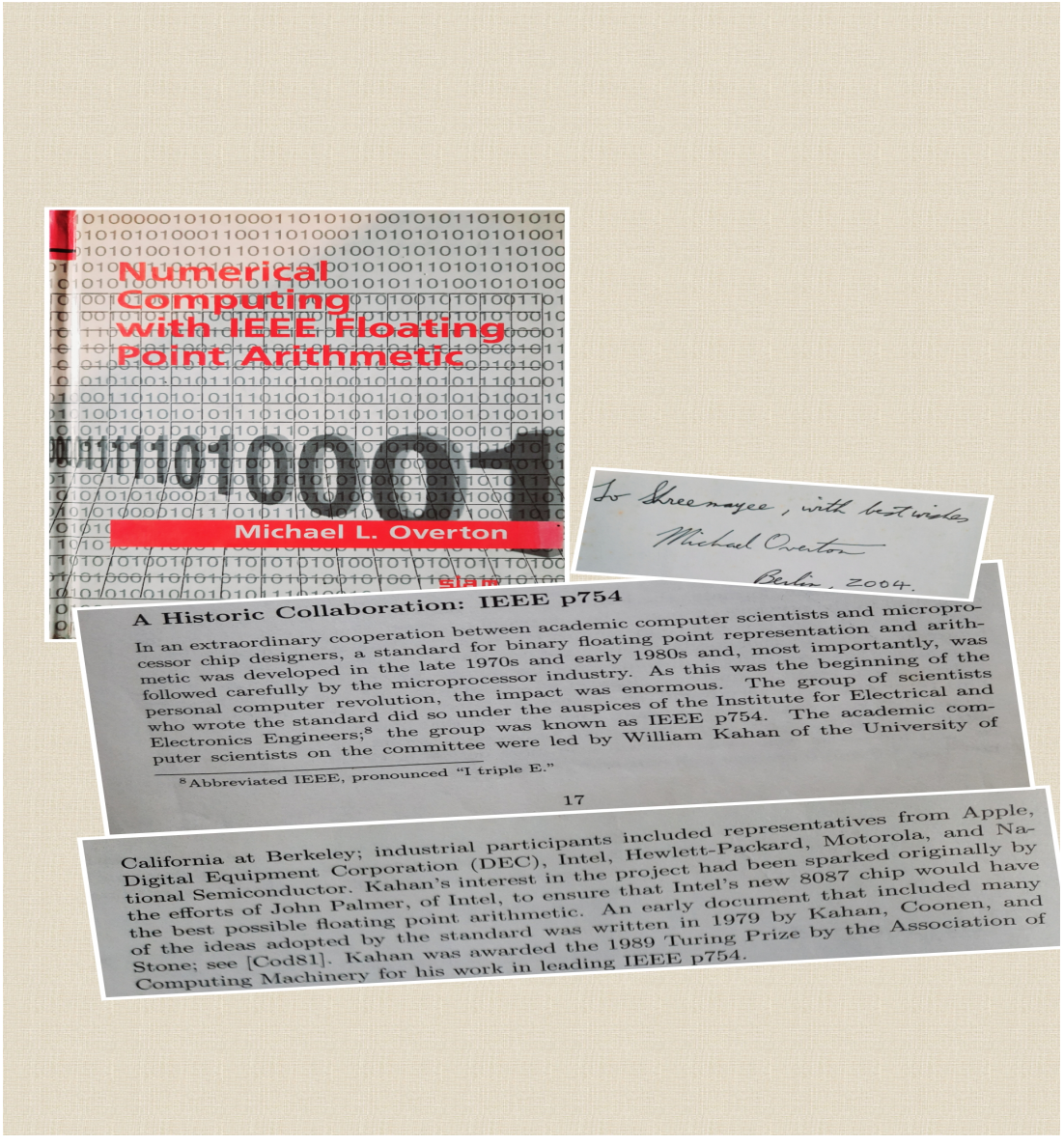
Therefore for all x with $N_{\min} \leq |x| \leq N_{\max}$,

$$fl(x) = x(1 + \delta) \text{ where } |\delta| \leq u.$$

IEEE finite precision arithmetic



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- This is a binary ($\beta = 2$) system.
- In single precision there are 32 bits to store a number divided into 1 bit for the sign, 23 bits for s and 8 bits for E.
- In double precision there are 64 bits to store a number divided 1 bit for the sign, 52 bits for s and 11 bits for E.

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To care of both positive and negative exponents, the exponent string stores the binary representation of $E + 127$ in single and $E + 1023$ in double precision. This is called *biased representation*.

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For example if the exponent field stores 1000001 the exponent stored is $129 - 127 = 2$ and if the exponent field stores 0000011, then the exponent stored is -124.

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How is zero stored?

IEEE storage tables: Single precision

Table 4.1: IEEE Single Format

$\pm$	$a_1 a_2 a_3 \ldots a_8$	$b_1 b_2 b_3 \ldots b_{23}$

If exponent bitstring $a_1 \ldots a_8$ is	Then numerical value represented is
$(00000000)_2 = (0)_{10}$	$\pm(0.b_1 b_2 b_3 \ldots b_{23})_2 \times 2^{-126}$
$(00000001)_2 = (1)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{23})_2 \times 2^{-126}$
$(00000010)_2 = (2)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{23})_2 \times 2^{-125}$
$(00000011)_2 = (3)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{23})_2 \times 2^{-124}$
$\downarrow$	$\downarrow$
$(01111111)_2 = (127)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{23})_2 \times 2^{0}$
$(10000000)_2 = (128)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{23})_2 \times 2^{1}$
$\downarrow$	$\downarrow$
$(11111100)_2 = (252)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{23})_2 \times 2^{125}$
$(11111101)_2 = (253)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{23})_2 \times 2^{126}$
$(11111110)_2 = (254)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{23})_2 \times 2^{127}$
$(11111111)_2 = (255)_{10}$	$\pm\infty$ if $b_1 = \cdots = b_{23} = 0$, NaN otherwise

Figure : Image courtesy *Numerical Computing with IEEE Floating Point Arithmetic by M. L. Overton*, page 19.

IEEE storage tables: Double precision

Table 4.2: IEEE Double Format

$\pm$	$a_1 a_2 a_3 \ldots a_{11}$	$b_1 b_2 b_3 \ldots b_{52}$

If exponent bitstring is $a_1 \ldots a_{11}$	Then numerical value represented is
$(00000000000)_2 = (0)_{10}$	$\pm(0.b_1 b_2 b_3 \ldots b_{52})_2 \times 2^{-1022}$
$(00000000001)_2 = (1)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{52})_2 \times 2^{-1022}$
$(00000000010)_2 = (2)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{52})_2 \times 2^{-1021}$
$(00000000011)_2 = (3)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{52})_2 \times 2^{-1020}$
$\downarrow$	$\downarrow$
$(01111111111)_2 = (1023)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{52})_2 \times 2^{0}$
$(10000000000)_2 = (1024)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{52})_2 \times 2^{1}$
$\downarrow$	$\downarrow$
$(11111111100)_2 = (2044)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{52})_2 \times 2^{1021}$
$(11111111101)_2 = (2045)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{52})_2 \times 2^{1022}$
$(11111111110)_2 = (2046)_{10}$	$\pm(1.b_1 b_2 b_3 \ldots b_{52})_2 \times 2^{1023}$
$(11111111111)_2 = (2047)_{10}$	$\pm\infty$ if $b_1 = \cdots = b_{52} = 0$, NaN otherwise

Figure : Image courtesy *Numerical Computing with IEEE Floating Point Arithmetic by M. L. Overton*, page 22.

Rounding in IEEE finite precision

- The gap between 1 and the next floating point number denoted by ϵ is called *machine epsilon*.
- The gap between any other pairs of normalized floating point numbers with exponent E is $\epsilon \times \beta^E$.
- The rounding mode is *round to nearest* and $u = \epsilon/2$.
- *Overflow* occurs for $|x| > N_{\max}$ as $fl(x) = \pm\infty$.
- *Underflow* occurs if $x \neq 0$ but $fl(x) = 0$.
- Subnormal numbers ensure very gradual underflow to 0.

IEEE single and double precision

Single precision (32 bit):

$$p = 24;$$
$$\epsilon = 2^{-23} \approx 1.2 \times 10^{-7}; \quad u \approx 0.6 \times 10^{-7};$$
$$E_{\min} = -126; \quad E_{\max} = 127;$$
$$N_{\min} = 2^{-126} \approx 1.2 \times 10^{-38};$$
$$N_{\max} = (2 - 2^{-23})2^{127} \approx 3.4 \times 10^{38}.$$

Double precision (64 bit):

$$p = 53;$$
$$\epsilon = 2^{-52} \approx 2.2 \times 10^{-16}; \quad u \approx 1.1 \times 10^{-16};$$
$$E_{\min} = -1022; \quad E_{\max} = 1023;$$
$$N_{\min} = 2^{-1022} \approx 2.2 \times 10^{-308};$$
$$N_{\max} = (2 - 2^{-52})2^{1023} \approx 1.8 \times 10^{308}.$$

IEEE finite precision system

Exercise: Consider the finite precision system $(2, 3, -1, 1)$. Suppose that the hidden bit technique is in use. Additionally assume that there is also a means of indicating that the hidden bit is zero and the exponent value is -1 so that it is a miniature version of the IEEE systems.

(a) List all the numbers in the system including the subnormal numbers in both binary and decimal forms.

(b) Find ϵ, u, $N_{\min}$ and $N_{\max}$.

(c) Noting that for any three numbers a, b, and c in the system,

$$fl(a+b+c) = fl(fl(a+b)+c) \text{ and } fl(c+b+a) = fl(fl(c+b)+a),$$

find three numbers x, y, z such that $fl(x+y+z) \neq fl(z+x+y)$ in round to nearest rounding mode.