

Gaussian Elimination & LU Decompositions

Square System of Equations

Consider

$$Ax = b$$

where

$$A = [a_{ij}] \longleftarrow n \times n \text{ matrix}$$

$$b \longleftarrow n \times 1 \text{ vector.}$$

In expanded form,

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n$$

Gaussian Elimination



Carl Friedrich Gauss
(1777-1855)

A Summary of the Evolution of Gaussian Elimination

Gaussian Elimination With No Pivoting (GENP)

$A \longrightarrow A^{(1)} \longrightarrow \dots \longrightarrow A^{(n-1)} =: U$ (upper triangular).

$b \longrightarrow b^{(1)} \longrightarrow \dots \longrightarrow b^{(n-1)}.$

Gaussian Elimination With No Pivoting (GENP)

$$A \longrightarrow A^{(1)} \longrightarrow \dots \longrightarrow A^{(n-1)} =: U \text{ (upper triangular).}$$
$$b \longrightarrow b^{(1)} \longrightarrow \dots \longrightarrow b^{(n-1)}.$$

Step 1: Create zeros in the first column of A :

$$A \longrightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}}_{=:A^{(1)}}; \quad b \longrightarrow \underbrace{\begin{bmatrix} b_1 \\ b_2^{(1)} \\ \vdots \\ b_n^{(1)} \end{bmatrix}}_{=:b^{(1)}}$$

where

$$a_{ij}^{(1)} = a_{ij} - \underbrace{\frac{a_{i1}}{a_{11}}}_{=:m_{i1}} a_{1j}; \quad b_i^{(1)} = b_i - \frac{a_{i1}}{a_{11}} b_1; \quad i = 2 : n, j = 2 : n.$$

Here $a_{11} \leftarrow$ pivot (assumed non zero); $m_{i1} \leftarrow$ multipliers;

GENP

Step k: Create zeros in column k of $A^{(k-1)}$:

$$A^{(k-1)} = \left[\begin{array}{cccc|ccc} a_{11} & \cdots & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1k} \\ & a_{22}^{(1)} & \cdots & a_{2k}^{(1)} & a_{2,k+1}^{(1)} & \cdots & a_{2n}^{(1)} \\ & & \ddots & \vdots & \vdots & \cdots & \vdots \\ & & & a_{kk}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{kn}^{(k-1)} \\ \hline & & & a_{k+1,k}^{(k-1)} & a_{k+1,k+1}^{(k-1)} & \cdots & a_{k+1,n}^{(k-1)} \\ & & & \vdots & \vdots & \cdots & \vdots \\ & & & a_{n,k}^{(k-1)} & a_{n,k+1}^{(k-1)} & \cdots & a_{nn}^{(k-1)} \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cccc|ccc} a_{11} & \cdots & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1k} \\ & a_{22}^{(1)} & \cdots & a_{2k}^{(1)} & a_{2,k+1}^{(1)} & \cdots & a_{2n}^{(1)} \\ & & \ddots & \vdots & \vdots & \cdots & \vdots \\ & & & a_{kk}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{kn}^{(k-1)} \\ \hline & & & & a_{k+1,k+1}^{(k)} & \cdots & a_{k+1,n}^{(k)} \\ & & & & \vdots & \cdots & \vdots \\ & & & & a_{n,k+1}^{(k)} & \cdots & a_{nn}^{(k)} \end{array} \right]$$

$=: A^{(k)};$

GENP

The same operations are performed on $b^{(k-1)}$:

$$b^{(k-1)} \longrightarrow \begin{bmatrix} b_1 \\ b_2^{(1)} \\ \vdots \\ b_k^{(k-1)} \\ b_{k+1}^{(k)} \\ \vdots \\ b_n^{(k)} \end{bmatrix} =: b^{(k)}$$

where for $i = k + 1 : n, j = k + 1 : n$,

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \underbrace{\frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}}_{=: m_{ik}} a_{kj}^{(k-1)}; \quad b_i^{(k)} = b_i^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} b_k^{(k-1)};$$

Here $a_{kk}^{(k-1)} \leftarrow$ pivot (assumed non zero); $m_{ik} \leftarrow$ multipliers;

Step $n - 1$: Create a zero in the $(n, n - 1)$ of $A^{(n-2)}$:

$$A^{(n-2)} \longrightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ & & \ddots & \vdots \\ & & & a_{nn}^{(n-1)} \end{bmatrix}}_{=: A^{(n-1)} \text{ (also called } U)}; \quad b^{(n-2)} \longrightarrow b^{(n-1)};$$

where assuming pivot $a_{n-1,n-1}^{(n-2)} \neq 0$ and using multiplier

$$m_{n,n-1} := \frac{a_{n,n-1}^{(n-2)}}{a_{n-1,n-1}^{(n-2)}},$$

$$a_{nn}^{(n-1)} = a_{nn}^{(n-2)} - m_{n,n-1} a_{n-1,n}^{(n-2)}; \quad b_n^{(n-1)} = b_n^{(n-2)} - m_{n,n-1} b_{n-1}^{(n-2)}.$$

GENP

Step $n - 1$: Create a zero in the $(n, n - 1)$ of $A^{(n-2)}$:

$$A^{(n-2)} \longrightarrow \underbrace{\begin{bmatrix} \mathbf{a_{11}} & a_{12} & \cdots & a_{1n} \\ & \mathbf{a_{22}^{(1)}} & \cdots & a_{2n}^{(1)} \\ & & \ddots & \vdots \\ & & & \mathbf{a_{nn}^{(n-1)}} \end{bmatrix}}_{=: A^{(n-1)} \text{ (also called } U)}; \quad b^{(n-2)} \longrightarrow b^{(n-1)};$$

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- ▶ The system is transformed to $Ux = b^{(n-1)}$.
- ▶ The pivots at each step are on the diagonal of U !
- ▶ All steps have to be repeated to solve any new system $Ax = c$ if the multipliers used in the GENP are not saved.

GENP $\equiv LU$ decomposition of A

Let

$$L = \begin{bmatrix} 1 & & & & & & & & \\ m_{21} & 1 & & & & & & & \\ m_{31} & m_{32} & \ddots & & & & & & \\ \vdots & \vdots & & \ddots & & & & & \\ m_{k1} & m_{k2} & \cdots & \cdots & 1 & & & & \\ m_{k+1,1} & m_{k+1,2} & \cdots & \cdots & m_{k+1,k} & \ddots & & & \\ \vdots & \vdots & & & \vdots & & 1 & & \\ m_{n1} & m_{n2} & \cdots & \cdots & m_{nk} & \cdots & m_{n,n-1} & 1 \end{bmatrix}.$$

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Then $A = LU$!

LU decomposition: A square matrix A is said to have an LU decomposition if there exists a unit lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

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This needs a proof!

LU decomposition: A square matrix A is said to have an LU decomposition if there exists a unit lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

GENP $\equiv LU$ decomposition of A

Proof: In step k of GENP

$$A^{(k)} = \left[\begin{array}{ccccccc} 1 & & & & & & \\ 0 & \ddots & & & & & \\ \vdots & \cdots & & 1 & & & \\ 0 & & -m_{k+1,k} & \ddots & & & \\ \vdots & & \vdots & & \ddots & & \\ 0 & \cdots & -m_{nk} & \cdots & & & 1 \end{array} \right] A^{(k-1)}$$

$\underbrace{\hspace{15em}}_{=:M_k}$

Then

$$U = A^{(n-1)} = M_{n-1}M_{n-2}\cdots M_k\cdots M_2M_1A$$

where M_k , $k = 1, \dots, n-1$ are the *multiplier* matrices or *Gauss transforms* of Gaussian Elimination.

Exercise: $b^{(n-1)} = M_{n-1}M_{n-2}\cdots M_1b$.

GENP $\equiv LU$ decomposition of A

Note that,

$$U = A^{(n-1)} = M_{n-1} \underbrace{\left(M_{n-2} \cdots \underbrace{\left(M_k \cdots \underbrace{\left(M_2 \underbrace{(M_1 A)}_{=A^{(1)}} \right)}_{=A^{(2)}} \right)}_{=A^{(k)}} \right)}_{=A^{(n-2)}}$$

and

$$M_k = I_n - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} e_k^T, \quad k = 1 : n-1,$$

are rank one updates of I_n . Here e_k is column k of I_n .

GENP $\equiv LU$ decomposition of A

Observe that

$$\blacktriangleright M_k^{-1} = I_n + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{kn} \end{bmatrix} e_k^T, \quad k = 1 : n-1, \text{ (Prove this!)}$$

\blacktriangleright For $i_1 < \dots < i_p$,

$$M_{i_1}^{-1} \dots M_{i_p}^{-1} = I_n + \sum_{i=i_1}^{i_p} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{i+1,i} \\ \vdots \\ m_{ni} \end{bmatrix} e_i^T, \text{ (Prove this!)}$$

GENP $\equiv LU$ decomposition of A

So $U = M_{n-1}M_{n-2} \cdots M_2M_1A$ implies,

$$A = M_1^{-1}M_2^{-1} \cdots M_{n-1}^{-1}U = \left(I_n + \sum_{k=1}^{n-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{pmatrix} e_k^T \right) U$$

$$= \begin{bmatrix} 1 & & & & & & & \\ m_{21} & 1 & & & & & & \\ m_{31} & m_{32} & \ddots & & & & & \\ \vdots & \vdots & & \ddots & & & & \\ m_{k1} & m_{k2} & \cdots & \cdots & 1 & & & \\ m_{k+1,1} & m_{k+1,2} & \cdots & \cdots & m_{k+1,k} & \ddots & & \\ \vdots & \vdots & & & \vdots & & 1 & \\ m_{n1} & m_{n2} & \cdots & \cdots & m_{nk} & \cdots & m_{n,n-1} & 1 \end{bmatrix} U = LU.$$

Algorithm for GENP/LU

```
for  $k = 1 : n - 1$ 
  if  $a_{kk} \neq 0$       (multiplier computation begins)
    for  $i = k + 1 : n$ 
       $a_{ik} = a_{ik} / a_{kk};$ 
    end
  else
    exit {'zero pivot encountered'}
  end      (multiplier computation ends)
  for  $i = k + 1 : n$  (matrix update begins)
    for  $j = k + 1 : n$ 
       $a_{ij} = a_{ij} - a_{ik} a_{kj}$ 
    end
  end      (matrix update ends)
end
end
```

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$L \longrightarrow I_n +$ strictly lower triangular part of output A .
 $U \longrightarrow$ upper triangular part of output A .

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```

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Exercise: Show that the flop count of LU decomposition of an $n \times n$ matrix is $\frac{2}{3}n^3 + O(n^2)$ flops.

Rank one updates in GENP

In Step k ,

$$A^{(k)} = M_k A^{(k-1)}$$

$$= \left(I_n - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} e_k^T \right) A^{(k-1)} = A^{(k-1)} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} \underbrace{e_k^T A^{(k-1)}}_{\text{row } k \text{ of } A^{(k-1)}}$$

$$= A^{(k-1)} - \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & a_{kk}^{(k-1)} & \dots & a_{kn}^{(k-1)} \end{bmatrix}}_{\text{rank one update of } A^{(k-1)}}$$

Rank one updates in GENP

Now

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & a_{kk}^{(k-1)} & \dots & a_{kn}^{(k-1)} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \hat{M}_k \end{array} \right],$$

$$\text{where } \hat{M}_k = \begin{bmatrix} m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} \begin{bmatrix} a_{kk}^{(k-1)} & \dots & a_{kn}^{(k-1)} \end{bmatrix}.$$

Rank one updates in GENP

Now

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & a_{kk}^{(k-1)} & \cdots & a_{kn}^{(k-1)} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \hat{M}_k \end{array} \right],$$

$$\text{where } \hat{M}_k = \begin{bmatrix} m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} \begin{bmatrix} a_{kk}^{(k-1)} & \cdots & a_{kn}^{(k-1)} \end{bmatrix}.$$

As

$$m_{ik} a_{kk}^{(k-1)} = \left(a_{ik}^{(k-1)} / a_{kk}^{(k-1)} \right) a_{kk}^{(k-1)} = a_{ik}^{(k-1)}, \quad i = k+1 : n,$$

the first column of \hat{M}_k is

$$\begin{bmatrix} a_{k+1,k}^{(k-1)} \\ \vdots \\ a_{nk}^{(k-1)} \end{bmatrix}.$$

Rank one updates in GENP

Therefore,

$$A^{(k)} = A^{(k-1)} - \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \widehat{M}_k \end{array} \right]$$

$$= \left[\begin{array}{c|c} A_{11}^{(k-1)} & A_{12}^{(k-1)} \\ \hline A_{21}^{(k-1)} & A_{22}^{(k-1)} \end{array} \right]$$

$$- \left[\begin{array}{cccc|cccc} 0 & & 0 & 0 & 0 & & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & & 0 & 0 & 0 & & \dots & 0 \\ \hline 0 & & 0 & a_{k+1,k}^{(k-1)} & \left[\begin{array}{c} m_{k+1,k} \\ \vdots \\ m_{nk} \end{array} \right] & \left[\begin{array}{ccc} a_{k,k+1}^{(k-1)} & \dots & a_{kn}^{(k-1)} \end{array} \right] \\ \vdots & \dots & \vdots & \vdots & & & & \\ 0 & & 0 & a_{nk}^{(k-1)} & & & & \end{array} \right]$$

where

$$A_{11}^{(k-1)} \rightarrow k \times k; \quad A_{21}^{(k-1)} \rightarrow k \times (n-k);$$

$$A_{21}^{(k-1)} \rightarrow (n-k) \times k; \quad A_{22}^{(k-1)} \rightarrow (n-k) \times (n-k).$$

Rank one updates in GENP

$$\text{As } A_{21}^{(k-1)} = \begin{bmatrix} 0 & \cdots & 0 & a_{k+1,k}^{(k-1)} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nk}^{(k-1)} \end{bmatrix}, \text{ therefore,}$$

$$A^{(k)} = \left[\begin{array}{c|c} A_{11}^{(k-1)} & A_{12}^{(k-1)} \\ \hline & \mathbf{A}_{22}^{(k)} \end{array} \right],$$

where

$$A_{22}^{(k)} = A_{22}^{(k-1)} - \underbrace{\begin{bmatrix} m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} \begin{bmatrix} a_{k,k+1}^{(k-1)} & \cdots & a_{kn}^{(k-1)} \end{bmatrix}}_{\text{rank one update of } A_{22}^{(k-1)}}$$

$$= \begin{bmatrix} a_{k+1,k+1}^{(k-1)} & \cdots & a_{k+1,n}^{(k-1)} \\ \vdots & \ddots & \vdots \\ a_{n,k+1}^{(k-1)} & \cdots & a_{nn}^{(k-1)} \end{bmatrix} - \begin{bmatrix} a_{k+1,k}^{(k-1)} \\ \vdots \\ a_{nk}^{(k-1)} \end{bmatrix} \begin{bmatrix} a_{k,k+1}^{(k-1)} & \cdots & a_{kn}^{(k-1)} \end{bmatrix}$$

$\approx - \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \end{bmatrix}$

Algorithm for GENP/LU with higher level BLAS

```
for k = 1:n-1
    if A(k,k)  $\neq$  0      (multiplier computation begins)
        A(k+1:n,k) = A(k+1:n,k)/A(k,k);
    else
        exit {'zero pivot encountered'}
    end      (multiplier computation ends)

    (matrix update)
    A(k+1:n,k+1:n) = A(k+1:n,k+1:n) - A(k+1:n,k)*A(k,k+1:n);
end
```

GENP for solving systems of equations

Pseudocode for solving $n \times n$ system $Ax = b$:

1. Find LU decomposition of A . $(\frac{2}{3}n^3 + O(n^2)$ flops)
2. Solve $Ly = b$ for y . (n^2) flops)
3. Solve $Ux = y$ for x . (n^2) flops)

Total flops: $\frac{2}{3}n^3 + O(n^2)$ flops.

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First step need NOT be repeated for solving other systems with same A .

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Theorem: A nonsingular square matrix has an LU decomposition if and only if all its leading principal submatrices are nonsingular.

Additionally for such matrices, the LU decomposition is unique.

$A \rightarrow$ nonsingular.

$$\text{Suppose } A = LU \Rightarrow \begin{matrix} k & n-k \\ A_{11} & A_{12} \\ \hline n-k & A_{21} & A_{22} \end{matrix} = \begin{matrix} k & n-k \\ L_{11} & \\ \hline n-k & L_{12} & L_{22} \end{matrix} \begin{matrix} k & n-k \\ U_{11} & U_{12} \\ \hline & U_{22} \end{matrix}$$

$$\Rightarrow A_{11} = L_{11} U_{11}$$

$$\therefore \det A_{11} = \det(L_{11} U_{11}) = \underbrace{\det L_{11}}_1 \det U_{11} = \det U_{11} \quad \text{--- (1)}$$

$$\text{But } 0 \neq \det A = \det(LU) = \underbrace{\det L}_1 \det U = \det U_{11} \det U_{22}$$

$$\Rightarrow \det U_{11} \neq 0$$

$\therefore \text{(1)} \Rightarrow \det A_{11} \neq 0 \Rightarrow$ leading principal submatrices of A are nonsingular.

Conversely let all leading principal submatrices of A be nonsingular, for $n=1$, $A = [a]$, $a \neq 0$.

$$\& A = [a] = \underbrace{[1]}_{=: L} \underbrace{[a]}_{=: U},$$

Suppose that A has an LU decomposition for $\text{size} < n$,

$$A = \begin{bmatrix} \hat{A} & b \\ c^T & a_{nn} \end{bmatrix}, \quad \text{So } \hat{A} \text{ has an LU decomposition say } \hat{A} = \hat{L} \hat{U}.$$

$$= \begin{bmatrix} \hat{L} \hat{U} & b \\ c^T & a_{nn} \end{bmatrix} = \begin{bmatrix} \hat{L} \hat{U} & \hat{L} \hat{L}^{-1} b \\ c^T & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{L} & \\ l^T & 1 \end{bmatrix} \begin{bmatrix} \hat{U} & \hat{L}^{-1} b \\ & u_{nn} \end{bmatrix}$$

where $l^T \hat{U} = c^T$ and $u_{nn} + l^T \hat{L}^{-1} b = a_{nn}$
 i.e. $l = (\hat{U}^T)^{-1} c$ and $u_{nn} = a_{nn} - l^T (\hat{L}^{-1} b)$

Gaussian Elimination With Partial Pivoting (GEPP)

1. Checking A for existence of LU decomposition is not possible in practice.
 - (i) Numerically it is only possible to ascertain how close A and its leading principal submatrices are to being singular.
 - (ii) Ascertaining the proximity of A and its leading principal submatrices to a singular matrix will cost more flops than finding the LU factors.

Gaussian Elimination With Partial Pivoting (GEPP)

1. Checking A for existence of LU decomposition is not possible in practice.
2. Even if A has an LU decomposition, computing it is a numerically unstable process.
 - ▶ Small pivots can lead to large multipliers and result in instability in finite precision arithmetic.

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Try Gaussian Elimination with row exchanges also called **Gaussian Elimination with Partial Pivoting (GEPP)**!

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$$PA = A(p, :) = LU$$

2. Even if A has an LU decomposition, computing it is a numerically unstable process.

$$P = I(p, :)$$

Any remedies?

Try Gaussian Elimination with row exchanges also called **Gaussian Elimination with Partial Pivoting (GEPP)**!

What is this?

$$p = [1:n]'$$

For each $k = 1 : n - 1$

1. Find $a_{pk}^{(k-1)}$ such that $|a_{pk}^{(k-1)}| = \max_{k \leq j \leq n} |a_{jk}^{(k-1)}|$.

$$[k, m] = \max(\text{abs}(A(k:n, k)))$$

$k \rightarrow$ largest abs. entry
 $m \rightarrow$ position where it occurs

2. If $p \neq k$ interchange rows k and p .

3. Perform the usual GE steps to create zeros in column k .

if $m \neq k$ $A([k, m], :) = A([m, k], :)$ end
 $p([k, m]) = p([m, k])$

GEPP

$$A^{(k-1)} = \left[\begin{array}{cccc|cccc} a_{11} & \cdots & \cdots & a_{1k} & a_{1,k+1} & \cdots & \cdots & a_{1n} \\ m_{21} & a_{22}^{(1)} & \cdots & a_{2k}^{(1)} & a_{2,k+1}^{(1)} & \cdots & \cdots & a_{2n}^{(1)} \\ \vdots & m_{32} & \ddots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & a_{kk}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & \cdots & a_{kn}^{(k-1)} \\ \vdots & \vdots & \vdots & a_{k+1,k}^{(k-1)} & a_{k+1,k+1}^{(k-1)} & \cdots & \cdots & a_{k+1,n}^{(k-1)} \\ m_{n1} & m_{n2} & \vdots & a_{n,k}^{(k-1)} & a_{n,k+1}^{(k-1)} & \cdots & \cdots & a_{nn}^{(k-1)} \end{array} \right]$$

GEPP $\equiv LU$ decomposition of row permuted A

Permutation Matrices: An $n \times n$ permutation matrix P is obtained by interchanging rows and/or columns of I_n .

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2. Transpositions are symmetric matrices. $P^T = P$
3. Transpositions are their own inverses. $P^T = P = P^{-1}$

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Theorem Given any $n \times n$ matrix A , there exists a permutation P such that PA has an LU decomposition.

GEPP $\equiv LU$ decomposition of row permuted A

Recall that GENP requires multiplier matrices M_1, \dots, M_{n-1} such that

$$U = M_{n-1} M_{n-2} \cdots M_1 A.$$

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Now GEPP requires finding transpositions P_1, \dots, P_{n-1} and multiplier matrices M_1, \dots, M_{n-1} , such that

$$U = \left\{ M_{n-1} P_{n-1} \left(M_{n-2} P_{n-1} \cdots \left(M_k P_k \cdots \left(M_2 P_2 \underbrace{(M_1 P_1 A)}_{=A^{(1)}} \right) \right) \right) \right\}$$

$\underbrace{\hspace{15em}}_{=A^{(2)}} \quad \underbrace{\hspace{10em}}_{=A^{(k)}} \quad \underbrace{\hspace{15em}}_{=A^{(n-2)}} \quad \underbrace{\hspace{15em}}_{=A^{(n-1)}}$

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Here for $k = 1, \dots, n - 1$,

1. $P_k = \left[\begin{array}{c|c} I_{k-1} & \\ \hline & \hat{P}_k \end{array} \right]$, \hat{P}_k being a $(n - k + 1) \times (n - k + 1)$ transposition.

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2. $M_k = I_n - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} e_k^T$, with $m_{jk} = a_{jk}^{(k-1)} / a_{kk}^{(k-1)}$,
 $j = k+1 : n$.

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1. Let $\mathcal{P}_k = P_{k+1} \cdots P_{n-1}$, $k = 1, \dots, n-2$. Then,

$$\mathcal{P}_k = \left[\begin{array}{c|c} I_k & \\ \hline & \tilde{P}_{k+1} \cdots \tilde{P}_{n-1} \end{array} \right]$$

where for all $j = k+1, \dots, n-1$, \tilde{P}_j are transpositions of size $n-k \times n-k$.

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2. Let $\tilde{M}_k = \mathcal{P}_k^T M_k \mathcal{P}_k$, $k = 1, \dots, n-2$. Then,

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where

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3. $U = M_{n-1} \tilde{M}_{n-2} \cdots \tilde{M}_1 P_{n-1} P_{n-2} \cdots P_1 A$.

Prove this!