DEPARTMENT OF MATHEMATICS
IIT GUWAHATI

MA 473 **Computational Finance** **Lab – III** **Date: 13.08.2024**

1. Consider the following Black-Scholes PDE for European call:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0, \quad (0,\infty)\times(0,T],\ T>0 \\ V(S,t) = 0, \quad \text{for } S = 0, \\ V(S,t) = S - Ke^{-r(T-t)}, \quad \text{for } S \to \infty \\ \text{with the terminal condition } V(S,T) = \max\left(S-K, 0\right). \end{cases}$$

Since the given final condition is not smooth, the resulting solution is not smooth enough for the convergence of finite difference approximations. Hence we need to modify the above model as defining $\pi_\varepsilon(y)$ such that

$$\pi_\varepsilon(y) = \begin{cases} y, & y \geq \varepsilon, \\ c_0 + c_1 y + \cdots + c_9 y^9, & -\varepsilon < y < \varepsilon, \\ 0, & y \leq -\varepsilon, \end{cases}$$

where $0 < \varepsilon \ll 1$ is a transition parameter and $\pi_\varepsilon(y)$ is a function which smooths out the original max $(y, 0)$ around $y = 0$. This requires that $\pi_\varepsilon(y)$ satisfies

$$\pi_\varepsilon(-\varepsilon) = \pi_\varepsilon'(-\varepsilon) = \pi_\varepsilon''(-\varepsilon) = \pi_\varepsilon'''(-\varepsilon) = \pi_\varepsilon^{(4)}(-\varepsilon) = 0,$$

$$\pi_\varepsilon(\varepsilon) = \varepsilon, \quad \pi_\varepsilon'(\varepsilon) = 1, \quad \pi_\varepsilon''(\varepsilon) = \pi_\varepsilon'''(\varepsilon) = \pi_\varepsilon^{(4)}(\varepsilon) = 0.$$

Using these ten conditions we can easily find that

$$c_0 = \frac{35}{256}\varepsilon, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{35}{64\varepsilon}, \quad c_4 = -\frac{35}{128\varepsilon^3},$$
$$c_6 = \frac{7}{64\varepsilon^5}, \quad c_8 = -\frac{5}{256\varepsilon^7}, \quad c_3 = c_5 = c_7 = c_9 = 0.$$

Replacing $\max(S-K, 0)$ in the given terminal condition by the fourth-order smooth function $\pi_\varepsilon(S-K)$ we obtain

$$\begin{cases} -\dfrac{\partial U}{\partial t} - \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 U}{\partial S^2} - (r-\delta) S \dfrac{\partial U}{\partial S} + rU = 0, \quad (0,\infty)\times(0,T],\ T>0 \\ U(S,t) = 0, \quad \text{for } S = 0, \\ U(S,t) = S - Ke^{-r(T-t)}, \quad \text{for } S \to \infty \\ \text{with the terminal condition } U(S,T) = \pi_\varepsilon(S-K). \end{cases}$$

A) Solve the above modified Black-Scholes PDE for $\varepsilon = 10^{-2}$, 10^{-4} and 10^{-6} by the following schemes:

   (i) Forward-Euler for time & central difference for space (FTCS) scheme.

   (ii) Backward-Euler for time & central difference for space (BTCS) scheme.

   (iii) Crank-Nicolson finite difference scheme

B) Also, calculate the following Greeks

$$\Delta\,(\text{Delta}) = \frac{\partial U}{\partial S},\ \Gamma\,(\text{Gamma}) = \frac{\partial^2 U}{\partial S^2},\ \nu\,(\text{Vega}) = \frac{\partial U}{\partial \sigma},\ \Theta\,(\text{Theta}) = \frac{\partial U}{\partial t}\ \text{ and }\ \rho\,(\text{Rho}) = \frac{\partial U}{\partial r}.$$

The values of the parameters are $T = 1$, $K = 20$, $r = 0.07$, $\sigma = 0.2$ and $\delta = 0.01$.

The output files should contain the following for above problem:

a) Plot the numerical solutions of A) and B) at the different times $t = 0,\ 0.25,\ 0.5,\ 0.75$ and 1.

b) Draw the surface plot of the numerical solutions.
