Case 2

$rank(A) = r < m$

$(AP)e_1 \quad (AP)e_2$

linear ind why?

$$AP = QR = [\hat{Q}_1 \quad Q_2] \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} Q_1R_1 & Q_1R_2 \\ 0 & 0 \end{bmatrix}$$

$Q_1R_1 \rightarrow$ will preserve rank of R_1 & ($\because Q_1$ is orthogonal)

$\downarrow$

orthogonal matrix preserves rank of a matrix on operat

$$[(AP)_1 \dots (AP)_r] = \begin{bmatrix} Q_1R_1 \\ 0 \end{bmatrix}$$

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Kaliyuga : chaos/ lawlessness (in our context chaos in existing hierarchical structure)

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for non singular matrix A,

$Ax = b$

$x = A^{-1}b \rightarrow$ costly

$\{x^{(n)}\} \rightarrow \tilde{x}$

$n \rightarrow \infty \qquad x^{(n)} \simeq \tilde{x}$

For diagonal dominant

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$

Jacobi iteration

$$x_1^{(k+1)} = \frac{(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)})}{a_{11}}$$

$$x_2^{(k+1)} = (b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)})/a_{22}$$

$$x_3^{(k+1)} = \frac{(b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)})}{a_{33}}$$

Gauss seidel use $(k+1)^{th}$ values for x_2 & x_3.

A ~~→~~

$Ax = b$,

$Mx = (M - A)x + b$

M is non-singular, M = pre-conditioner ~~pre conditioned~~

$x = (I - M^{-1}A)x + M^{-1}b$

$x^{(n)} = g(x^{(n)})$ fixed pt iteration

$$x^{(k+1)} = \underbrace{(I - M^{-1}A)}_{B}x^{(k)} + M^{-1}b$$

x will converge iff $\rho(B) < 1 \iff \|B\|_2 < 1$ or $\|B\|_\infty < 1$

B: iteration matrix

$\rho(B) = \max |\lambda_i|$

spectral ~~radius~~ radius

$A = D - L - U$

~~for~~ For Jacobi $M = D$

Gauss Seidel, M:

Consider pre conditioner, $M = D$ then $M - D = L + U$

$$D x^{k+1} = (L+U) x^{(k)} + b$$

$$x^{k+1} = \underbrace{D^{-1}(L+U)}_{B} x^{(k)} + D^{-1} b \rightarrow \text{Jacobi}$$

where, $B = D^{-1}(L+U)$ $\rho(B) < 1$, provided $\|A\| \le 1$

If A is +ve definite

$M = D - L$

$M - A = U$, $(D - L) x^{(k+1)} = U x^{(k)} + b$

$$x^{(k+1)} = \underbrace{(D-L)^{-1} U}_{B} x^{(k)} + (D-L)^{-1} b \rightarrow \text{Gauss-Seidel}$$

$\rho(B) < 1$, for $\|A\| \le 1$

In order to speed up the conv of the Gauss Seidel we can introduce parameter ω_R

$$M = \frac{1}{\omega_R} D - L \quad \} \text{ SOR (Successive Over Relaxation)}$$

$$M - A = \left(1 - \frac{1}{\omega_R}\right) D + U$$

$$\left(\frac{D}{\omega_R} - L\right) x^{(k+1)} = \left[\left(1 - \frac{1}{\omega_R}\right) D + U\right] x^{(k)} + b$$

$$\Rightarrow \quad x^{(k+1)} = \left(\frac{D}{\omega_R} - L\right)^{-1} \left[\left(1 - \frac{1}{\omega_R}\right) D + U\right] x^{(k)} + \left(\frac{D}{\omega_R} - L\right)^{-1} b$$

<u>stopping criterion</u>,

$\|x - \tilde{x}\| < tol$

$\downarrow$

must be known before

$r^{(n)} = b - Ax^{(n)}$, if $\|r^{(n)}\| \to 0$ as $n \to 0$

if $\|r^{(n)}\| < TOL = 10^{-8}$ (stop)

if $\|x^{(n)} - x^{(n-1)}\| \to 0$ as $n \to \infty$

Cauchy seq.

B = iteration matrix corresponding to Jacobi iterat.

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(B)^2}}$$

The case of the American optn we have to impose the side condn i.e. $y \geq g$ at each iterat

<u>correctn vector</u> : $x^{(k)} - x^{(k-1)}$

$$r_i^{(k)} = b - \sum_{j=1}^{i} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}$$

$$x_i^{(k)} = x_i^{(k-1)} + \omega_k \frac{r_i^{(k)}}{a_{ii}}$$

Now,

$$Ax - \hat{b} = b - Ag$$

PSOR (projected SOR), $x_i^{(k)} = \max\{0, x_i^{(k-1)} + \omega_k \frac{r_i^{(k)}}{a_{ii}}\}$

$$\theta y_i = r_i^{(k)} + a_{ii}(x_i^{(k)} - x_i^{(k-1)})$$

Since the Cryer's problem has a unique min^m which ensures the convg PSOR.

$\omega(\xi)\ \phi_0(x_i, x_j)$

Date :- 18 <u>HS-217</u>

<u>Print & Censorship</u>

higher power / authorative feels threatned when there is a ~~potential~~ contest that can overthrow their influcence / control.

Initial ~~journalism~~ censorship started in India started ~~for~~ because years of allegations among the English press itself by the English press against the East India company by the English parliament.

~~of the~~ the

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Recap:

$$\begin{cases} -u'' = f & x \in (0,1) = \Omega \\ u(0) = 0 = u(1) \end{cases}$$

$v \in C_0^\infty(\Omega)$

u', v': weak derivatives

$$-\int u'' v \, dx = \int f v \, dx$$

$$\Rightarrow -\int v \, du' = \cancel{0}(u'v)_0^1 + \int_0^1 u' v' \, dx = \int f \, dx$$

$u \in L^p(\Omega)$ L: Lebesgue (a.e)

u: bdd measurable f^n

$$\underbrace{\int_a^b f(x) \, dx = \sum_{i=0}^n f(x^i) h}_{\text{Riemann integral}}$$

$$L^p(\Omega) = \{ u \in \Omega : \int_\Omega |u(x)|^p \, dx < \infty \}$$

$v \in L^2, v' \in L^2$

$v \in H^1(\Omega)$

$$H^1(\Omega) = \{ v \in L^2(\Omega), \ v' \in L^2(\Omega) \}$$

$$C_0^\infty(\Omega) \subseteq L^2(\Omega)$$

Supp $u \in C^{\infty}(\Omega)$, $v \in C_0^{\infty}(\Omega)$

① $\int_{\Omega} D^{\alpha} u(x)\, v(x)\, dx = (-1)^{|\alpha|} \int_{\Omega} u(x)\, D^{\alpha} v(x)\, dx$

(integrate by parts) $|\alpha| \leq k$;

$\forall\, v \in C_0^{\alpha}(\Omega)$

for $v \in L^1(\Omega)$

② $\int_{\Omega} w_{\alpha}(x)\, v(x)\, dx = (-1)^{|\alpha|} \int_{\Omega} u(x)\, D^{\alpha} v\, dx \quad \forall\, v \in C_0^{\infty}(\Omega)$

Comparing the RHS ① & ②

$w_{\alpha}(x) = D^{\alpha} u(x)$, since $u \in L^1(\Omega)$ the derivative is the weak derivative

is unown in the weak sense

$\Omega = (-1, 1)$

$u(x) = |x|$

$u'(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$

$\mathrm{sgn}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$

$$\int_{-1}^{1} |x|\, v\, dx = -\int_{-1}^{0} x\, v\, dx + \int_{0}^{1} x\, v\, dx$$

For $L^p(\Omega)$ f^ns

$k>0$, $p \in [1, \infty)$

$W_p^k(\Omega) = \{ u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega),\ |\alpha| \leq k \}$ is

known as Sobolov's space.

$$\|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)} \right)^{1/p}, \quad 1 \leq p \leq \infty$$

$p = 2$, $\|$

The norm will be defined through the inner prod.

$$W_2^k(\Omega) = \{ u \in L^2,\ D^\alpha u \in L^2 \}$$

$$(u, v) = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)$$

H: Hilbert space.

$$H^k(\Omega) = W_2^k(\Omega)$$

$$H^1(\Omega) = \{ u \in L^2(\Omega),\ \frac{\partial u}{\partial x_j} \in L^2(\Omega) \quad j = 1 \ldots m \}$$

$p=2$, $k=1$

here $\frac{\partial u}{\partial x_j}$ denotes the weak derivative of

u ~~belongs~~ $u \in L_2$

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2}\right)^{1/2}$$

$$H^2_0 = \{u \in H^2(\Omega) : u = 0, \partial\Omega\}$$

$$\begin{cases} -u'' = f \\ u(0) = 0, \quad u(1) = 1 \end{cases} \Rightarrow \quad H^1_E(\Omega) = \{u \in H^1(\Omega) \quad u(0) = u(1) = 1\}$$

abstract setting

$u'(1) = 5$

$u(0) + u'(0) = 1$

$u(1) + u'(1) = -5$

$$-\int u'' v \, dx = \int f v \, dx$$

$$= -(v u')_0^1 + \int v' u' \, dx = \int f v \, dx \quad \forall v \in H^1_E(\Omega)$$

$$+ \int v' u' \, dx = \int f v \, dx$$

$$v(1)u'(1) - v(0)u'(0)$$

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \sum_{j=1}^{n} \left\|\frac{\partial u}{\partial x_j}\right\|^2_{L^2}\right)^{1/2}$$

$$H^2_0 = \{u \in H^2(\Omega) \; ; \; u = 0, \partial\Omega\}$$

$$\begin{cases} -u'' = f \\ u(0) = 0, \; u(1) = 1 \end{cases} \Rightarrow \quad H^1_E(\Omega) = \{v \in H^1(\Omega) \; u(0) = u(1) = 1\}$$

abstract setting

$u'(1) = 5$

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$$-\int u'' v \, dx = \int f v \, dx$$

$$= -(v u')_0^1 + \int v' u' \, dx = \int f v \, dx \quad \forall v \in H^1_E(\Omega)$$

$$+ \int v' u' \, dx = \int f v \, dx$$

$$v(1) u'(1) - v(0) u'(0)$$

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- $u'' = f$

$-u'(0) = \alpha,$

$u'(1) = \beta$

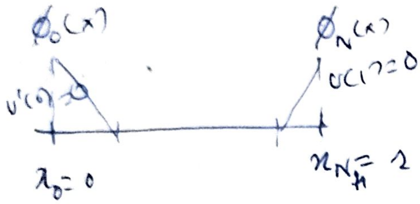
$-\int u'' v\,dx = \int f v\,dx$

$\forall v \in H^1_E(\Omega)$

$\int u'v'\,dx - [u'v]_0^1 = \int f v\,dx$

$H^1_E(\Omega) = \{u \in H^1(\Omega) :$

$-u'(1) = \alpha, u(1) = \beta \}$



$u_n(x) = \sum_{i=1}^{N} u_i \phi_i(x) \rightarrow$ ~~Dirich~~ Dirichlet B.V.P

$\phi_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$

$u_n(x) = \sum_{i=0}^{Nn} u_i \phi_i(x) \rightarrow$ Neumann

$\int_{x_0}^{x_1} \phi_0'(x) \phi_1, \quad \int_{x_N}^{x_{Nn}} \phi_{Nn} \phi_N \neq 0, \quad 0$ for other nodal pts.

Robin: $-u'' = f$

$\begin{cases} u(0) - u'(0) = \alpha \\ u(1) + u'(1) = \beta \end{cases}$

$H^1_R(\Omega) = \{u \in \Omega, u(0) - u'(0) = \alpha$
$u(1) + u'(1) = \beta \}$

And $v \in H^1_R(\Omega)$

if

$\int u'v'\,dx$

$= \int_\Omega f v\,dx + \ldots$

$\forall v \in H^1_R(\Omega)$

Obstacle problm :

$$\text{①}\begin{cases} \text{Find } u \in C^1(\Omega) \text{ st} \\ u''(u-g) = 0 \qquad -u'' \geq 0 \qquad u-g \geq 0 \\ u(-1) = u(1) = 0 \end{cases}$$

Consider the obstacle problem in order to solve the obstacle problm by finite element we will define the space by the following set

$K = \{ v \in C^0[-1,1] : v(-1) = v(1) = 0 \quad v(x) \geq g(x) \ \forall x \in \bar{\Omega}$

& v is piecewise C^1

(set of competing f's]

including
$\bar{\Omega}$: bddry pts.

Find $u \in K$

the sol^n u of obstacle problm ① $\Rightarrow u \in K$ &

for any $v \in K$ $\quad v - g \geq 0$, $-u'' \geq 0$

$\therefore -u''(v-g) \geq 0$

$\therefore$ The finite element formulam tells

$$\int_{-1}^{+1} -u''(v-g)\,dx \geq 0 \ \forall v \in K \text{ \& from ①} \rightarrow ②$$

$$\int_{-1}^{+1} u''(u-g)\,dx = 0 \rightarrow ③$$

$$② - ③ \Rightarrow \int_{-1}^{+1} -u''(v-u)\,dx \geq 0 \rightarrow ④$$

$\forall v \in K$

From ④ doesn't explicitly contain g where as
ineq
it is inside the space K

$$\int_{-1}^{+1} -d(u')(v-u)\,dx \geq 0,$$

$$\left[-u'(\cancel{v-u})^{\to 0}\right]_{-1}^{+1} + \int_{-1}^{1} v'(v-u)'\,dx \geq 0$$

since $u, v \in K$ $\quad u(-1) = v(-1)$
$\quad u(1) = v(1)$

weak formulation, find $u \in K$ st

⑤ $\left\{ \int u'(v-u)'\,dx \geq 0 \quad \forall v \in K \right.$

since ineq. in ⑤ is known ~~variational~~ variational inequality.

if we consider any approx soln $w \in K$ we will have

$$\int_{-1}^{+1} w'(v-w)'\,dx \geq 0 \quad \forall v \in K$$

$$\sum_{i=1}^{N} \int_{-1}^{+1} \phi_i'(x)\left(\phi_j'(x) - \phi_i'(x)\right) dx \geq 0 \quad \forall j \quad A_{ij}$$

$$w_n(x) = \sum_{i=0}^{N+1} w_i \phi_i(x)$$

(5) is a ~~minimazat~~ minimizatn problem

when $v = u$ $\quad \int u'(v-u)'\,dx = 0$

$\therefore$ ineq (5) is a minimizatn porblm

<u>Implementatn</u>

→ Discretize time

Consider the American,

$$(y_\tau - y_{xx})(y - g) = 0$$

$\cancel{y_\tau - y_{xx}}$ $\quad y_\tau - y_{xx} \geq 0 \quad \underbrace{y - g \geq 0}_{\text{main constraint}}$

$y(x, 0) = g(x, 0)$

$y(x_{min}, \tau) = g(x_{min}, \tau)$ $\quad y \in \mathcal{G}^1$, w.r.t x

$y(x_{max}, \tau) = g(x_{max}, \tau)$

(admissible fn's)

$\mathcal{K} = \{ v \in \mathcal{G}^0 : \frac{\partial v}{\partial x}$ v is piecewise $\mathcal{G}^0 \}$

$v(x, \tau) \geq g(x, \tau) \ \forall x, \tau$

$v(x, 0) = g(x, 0)$, $v(x_{min}, \tau) = g(x_{min}, \tau)$

$v(x_{max}, \tau) = g(x_{max}, \tau)$

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$$v \geq g; \quad \frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2} \geq 0$$

$$\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(v-g)\, dx \geq 0 \quad -(1)$$

$$\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y-g)\, dx = 0 \quad -(2)$$

(1) & (2) $\Rightarrow \displaystyle\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(v-y)\, dx \geq 0$

integration by parts gives

$$\Rightarrow \int_{x_{min}}^{x_{max}} \left[\frac{\partial y}{\partial x}(v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x}\right)\right] dx$$

$$\left[\frac{\partial y}{\partial x}\overset{0}{\cancel{(v-y)}}\right]_{x_{min}}^{x_{max}} \geq 0 \quad (\because y, v \in \mathcal{K} \quad v = y \text{ at bdry})$$

$$\int_{x_{min}}^{x_{max}} \left[\frac{\partial y}{\partial \tau}(v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x}\right)\right] dx \geq 0 \rightarrow (3)$$

if $v = y$ $\quad (*) = 0$, therefore (3) is a minimization problem.

$$I(y; v) = \int_{x_{min}}^{x_{max}} \left[\frac{\partial y}{\partial \tau}(v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x}\right)\right] dx \geq 0 \quad \forall\, v \in \mathcal{K}$$

$$\min_{y \in \mathcal{K}} I(y; v) = I(y, y) = 0$$

For Am opt^n we require the sol^n $\hat{y} \in \mathcal{K}$ is ξ^2

& we expect our soln $\hat{y}$ st

$$\min_{v \in \mathcal{K}} I(\hat{y}; v) = 0$$

$$\left.\begin{aligned} \hat{y} &= \sum \omega_i \phi_i(x) = \sum \omega_i(\tau)\phi_i(x) \\ v &= \sum v_i \phi_i(x) = \sum v_i(\tau)\phi_i(x) \end{aligned}\right\} \text{(4)}$$

Using (4) in (3), we get

$$\int_{x_{min}}^{x_{max}} \left[\sum \frac{d\omega}{d\tau}\phi_i \times \sum (v_i - \omega_i)\phi_i(x)\right.$$

$$\left. + \sum_i \omega_i(\tau)\phi_i'(x) \left(\sum_j (v_j(\tau) - \omega_j(\tau))\phi_j'(x)\right)\right] dx \geq 0$$

$$-\sum_i \sum_j \frac{d\omega_i}{d\tau}(v_i - \omega_j) \underbrace{\int \phi_i(x)\phi_j(x)\, dx}_{B} +$$

$$\sum_i \sum_j \omega_i(v_j - \omega_j) \underbrace{\int \phi_i' \phi_j'}_{A} dx \geq 0$$

$$\left(\frac{d\omega}{d\tau}\right)^T B(v - \omega) + \omega^T A (v - \omega) \geq 0$$

$$B = \int \phi_i \phi_j \, dx \qquad A = \int \phi_i' \phi_j' \, dx$$

$$\Rightarrow (v - \omega)^T \left[B\frac{d\omega}{d\tau} + A\omega\right] \geq 0 \rightarrow \text{(5)}$$

Ineq in (5) can be discretized by θ-method.

$\theta \in [0,1]$

$$\left(v^{(n+1)} - \omega^{(n+1)}\right)\left[B \frac{1}{\delta\tau}\left(\omega^{(n+1)} - \omega^{(n)}\right) + \theta A \omega^{(n+1)} + (1-\theta) A \omega^{(n)}\right] \geq 0$$

$\theta = 0 \rightarrow$ explicit Euler

$\theta = \frac{1}{2} \rightarrow$ Crank-Nicholson.

$\theta = 1 \rightarrow$ implicit. Euler.

$$\left(v^{(n+1)} - \omega^{(n+1)}\right)\left[\left(B + \delta\tau\theta A\right)\omega^{(n+1)} - \left(B - \delta\tau(1-\theta)A\right)\omega^{(n)}\right] \geq 0$$

$$r = \left(B - \delta\tau(1-\theta)A\right)\omega^{(n)}$$

$$C = B + \delta\tau\,\theta A$$

$$\Rightarrow \left(v^{(n+1)} - \omega^{(n+1)}\right)^T \left(C.\omega^{(n+1)} - r\right) \geq 0$$

For side cond^n, $\hat{y}(x,\tau) \geq g(x,\tau)$

$$\sum \omega_i(\tau)\,\phi_i(x) \geq g(x,\tau) \quad - (*)$$

$$\phi_i(x_j) = \begin{cases} 1 & i=j \\ 0 & o.w \end{cases}$$

$(*) \Rightarrow \quad \omega_j(\tau) \geq g(x_j,\tau)$

$$w^{(n)} \geq g^{(n)}$$

$\theta = 1/2$ $\quad w^{(0)}$

for $n = 1:N$

$$\left\{ \begin{array}{l} r = (B - \delta\tau(1-\theta)A)\, w^{(n)} \\ \quad + v \geq g \\ \Rightarrow (v - w)^T (a_0 - r) \geq 0,\ w \geq g. \end{array} \right.$$

In order to incorporate the side conditions one has to use the PSOR

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Sewing Machines & Modern India

informal sector → Small units especially in poorer areas where larger industries produce their products

$\Rightarrow$ ~~Only~~ un

$Av = \lambda v$ $\quad v^* A v = \lambda v^* v = \lambda \|v\|_2^2$

$\overline{v^* A v} = \bar{\lambda} \|v_2\|^2$

$v^* A^* v = \bar{\lambda} \|v\|_2^2$

$v^* A v = \bar{\lambda} \|v\|_2^2 = \lambda \|v\|_2^2 \Rightarrow \bar{\lambda} = \lambda$

$\Rightarrow \lambda \in \mathbb{R}$

Hermitian matrix $\Leftrightarrow$ Real eigen value

$\Rightarrow$ eigen vectors are also real

~~FD~~ FEM

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$r = (B - \Delta T (1-\theta) A) w^{(n)}$

$C = B + \Delta T \theta A$

$\forall v \geq g, \quad (v-w)^T (Cw - r) \geq 0 \quad w \geq g.$

$\Updownarrow$

Replica

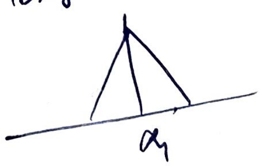
FDM : $A = C, \; b = r$ $\quad Aw - b \geq 0 \;, w \geq g$

$Cw - r \geq 0 \;, \quad w \geq g$ $\quad (Aw-b)^T (w-g) = 0$

$(Cw - r)^T (w - g) = 0$

trapezoidal

$\phi_i = \begin{cases} \dfrac{x_{i+1} - x_i}{x_{i+1} - x_i} & x_i \leq x \leq x_{i+1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \leq x \leq x_i \\ 0 & \text{o.w} \end{cases}$



FDM $\Leftrightarrow$ FEM

$(\Leftarrow)$ $\forall v \geq g$ $\quad Cw - r \geq 0, \; w \geq g$

$(v-w)^T (Cw - r) \geq 0 \;\; w \geq g$ $\quad (w-g)^T (Cw - r) \geq 0$

~~Supp soln of FEM~~

Supp assume w satisfies FDM $\Rightarrow w \geq g$ (side cond^n is true)

$(\Leftarrow)$

$$(v-w)^T (Cw - r) = (v-w)^T (Cw - r) - (w-g)^T (Cw - r) \geq 0$$

$$\Rightarrow (v-w)^T (Cw - r) \geq 0$$

FDM $\Rightarrow$ FEM

$(\Rightarrow)$ Supp w is a soln of FEM $\Rightarrow w \geq g$

$(Cw - r)^T (w \geq g)$ & $(v-w)^T (Cw - r) \geq 0$

$$v^T (Cw - r) \geq w^T (Cw - r)$$

~~$v \geq g$~~

$$g^T (Cw - r) \geq w^T (Cw - r)$$

(i) we have to prove $Cw - r \geq 0$

Supp k^{th} of $(Cw - r)$ is $-ve$ & k^{th} comp of v is as large as possible

$$\underbrace{v^T (Cw - r)}_{\text{becomes as small as poss}}$$

$\therefore$ LHS is as small as possible which is a contradiction

$\therefore$ $(Cw - \lambda) \geq 0$

(ii) $(Cw - \lambda)^T (w - g) = 0$

$w \geq g$ $\quad (w - g)^T (Cw - \lambda) \geq 0 \rightarrow$ ①

replace $v = g$.

$(g - w)^T (Cw - \lambda) \geq 0 \Rightarrow (w - g)^T (Cw - \lambda) \leq 0$ — ②

① & ② $\Rightarrow$ $(w - g)^T (Cw - \lambda) = 0$

which tells the FEM & FDM are ~~same~~ giving the same soln for the basis elements.

FDM we use C-N & num. quad (Simpsons ~~trapezoidal~~)

FE we used basis elem

In the implementatn of FEM, we have to solve $Cw = \lambda$ st $w \geq g$ using PSOR (ensures side condn)

Exotic optns (exercise)

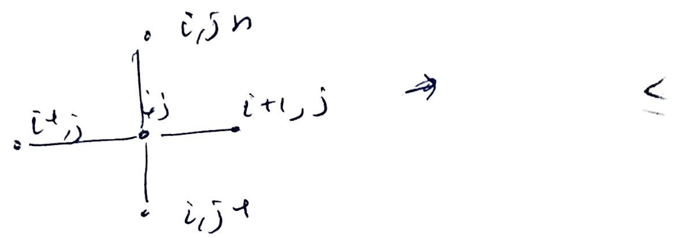
$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \omega\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + bu = f(x,y,t)$$

↓

Implicit $\theta \in [0,1]$

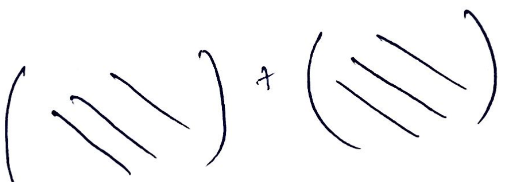
beaux of 2-D

→ at each time level



5-pt tencil (exercise)

ADI $(t^n, t^{n+1}), (t^n, t^{n+1/2}), (t^{n+1/2}, t^{n+1})$



MA-423 Lab

4.2.20

A = randn(9,5)

rank(A) : numerical rank but very often actual rank.

>>svd(A) → 5 non-zero ~~others~~ rows

$\mathcal{E} = 2\max(n,m)\, u \|A\|_2$

A(:, 6:7) = A(:, [2,4]) + A(:, [3,5])

size(A) → (9 x 7)

$x^* y = (\cancel{x_1 + ix_2})$

$y^a x = y_1 x_1 + y_2$

$[x_1, +x_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$= x_1 y_1 + x_2 y_2$

MA-473 Date: 14/10/24

Power method & its variants

let $[x_1 \; x_n]^T \in \mathbb{C}^n \setminus \{0\}$ be arbitrarily chosen. set $q_0 = x/s_0$ where $s_0 = x_i$ st $|x_i| = \|x\|_\infty$

$$q_1 = \frac{Aq_0}{s_1} = \frac{A(x/s_0)}{s_1} = \frac{Ax}{s_0 s_1} = \frac{\sum_{k=1}^{n} \lambda_k c_k v_k}{s_0 s_1}$$

$$= \frac{\lambda_1}{s_0 s_1}\Big[\underbrace{c_1 v_1 + \sum_{k=2}^{n} \frac{\lambda_k}{\lambda_1} c_k v_k}_{\frac{Ax}{\lambda_1}}\Big]$$

let $\tilde{q}_j = \frac{A^j x}{\lambda_1^j}$, $j = 1, \dots m$

$s_i = \tilde{q}_j(i)$ st $|\tilde{q}_j(i)| = \|\tilde{q}_j\|_\infty$

$$q_i = \frac{Aq_{i-1}}{s_i}$$

$$q_1 = \frac{\lambda_1}{s_0 s_1} \tilde{q}_1, \quad q_2 = \frac{\lambda_1^2}{s_0 s_1 s_2} \tilde{q}_2$$

$$q_j = \frac{\lambda_1^j}{\prod_{k=0}^{j} s_k} \tilde{q}_j, \quad \text{let } \mu_j = \underbrace{\prod_{k=0}^{n} s_k / \lambda_1^j}_{} \quad j = 0, 1, \dots$$

$(*)$ $q_j = \frac{\tilde{q}_j}{\mu_j}$ is st that $\|q_j\|_\infty = 1$

Then $q_j = \tilde{q}_j / \mu_j$ $j = 0, 1, \dots$

let $i_j = \min_i \{1 \dots n : |Aq_{j-1}(i)| = \|Aq_{j-1}\|_\infty\}$

$\Rightarrow i_j = \min \{1, \dots n ; q_j(i) = 1\}$

$(*)$ $\frac{\tilde{q}_j(i_j)}{\mu_j} = q_j(i_j) = 1 \Rightarrow \mu_j = \tilde{q}_j(i_j)$

$$= \left[\frac{A^j x}{\lambda_1^j}\right](i_j)$$

$$= \Big[c_1 v_1 + \underbrace{\sum_{k=2}^{n} \left(\frac{\lambda_k}{\lambda_1}\right)^j c_k v_k}_{\downarrow \, 0 \text{ as } j \to \infty}\Big](i_j)$$

$\therefore \lim_{j\to\infty} \mu_j = \lim_{j\to\infty} c_1 v_1(i_j)$

$$\Rightarrow \lim_{j\to\infty} \frac{\tilde{q}_j}{\mu_j} = \lim_{j\to\infty} \frac{c_1 v_1}{c_1 v_1(i_j)} = \lim_{j\to\infty} \frac{v_1}{v_1(i_j)}$$

$$\lim_{j\to\infty}\left\|\frac{v_1}{v_1(i_j)}\right\|_\infty = \lim_{j\to\infty}\|q_j\|_\infty = 1$$

$$\lim_{j\to\infty} Aq_j = A\lim_{j\to\infty} q_j = A\lim_{j\to\infty}\frac{v_1}{v_1(i_j)} = \lim_{j\to\infty}\frac{Av_1}{v_1(i_j)}$$

$$= \lambda_1 \lim_{j\to\infty}\frac{v_1}{v_1(i_j)}$$

$$\therefore \lim_{j\to\infty} Aq_j = \lambda_1 \lim_{j\to\infty} q_j = \lambda_1 \lim_{j\to\infty} q_j$$

$\therefore$ For large enough j, $Aq_j \approx \lambda_1 q_j$

$\Rightarrow \|Aq_j\|_\infty \simeq |\lambda_1| \to ①$, $|Aq_j(i_j)| \approx |\lambda_1 \underbrace{q_j(i_j)}_{1}| = |\lambda_1| \to ②$

$|Aq_j(i)| \simeq |\lambda_1 \underbrace{q_j(i)}_{(\leq 1)}| \leq |\lambda_1| \simeq \|Aq_j\|_\infty \quad \forall\, i = 1, \dots m \to ③$

For ① & ② $|Aq_j(i_j)| \simeq \|Aq_j\|_\infty$ for $i \neq i_j$

& ③ $\Rightarrow |Aq_j(i)| \leq |\lambda_1| \simeq \|Aq_j\|_\infty$

If $i_j \neq 1$ by defn of i_j, $|Aq_j(i)| \simeq |\lambda_1| |q_j(i)| < |\lambda_1| \quad \forall\, i = 1, \dots i_j - 1$

$\therefore i_{j+1} = i_j$ for large enough j (i_j starts repeating for large enough j)

supp $\lim_{j\to\infty} i_j = i_0$

Then $\lim_{j\to\infty} q_j = \lim \frac{v_1}{v_1(i_j)} = v_1 / v_1(i_0) = \hat{v}_1$ (dominant e-vector of A)

ii) $\lim_{j\to\infty} s_j = \lambda_1$, $\lim_{j\to\infty} q_j = \hat{v}_1 \Rightarrow \lim_{j\to\infty} Aq_j = A \lim_{j\to\infty} q_j = A\hat{v}_1 = \lambda_1 \hat{v}_1$

$$\Rightarrow \lim_{j\to\infty} \underbrace{\frac{Aq_j}{s_j}}_{q_{j+1}} = \lim \frac{\lambda_1 \hat{v}_1}{s_j}$$

$\Rightarrow$ ~~$\lim_{j\to\infty} \frac{\lambda_1 \hat{v}_1}{s_j}$~~ $\lim_{j\to\infty} q_{j+1} = \frac{\lambda_1 \hat{v}_1}{\lim_{j\to\infty} s_j} \Rightarrow \hat{v}_1 = \frac{\lambda_1 \hat{v}_1}{\lim_{j\to\infty} s_j}$ ⊔

$\lim_{j\to\infty} q_j = \hat{v}_1$

* rounding error can make the const. corresp to $v_1 \neq 0$ even if $c_1 = 0$ due to rounding error.

MA-423
Matlab Lab

$A_{n\times m} = U \Sigma V^*$ ~~$A = U_{n\times}$~~ $_{n\times m}$

$n \geq m$, $A = U V^* (V \Sigma V^*)$ $V^* V = I$

$n < m$, $A = (U \Sigma U^*)(U V^*)$

$\quad$ R

$U \Sigma U V^*$

$U \Sigma U^* U V$

$\underbrace{U \Sigma U^*}_{R} U V^*$

$U_{18\times 6}$ $S_{6\times 6}$ $V_{6\times 6}$

$n = 18$, $m = 6$ $\quad n \times m$

$W = U_{18\times 6} \times V'_{6\times 6}$

$R = U_{18\times 6} \times S_{6\times 6} \times U'_{6\times 18}$

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Exotic options

$V(S, t) \to \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0$

$V(S_1, S_2, t) \to ?$

$dX_t = a \quad dt + \quad dW$

$dS_1 = \mu_1 S_1 dt + a_1 S_1 dW_t^{(1)}$

$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_t^{(2)}$

$E(dW^{(1)}, dW^{(2)}) = \rho dt$

$\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \underbrace{\sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j}}_{} +$

$\sum_{i=1}^{n} (r - \delta_i) S_i \frac{\partial V}{\partial S_i} - rV = 0$

$\Omega = (S^1_{min}, S^2_{min}) \times (S^1_{max}, S^2_{max})$

$\Omega \times (0, T]$

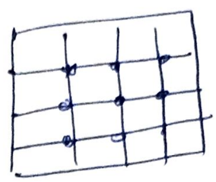
$\mathrm{cov}\left(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}\right)$

$= E(a_1 dW^{(1)}, a_2 dW^{(2)}) = \rho a_1 a_2 dt$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum \rho_{12}\, a_1 a_2\, S_1 S_2 \left(\frac{\partial^2 V}{\partial S_1^2} + \frac{\partial^2 V}{\partial S_2^2} + \frac{\partial^2 V}{\partial S_1 \partial S_2}\right) + \dots$$

eq (1)

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \to eq\ (2)$$



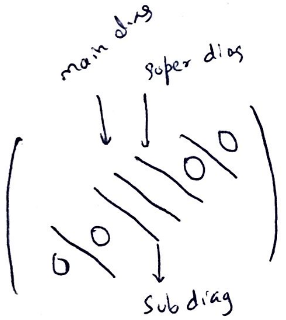
$h = 1/4$

$(Mn)^2 \times (Nn)^2$

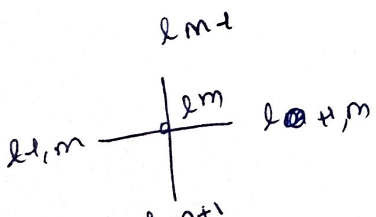
Discretization using CTCS, BTCS, CN.

$$\frac{U^{n+1}_{lm} - U^{n}_{lm}}{\delta t} = \frac{\left(U^{n+1}_{l+1,m} - 2U^{n+1}_{l,m} + U^{n+1}_{l-1,m}\right)}{h^2} + \frac{\left(U^{n+1}_{l,m+1} - 2U^{n+1}_{l,m} + U^{n+1}_{l,m-1}\right)}{h^2}$$

A =



$$\frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{\left(U_{l+1,m} - 4U_{lm} + U_{l-1,m} + U_{l,m+1} + U_{l,m-1}\right)}{h^2}$$



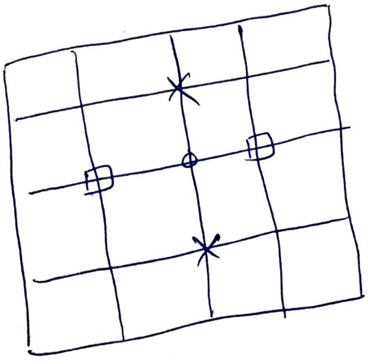
$$\delta^2_{xy} U^{n}_{lm} = \delta^2_x U^{n+1}_{lm} + \delta^2_y U^{n+1}_{lm}$$

$U^{n+1}_{l,m}$

$$\frac{\partial u}{\partial x} = \frac{\hat{U}^{n}_{l+1,m} - \hat{U}^{n}_{l-1,m}}{2h} \quad \text{(Cnt Diff)}$$

If we discretize the pde's corresponding to the exotic eqn given in ① or the 2D Heat condn given in ② by any implicit scheme like BTCS or C-N then the bandwidth of the matrix A, ($AU = f$) will be Large, therefore it takes enormous CPU time & memory to ~~the~~ solve the sys of ∴ ~~It takes~~ linear algb. eqns. In order to overcome these computatnal complexities A.D.I Type (Alternating Direction Implicit Scheme)

$v(x, y, 0) = \phi(x, y)$ I.C



$t^n \to t^{n+1}$

or x : explicit
y : implicit
or vice versa

$t^n \to t^{n+1/2} \to t^{n+1}$

Fractional time step)

x : implicit
y : explicit

When we progress from time t^n to t^{n+1} we introduce an artifical time step at $t^{n+1/2}$ & the whole scheme will be returned in the following manner.

$t^n \longrightarrow t^{n+1}$ x : explicit, y : implicit } or vice versa.

$t^{n+1/2} \longrightarrow t^{n}$ ~~y : impl~~ x : implicit, y : explicit

$$\frac{U_{lm}^{n+1/2} - U_{lm}^{n}}{(\delta t/2)} = \left(\frac{U_{l+1,m}^{n+1/2} - 2U_{lm}^{n+1/2} + U_{l-1,m}^{n+1/2}}{h^2}\right) + \left(\frac{U_{l,m+1}^{n} - 2U_{lm}^{n} + U_{lm-1}^{n}}{h^2}\right) \longrightarrow (3)$$

$$\frac{U^{n+1} - U^{n+1/2}}{\delta t/2} = \left(\frac{U_{l+1,m}^{n+1/2} - 2U_{l,m}^{n+1/2} + U_{l-1,m}^{n+1/2}}{h^2}\right) + \left(\frac{U_{l,m+1}^{n+1} - 2U_{lm}^{n+1} + U_{lm-1}^{n+1}}{h^2}\right) \longrightarrow (4)$$

Starting from the given I.C's we can solve (3), by using the soln of (3) as the ~~I.V~~ initial value for (4)

$$-\delta^2 v(v^T \hat{A} v)\theta^T = -\cancel{\delta(\theta^T \hat{A} v)} - \delta v(\underbrace{-\delta \hat{A} v}_{\theta})\hat{\theta}^T$$

$$= \cancel{-\delta v v \hat{v}}$$

$$= \delta v v^T v \hat{v}$$

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<u>recap:</u>

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$t_n \to t_{n+1/2} \to t_{n+1}$

$$\delta_x^2 u_{ij}^n = u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n$$

$$\delta_y^2 u_{ij}^n = u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n$$

$$\frac{U_{lm}^{n+1/2} - U_{lm}^n}{(\delta\tau/2)} = \frac{1}{h^2}\left[\delta_x^2 U_{lm}^{n+1/2} + \delta_y^2 U_{lm}^n\right] \to (5)$$

$$\frac{U_{lm}^{n+1} - U_{lm}^{n+1/2}}{\delta\tau/2} = \frac{1}{h^2}\left[\delta_x^2 U_{lm}^{n+1/2} + \delta_y^2 U_{lm}^{n+1}\right] \to (6)$$

$\delta \quad U_{lm}$

Form eq ③ of prv do class.

$$\frac{U_{lm}^{n+1/2} - U_{lm}^{n}}{\delta t/2} = \frac{1}{h^2}\left[U_{l+1,m}^{n+1/2} - 2U_{lm}^{n+1/2} + U_{l-1,m}^{n+1/2} + U_{l,m+1}^{n} - 2U_{lm}^{n} + U_{l,m-1}^{n} \right]$$

$$U_{lm}^{n+1/2} - U_{lm}^{n} = \frac{\delta t}{2h^2}\left[U_{l+1,m}^{n+1/2} - 2U_{lm}^{n+1/2} + U_{l-1,m}^{n+1/2} + U_{l,m+1}^{n} - 2U_{lm}^{n} + U_{l,m-1}^{n} \right]$$

$$U_{lm}^{n+1/2} - U_{lm}^{n} = \lambda U_{l+1,m}^{n+1/2} - 2\lambda U_{lm}^{n+1/2} + \lambda U_{l-1,m}^{n+1/2} + \lambda U_{l,m+1}^{n} - 2\lambda U_{lm}^{n} + \lambda U_{l,m-1}^{n}$$

$$-\lambda U_{l,m+1}^{n} + (1+2\lambda) U_{lm}^{n} - \lambda U_{l,m-1}^{n} = \lambda U_{l+1,m}^{n+1/2} + (1-2\lambda) U_{lm}^{n+1/2} + \lambda U_{l-1,m}^{n+1/2}$$

Form eq ④

$$\frac{U_{lm}^{n+1} - U_{lm}^{n+1/2}}{\delta t/2} = \frac{1}{h^2}\left[U_{l+1,m}^{n+1/2} - 2U_{l,m}^{n+1/2} + U_{l-1,m}^{n+1/2} + U_{l,m+1}^{n+1} - 2U_{l,m}^{n+1} + U_{l,m-1}^{n+1} \right]$$

$$-\lambda U_{l,m+1}^{n} + (1+2\lambda) U_{lm}^{n+1} - \lambda U_{l,m-1}^{n+1} = \lambda U_{l+1,m}^{n+1/2} + (1-2\lambda) U_{l,m}^{n+1/2} + \lambda U_{l-1,m}^{n+1/2}$$

$$A_1 U^{n+1/2} = B_1 U^{n}$$

$$A_2 U^{n+1} = B_2 U^{n+1/2}$$

If we eliminate $U_{lm}^{n+1/2}$ ~~of the~~ ~~form~~ the two eq^ns.

(7) $$\left(1-\frac{\lambda}{2}\delta_x^2\right) U_{lm}^{n+1/2} = \left(1+\frac{\lambda\delta_y^2}{2}\right) U_{lm}^{n}$$

(8) $$\left(1-\frac{\lambda\delta_y^2}{2}\right) U_{lm}^{n+1} = \left(1+\frac{\lambda\delta_x^2}{2}\right) U_{lm}^{n+1/2}$$

then we get,

$$\left(1-\frac{\lambda}{2}\delta_x^2\right)\left(1-\frac{\lambda}{2}\delta_y^2\right) U_{lm}^{n+1} = \left(1+\frac{\lambda}{2}\delta_x^2\right)\left(1+\frac{\lambda}{2}\delta_y^2\right)U_{lm}^{n}$$

$$\Rightarrow \left[1-\frac{\lambda}{2}\left(\delta_x^2+\delta_y^2\right)+\frac{\lambda^2}{4}\delta_x^2\delta_y^2\right] U_{lm}^{n+1}$$

$$= \left[1+\frac{\lambda}{2}\left(\delta_x^2+\delta_y^2\right)+\frac{\lambda^2}{4}\delta_x^2\delta_y^2\right] U_{lm}^{n} \rightarrow (9)$$

for ~~θ~~ $=1/2$

θ- scheme

$$\theta \left[\frac{U_{lm}^{n+1/2} - U_{lm}^{n}}{\delta t/2} \right] + (1-\theta) \left[\frac{U_{lm}^{n+1/2} - U_{lm}^{n}}{\delta t/2} \right]$$

$$= \frac{\theta}{h^2} \left[U_{l+1,m}^{n+1/2} - 2U_{lm}^{n+1/2} + U_{l-1,m}^{n+1/2} + U_{l,m+1}^{n} - 2U_{lm}^{n} + U_{l,m-1}^{n} \right]$$

$$\frac{U_{lm}^{nn} - U_{lm}^{n}}{\delta t} = \theta (\delta_x^2 U_{lm}^{n} + \delta_y^2 U_{lm}^{n})/h^2 + (1-\theta)(\delta_x^2 U_{lm}^{n+1/2} + \delta_y^2 U_{lm}^{nn})/h^2$$

$\theta = 1/2 \rightarrow C.N$, $\lambda = \delta t/h^2$

$$\left[1 - (1-\theta)(\delta_x^2 + \delta_y^2) U_{lm}^{nn} \right] = \left[1 + \theta \frac{\lambda}{2} (\delta_x^2 + \delta_y^2) U_{lm}^{n} \right] \rightarrow (10)$$

Comparing (9) & (10)

We have $\frac{\lambda^2}{4} \delta_x^2 \delta_y^2$ as an extra term in (9)

which presents in the ADI scheme.

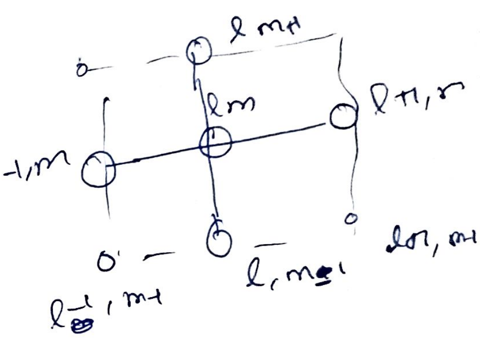
The ~~addition~~ additional term which presents in the ADI scheme proportional to λ^2 $O(k^2+h^2)$ which is the same order as C.N $[O(k^2+h^2)]$ scheme which is 2nd order in time & space & the scheme (9) is known as the approximate factorization

$$\frac{\partial V}{\partial t} = \frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial S_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial S_1 \partial S_2} + (r-\delta_1)S_1\frac{\partial V}{\partial S_1}$$
$$+ (r-\delta_2)S_2\frac{\partial V}{\partial S_2} - rV = 0$$

for $V(S_1, S_2, t)$

~~we can~~ give $\rho = 0$ when $\rho = Cov(dW_t^{(1)}, dW_t^{(2)})$

$$\delta_{S_1}\delta_{S_2} V_{lm}^n = ?$$



$$D^+ D^- V_{lm}^n = D^+\left(\frac{V_{l+1,m}^n - V_{lm}^n}{h}\right)$$

$$= \frac{V_{l+1,m}^{n+1} \cancel{/ V_{l+1,m}^n}}{(\delta t)h} - \frac{V_{lm}^{n+1} - V_{lm}^n}{(\delta t)h}$$

$$\delta_{S_1}\delta_{S_2} = \delta_{S_1}\left(\frac{V_{l,m+1} - V_{l,m-1}}{2h}\right)$$

$$= \frac{1}{4h^2}\left(V_{l+1,m+1} - V_{l+1,m-1} - V_{l-1,m+1} + V_{l-1,m-1}\right)$$

Asian

$D^+ V_\ell = \frac{V_{\ell n} - V_\ell}{h}$ $D^- V_\ell = (V_\ell - V_{\ell-1})/h$

$V(S,t) = V^n_m$

$D_t U_\ell = \frac{V_{\ell n} - V_{\ell-1}}{2h}$

$V(S, A_{rg}, t)$
$t < T$

$t_1 \; - - \; t_m$

cts : $\frac{1}{T}\int_0^T S_\theta \, d\theta$

discrete : $\frac{1}{m}\sum_{i=1}^{m} S_{ti}$

$V(S, A, t)$

$$\frac{\partial v}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial S^2} + (r - S)\frac{\partial v}{\partial S} + S\frac{\partial v}{\partial A} - rv = 0$$

if we evaluate the underlying at diff time instances $t_1 - - t_m$ the corresponding vals S_{t_1} S_{tm} then one can take either the arithmetic or the geometric mean. ~~If we consider the~~ integr.

~~cts time we have~~

$$R Q_1 = \begin{bmatrix} x_{11} & \\ & x_{nn} \end{bmatrix} \left[\begin{array}{c|c} Q_1 & \\ \hline & I_{n-2} \end{array} \right]$$

Applying this tech. on the ith we get ~~$O(n^2)$~~ A_i as upp Hess ~~...~~ A_1 as upp Hess. Q so matrix on every ith

<u>MA-473</u> If we sample the underlying asset S_t at discrete time intervals with equidist time interval. Then we we obtain $S_{t_1}, \ldots, S_{t_{n-1}}, S_{t_n}$. Then we can consider avg as arithmetic mean $\forall n \in \mathbb{N}$

$$\frac{1}{n} \sum_{i=1}^{n} S_{t_i} = \frac{h}{T} \sum_{i=1}^{n} S_{t_i} := A \to (1)$$ or if the time interval is cts then ~~we can have~~ the avg will be

$$\boxed{h = T/n}$$

$$\hat{S} = \frac{1}{T} \int_0^T S_\theta d\theta \to (2)$$

Instead of the arithematic mean we can consider the geo. mean for both discrete & cts.

$$\left(\prod_{t=1}^{n} S_{t_i} \right)^{1/n} = \exp\left(\frac{1}{n} \log \prod_{i=1}^{n} S_{t_i} \right)$$

Discrete:

$$= \exp\left(\frac{1}{n} \sum_{i=1}^{n} \log S_{t_i} \right) \to (3)$$

Cts:

$$\hat{S} = \exp\left(\frac{1}{T} \int_0^T \log S_t \, dt \right) \to (4)$$

$$A = \hat{S} = S_t = \frac{1}{t}\int_0^t f(S, \theta)\, d\theta$$

$V(S, t) \rightarrow$ European $\quad V(S, A, t) \rightarrow$ Asian.

In the Asian optn with avg $\hat{S}$ with the underlying ~~pay~~ asset

S_t, $t < T$

$(\hat{S} - K)^+ \rightarrow$ avg price call

$(K - \hat{S})^+ \rightarrow$ avg price put

$(S_T - \hat{S}) \rightarrow$ avg strike call

$(\hat{S} - S_T) \rightarrow$ avg strike put.

Consider cts case 1st:

Let us denote the avg $A_t = \int_0^t f(S_\theta, \theta)\, d\theta$

Instead of reg SDE, we'll have.

$$dA_t = a_A(t)\,dt + b_A\, dW_t \qquad a_A = f(S_t, t) \quad b_A = 0$$

then

$$dV_t = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t)\frac{\partial V}{\partial A}\right) dt + \sigma S \frac{\partial V}{\partial S}\, dW_t$$

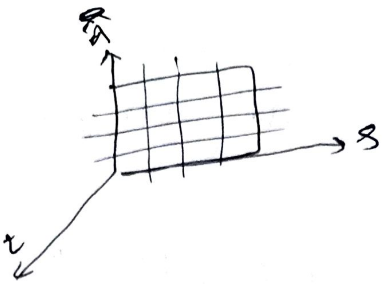
$$\Rightarrow \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \underbrace{f(S,t)\frac{\partial V}{\partial A}}_{\text{extra term}} - rV = 0 \quad (**)$$

- Avs.

Black-Scholes like pde

Apply

BTCS, ~~CCS~~ C-N



$\delta S,\ \delta A,\ \delta t$

We can consider simplest case.

BTCS:

V^{n+1}_{m} V^{n}_{m}

$$\frac{V^{n+1}_{l,m} - V^{n}_{l,m}}{\delta t} + \frac{\sigma^2}{2} S_m^2 \left(\frac{V^{n+1}_{l,m+1} - 2V^{n+1}_{l,m} - V^{n+1}_{l,m-1}}{\delta S^2}\right) + f(S_m, t_{n+1}) \frac{V^{n+1}_{l,m} - V^{n+1}_{l,m}}{\delta A}$$

$$+ r S_m \frac{V^{n+1}_{l,m+1} - V^{n+1}_{l,m-1}}{\delta S}$$

$$- r V^{n+1}_{l,m} = 0$$

$$\frac{V^{n+1}_{l,m} - V^{n}_{l,m}}{\delta t} + \frac{\sigma^2}{2} S_m^2 \left(\frac{V^{n+1}_{l,m+1} - 2V^{n+1}_{l,m} - V^{n+1}_{l,m-1}}{\delta S^2}\right)$$

$$- f(S_m, t_{n+1}) \frac{V^{n+1}_{l,m+1} - V^{n+1}_{l,m-1}}{2\delta A}$$

$$+ r S_m \frac{V^{n+1}_{l,m+1} - V^{n+1}_{l,m-1}}{\delta S}$$

$$- r V^{n+1}_{l,m} = 0$$

when $f(S,t)$ is linear then we can reduce the 2D B-S like pde into 1D pde (dim-red^n)

Consider the B-S like pde (++) in 2D domain

$V(S,A,t)$ $S>0$, $A>0$, $0 \le t \le T$

in the particular case when $f(S,t)=S$ we can reduce the dim to 1D. Let us consider the European Arithematic Avg strike call with payoff $(S_T - \hat{S})^+ = (S_T - \frac{1}{T}A_T)^+ = S_T\left(1 - \frac{1}{TS_T}\int_0^T S_\theta d\theta\right)^+$

Let us denote $R_t = \frac{1}{S_t}\int_0^t S_\theta d\theta$, $A_t = R_t / S_t$

$S_T\left(1 - \frac{1}{T}R_T\right)^+ = S_T H(R,T)$

$V(S,A,t) = S \cdot H(R,t)$

where H is a f^n of R & t & R is an independent variable

$R_{t+\delta t} - R_t$

$dR_t =$

$R_{t+\delta t} = dR_t + R_t$

$\therefore \begin{cases} dS_t = \mu S_t\,dt + a S_t\,dw_t \\ dR_t = (1 + (a^2 - \mu) R_t)dt - a R_t\,dw_t \end{cases}$

HSS-217 Date: 23/10/24

Everyday Tech. Grammaphone

Q) What made the new sound space modern?
→ ① Its ethimaltry, in able to listen to any sound anywhere

② became a commodi

③ in a phy env you need to be at a ~~specific~~ particular place & time to listen to a particular sound, ~~the sound spc~~ the new soundspace transformed this relationships ~~with~~ with ~~time~~ space & time by ~~allowing~~ making it possible to listen to any sound ~~at~~ anywhere at any time

MA-473 Date: 22/10/24

From prev class,

$A_t = S_t R_t$

$V(S, A, t) = S \cdot H(R, t)$

$A = S \cdot R$

Transformation: $V(S, A, t) = \tilde{V}(S, R, t) = S H(R, t)$, $R = \frac{A}{S}$

$V_t = S H_t$

$V_s = H(R, t) + S H_R$

$V_{ss} = \frac{\partial H}{\partial R}\left(-\frac{A}{S^2}\right) + 1 = -\frac{A}{S^2}\frac{\partial H}{\partial R} + 1$

$\frac{\partial V}{\partial t} + \Big($

$V_{ss} = H_R\left(-\frac{A}{S^2}\right) + H_R + S \cdot H_{RR}\left(-\frac{A}{S^2}\right)$

$= -\frac{H_R}{S^2} + H_R - \frac{H_{RR} A}{S}$

$V_A = S \; H_R \cdot \frac{1}{S} = H_R$

Substituting in original eqn

$$\boxed{\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} R^2 \frac{\partial^2 H}{\partial R^2} + (1 - r R)\frac{\partial H}{\partial R} = 0}$$

For the bdd'y condn $R \to \infty$ we obtain from the

payoff $H(R, T) = \left(1 - \frac{1}{T} R_T\right)^+$

$$\lim_{R_T \to \infty} H(R_T, T) = 0$$

$R_t = \frac{1}{S_t} \int_0^t S_\theta \, d\theta$, the integral R_t is bdd'd

$\therefore$ As $S \to \infty$, $R \to \infty$

This is the European call as $S \to 0$ we cannot exercise the European call $\Rightarrow$ $H(R, t) = 0$, $R \to \infty$

Date : 25/10/24

$A_0 = A$, $A_0 = Q_0 R_0$ $(R_0(i,i) \geq 0 \ \forall i = 1, \dots, n)$ $A_1 := R_0 Q_0$

$= Q_0^T A_0 Q_0$

$A_1 = Q_1 R_1$ $(R_1(i,i) > 0)$ to

$A_2 = R_1 Q_1$

$= Q_1^T A_1 Q_1$

$A_1, A_2 \xrightarrow{\text{converge}}$ Schur form of A

Date: 25/10/24

Recap: $\dfrac{\partial H}{\partial t} + \dfrac{\alpha^2}{2} R^2 \dfrac{\partial^2 H}{\partial R^2} + (1-\lambda R)\dfrac{\partial H}{\partial R} = 0$

bdd^{ry} condⁿ : $R=0$, $R\to\infty$

$R_1 \to \infty$ $R_T \to \infty$

$R\to\infty$, $S\to\infty$

$R\to 0$?

$dR_t = (1+(\alpha^2-\mu)R_t)\,dt + aR_t\,dw_t \to (1)$

$\Rightarrow dR_0 = dt$ $\boxed{R_T=0}$

$R\to\infty$

For the left hand bdd^{ry} conditn when $R=0$, we face some difficulties for eg $R_0=0$ then from (1) we can obtain $dR_0 = dt$ $\Rightarrow$ R_T won't remain at 0

$\therefore$ we can expect R_T to be 0.

Therefore to obtain the bdd^{ry} conditn at $R=0$, we use the pde directly.

(assmd $\dfrac{\alpha^2 R^2}{2}\dfrac{\partial^2 H}{\partial R^2} \to 0$)

(1) $\Rightarrow$ $\dfrac{\partial H}{\partial t} + \dfrac{\partial H}{\partial R} = 0$

provided the 2nd derivative of H or H is bdded we can draw the conclusion that

if it is bdded $\dfrac{\alpha^2 R^2}{2}\dfrac{\partial^2 H}{\partial R^2} \to 0$

Supp $\frac{\partial \tilde{H}}{\partial R^2}$ Supp $\frac{\sigma^2 R^2}{2} \frac{\partial \tilde{H}}{\partial R^2} = \underset{\text{const}}{\theta}(\neq 0)$

$$\frac{\partial H}{\partial R^2} = O\left(\frac{1}{R^2}\right)$$

$$\frac{\partial \tilde{H}}{\partial R^2} = \frac{c}{R^2}$$

$$\frac{\partial H}{\partial R} = c \int R^{-2}\, dR = \frac{c R^{-1}}{(-1)} + c_1$$

$$H = (-c) \log R + c_1 R$$

as $R \to 0$ H is unbdd, but we assumed it's bdded which is a contradn

$\therefore \frac{\sigma^2 R^2}{2} \frac{\partial^2 H}{\partial R^2} \to 0$ as $R \to 0$ (when H is bdded)

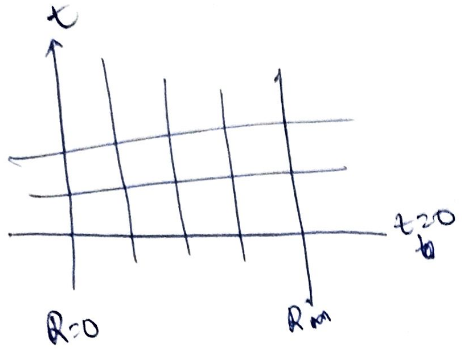
$R \to \infty$ $H(R,t) = 0$

$R = 0$ $\frac{\partial H}{\partial R} + \frac{\partial H}{\partial t} = 0$, (*)

$$H(R_T, T) = \left(1 - \frac{R_T}{T}\right)^+$$

$$\frac{H_m^{n} - H_m^{n+1}}{\delta t} + \frac{H_{m n}^{n+1} - H_m^{n+1}}{\delta R} = 0$$

~~Date: 25/10/29~~



$$D^+ D^+ H_m^n = D^+ \left(\frac{H_{m}^{n} - H_{m}^{n}}{\delta R} \right)$$

$$\left. \frac{\partial H}{\partial R} \right|_{R=0} = \frac{-3H_0^n + 4H_1^n - H_2^n}{2\delta R} + O(\delta R^2)$$

once we obtain H

$V(S, A, t) = S\,H(R, t)$

When we have discrete avg $t_1\ t_2$ t_m

S_{t_1} S_{t_m}

The past hist on discrete t_i.

If supp, we determine the underlying at the then

discrete t_i inst at $t_1, t_2 \ldots t_m$

the arithmetic avg $A_{t_k} = \frac{\sum_{i=1}^{k} S_{t_i}}{k}$ $k = 1, \ldots M$

$(*)$

(*) can be re written as an update form the prev one, $A_{t_k} = \frac{A_{t_{k-1}}(k-1) + S_{t_k}}{k}$

$$= A_{t_{k-1}} + \frac{1}{k}(S_{t_k} - A_{t_{k-1}}) \rightarrow (1)$$

$t = T \rightarrow t = 0$ (top to btm)

It is more appropriate to have the integration in backward form,

~~$A_{t_{k-1}} = A_{t_k}$~~ $+ \frac{1}{(k-1)}$ ~~$(A_{t_k}$~~ $- S_{t_k})$

$$A_{t_{k-1}} = A_{t_k} + \frac{1}{(k-1)}(A_{t_{k-1}} - S_{t_k}) \rightarrow (2)$$

from (1) & (2) we can obs that A_{t_k} is const b/w the sampling times & it jumps at t_k with $\frac{1}{k-1}(A_{t_k} - S_{t_k})$

$S = S_k$

$$A^-(S) = A^+(S) + \left(\frac{1}{k-1}\right)(A^+(S) - S)$$

where A^-, A^+ denote the values of A immediately before & after sampling of t_k.

From the No-Arbitrage principle we can note that the continuity of $V(S_t, A_t, t)$ for any realization of random walk.

$$V(S, A^+, t_k) = V(S, A^-, t_k) \rightarrow (+)$$

For any fixed S, A the eq (+) defin a jmp of v

at the sampling t_k.

~~$(A, Q(2))$~~ $\frac{\partial v}{\partial A} = 0$ of the jump conditn

For the numerical calculatns

if we discretize A-axis into discrete values $A_1 \dots A_J$

then for each time period b/w two consecutive

samplings ~~$t_k \to t_{k+1}$~~ $t_{k n} \to t_k$ the optn value v in the discretizatn of

is independent of A, because $\frac{\partial v}{\partial A} = 0$

A_t is piecewise const, therefore, $\frac{\partial v(A)}{\partial A}$

Therefore we obtain $1, \dots J$ no. of 1-D B-S pde, all can be

integrated sepuately / independently for $t_{kn} \to t_k$.

which is more suitable for parallel computing

Date: 28/10/24

SDE (Stochastic differential eqn's)

derivative form:

$$\begin{cases} y' = f(y) & , t \in [0, 1] \\ y(0) = \alpha \end{cases}$$

Differential form (since derivative may not exists)

$M(x, y)\,dx$

$+ N(x, y)\,dy = 0$

SDE

$$\begin{cases} dX(t) = a(t,X)\, dt + b(t,x)\, dw_t \\ X(0) = X_0 \end{cases}$$

$$\int_0^t dX(s) = \int_0^t a(s,x)\, ds + \int_0^t a(s,x)\, dw_s$$

$$X(t) - X(0) = \int_0^t a(s,x)\, ds + \int_0^t a(s,x)\, dw_s$$

B-S SDE: $dx = \mu x\, dt + \alpha x\, dw_t$

$$x(t) = (\mu t + \frac{\alpha^2}{2})$$

$$\int_a^b f(x)\, dx = \sum_{i=0}^{n} f(x^*)\, \delta x$$ Riemann Integrand

$x^* \in [x_i, x_{i+1}]$

$\delta w_i = w_{i+1} - w_i$

$= \xi_i \sqrt{\Delta t}$

$$\int_a^b f(x)\, dw_t = \sum_{i=0}^{n} f(t_{i+1})\, \Delta w_i$$

$\xi_i \in N(0,1)$

f: deterministic

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(s)\, dw_s$$

Diffusion eqn

B.S ~~SDE~~: $dX = \mu X\, dt + \alpha X\, dw_t$

$\cancel{X(t)}$

$$X(t) = \exp\left\{ \left(\mu - \frac{\alpha^2}{2}\right) X_t + \alpha t w_t \right\}$$

$I = \int_0^t f(t)\, dw_t$ (Itos integral)

$dI = f\, dw_t$

$Y = f(t, x)$

$dY = f_t\, dt + f_x\, dx + \frac{1}{2} f_{xx}\, dx\, dx$

where $dx\,dx$ can be interpreted by using identies

$dt\,dt = 0$, $dt\,dw = 0$ $dw_t\,dw_t = dt$

B.S diffusion eqn soln :

$X(t) = X_0\, e^{(\mu - \frac{a^2}{2})t + a w_t}$ $Y = (\mu - \frac{a^2}{2})t + a w_t$

$dx(t) = X_0 e^{Y}$

$dY = (\mu - \frac{a^2}{2})\, dt + a\, dw_t$

$dY\, dY = a^2\, dt$

~~dx~~ ~~x = f~~

putting

$dx = \cancel{f_t\, dt} + f_x\, dx + \frac{1}{2} f_{xx}\, dx\, dx$

$dx = X_0 e^{Y} dY + X_0 e^{Y} dY\, dY$

$= X_0 e^{Y}\left[(\mu - \frac{a^2}{2})dt + a\, dw_t\right] + X_0 e^{Y} a^2 dt$

$X_0 e^{(\mu - \frac{a^2}{2})t + a dW_t} \left[(\mu - \frac{a^2}{2}) dt + a\, dW_t + a^2 dt \right]$

$dX_{(t)} = X_0\, e^{(\mu - a^2/2)t + a W_t} \left[(\mu + \frac{a^2}{2}) dt + a^2 dW_t \right]$

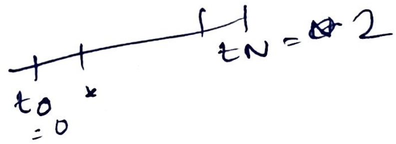
$+ X_0$

In principle it is difficult to obtain closed form analytical soln's for SDEs, therefore one has to seek numerical approximate soln for SDEs.

[Seidel 3rd chp]

$h = 1/N \qquad h = t_{n+1} - t_n$

$$\begin{cases} y'(t) = f(t, y) ; & t \in (0,1) \\ y(0) = \alpha \end{cases}$$



$$\int_{t_n}^{t_{n+1}} y'(s)\, ds = \int_{t_n}^{t_{n+1}} f(s, y(s))\, ds$$

$$\Rightarrow y_{n+1} - y_n = f(t_n, y_n) h$$

Euler Maryama

$$\begin{cases} dX(t) = a(t, X)\, dt + b(t, X)\, dW_t , & t \in (0, T] \\ X(0) = X_0 \end{cases}$$

$h = T/N$

Integrate b/w t_n to t_{n+1}.

$$\int_{t_n}^{t_{n+1}} dX(s) = \int_{t_n}^{t_{n+1}} \cancel{dX(s)}\, a(s, X(s))\, ds + \int_{t_n}^{t_{n+1}} b(s, X(s))\, dW_s$$

$$\boxed{X(t_{n+1}) - X(t_n) = a(t_n, X_n)\, h + b(t_n, X_n)\, \Delta W_{n+1}}$$

$Z_i \in N(0, 1)$

$\Delta W_{in} = Z_i \sqrt{\Delta t}$

randn()

<u>Euler</u> T.E $= O(h)$

(LHS - RHS)

~~T.E Euler Maryama~~

$\Delta W = O(\sqrt{t})$

Because of the jammed B.M the order of Euler Maryama scheme gets diminished by ½.

Next goal: $O(\sqrt{h}) \xrightarrow{?} O(h)$