Analisi Matematica - 15.3.2019 - prima parte

Thursday, March 14, 2019 16:17

Proprietà dei limiti di successioni

<u>Teorema</u>: Se $a_n \to \ell \in \mathbb{R}$ allora $\{a_n\}$ è limitata.

<u>Dim</u>: Uso la def. di lim. con $\varepsilon = 1$, e quindi esiste $n_0 \in \mathbb{N}$ t.c.

$$\forall n \geq n_0 \quad \ell - 1 \leq a_n \leq \ell + 1$$

Ma allora definisco

$$M = \max\{a_0, a_1, \dots, a_{n_0-1}, \ell+1\}$$
$$m = \min\{a_0, a_1, \dots, a_{n_0-1}, \ell-1\}$$

e ottengo che

$$m \leq a_n \leq M \quad \forall n \in \mathbb{N}$$

■

Quindi

$$\text{conv} \Rightarrow \text{limitata}$$
$$\not\Leftarrow \qquad a_n = (-1)^n$$ è limitata ma non convergente

Teorema (della permanenza del segno) Sia $\{a_n\}$ una successione. Allora

- Se $a_n \to \ell \in \mathbb{R}$, $\ell > 0$ allora $a_n > 0$ definit.
- Se $a_n \to \ell \in \mathbb{R}$, $\ell < 0$ allora $a_n < 0$ definit.

<u>Dim</u>: Supp. che $a_n \to \ell \in \mathbb{R}$, $\ell > 0$. Uso la def. di limite con $\varepsilon = \frac{\ell}{2} > 0$ e ho quindi che

$$0 < \frac{\ell}{2} = \ell - \frac{\ell}{2} \leq a_n \leq \ell + \frac{\ell}{2} \qquad \text{definit.}$$

quindi

$$0 < a_n \quad \text{definit.}$$

■

Teorema (analogo al prec.) $\{a_n\}$
- Se $a_n \to +\infty$ allora $a_n > 0$ def.
- Se $a_n \to -\infty$ allora $a_n < 0$ def.

Dim: Uso la def. di lim. con $M = 10$

$a_n \to +\infty \Rightarrow$

$$a_n \geq 10 > 0 \quad \text{defin.}$$

oss: Se $l = 0$ non si può dire nulla

$a_n = \frac{1}{n}$ $\quad a_n \to 0$, $\quad a_n > 0$

$a_n = -\frac{1}{n}$ $\quad a_n \to 0$, $\quad a_n < 0$

$a_n = \frac{(-1)^n}{n}$ $\quad a_n \to 0$, oscilla tra val. > 0 e val. < 0

Conseguenza del teo. precedente:

Teo: $\{a_n\}$ $a \in \mathbb{R}$

$$\left.\begin{array}{l} a_n \to a \\ a_n \geq 0 \text{ def unit.} \end{array}\right\} \Rightarrow a \geq 0$$

Dim: Se per assurdo $a < 0$ allora per il teo della perm. del segno

$$a_n < 0 \text{ def.} \quad !! \text{ Assurdo } \quad a_n \geq 0 \text{ def.}$$

Teoremi di confronto

Caso a due successioni

Teorema: $\{a_n\}, \{b_n\}$ succ tali che

$$a_n \leq b_n \text{ definitivamente} \quad (*)$$

Allora

$a_n \to +\infty \Rightarrow b_n \to +\infty$
$b_n \to -\infty \Rightarrow a_n \to -\infty$

<u>Dim</u>: Hp: $a_n \to +\infty$ Th: $b_n \to +\infty$
Prendo $n_0 \in \mathbb{N}$ t.c.

$$a_n \leq b_n \quad \forall n \geq n_0. \quad \text{(esiste per } (*)\text{)}$$

<u>Fisso $M \in \mathbb{R}$</u>, poiché $a_n \to +\infty$
$\exists n_1 \in \mathbb{N} : \quad a_n \geq M \quad \forall n \geq n_1.$

Ho allora $\forall n \geq \max\{n_0, n_1\}$

$$\underbrace{b_n \geq a_n}_{n \geq n_0} \underbrace{\geq M}_{n \geq n_1}$$

quindi

$$\underline{b_n \geq M \quad \text{def.}}$$

∎

OSS: Non valgono altre implicazioni oltre a quelle di sopra. $a_n \leq b_n$

Se $a_n \to -\infty$ b_n può fare quello che vuole.
Se $b_n \to +\infty$ a_n " " " " ".

Caso a tre successioni
<u>Teorema</u> Siano $\{a_n\}, \{b_n\}, \{c_n\}$ succ. tali che
(i) $a_n \leq b_n \leq c_n$ definit.
(ii) $a_n \to \ell \in \mathbb{R}$, $c_n \to \ell \in \mathbb{R}$ (stesso ℓ)

Allora anche $b_n \to \ell$.
(In particolare è di tipo 1)
<u>Dim</u>: Tesi: $\forall \varepsilon > 0 \quad \ell - \varepsilon \leq b_n \leq \ell + \varepsilon$ defin.

Sia $\varepsilon > 0$

(i) $\Rightarrow$

$$\exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \quad \underbrace{a_n \leq b_n \leq c_n}_{2.}$$

(ii) $a_n \to \ell \Rightarrow$

$$\exists n_a \in \mathbb{N} \ \forall n \geq n_a \quad \underbrace{\ell - \varepsilon \leq a_n}_{1.} \leq \ell + \varepsilon$$

(ii), $c_n \to \ell \Rightarrow$

$$\exists n_c \in \mathbb{N} \ \forall n \geq n_c \quad \ell - \varepsilon \leq \underbrace{c_n \leq \ell + \varepsilon}_{3.}$$

Ho allora $\forall n \geq \underline{\max\{n_0, n_a, n_c\}}$ valgono 1. 2. 3. quindi

$$\ell - \varepsilon \underset{1.}{\leq} a_n \underset{2.}{\leq} b_n \underset{2.}{\leq} c_n \underset{3.}{\leq} \ell + \varepsilon$$

quindi

$$\underline{\ell - \varepsilon \leq b_n \leq \ell + \varepsilon}$$

■

<u>OSS</u>: Se a_n e c_n non convergono o convergono a limiti diversi non è detto che b_n converga:

$$-1 \leq (-1)^n \leq 1 \quad \forall n$$
$$a_n \quad b_n \quad c_n$$

$a_n \to -1$, $c_n \to 1$ ma b_n è tipo 4.

OSS Ogni succ. costante $a_n = c$ converge a c

$\forall \varepsilon > 0 \quad |a_n - c| < \varepsilon$
$$\overset{\parallel}{0}$$

Applicazione: $\lim_{n \to +\infty} 2^n = +\infty$.

- Si può usare la def. di lim.

- Dim. più semplice

  Provo che $n \leq 2^n$ def.

  (facile per induzione)

  So che $a_n = n \to +\infty$ allora anche

  $b_n = 2^n \to +\infty$ (teo. di confronto)

Analisi Matematica - 15.3.2019 - seconda parte

Successioni monotone

Queste succ. non sono mai irregolari.

Def: Una succ. $\{a_n\}_{n \geq n_0}$ si dice

- **monotona crescente** se
$$a_n \leq a_{n+1} \quad \forall n \geq n_0$$
- **strettamente crescente** se
$$a_n < a_{n+1} \quad \forall n \geq n_0$$
- **monotona decrescente** se
$$a_n \geq a_{n+1} \quad \forall n \geq n_0$$
- **strettamente decrescente** se
$$a_n > a_{n+1} \quad \forall n \geq n_0$$

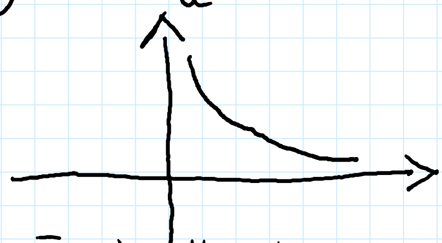
OSS: Se a_n è del tipo $a_n = f(n)$ $n \geq n_0$, ha lo stesso tipo di monotonia di f

Per esempio f crescente $\Rightarrow \{a_n\}$ crescente

$$n < n+1 \underset{f \text{ crescente}}{\Rightarrow} f(n) \leq f(n+1) \Rightarrow a_n \leq a_{n+1}$$

- $\{n^3\}$ $\quad f(x) = x^3$ strett. crescente
  $n^3 = f(n)$ $\quad \{n^3\}$ strett. cresc.

- $\{\frac{1}{n}\}$ $\quad \frac{1}{n} = f(n)$ $\quad n \geq 1$ ove $f(x) = \frac{1}{x}$ $\quad x > 0$



f è strett. decrescente

$\{\frac{1}{n}\}$ è strett. decrescente

Succ. strett. crescenti: $\{\log n\}, \{e^n\}, \{e^n\}$
$\{n^2\}$

Succ. non monotona: $\{(-1)^n\}$

<u>Teorema</u>: Sia $\{a_n\}$ una succ. monotona. Allora sono possibili due tipi di comportamento
(a) Se $\{a_n\}$ è non. crescente (se strett. crescente OK)
$a_n \to +\infty$ oppure $a_n \to \ell \in \mathbb{R}$
e in entrambi i casi il limite coincide con
$\sup\{a_n \mid n \in \mathbb{N}\}$ (event. $+\infty$)

(b) Se $\{a_n\}$ è non decrescente (se strett. decrescente OK)
$a_n \to -\infty$ oppure $a_n \to \ell \in \mathbb{R}$
e in entrambi i casi il limite coincide con
$\inf\{a_n \mid n \in \mathbb{N}\}$. (event. $-\infty$)

Applicazione
Il numero di Nepero (il numero e)
Consideriamo la successione

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad n \geq 1$$

<u>Teorema</u>: La succ. $\{e_n\}_{n \geq 1}$ verifica

1. $2 \leq e_n \leq 3 \quad \forall n \geq 1$
2. $\{e_n\}$ è crescente

Dim:
- $e_n \geq 2 \quad \forall n \geq 1$

Bernoulli: $(1+x)^n \geq 1 + nx \quad \forall x > -1$
$\forall n \in \mathbb{N}$

$\forall n \in \mathbb{N}$

$$e_n = \left(1+\underset{\substack{\parallel \\ x > -1}}{\frac{1}{n}}\right)^n \geq 1 + n \cdot \frac{1}{n} = 1+1 = 2$$

- $e_n \leq 3 \quad \forall n \geq 1$

Binomio di Newton:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

$$e_n = \left(1+\frac{1}{n}\right)^n = 1 + \sum_{k=1}^{n} \binom{n}{k} \frac{1}{n^k} \qquad (1)$$

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} \to k \text{ fattori tutti} \leq n$$

$$\{n-i \leq n\}$$

$$\leq \frac{n^k}{k!}$$

Da (1)

$$e_n \leq 1 + \sum_{k=1}^{n} \frac{\cancel{n^k}}{k!} \cdot \frac{1}{\cancel{n^k}} = 1 + \sum_{k=1}^{n} \frac{1}{k!}$$

$$\leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} =$$

$$\frac{1}{k!} \leq \frac{1}{2^{k-1}}$$

(per induzione)

$2^{k-1} \leq k!$

$$= 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$$

$$\leq 1 + \frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{\frac{1}{2}} = 1+2 = 3$$

$$2^{k-1} \leq k! \quad \forall k \geq 1$$

$k=1 \qquad 1 \leq 1$

$K \qquad 2^{K-1} \le K!$ Hp.

$K+1 \qquad 2^{K} \le (K+1)!$ Tes.

$$2^K = 2 \cdot 2^{K-1} \underset{Hp}{\le} 2 \cdot K! \le (K+1)K! = (K+1)!$$

- $\{e_n\}$ è crescente cioè $\forall n \ge 2 \quad e_n \ge e_{n-1}$

$$e_n \ge e_{n-1}$$

$$\Longleftrightarrow \quad \left(1+\frac{1}{n}\right)^n \ge \left(1+\frac{1}{n-1}\right)^{n-1} \qquad \left\{ a^{n-1} = \frac{a^n}{a} \right\}$$

$$\Longleftrightarrow \quad \left(\frac{n+1}{n}\right)^n \ge \left(\frac{n}{n-1}\right)^{n-1}$$

$$\Longleftrightarrow \quad \left(\frac{n+1}{n}\right)^n \ge \left(\frac{n}{n-1}\right)^{n} \frac{n-1}{n}$$

$$\Longleftrightarrow \quad \left(\frac{n+1}{n}\right)^n \cdot \left(\frac{n-1}{n}\right)^{n} \ge \frac{n-1}{n}$$

$$\Longleftrightarrow \quad \left(\frac{n^2-1}{n^2}\right)^n \ge \frac{n-1}{n}$$

$$\Longleftrightarrow \quad \left(1-\frac{1}{n^2}\right)^n \ge 1-\frac{1}{n}$$ vera per Bernoulli:

$n \ge 2 \qquad x = -\frac{1}{n^2} > -1$

$$\left(1-\frac{1}{n^2}\right)^n \ge 1 + n\left(-\frac{1}{n^2}\right) = 1 - \frac{1}{n}$$

$\blacksquare$

Conseguenza: $\{e_n\}$ o tende a $+\infty$ oppure tende ad $e \in \mathbb{R}$ (per il teorema sulle succ. crescenti)

Ma $2 \le e_n \le 3 \Rightarrow e_n \to e \in \mathbb{R}$

Tale limite si chiama numero di Nepero e si indica con e

$$e = \sup\{e_n \mid n \geq 1\}$$

$$2 \leq e_n \leq e \leq 3$$

$$e_n \leq 3 \ \forall n \geq 1$$

Si prova che $e \in \mathbb{R} \setminus \mathbb{Q}$ (irrazionale)

$$e = 2.718\ldots$$

Analisi Matematica - 15.3.2019 - terza parte

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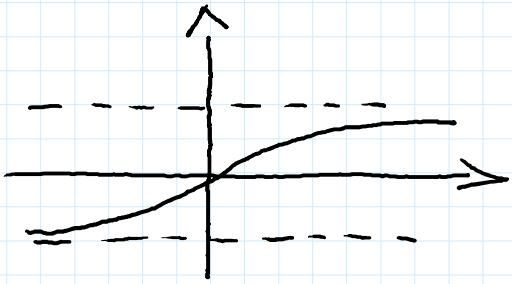
Dominio di funzioni

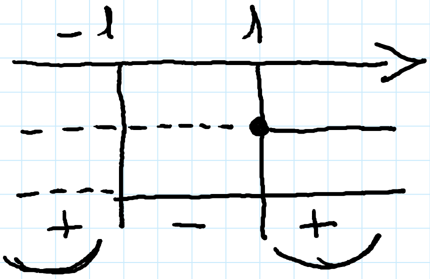
- $f(x) = \sqrt{\operatorname{arctg}\left(\frac{x-1}{x+1}\right)}$

$$\begin{cases} x+1 \neq 0 \\ \operatorname{arctg}\frac{x-1}{x+1} \geq 0 \end{cases}$$



$$\begin{cases} x+1 \neq 0 \qquad x \neq -1 \\ \frac{x-1}{x+1} \geq 0 \end{cases}$$

$$\frac{x-1}{x+1} \geq 0 \qquad \begin{matrix} x-1 \geq 0 & x \geq 1 \\ x+1 > 0 & x > -1 \end{matrix}$$

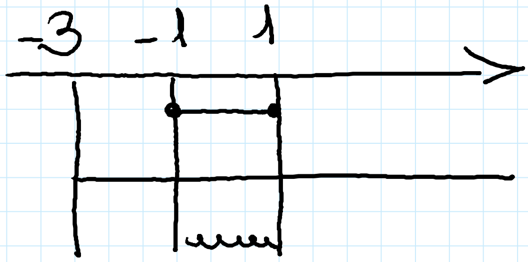


$$\operatorname{dom} f = (-\infty, -1) \cup [1, +\infty)$$

- $f(x) = \frac{\operatorname{arcsen} x}{\sqrt{x+3}}$

$$\begin{cases} -1 \leq x \leq 1 \\ \sqrt{x+3} \neq 0 \\ x+3 \geq 0 \end{cases} \quad x+3>0 \qquad \begin{cases} -1 \leq x \leq 1 \\ x+3 > 0 \end{cases}$$

$$\begin{cases} -1 \leq x \leq 1 \\ x > -3 \end{cases}$$



$$\operatorname{dom} f = [-1, 1]$$

- $f(x) = \frac{2^x - 3}{x^2+x-2}$

$x^2+x-2 \neq 0 \qquad x \neq -2, \; x \neq 1$

$x^2 + x - 2 \neq 0 \qquad x \neq -2,\ x \neq 1$

$$x = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \begin{cases} -2 \\ 1 \end{cases}$$

$\text{dom } f = \mathbb{R} \setminus \{-2, 1\}$

• $f(x) = \dfrac{1}{\sqrt{\log_{\frac{1}{2}} x^2 - 2}}$

1. $\sqrt{\log_{\frac{1}{2}} x^2 - 2} \neq 0 \quad \Rightarrow \quad \log_{\frac{1}{2}} x^2 - 2 > 0$

   $\log_{\frac{1}{2}} x^2 - 2 \geq 0$

2. $x^2 > 0$

$$\begin{cases} x^2 > 0 \\ \log_{\frac{1}{2}} x^2 - 2 > 0 \end{cases}$$

• $x^2 > 0 \qquad x \neq 0$

• $\log_{\frac{1}{2}} x^2 - 2 > 0$

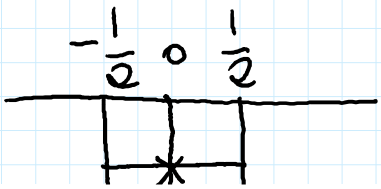
$\log_{\frac{1}{2}} x^2 > 2 = \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^2 \Rightarrow \log_{\frac{1}{2}}$ strett. decr.

$x^2 < \left(\frac{1}{2}\right)^2$

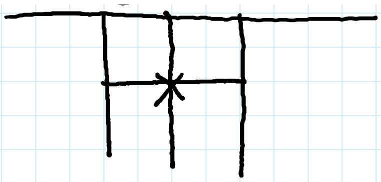
$x^2 - \left(\frac{1}{2}\right)^2 < 0$

$-\frac{1}{2} < x < \frac{1}{2}$

$$\begin{cases} x \neq 0 \\ -\frac{1}{2} < x < \frac{1}{2} \end{cases}$$



$$\begin{cases} x \neq 0 \\ -\frac{1}{2} < x < \frac{1}{2} \end{cases}$$



$$\text{dom} f = \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$$

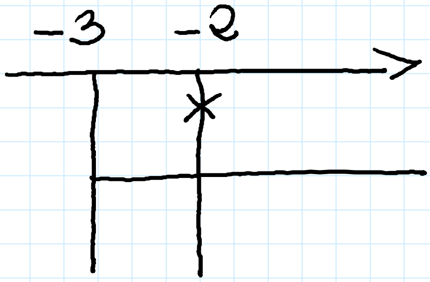
[Altro modo: $\log_{\frac{1}{2}} x^2 = 2 \log_{\frac{1}{2}} |x|$

$2 \log_{\frac{1}{2}} |x| - 2 > 0 \iff \log_{\frac{1}{2}} |x| > 1 = \log_{\frac{1}{2}} \frac{1}{2} \iff |x| < \frac{1}{2}$

$\iff -\frac{1}{2} < x < 1$]

- $f(x) = \dfrac{x^2 + x}{\log_2 (x+3)}$

$$\begin{cases} \log_2 (x+3) \neq 0 \\ x + 3 > 0 \end{cases} \qquad \begin{cases} x \neq -2 \\ x > -3 \end{cases}$$

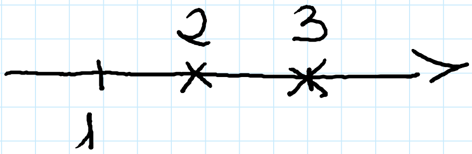


$\log_2 (x+3) \neq 0$
$x + 3 \neq 1$
$x = -3 + 1 \neq -2$

$$\text{dom} f = (-3, -2) \cup (-2, +\infty)$$

- $f(x) = \dfrac{\log_{1/2} (x-1)}{(x-2)(x-3)}$

$$\begin{cases} x - 1 > 0 \\ x - 2 \neq 0 \\ x - 3 \neq 0 \end{cases} \qquad \begin{cases} x > 1 \\ x \neq 2 \\ x \neq 3 \end{cases}$$



$$\text{dom} f = (1, 2) \cup (2, 3) \cup (3, +\infty) = (1, +\infty) \setminus \{2, 3\}$$

- $f(x) = \sqrt{e^x - 1} + \text{arcsen} \dfrac{x}{2}$

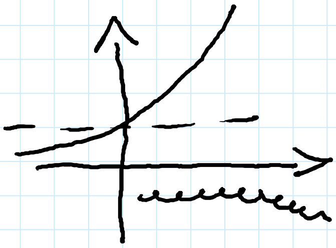
- $f(x) = \sqrt{e^x - 1} + \arcsin \frac{x}{2}$

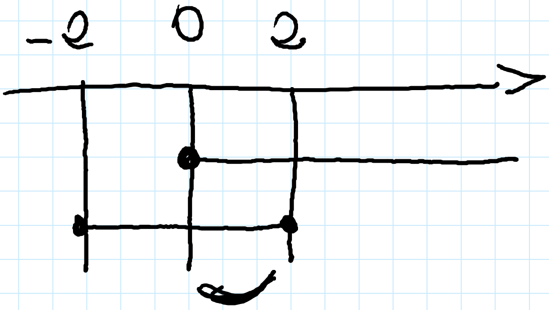
$$\begin{cases} e^x - 1 \geq 0 \\ -1 \leq \frac{x}{2} \leq 1 \end{cases} \qquad \begin{cases} x \geq 0 \\ -2 \leq x \leq 2 \end{cases}$$

$e^x - 1 \geq 0$

$e^x \geq 1 = e^0 \iff x \geq 0$

e^x strett. cresc.

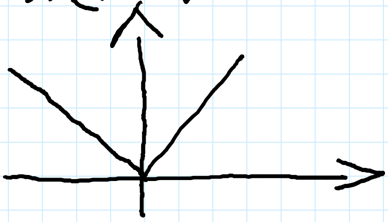




$\text{dom } f = [0, 2]$

- $f(x) = \sqrt{\log_{\frac{1}{2}} |x+2|}$

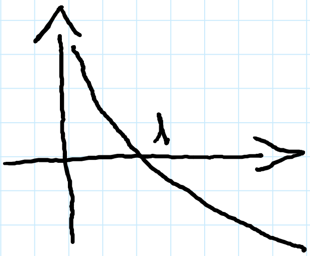
$$\begin{cases} 1.\ |x+2| > 0 \\ 2.\ \log_{\frac{1}{2}} |x+2| \geq 0 \end{cases} \qquad \begin{cases} x \neq -2 \\ -3 \leq x \leq -1 \end{cases}$$



1. $|x+2| > 0$

   $x + 2 \neq 0$

   $x \neq -2$



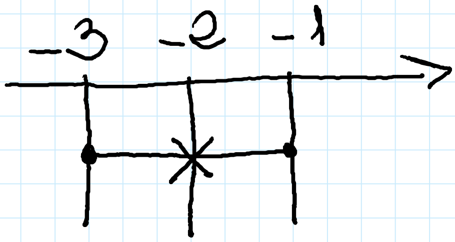
2. $\log_{\frac{1}{2}} |x+2| \geq 0 = \log_{\frac{1}{2}} 1 \iff$

   $|x+2| \leq 1$

   $-1 \leq x + 2 \leq 1$

   $-1 - 2 \leq x \leq -2 + 1$

   $-3 \leq x \leq -1$



$\text{dom } f = [-3, -2) \cup (-2, -1]$

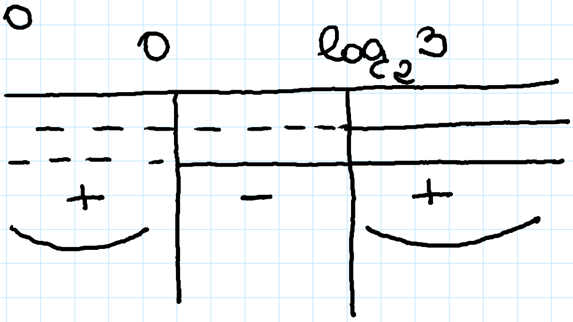
- $f(x) = \log_2 \left(\dfrac{2^x - 3}{\operatorname{arctg} x} \right)$

$\operatorname{arctg} x \neq 0 \qquad x \neq 0$

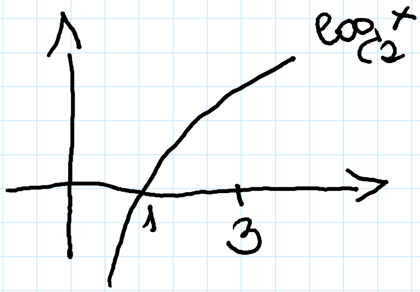
$\dfrac{2^x - 3}{\operatorname{arctg} x} > 0$

N $\quad 2^x - 3 > 0 \qquad 2^x > 3 = 2^{\log_2 3} \qquad x > \log_2 3$

D $\quad \operatorname{arctg} x > 0 \qquad x > 0$



$\operatorname{dom} f = (-\infty, 0) \cup (\log_2 3, +\infty)$



- $f(x) = \sqrt{\dfrac{x^2 - 3x + 2}{(x+6)(x^2+1)}}$

$(x+6)\underbrace{(x^2+1)}_{\to \Delta < 0} \neq 0 \qquad x + 6 \neq 0 \qquad x \neq -6$

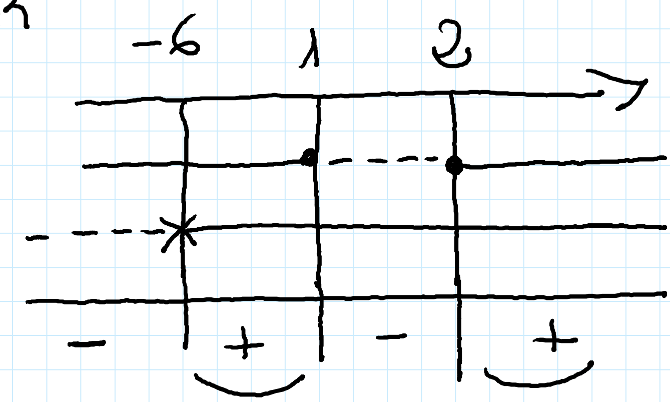
$\dfrac{x^2 - 3x + 2}{(x+6)(x^2+1)} \geq 0$

- $x^2 - 3x + 2 \geq 0 \qquad x \leq 1 \qquad x \geq 2$

$x = \underline{3 \pm \sqrt{9 - 8}} = \underline{3 \pm 1} \nearrow 2$

$$x = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} \; \begin{matrix} \nearrow 2 \\ \searrow 1 \end{matrix}$$

- $x+6>0 \qquad x>-6$
- $x^2+1>0 \qquad \forall x \in \mathbb{R}$



$$\text{dom } f = (-6, 1] \cup [2, +\infty)$$