Analisi Matematica - 9.5.2019 - prima parte

Relazioni tra "o piccolo", "O grande", "asintotico".

Per $x \to x_0$

$f(x) \sim g(x) \iff f(x) = \omega(x) \cdot g(x) \quad \omega(x) \to 1$

$f(x) = o(g(x)) \iff f(x) = \omega(x) \cdot g(x) \quad \omega(x) \to 0$

$f(x) = O(g(x)) \iff f(x) = \omega(x) g(x) \quad \omega(x)$ limitata def per $x \to x_0$

Proprietà:

$o(g) \pm o(g) = o(g)$

$o(f) \cdot o(g) = o(fg)$

$\alpha\, o(g) = o(g) \quad \alpha \in \mathbb{R}$

$o(\alpha g) = o(g)$

1. $f \sim g \iff f = g + o(g)$
2. $f = o(g) \implies f = O(g)$
3. $f \sim g \implies f = O(g)$

Esempi: VERO o FALSO?

- $\sin x \sim x$ VERO $x \to 0$

  $\sin x = x \cdot \underbrace{\frac{\sin x}{x}}_{\to 1}$

- $\sin x = O(x)$ VERO

  prop. 3

- $\sin x^2 = o(x)$ $x \to 0$ VERO

  $\sin x^2 = x \cdot \underbrace{\frac{\sin x^2}{x}}_{\omega(x) \sim \frac{x^2}{x} = x \to 0}$

$$\omega(x) \sim \frac{x^2}{x} = x \to 0$$

- $\operatorname{sen} x^2 = O(x)$ VERO

propr. 2.

- $\operatorname{sen} x^2 = O(x^2)$ $x \to 0$ VERA

$$\operatorname{sen} x^2 = x^2 \cdot \underbrace{\boxed{\frac{\operatorname{sen} x^2}{x^2}}}_{\to 1}$$

Di più:

$$\operatorname{sen} x^2 \sim x^2 \quad x \to 0$$

- $\operatorname{sen} x^2 = o(x^2)$ FALSO

$$\operatorname{sen} x^2 = x^2 \cdot \underbrace{\boxed{\frac{\operatorname{sen} x^2}{x^2}}}_{\to 1}$$

- $\operatorname{sen} x^2 = x^2 + o(x^2)$ $x \to 0$ VERO

$\operatorname{sen} x^2 \sim x^2$ + prop. 1.

- Per $x \to 0$

$$f(x) = O(x^a),\ a > 0 \Rightarrow f(x) = o(x^b)\ \forall b < a$$

$f(x) = \omega(x) \cdot x^a$ $\quad \omega(x)$ lim. def. $x \to 0$

$$f(x) = \omega(x) \cdot \frac{x^a}{x^b} \cdot x^b = \underbrace{\omega(x)\, x^{a-b}}_{\omega_1(x)} \cdot x^b$$

$$\omega_1(x) = \omega(x) \cdot x^{a-b} \qquad a - b > 0$$

$\omega(x) \to$ lim. per $x \to 0$; $\quad x^{a-b} \to 0$ $\quad x \to 0$

$$\Rightarrow \omega_1(x) \to 0 \Rightarrow f(x) = o(x^b) \quad x \to 0$$

- Se $b > a$ Non si può dire nulla

- Studiare per $x \to 0$ $x - \operatorname{sen} x = f(x)$

$f(x) \to 0$ $x \to 0$

$f(x) = x - x + o(x) = o(x)$ $\operatorname{sen} x = x + o(x)$

$\Rightarrow f(x) = o(x^b)$ $\forall b \in (0, 1]$

Se $a > 1$? Non so, non ho abbastanza info.

$$f(x) = x - \left(x - \frac{x^3}{3!} + o(x^3)\right)$$

$$= \frac{x^3}{3!} + o(x^3) = \frac{x^3}{3!} + o\left(\frac{x^3}{3!}\right) \qquad o(\alpha g) = o(g)$$

$f = g + o(g)$

$$\Rightarrow f(x) \sim \frac{x^3}{3!} = \frac{x^3}{6} \qquad x \to 0$$

$$f(x) = O(x^3) \Rightarrow f(x) = o(x^b) \ \forall b < 3$$

- $$\lim_{x \to 0} \frac{x e^{-x^2} - \operatorname{sen} x + \frac{5}{6} x^3}{(x + 2x^2)^2 \log^3\left(1 + \frac{x}{2}\right)} = ? \qquad \left[\frac{0}{0}\right]$$

Numeratore: $x \to 0$

$x e^{-x^2} = x + o(x)$

$\sim x$

$\operatorname{sen} x = x + o(x)$

$$NUM = \cancel{x} + o(x) - \cancel{x} + o(x) + \frac{5}{6} x^3$$

$$= \frac{5}{6} x^3 + o(x) = o(x)$$

$$\frac{5}{6} x^3 = x \cdot \underbrace{\frac{5}{6} x^2}_{\substack{\omega(x) \\ \downarrow \\ 0}}$$

$$DEN: \quad (x + 2x^2)^2 \log^3\left(1 + \frac{x}{2}\right) =$$

$$= \left[x(1 + 2x)\right]^2 \log^3\left(1 + \frac{x}{2}\right)$$

$$= x^2 \underbrace{(1 + 2x)^2}_{\downarrow 1} \log^3\left(1 + \frac{x}{2}\right)$$

$$= x^2 (1+2x)^{\overset{\downarrow}{2}}_{\to 1} \log_C^3\left(1+\frac{x}{2}\right)$$

$$\sim x^2 \cdot 1 \cdot \left(\frac{x}{2}\right)^3 = \frac{x^5}{8}$$

$$\ell = \lim_{x\to 0} \frac{o(x)}{\frac{x^5}{8}} \quad \text{non so}$$

Idea: sviluppare il num. almeno fino al 3° ordine

$$e^t = 1 + t + o(t) \quad t \to 0$$

$$-x^2 \to 0$$

$$e^{-x^2} = 1 - x^2 + o(-x^2) = 1 - x^2 + o(x^2)$$

$$x e^{-x^2} = x - x^3 + \underbrace{x\, o(x^2)}_{x \cdot \omega(x) \cdot x^2} = x - x^3 + o(x^3)$$

$$\omega(x) \to 0$$

$$\text{NUM} = \cancel{x} - \cancel{x^3} + o(x^3) - \left(\cancel{x} - \cancel{\frac{x^3}{3!}} + o(x^3)\right) + \cancel{\frac{5}{6}x^3} = o(x^3)$$

$$\text{coeff. di } x^3: \quad -1 + \frac{1}{6} + \frac{5}{6} = 0$$

$$\ell = \lim_{x\to 0} \frac{o(x^3)}{\frac{x^5}{8}} \quad \text{non so}$$

Si cancellano anche i termini del 3° ordine!

Devo sviluppare fino al 5° ordine:

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2) \quad t \to 0 \qquad \underbrace{x e^{-x^2}}$$

$$x^2 \to 0$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} + o(x^4) \quad x \to 0$$

$$x e^{-x^2} = x - x^3 + \frac{x^5}{}$$

$$xe^{-x^2} = x - x^3 + \frac{x^5}{2} + \underbrace{x\cdot o(x^4)}_{o(x^5)}$$

$NUM =$

$$\cancel{x} - \cancel{x^3} + \frac{x^5}{2} + o(x^5) \overbrace{- \cancel{x} + \cancel{\frac{x^3}{3!}} - \frac{x^5}{5!} + o(x^5)}^{-\sin x} + \cancel{\frac{5}{6}x^3}$$

$$= \left(\frac{1}{2} - \frac{1}{5!}\right)x^5 + o(x^5) = \frac{59}{120}x^5 + o(x^5)$$

$$\ell = \lim_{x\to 0} \frac{\frac{59}{120}x^5 + o(x^5)}{\frac{1}{8}x^5} = \lim_{x\to 0} \frac{\frac{59}{120}x^5}{\frac{1}{8}\cancel{x^5}}$$

$$= \frac{59}{120}\cdot 8 = \frac{59}{15}$$

Analisi Matematica - 9.5.2018 - seconda parte

- $f(x) = \dfrac{2x^2+7x+7}{e^x}$

Dominio: $e^x \neq 0 \quad \forall x \in \mathbb{R} \qquad \text{dom} f = \mathbb{R}$

$f(0) = \dfrac{7}{e^0} = 7 \qquad (0,7)$

$f(x)=0 \qquad 2x^2+7x+7=0 \qquad \Delta = 48-56 < 0$

non si annulla mai

$f(x)>0 \quad \forall x \in \mathbb{R} \quad : \quad \begin{cases} 2x^2+7x+7>0 & \forall x \in \mathbb{R} \\ e^x>0 \end{cases}$

Limiti: $\pm\infty$

$x \to -\infty \qquad 2x^2+7x+7 \sim 2x^2 \to +\infty$

$e^x \to 0$

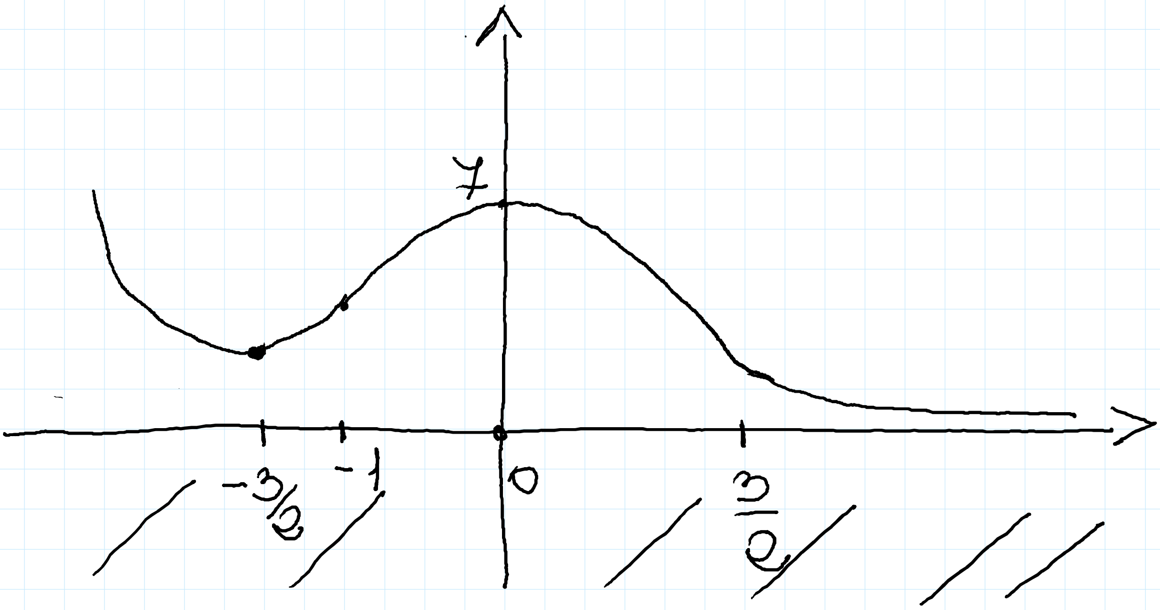
$\left[\dfrac{+\infty}{0^+}\right] \qquad f(x) \to +\infty$

$\dfrac{f(x)}{x} \sim \dfrac{2x^2}{e^x \cdot x} = \dfrac{2x}{e^x} \to -\infty$

$\left[\dfrac{-\infty}{0^+}\right]$

$x \to +\infty \qquad \dfrac{+\infty}{+\infty} \qquad f(x) \sim \dfrac{2x^2}{e^x} \to 0$

$y=0$ asintoto orizzontale $x \to +\infty$



Derivata prima:

$\forall x \in \mathbb{R}$

$$f'(x) = \frac{(4x+7)\cancel{e^x} - (2x^2+7x+7)\cancel{e^x}}{e^{\cancel{2x}\,x}}$$

$$= \frac{4x+\cancel{7}-2x^2-7x-\cancel{7}}{e^x} = \frac{-2x^2-3x}{e^x}$$

$$= \underset{<0}{-}\ \frac{2x^2+3x}{\underset{>0}{e^x}}$$

$$f'(x) \geq 0 \iff \underset{\substack{\parallel \\ x(2x+3)}}{2x^2+3x} \leq 0 \iff -\frac{3}{2} \leq x \leq 0$$

$x=0 \quad x=-\frac{3}{2}$

f'	$-\frac{3}{2}$		0	
	$---$	$++++$		$----$
	↘	↗		↘

$x = -\frac{3}{2}$ p.to di min. rel.

$x = 0$ p.to di max rel.

Derivata seconda

$$f''(x) = -\frac{(4x+3)\cancel{e^x} - (2x^2+3x)\cancel{e^x}}{e^{\cancel{2x}\,x}}$$

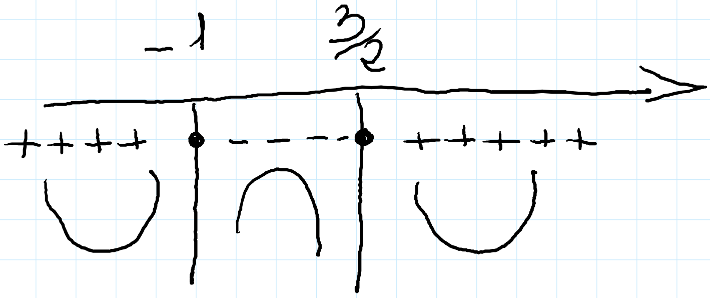
$$= -\frac{4x+3-2x^2-3x}{e^x} = -\frac{-2x^2+x+3}{e^x}$$

$$= \frac{2x^2-x-3}{\underbrace{e^x}_{>0}}$$

$$f''(x) \geq 0 \iff 2x^2-x-3 \geq 0 \iff x \leq -1 \quad x \geq \frac{3}{2}$$

$$x = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm 5}{4} \begin{cases} -1 \\ \frac{3}{2} \end{cases}$$

f'' :



$x = -1$, $x = \frac{3}{2}$ p.ti di flesso

- $\mathrm{Im}\, f = (0, +\infty)$
- $\inf_{\mathbb{R}} f = 0 \qquad \sup_{\mathbb{R}} f = +\infty$

$\downarrow$

non ha minimo

- $f(x) = \lambda$ ha

0 sol. se $\lambda \leq 0$,

1 sol se $0 < \lambda < f(-\frac{3}{2})$

2 sol. se $\lambda = f(-\frac{3}{2})$

3 sol. se $f(-\frac{3}{2}) < \lambda < f(0) = 4$

(1 sol se $\lambda > 4$

1 sol se $\lambda > f$

2 sol se $\lambda = f(0) = 1$

- $f(x) = \dfrac{x}{\log_e x - 2}$

$\log_e \rightarrow x > 0$

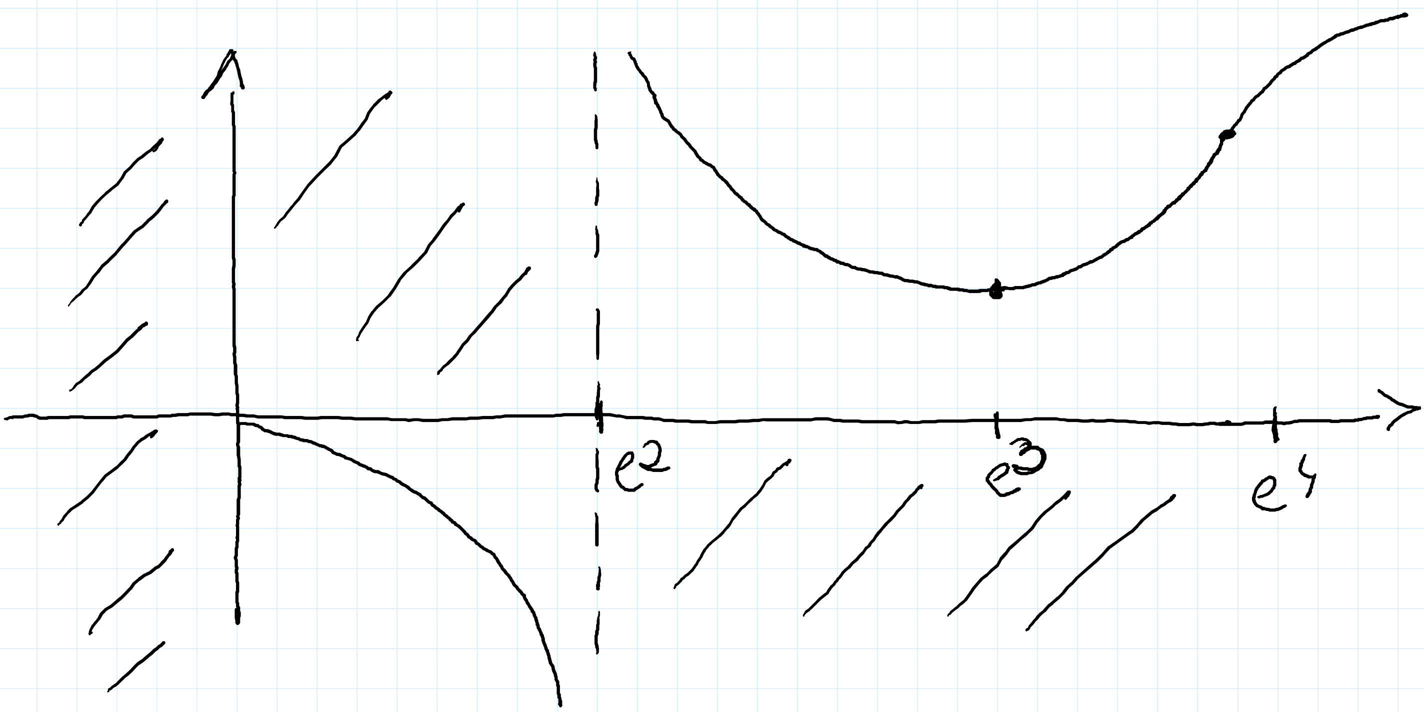
Den $\rightarrow \log_e x - 2 \neq 0 \qquad \log_e x \neq 2 = \log_e e^2$

$x \neq e^2$

$\text{dom } f = (0, e^2) \cup (e^2, +\infty)$

$0 \notin \text{dom } f, \quad f(x) = 0 \quad x = 0 \notin \text{dom } f$

$f(x) > 0 \underset{x>0}{\iff} \log_e x - 2 > 0 \iff \log_e x > 2 \iff x > e^2$



Limiti: $\quad 0, \; e^2, \; +\infty$

$x \rightarrow 0 \quad \left[\dfrac{0}{-\infty}\right] \quad f(x) \rightarrow 0$

$x \rightarrow e^2 \quad \left[e^2 \right.$

$\left[-\infty\right]$

$x \to e^2 \quad \left[\dfrac{e^2}{0}\right]$

$x \to (e^2)^+ \quad \left[\dfrac{e^2}{0^+}\right] \quad f(x) \to +\infty \qquad x = e^2$ asintoto verticale

$x \to (e^2)^- \quad \left[\dfrac{e^2}{0^-}\right] \quad f(x) \to -\infty$

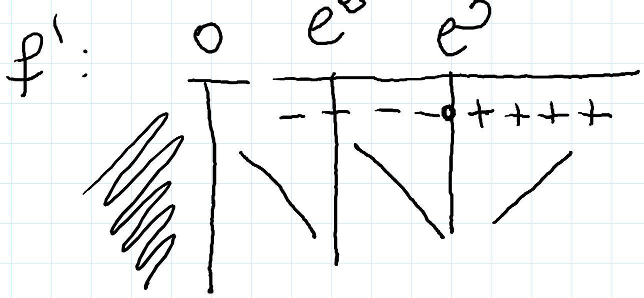
$x \to +\infty \quad \left[\dfrac{+\infty}{+\infty}\right] \quad f(x) \sim \dfrac{x}{\log_e x} \to +\infty$

$\dfrac{f(x)}{x} \sim \dfrac{1}{\log_e x} \to 0$ No as. obliquo

Derivata prima:

$$f'(x) = \frac{\log_e x - 2 - \cancel{x}\,\frac{1}{\cancel{x}}}{(\log_e x - 2)^2} = \frac{\log_e x - 3}{(\log_e x - 2)^2}$$

$f'(x) \geq 0 \quad \log_e x - 3 \geq 0 \quad \log_e x \geq 3 \quad x \geq e^3$



$x = e^3$ pto di min rel.

Derivata seconda:

$$f''(x) = \frac{\frac{1}{x}(\log_e x - 2)^{\cancel{2}} - (\log_e x - 3)\,2\,\cancel{(\log_e x - 2)} \cdot \frac{1}{x}}{(\log_e x - 2)^{\cancel{4}\,3}}$$

$$= \frac{1}{x}\,\frac{\log_e x - 2 - 2\log_e x + 6}{(\log_e x - 2)^3}$$

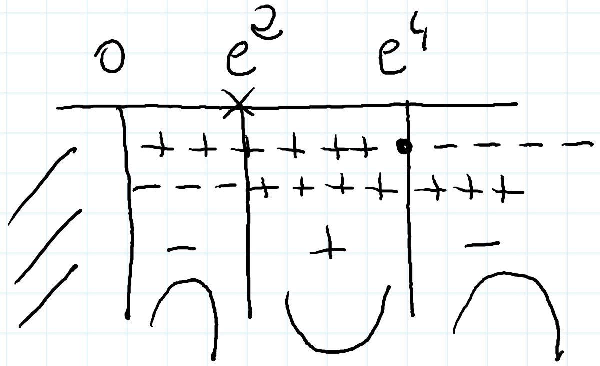
$$(\log_e x - 2)^3$$

$$= \frac{1}{x} \; \frac{-\log_e x + 4}{(\log_e x - 2)^3}$$

$f''(x) \geq 0 \qquad 4 - \log_e x \geq 0 \qquad \log_e x \leq 4 \qquad x \leq e^4$

$\qquad\qquad\qquad\; \log_e x - 2 > 0 \qquad \log_e x > 2 \qquad x > e^2$

f''



$x = e^4$ p.to di flesso

- $\inf f = -\infty$, $\sup f = +\infty$

$\operatorname{Im} f = (-\infty, 0) \cup (f(e^3), +\infty)$