Analisi Matematica - 22.5.2019 - prima parte

Integrazione per parti:

$$\int_a^b f(x)g'(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

- $$\int_0^1 x e^x \,dx = \int_0^1 x De^x \,dx = \Big[x e^x\Big]_0^1 - \int_0^1 e^x \cdot 1 \,dx$$
$$= e - \Big[e^x\Big]_0^1 = e - (e-1) = 1$$

- $$\int x^2 e^x \,dx = \int x^2 De^x \,dx =$$
$$= e^x x^2 - 2\int \underset{=De^x}{e^x} \cdot x \,dx$$
$$= x^2 e^x - 2\Big[e^x \cdot x - \int e^x dx\Big]$$
$$= x^2 e^x - 2x e^x + 2e^x + c \qquad c \in \mathbb{R}$$

- $$\int \log x \,dx = \int 1 \cdot \log x \,dx = \int Dx \cdot \log x \,dx$$
$$= x \log x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \,dx$$
$$= x \log x - x + c \qquad c \in \mathbb{R}$$

- $$\int \operatorname{arctg} x \,dx = \int Dx \cdot \operatorname{arctg} x \,dx$$
$$= x \operatorname{arctg} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx \quad = \int \frac{f'}{f}\,dx$$
$$= x \operatorname{arctg} x - \frac{1}{2}\log(1+x^2) + c$$

- $$\int x^m \log x \,dx = \int D\frac{x^{m+1}}{} \cdot \log x \,dx$$

- $\int x^n \log x \, dx = \int D \frac{x^{n+1}}{n+1} \cdot \log x \, dx$

$$= \frac{x^{n+1}}{n+1} \cdot \log x - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} \, dx$$

$$= \frac{1}{n+1} x^{n+1} \log x - \frac{1}{n+1} \int x^n \, dx$$

$$= \frac{1}{n+1} x^{n+1} \log x - \frac{1}{(n+1)^2} x^{n+1} + c$$

- $\int_0^1 \log(1+x) dx = \int_0^1 Dx \cdot \log(1+x) dx$

$$= \Big[x \log(1+x)\Big]_0^1 - \int \frac{x \, + 1 \cancel{+} 1}{1+x} dx \qquad \int \frac{f'}{f} \quad f'=1, \; f=1+x$$

$$= \Big[x \log(1+x)\Big]_0^1 - \int_0^1 \left(1 - \frac{1}{1+x}\right) dx$$

$$= \Big[x \log(1+x)\Big]_0^1 - \Big[x - \log|1+x|\Big]_0^1$$

$$= \log 2 - 1 + \log 2 = 2 \log 2 - 1$$

Integrazione di funzioni razionali fratte

Calcolo di

$$\int \frac{P_n(x)}{Q_m(x)} dx$$

ove P_n è un polinomio di grado n e Q_m è un polinomio di grado m.

Regola: se $n \geq m$ eseguire la divisione tra polinomi.

$$I = \int \frac{x^3 + x}{x^2 + x + 1} dx$$

$$I = \int \frac{x^3+x}{x^2+x+1}\,dx$$

$$\begin{array}{l|l} x^3 \qquad +x & x^2+x+1 \\ \hline -x^3-x^2-x & x-1 \\ \hline // \; -x^2 \; // & \\ \quad +x^2+x+1 & \\ \hline \quad // \quad x+1 & \end{array}$$

$x+1 \rightarrow$ Mi fermo poiché $\deg(x+1) < \deg(x^2+x+1)$

$$I = \int \left(x - 1 + \underbrace{\frac{x+1}{x^2+x+1}}_{?} \right) dx$$

In generale $\dfrac{P_n(x)}{Q_m(x)} = \underbrace{S(x)}_{\text{polinomio quoziente}} + \dfrac{R(x)}{Q_m(x)}$ $\quad R(x) \to$ resto

Supponiamo che $n < m$. Distinguiamo vari casi:

1) $m = 1$

$$I = \int \frac{c}{ax+b}\,dx = c \int \frac{dx}{ax+b}$$

$$= \frac{c}{a} \int \underbrace{\frac{a}{ax+b}}_{\hookrightarrow \frac{f'}{f}}\,dx = \frac{c}{a} \log|ax+b| + K \qquad K \in \mathbb{R}$$

$$\int \frac{2}{3x+5}\,dx = \frac{2}{3} \int \frac{3}{3x+5}\,dx = \frac{2}{3} \log|3x+5| + c$$

2) … $I = \int$ … $dx + e$ …

2) $n = 2 \quad I = \int \frac{}{ax^2+bx+c}\,dx$

$\Delta = b^2 - 4ac$

2a) $\Delta > 0$

$$I = \int \frac{x+2}{x^2+x-6}\,dx \qquad x^2+x-6 = (x-2)(x+3)$$

$$x = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} \begin{matrix} \nearrow -3 \\ \searrow 2 \end{matrix}$$

Si scompone la funzione integranda:

$$\frac{x+2}{x^2+x-6} = \frac{a}{x-2} + \frac{b}{x+3}$$

a b da determinare

$$= \frac{ax+3a+bx-2b}{(x-2)(x+3)} \Rightarrow$$

$$x+2 = (a+b)x + 3a - 2b \iff$$

$$\begin{cases} a+b = 1 \\ 3a-2b = 2 \end{cases} \quad \begin{cases} a = 1-b \\ 3-3b-2b = 2 \end{cases} \quad \begin{cases} // \\ -5b = -1 \end{cases}$$

$$\begin{cases} a = 1 - \frac{1}{5} = \frac{4}{5} \\ b = \frac{1}{5} \end{cases}$$

$$I = \int \left(\frac{4}{5} \cdot \frac{1}{x-2} + \frac{1}{5}\,\frac{1}{x+3} \right) dx$$

$$= \frac{4}{5} \int \frac{1}{x-2}\,dx + \frac{1}{5} \int \frac{dx}{x+3}$$

$$= \frac{4}{5} \log_e |x-2| + \frac{1}{5} \log_e |x+3| + c$$

2b) $\Delta = 0 \qquad ax^2+bx+c = a(x-x_0)^2$

$$I = \int \frac{x}{x^2+2x+1}\,dx$$

$x^2+2x+1 = (x+1)^2$

Pongo $\varphi(x) = x+1$ e determino f tale che

$f(\varphi(x)) = \dfrac{x}{(x+1)^2} \qquad \varphi'(x) = 1$

$$I = \int f(\varphi(x))\cdot\varphi'(x)\,dx$$

Det. f: $\quad \dfrac{x}{(x+1)^2} = \dfrac{(x+1)-1}{(x+1)^2} = \dfrac{1}{x+1} - \dfrac{1}{(x+1)^2}$

$f(t) = \dfrac{1}{t} - \dfrac{1}{t^2}$

$$I \underset{\substack{\Downarrow \\ \text{int. per sostituzione}}}{=} \left[\int \left(\frac{1}{t} - \underbrace{\frac{1}{t^2}}_{=t^{-2}}\right)dt\right]_{t=\varphi(x)=x+1}$$

$$= \left[\log|t| - \frac{t^{-2+1}}{-2+1}\right]_{t=x+1} + c$$

$$= \log|x+1| + \frac{1}{x+1} + c$$

- $I = \displaystyle\int \frac{x+1}{(3x+2)^2}\,dx$

$\varphi(x) = 3x+2 \qquad \varphi'(x) = 3$

Determino f tale che $f(\varphi(x)) = \dfrac{x+1}{(3x+2)^2}$

$x+1 \quad = 1 \quad 3x+3 \quad = 1 \quad (3x+2)+1$

$$\frac{x+1}{(3x+2)^2} = \frac{1}{3}\,\frac{3x+3}{(3x+2)^2} = \frac{1}{3}\,\frac{(3x+2)+1}{(3x+2)^2} =$$

$$= \frac{1}{3}\left(\frac{1}{3x+2} + \frac{1}{(3x+2)^2}\right)$$

$$f(t) = \frac{1}{3}\left(\frac{1}{t} + \frac{1}{t^2}\right)$$

$$I = \frac{1}{3}\int \overset{=\varphi'(x)}{3}\, f(\varphi(x))\,dx = \frac{1}{3}\int f(\varphi(x))\varphi'(x)\,dx$$

$$\underset{\text{sost.}}{=} \frac{1}{3}\left[\int \frac{1}{3}\left(\frac{1}{t} + \frac{1}{t^2}\right)dt\right]_{t=3x+2}$$

$$= \frac{1}{9}\left[\log_e|t| - \frac{1}{t}\right]_{t=3x+2} + c$$

$$= \frac{1}{9}\left(\log_e|3x+2| - \frac{1}{3x+2}\right) + c$$

2c) $\Delta < 0$

Idea: $\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + c$

$\Delta < 0$

$$I = \int \frac{dx}{3+x^2} = \frac{1}{3}\int \frac{dx}{1+\frac{x^2}{3}} = \frac{1}{3}\int \frac{dx}{1+\left(\frac{x}{\sqrt{3}}\right)^2}$$

$3 = \sqrt{3}\cdot\sqrt{3}$

$$= \frac{1}{\sqrt{3}}\int \frac{\frac{1}{\sqrt{3}}}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx$$

$\varphi(x) = \frac{x}{\sqrt{3}}$

$\varphi'(x) = \frac{1}{\sqrt{3}}$

$$= \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x}{\sqrt{3}} + c$$

$f(t) = \frac{1}{1+t^2}$

In generale: se $a>0$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + c$$

Caso generale:

- $I = \int \frac{x}{x^2+2x+4}\,dx \quad \to \Delta = 4-16$

Al numeratore si fa comparire la derivata del denominatore: $2x+2$

$$x = \frac{1}{2}\,2x = \frac{1}{2}(2x+2-2) = \frac{1}{2}(2x+2) - 1$$

Il denominatore si scrive come somma di quadrati, (completamento del quadrato)

$$x^2 + 2x + 4 = x^2 + 2\cdot 1\,x + 1 - 1 + 4 =$$
$$= (x+1)^2 + 3$$

$$= (x+1)^2 + 3$$

$$I = \int \frac{\frac{1}{2}(2x+2) - 1}{x^2+2x+4}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+4}\,dx \; f'/f \; - \int \frac{1}{3+(x+1)^2}\,dx$$

$$= \frac{1}{2}\log(x^2+2x+4) - \frac{\cancel{1}\sqrt{3}}{3}\int \frac{1 \cdot \frac{1}{\sqrt{3}}}{1+\left(\frac{x+1}{\sqrt{3}}\right)^2}\,dx \qquad 3 = \sqrt{3}\cdot\sqrt{3}$$

$$= \frac{1}{2}\log\left(x^2+2x+4\right) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x+1}{\sqrt{3}} + c$$

- $$I = \int \frac{1-2x}{x^2+2x+5}\,dx \qquad \Delta = 4 - 20$$

$$\frac{1-2x}{x^2+2x+5} = -\frac{2x-1}{x^2+2x+5} = -\frac{2x+2-2-1}{x^2+2x+5}$$

$$= -\frac{2x+2}{x^2+2x+5} + \frac{3}{x^2+2x+5}$$

$$x^2+2x+5 = x^2+2x+1-1+5 = (x+1)^2+4$$

$$I = -\int \frac{2x+2}{x^2+2x+5}\,dx + 3\int \frac{1}{4+(x+1)^2}\,dx$$

$$= -\log\left(x^2+2x+5\right) + \frac{3\cdot 2}{4}\int \frac{1 \cdot \frac{1}{2}}{1+\left(\frac{x+1}{2}\right)^2}\,dx$$

$$= -\log(x^2+2x+5) + \frac{3}{2}\operatorname{arctg}\frac{x+1}{2} + c$$

$$\bullet\ I = \int \operatorname{arctg} \frac{x}{x-2}\, dx = \int Dx \cdot \operatorname{arctg} \frac{x}{x-2}\, dx$$

$$= x \operatorname{arctg} \frac{x}{x-2} - \int x \frac{1}{1+\left(\frac{x}{x-2}\right)^2} \cdot \frac{\cancel{x}-2-\cancel{x}}{(x-2)^2}\, dx$$

$$= x \operatorname{arctg} \frac{x}{x-2} + 2\int x \frac{\cancel{(x-2)^2}}{(x-2)^2+x^2} \cdot \frac{1}{\cancel{(x-2)^2}}\, dx$$

$$\left[\frac{1}{1+\frac{x^2}{(x-2)^2}} = \frac{(x-2)^2}{(x-2)^2+x^2} \right]$$

$$= x \operatorname{arctg} \frac{x}{x-2} + 2\int \frac{x}{2x^2-4x+4}\, dx$$

$$= x \operatorname{arctg} \frac{x}{x-2} + \int \frac{x}{x^2-2x+2}\, dx$$

$$I_1 = \int \frac{x}{x^2-2x+2}\, dx$$

$$\frac{x}{x^2-2x+2} = \frac{1}{2} \frac{(2x-2)+2}{x^2-2x+2} = \frac{1}{2} \frac{2x-2}{x^2-2x+2} + \frac{1}{x^2-2x+2}$$

$$x^2-2\cdot x+2 = (x^2-2x+1)+1 = (x-1)^2+1$$

$$I_1 = \frac{1}{2}\int \frac{2x-2}{x^2-2x+2}\, dx + \int \frac{dx}{1+(x-1)^2}$$

$$= \frac{1}{2} \log(x^2-2x+2) + \operatorname{arctg}(x-1) + c$$

3) $m > 2$ $\quad I = \int \frac{P_n(x)}{Q_m(x)}\, dx$

- 2. scomporre Q_m in fattori di primo e secondo grado irriducibili
- 2. scomporre $\frac{P_n}{Q_m}$ in somma di frazioni più semplici

$$I = \int \frac{dx}{x^2(x+3)}$$

$$\frac{1}{x^2(x+3)} = \frac{ax+b}{x^2} + \frac{c}{x+3}$$

Regola: i numeratori hanno grado inferiore di 1 rispetto al proprio denominatore

$$\frac{1}{x^2(x+3)} = \frac{ax^2+3ax+bx+3b+cx^2}{x^2(x+3)}$$

$$\begin{cases} a+c=0 \\ 3a+b=0 \\ 3b=1 \end{cases} \qquad \begin{cases} c=-a=\frac{1}{9} \\ a=-b/3=-\frac{1}{9} \\ b=\frac{1}{3} \end{cases}$$

$$I = \int \left(\frac{-\frac{1}{9}x+\frac{1}{3}}{x^2} + \frac{1}{9}\,\frac{1}{x+3} \right) dx$$

$$= -\frac{1}{9}\int \frac{dx}{x} + \frac{1}{3}\int \frac{1}{x^2}\,dx + \frac{1}{9}\int \frac{dx}{x+3}$$

$$= -\frac{1}{9}\log_e|x| - \frac{1}{3}\,\frac{1}{x} + \frac{1}{9}\log_e|x+3| + c$$