Serie di potenze

- $\sum_{n=0}^{\infty} (\sqrt{n})^n x^n$

$a_n = (\sqrt{n})^n = n^{n/2} > 0$

$\rho = \lim_{n\to+\infty} \sqrt[n]{|a_n|} = \lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \left(n^{n/2}\right)^{\frac{1}{n}}$

$= \lim_{n\to+\infty} n^{\frac{1}{2}} = +\infty$

$R = 0$ L'ins. di conv. è $I = \{0\}$

- $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{2^n} (x-2)^n$

$x_0 = 2$ $a_n = \frac{1}{n} \cdot \frac{1}{2^n} > 0$

$\sqrt[n]{|a_n|} = \sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n}} \cdot \frac{1}{2} \to \frac{1}{1 \cdot 2} = \frac{1}{2} = \rho$

$R = 2$

La serie conv. assolutamente se $-2 < x-2 < 2$
$0 < x < 4$

La serie non converge se $x < 0$ o $x > 4$

$x=0$, $x=4$ da studiare

$x=0$: $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{\cancel{2^n}} \underbrace{(-2)^n}_{=(-1)^n \cdot \cancel{2^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

- Non conv. assolutamente, ma converge per Leibnitz
$b_n = \frac{1}{n}$

$x = 4$ $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{\cancel{2^n}} \cancel{2^n} = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$

Insieme di conv. assoluta: $(0,4)$
" " convergenza $[0,4)$

- $\sum_{n=0}^{\infty} n^2 x^n$ $a_n = n^2$ $x_0 = 0$

$\frac{|a_{n+1}|}{|a_n|} = \frac{1}{(n+1)^2} \cdot n^2 \sim \frac{n^2}{n^2} = 1$ $\rho = 1$

$R = 1$

La serie conv. ass. se $-1 < x < 1$ $(-1, 1)$

" " non converge se $x < -1$ o $x > 1$

$x = -1$, $x = 1$

$x = -1$ $\sum_{k=0}^{\infty} k^2 (-1)^k$

$k^2 \to +\infty \Rightarrow (-1)^k k^2$ non può tendere a 0

$\Rightarrow$ non converge

$x = 1$ $\sum_{k=0}^{\infty} k^2 = +\infty$ $b_k = k^2 \to +\infty \Rightarrow$
$\sum b_k$ non converge $\overset{b_k \geq 0}{\Longrightarrow}$
$\sum b_k = +\infty$

Integrale indefinito

<u>Def</u>: Sia $f : [a,b] \to \mathbb{R}$. L'insieme i cui elementi sono le (eventuali) primitive di f si chiama integrale indefinito di f e si denota con

$$\int f(x)\,dx .$$

oss: $\int f(x)\,dx$ o è l'insieme vuoto o ha infiniti elementi.

- $$\underbrace{\int_a^b f(x)\,dx}_{\text{numero}} \neq \underbrace{\int f(x)\,dx}_{\text{insieme}}$$

- Poiché le primitive di f, se esistono, differiscono per una costante, si scrive

$$\int f(x)\,dx = G(x) + c \quad \text{se } G' = f$$
(G primitiva di f)

invece che

$$\int f(x)\,dx = \{ G(x) + c \,/\, c \in \mathbb{R} \}$$

- $\int x\,dx = \frac{x^2}{2} + c$ $c \in \mathbb{R}$

Integrali indefiniti immediati: $c \in \mathbb{R}$

- $\int K\,dx = Kx + c \qquad K \in \mathbb{R}$
- $\int x^p\,dx = \frac{1}{p+1}x^{p+1} + c \qquad p \in \mathbb{R} \setminus \{-1\}$
- $\int \frac{1}{x}\,dx = \log|x| + c \qquad x<0 \text{ o } x>0$
- $\int a^x\,dx = \frac{a^x}{\log a} + c$
- $\int e^x\,dx = e^x + c$
- $\int \operatorname{sen} x\,dx = -\cos x + c$
- $\int \cos x\,dx = \operatorname{sen} x + c$
- $\int \frac{1}{\cos^2 x}\,dx = \int (1+\operatorname{tg}^2 x)\,dx = \operatorname{tg} x + c$
- $\int \frac{dx}{1+x^2} = \operatorname{arctg} x + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \operatorname{arcsen} x + c$

Integrazione per scomposizione

f, g continue, $\alpha, \beta \in \mathbb{R}$

$\left.\begin{array}{l} F \text{ prim. di } f \\ G \text{ prim. di } g \end{array}\right\} \Rightarrow \alpha F + \beta G \text{ è prim. di } \alpha f + \beta g$

$[(\alpha F + \beta G)' = \alpha F' + \beta G' = \alpha f + \beta g]$

$$\boxed{\int (\alpha f + \beta g)(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx}$$

Si usa per l'integrazione dei polinomi

$$\int (4x^2 + 5x + 1)\,dx = 4\int x^2\,dx + 5\int x\,dx + \int 1\,dx$$
$$= 4\,\frac{x^{2+1}}{2+1} + 5\,\frac{x^{1+1}}{1+1} + 1 \cdot x + c$$

$$= \underbrace{\frac{4}{3}x^3 + 5\frac{x^2}{2} + x}_{G} + c$$

$$\int_0^1 (4x^2+5x+1)\,dx = \left[\frac{4}{3}x^3 + \frac{5}{2}x^2 + x\right]_0^1$$
$$= \frac{4}{3} + \frac{5}{2} + 1$$

Integrazione per sostituzione

$\varphi : [a,b] \to \mathbb{R}$ derivabile, φ' continua e $\varphi([a,b]) \subseteq I$
I, intervallo. Se $f : I \to \mathbb{R}$ continua
($f \circ \varphi : [a,b] \to \mathbb{R}$ è ben definita ed è continua)
Allora

$$\boxed{\int f(\varphi(x))\varphi'(x)\,dx = \left[\int f(t)\,dt\right]_{t=\varphi(x)}}$$

Se G è una prim. di $f \Rightarrow G \circ \varphi$ è prim. di $(f\circ\varphi)\cdot\varphi'$

$$\frac{d}{dx}(G\circ\varphi)(x) = G'(\varphi(x))\cdot\varphi'(x) = f(\varphi(x))\varphi'(x)$$

• $$I = \int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{\overbrace{e^x}^{=\varphi'(x)}}{\underbrace{1+(e^x)^2}_{=f(\varphi(x))}}\,dx$$

$\varphi(x) = e^x \qquad \varphi'(x) = e^x$

$$f(t) = \frac{1}{1+t^2} \Rightarrow f(\varphi(x)) = \frac{1}{1+(e^x)^2}$$

$$I = \left[\int \frac{1}{1+t^2}\,dt\right]_{t=e^x} = \left[\operatorname{arctg} t + c\right]_{t=e^x}$$
$$= \operatorname{arctg} e^x + c$$

Integrale definito per sostituzione

$$\boxed{\int_a^b f(\varphi(x))\varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx}$$

G prim. di f

$$\int_a^b f(\varphi(x))\varphi'(x)\,dx = \left[G(\varphi(x))\right]_a^b$$
$$= G(\varphi(b)) - G(\varphi(a))$$
$$\varphi(b)$$

$$= \int_{\varphi(a)} f(x)\,dx$$

- $I = \int_0^{\pi/2} \underset{\overset{\shortparallel}{D\,\mathrm{sen}\,x}}{\cos x} \cdot \mathrm{sen}^2 x \, dx =$

$\varphi'(x) = \cos x \qquad \varphi(x) = \mathrm{sen}\,x \qquad f(t) = t^2$

$f(\varphi(x)) = (\mathrm{sen}\,x)^2 = \mathrm{sen}^2 x$

$\varphi(0) = \mathrm{sen}\,0 = 0 \qquad \varphi\left(\frac{\pi}{2}\right) = \mathrm{sen}\frac{\pi}{2} = 1$

$$I = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

Altra possibilità: calcolare prima

$$\int \underset{\varphi'}{\underset{\downarrow}{\cos x}} \cdot \underset{f(\varphi(x))}{\underset{\downarrow}{\mathrm{sen}^2 x}}\,dx = \left[\int t^2\,dt\right]_{t=\mathrm{sen}\,x}$$

$$= \left[\frac{t^3}{3}\right]_{t=\mathrm{sen}\,x} = \frac{1}{3}\mathrm{sen}^3 x$$

$$I = \left[\frac{1}{3}\mathrm{sen}^3 x\right]_0^{\pi/2} = \frac{1}{3} - 0 = \frac{1}{3}$$

- $I = \int \frac{dx}{x \log_e x}$

$\varphi'(x) = \frac{1}{x} \qquad \varphi(x) = \log_e x \qquad f(t) = \frac{1}{t} \implies$

$$I = \int \varphi'(x) f(\varphi(x))\,dx$$

$$I = \left[\int \frac{1}{t}\,dt\right]_{t=\log_e x} = \left[\log_e|t| + c\right]_{t=\log_e x}$$

$$= \log_e|\log_e x| + c$$

- $I = \int_0^2 \frac{dx}{\sqrt{x}\,(x+1)} = 2\int_0^2 \frac{dx}{2\sqrt{x}\,((\sqrt{x})^2+1)}$

$\varphi(x) = \sqrt{x} \qquad \varphi'(x) = \frac{1}{2\sqrt{x}} \qquad f(t) = \frac{1}{t^2+1}$

$\varphi(0) = 0$
$\varphi(2) = \sqrt{2}$

$$I = 2 \int_0^{\sqrt{2}} \frac{1}{1+x^2}\, dx = 2 \left[\operatorname{arctg} x \right]_0^{\sqrt{2}}$$
$$= 2 \operatorname{arctg} \sqrt{2} - \cancel{2 \operatorname{arctg} 0}$$

- $$I = \int \operatorname{tg} x \, dx = -\int \frac{-\operatorname{sen} x}{\cos x}\, dx$$

$\varphi'(x) = -\operatorname{sen} x \qquad \varphi(x) = \cos x$
$f(t) = \frac{1}{t}$

$$I = -\left[\int \frac{1}{t}\, dt \right]_{t=\cos x} = -\left[\log |t| + c \right]_{t=\cos x}$$
$$= -\log|\cos x| + c$$

In generale

$$\int \frac{f'(x)}{f(x)}\, dx = \log |f(x)| + c$$

$$\int f(x)^{\alpha} f'(x)\, dx = \left[\int t^{\alpha}\, dt \right]_{t=f(x)} \qquad \alpha \neq -1$$
$$= \left[\frac{t^{\alpha+1}}{\alpha+1} \right]_{t=f(x)}$$
$$= \frac{1}{\alpha+1} f(x)^{\alpha+1} + c$$

- $$I = \int x^2 \sqrt{2+x^3}\, dx = \frac{1}{3} \int 3x^2 \cdot \sqrt{2+x^3}\, dx$$

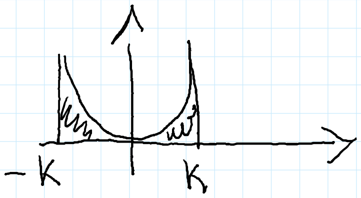
$$\left[3x^2 = \varphi'(x) \quad \varphi(x) = 2+x^3 \quad f(t) = \sqrt{t} = t^{\frac{1}{2}} \right]$$

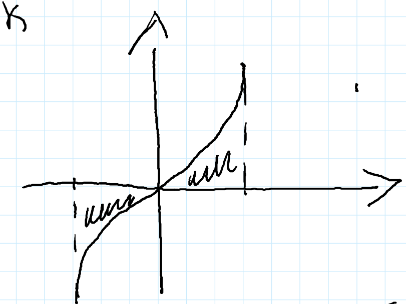
$$= \frac{1}{3} \left[\int \sqrt{t}\, dt \right]_{t=2+x^3}$$
$$= \frac{1}{3} \left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_{t=2+x^3} = \frac{1}{3} \frac{(2+x^3)^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \frac{2}{9} (2+x^3)^{3/2} + c$$

Simmetrie

$f: [-K, K] \to \mathbb{R}$ integrabile

- Se f è pari $\displaystyle\int_{-K}^{K} f(x) = 2\int_{0}^{K} f(x)\,dx$
- Se f è dispari $\displaystyle\int_{-K}^{K} f(x)\,dx = 0$





- $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_{0}^{\pi/2} \cos x\,dx = 2\left[\operatorname{sen} x\right]_{0}^{\pi/2} = 2\cdot 1 = 2$
- $\displaystyle\int_{-1}^{1} \underbrace{e^{|x|}}_{\text{pari}}\,dx = 2\int_{0}^{1} e^{\overset{= x \text{ perché } x\in[0,1]}{|x|}}\,dx = 2\int_{0}^{1} e^{x}\,dx$

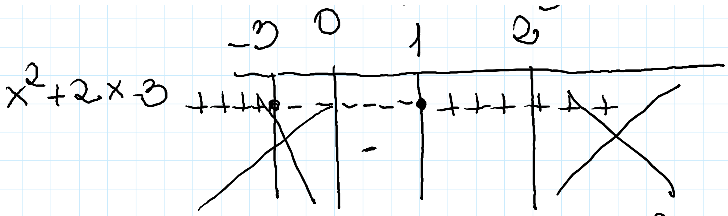
$$= 2\left[e^{x}\right]_{0}^{1} = 2(e-1)$$

$$\int_{-1}^{1} e^{|x|}\,dx = \underbrace{\int_{-1}^{0} e^{-x}\,dx}_{\text{sost.}} + \underbrace{\int_{0}^{1} e^{x}\,dx}_{\text{immediato}}$$

- $\displaystyle\int_{-\pi/2}^{\pi/2} \operatorname{sen} x\,dx = 0$ ↳ dispari
- $\displaystyle\int_{-1}^{1} \frac{x^3}{1+x^2}\,dx = 0$ ↳ dispari
- $I = \displaystyle\int_{0}^{2} |\underbrace{x^2+2x-3}_{=(x-1)(x+3)}|$

$-3 \quad 0 \quad 1 \quad 2$

x^2+2x-3



$$I = \int_0^1 (-x^2-2x+3)\,dx + \int_1^2 (x^2+2x-3)\,dx$$

$$= \left[-\frac{x^3}{3} - \cancel{2}\,\frac{x^2}{\cancel{2}} + 3x\right]_0^1 + \left[\frac{x^3}{3} + 2\,\frac{x^2}{2} - 3x\right]_1^2$$

Integrazione per parti

f, g derivabili in $[a,b]$, con derivata continua

$$(f g)' = f' \cdot g + f \cdot g'$$

$$f \cdot g' = (f \cdot g)' - f' g$$

Da cui

$$\boxed{\begin{aligned} \int f \cdot g' \, dx &= \int (f \cdot g)' dx - \int f' g \, dx \\ &= f \cdot g - \int f' g \, dx \end{aligned}}$$

Integrale definito :

$$\boxed{\begin{aligned} \int_a^b f(x) g'(x) \, dx &= \left[f \cdot g - \int f' g \, dx \right]_a^b \\ &= \left[f(x) \cdot g(x) \right]_a^b - \int_a^b f'(x) g(x) \, dx \end{aligned}}$$

- $$\begin{aligned} \int x \operatorname{sen} x \, dx &= \int \overset{f}{x} \cdot D(\overset{g}{-\cos x}) \, dx \\ &= -x \cos x - \int 1 \cdot (-\cos x) \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \operatorname{sen} x + c \quad c \in \mathbb{R} \end{aligned}$$

Serie armonica generalizzata e integrali

$$\sum_{n=1}^{\infty} \frac{1}{n^a} \quad \text{se } a > 1, \quad \text{è convergente}$$

Lo ho dim. usando la dis.

$$S_n \leq 1 + \frac{1}{a-1}\left(1 - \frac{1}{n^{a-1}}\right) \quad \forall n \geq 1 \quad (D)$$

ove $S_n = 1 + \frac{1}{2^a} + \dots + \frac{1}{n^a} = \sum_{k=1}^{n} \frac{1}{n^k}$

Perché è vera (D)?

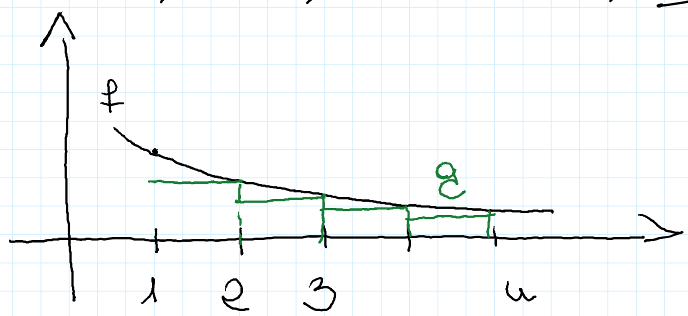
$$S_n = 1 + \sum_{k=2}^{n} \frac{1}{k^a}$$ quindi (D) è equivalente a

$$\sum_{k=2}^{n} \frac{1}{k^a} \leq \frac{1}{a-1}\left(1 - \frac{1}{n^{a-1}}\right) \quad \forall n \geq 2 \quad (D')$$

Dimostro (D'): fisso $n \geq 2$

$f : [1, n] \to \quad f(x) = \frac{1}{x^a}$

g : costante a tratti
$\forall k = 2, \dots, n, \ \forall x \in [k-1, k] \quad g(x) = \frac{1}{k^a}$



Poiché f è decrescente, $g(x) \leq f(x) \ \forall x \in [1, n]$, f e g sono integrabili in $[1, n]$ e per la proprietà di monotonia dell'integrale

$$\int_1^n g(x)\,dx \leq \int_1^n f(x)\,dx$$

$$\sum_{k=2}^{n} 1 \cdot \frac{1}{k^a} \leq \int_1^n x^{-a}\,dx$$

$$= \left[\frac{x^{-a+1}}{1-a}\right]_1^n$$

$$\left[\frac{\quad}{1-a} \right]_1$$

$$= \frac{1}{1-a}\left(u^{1-a} - 1 \right)$$

$$= \frac{1}{1-a}\left(\frac{1}{u^{a-1}} - 1 \right)$$

$$= \frac{1}{a-1}\left(1 - \frac{1}{u^{a-1}} \right)$$