Analisi Matematica - 27.5.2019 - prima parte

Integrazione per sostituzione

$$\int f(\varphi(x))\varphi'(x)\,dx = \left[\int f(t)\,dt\right]_{t=\varphi(x)}$$

Pongo $\varphi(x) = t$ "metto al posto di $\varphi(x)$ la var. t"

$\varphi'(x)\,dx = dt$ " " " " $\varphi'(x)dx$ dt"

$$I = \int \frac{e^x}{e^{2x}+2e^x+1}\,dx \qquad \begin{array}{l}\varphi(x) = e^x \\ \varphi'(x)dx = e^x dx\end{array}$$

$e^{2x} = (e^x)^2$

$$f(t) = \frac{1}{t^2+2t+1} = \frac{1}{(t+1)^2}$$

$$I = \left[\int \frac{1}{(t+1)^2}\,dt\right]_{t=e^x} = \left[\frac{(t+1)^{-2+1}}{-2+1}\right]_{t=e^x}$$

$$= \left[-\frac{1}{(t+1)}\right]_{t=e^x} = -\frac{1}{e^x+1} + c$$

$$I_d = \int_0^1 \frac{e^x}{e^{2x}+2e^x+1}\,dx = \int_1^e \frac{dt}{(t+1)^2} = \left[-\frac{1}{t+1}\right]_1^e$$

$\varphi(0) = 1$, $\varphi(1) = e$

- $$I = \int_1^{+\infty} \frac{\log x}{(x+1)^2}\,dx$$

$$I_{ind} = \int \frac{\log x}{(x+1)^2}\,dx = \int D\left(-\frac{1}{x+1}\right)\log x\,dx$$

$$= -\frac{1}{x+1}\cdot \log x + \int \frac{1}{x+1}\,\frac{1}{x}\,dx$$

$$= -\frac{1}{x+1}\cdot \log_e x + \int \frac{1}{x+1}\cdot\frac{1}{x}\,dx$$

$$\frac{1}{x(x+1)} = \frac{a}{x} + \frac{b}{x+1} = \frac{ax + a + bx}{x(x+1)}$$

$$\begin{cases} a+b=0 \\ a=1 \end{cases} \qquad \begin{cases} b=-a=-1 \\ a=1 \end{cases}$$

$$I_{ind} = -\frac{\log_e x}{x+1} + \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$$

$$= -\frac{\log_e x}{x+1} + \underbrace{\log_e|x| - \log_e|x+1|}_{} + C$$

$$I = \lim_{B\to+\infty}\int_1^B \frac{\log_e x}{(x+1)^2}\,dx =$$

$$= \lim_{B\to+\infty}\left[-\frac{\log_e x}{(x+1)} + \log_e\left|\frac{x}{x+1}\right|\right]_1^B$$

$$= \lim_{B\to+\infty}\left(-\frac{\log_e B}{B+1} + \log_e\frac{B}{B+1} - \log_e\frac{1}{2}\right)$$

$$\frac{\log_e B}{B+1} \sim \frac{\log_e B}{B} \to 0 \qquad \log_e\frac{B}{B+1} \to \log_e 1$$

$$= -\log_e\frac{1}{2} = \log_e 2$$

- $$\int_0^1 \log_e x\,dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^1 \log_e x\,dx =$$

$$= \lim_{\varepsilon\to0^+}\int_\varepsilon^1 Dx\cdot\log_e x\,dx = \lim_{\varepsilon\to0^+}\left[x\log_e x - \int\frac{x\,dx}{x}\right]_\varepsilon^1$$

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$\varepsilon \to 0^+$ $\quad \varepsilon \qquad \varepsilon \to 0^+ \qquad \varepsilon$

$$= \lim_{\varepsilon \to 0^+} \Big[x \log x - x \Big]_{\varepsilon}^{1} =$$

$$= \lim_{\varepsilon \to 0^+} \Big(-1 - \underbrace{\varepsilon \log \varepsilon}_{\to 0} + \underbrace{\varepsilon}_{\to 0} \Big) = -1$$

- $$I = \int \frac{\overbrace{dx}^{dt}}{\operatorname{tg}^4 x \, \cos^2 x} =$$

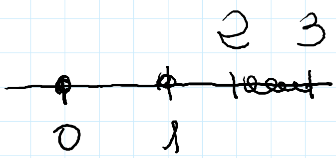
$$\varphi(x) = \operatorname{tg} x \qquad \varphi'(x) = \frac{1}{\cos^2 x}$$

$$f(t) = \frac{1}{t^4} \; : \; f(\varphi(x)) = \frac{1}{\operatorname{tg}^4 x}$$

$$I = \left[\int \frac{1}{t^4} \, dt \right]_{t = \operatorname{tg} x} = \left[\frac{t^{-4+1}}{-4+1} \right]_{t = \operatorname{tg} x}$$

$$= \left[-\frac{1}{3} \, \frac{1}{t^3} \right]_{t = \operatorname{tg} x} = -\frac{1}{3} \, \frac{1}{\operatorname{tg}^3 x} + c$$

- $$\int_2^3 \frac{1}{\sqrt{x}\,(x-1)} \, dx = I$$



$$I = 2 \int_2^3 \frac{1}{2\sqrt{x}\,(x-1)} \, dx = 2 \int_2^3 \frac{1}{2\sqrt{x}} \, \frac{1}{(\sqrt{x})^2 - 1} \, dx$$

$$\varphi(x) = \sqrt{x} \qquad \varphi'(x) = \frac{1}{2\sqrt{x}}$$

$$f(t) = \frac{1}{t^2 - 1} \qquad \begin{aligned} \varphi(2) &= \sqrt{2} \\ \varphi(3) &= \sqrt{3} \end{aligned}$$

$$I = 2 \int^{\sqrt{3}} \frac{dt}{\;}$$

$$\sqrt{2} \qquad t^2-1$$

$$\frac{1}{t^2-1} = \frac{a}{t-1} + \frac{b}{t+1} = \frac{at+a+bt-b}{t^2-1}$$

$$\begin{cases} a+b=0 \\ a-b=1 \\ \hline 2a+\cancel{\ } = 1 \end{cases} \longrightarrow \begin{cases} a = \frac{1}{2} \\ b = -a = -\frac{1}{2} \end{cases}$$

$$I = \cancel{2}\int_{\sqrt{2}}^{\sqrt{3}} \left(\frac{1}{\cancel{2}} \frac{1}{t-1} - \frac{1}{\cancel{2}} \frac{1}{t+1} \right) dt$$

$$= \Big[\log_e |t-1| - \log_e |t+1| \Big]_{\sqrt{2}}^{\sqrt{3}}$$

$$= \left[\log_e \left| \frac{t-1}{t+1} \right| \right]_{\sqrt{2}}^{\sqrt{3}}$$

$$= \log_e \frac{\sqrt{3}-1}{\sqrt{3}+1} - \log_e \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$$I = \int \frac{3x+1}{x^2+6x+12}\, dx$$

$$\Delta = 36 - 4\cdot 12 < 0$$

$$x^2+6x+12 = x^2+2\cdot 3x+9\underbrace{-9+12} = (x+3)^2+3$$

$$\frac{3x+1}{x^2+6x+12} = 3\,\frac{x+\frac{1}{3}}{x^2+6x+12} = \frac{3}{2}\,\frac{2x+\frac{2}{3}+6-6}{x^2+6x+12}$$

$$= \frac{3}{2}\,\frac{2x+6}{x^2+6x+12} + \frac{3}{2}\,\frac{-\frac{16}{3}}{(x+3)^2+3}$$

$$2 \qquad x^2+6x+12 \qquad 2 \qquad (x+3)^2+3$$

$$= \frac{3}{2}\,\frac{2x+6}{x^2+6x+12} - \frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{16}^{\,8}}{\cancel{3}}\,\frac{1}{(x+3)^2+3}$$

$$I = \frac{3}{2}\int\frac{2x+6}{x^2+6x+12}\,dx - 8\int\frac{dx}{(x+3)^2+3}$$

$$\int\frac{f'}{f}$$

$$= \frac{3}{2}\log_e(x^2+6x+12) - \frac{8\sqrt{3}}{\cancel{3}\sqrt{3}}\int\frac{dx\cdot\frac{1}{\sqrt{3}}}{1+\left(\frac{x+3}{\sqrt{3}}\right)^2}$$

$$= \frac{3}{2}\log_e(x^2+6x+12) - \frac{8}{\sqrt{3}}\operatorname{arctg}_e\frac{x+3}{\sqrt{3}} + c$$

↳ der. è $\frac{1}{\sqrt{3}}$

$f(x) = (x^2-3)e^{-x}$

dom $f = \mathbb{R}$

$f(0) = -3\cdot 1 = -3 \qquad (0,-3)$

$f(x) = 0 \iff x^2-3=0 \iff x = \pm\sqrt{3} \qquad (\pm\sqrt{3}, 0)$
$e^{-x} \neq 0$

$f(x) > 0 \iff x^2-3>0 \iff x<-\sqrt{3} \text{ o } x>\sqrt{3}$
$e^{-x} > 0$

Limiti agli estremi: $\pm\infty$

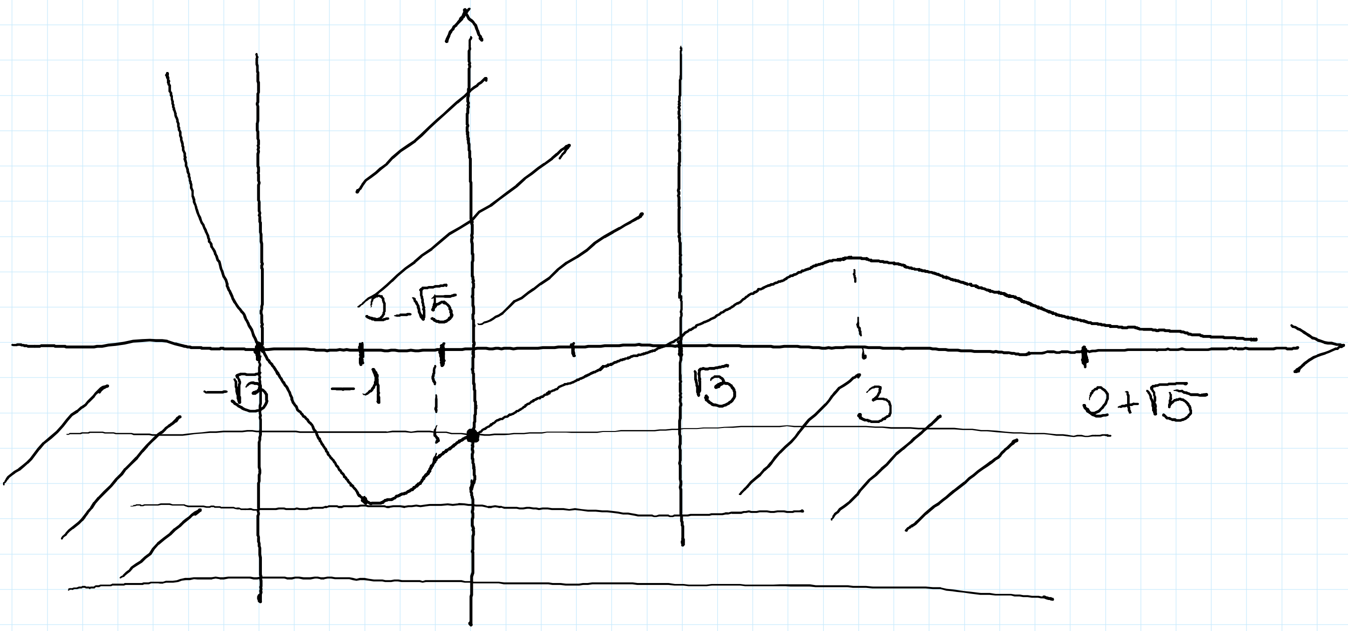
$x \to -\infty \qquad f(x) \sim x^2 e^{-x} = +\infty \qquad [+\infty \cdot +\infty]$

$\frac{f(x)}{x} \sim \underset{-\infty}{x}\ \underset{+\infty}{e^{-x}} \to -\infty \implies$ no asintoto obliquo

$x \to +\infty \qquad f(x) \sim x^2 e^{-x} \qquad [+\infty \cdot 0] \qquad e^{-x} \to 0$

$= \frac{x^2}{e^x} \to 0 \qquad \left[\frac{+\infty}{+\infty}\right]$

$y=0$ asintoto orizzontale a $+\infty$



Derivata prima: $\forall x \in \mathbb{R}$

$f'(x) = 2x\, e^{-x} + (x^2-3)\, e^{-x} \cdot D(-x)$

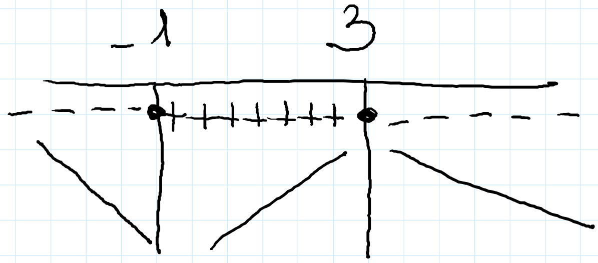
$$= 2x e^{-x} - (x^2-3)e^{-x}$$

$$= e^{-x}(2x - x^2 + 3) = \underset{\vee 0}{e^{-x}}(-x^2+2x+3)$$

$$f'(x) \geq 0 \iff -x^2+2x+3 \geq 0 \iff x^2-2x-3 \leq 0 \iff$$

$$x = 1 \pm \sqrt{1+3} = 1 \pm 2 \quad \begin{matrix} \nearrow -1 \\ \searrow 3 \end{matrix} \qquad -1 \leq x \leq 3$$

f':



$x = -1$ p.to di min. relativo

$x = 3$ " " max. relativo

$$f(-1) = -2e \qquad f(3) = 6e^{-3}$$

Derivata seconda: $\forall x \in \mathbb{R}$

$$f''(x) = -e^{-x}(-x^2+2x+3) + e^{-x}(-2x+2)$$

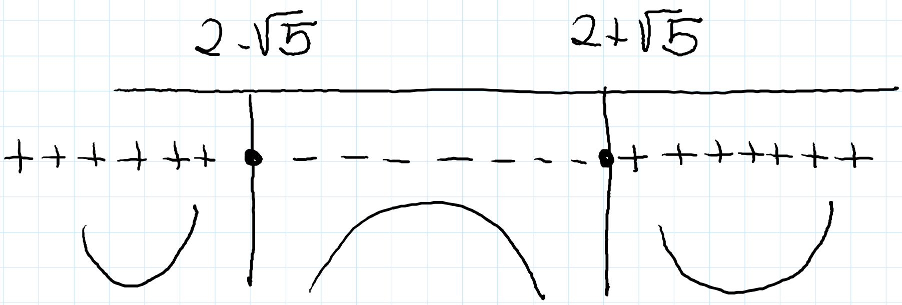
$$= e^{-x}(x^2-2x-3-2x+2)$$

$$= e^{-x}(x^2-4x-1)$$

$$f''(x) \geq 0 \iff x^2-4x-1 \geq 0 \iff x \leq 2-\sqrt{5} \text{ o } x \geq 2+\sqrt{5}$$

$$x = 2 \pm \sqrt{4+1} = 2 \pm \sqrt{5}$$

f''



Estremi di f: $\sup_{\mathbb{R}} f = +\infty$

$$\min f = f(-1)$$

$\operatorname{Im} f = [f(-1), +\infty)$ $^{(1)}$

L'eq. $f(x) = \lambda$ ha

0 sol se $\lambda < f(-1)$

1 sol se $\lambda = f(-1)$

⋮

(completare)

- $\displaystyle \lim_{x \to 0} \frac{e^{2x} - 1 + x \operatorname{sen} x}{\operatorname{tg} x + 1 - \cos x} = ?$ $\left[\frac{0}{0}\right]$

$$\underbrace{e^{2x} - 1}_{\sim 2x} + \underbrace{x \operatorname{sen} x}_{\sim x^2} = (e^{2x} - 1)\Bigg(1 - \underbrace{\frac{x \operatorname{sen} x}{e^{2x} - 1}}_{\sim \frac{x^2}{2x} \to 0}\Bigg) \quad (*)$$

$$(*) \sim \underbrace{e^{2x} - 1}_{2x \to 0} \sim 2x$$

$$\underbrace{\operatorname{tg} x}_{\sim x} + 1 - \underbrace{\cos x}_{\sim \frac{1}{2}x^2} = \operatorname{tg} x \Bigg(1 + \underbrace{\frac{1 - \cos x}{\operatorname{tg} x}}_{\sim \frac{\frac{1}{2}x^{\cancel{2}}}{\cancel{x}} \to 0}\Bigg) = \operatorname{tg} x \cdot \underbrace{\omega(x)}_{\downarrow 1}$$

$$\sim \operatorname{tg} x \sim x$$

$$? = \lim_{x \to 0} \frac{2x}{x} = 2$$

- $\displaystyle I = \int \frac{e^x - 1}{e^x + 1}\, dx = \int \frac{e^x - 1}{(e^x + 1)e^x} \cdot \underbrace{e^x dx}_{dt}$

$$e^{x}+1 \qquad\qquad \int \frac{}{(e^{x}+1)e^{x}} \, dt$$

$$\varphi(x) = e^{x} \qquad t = e^{x}$$
$$dt = e^{x} dx$$

$$= \left[\int \frac{t-1}{(t+1)t} \, dt \right]_{t=e^{x}}$$

$$\frac{t-1}{(t+1)t} = \frac{a}{t} + \frac{b}{t+1}$$
$$= \frac{at + a + bt}{t(t+1)}$$

$$\begin{cases} a+b=1 \\ a=-1 \end{cases} \qquad \begin{cases} b = 1-a = 1+1 = 2 \\ a=-1 \end{cases}$$

$$I = \left[-\int \frac{1}{t} \, dt + 2 \int \frac{1}{t+1} \, dt \right]_{t=e^{x}}$$
$$= \left[-\log_e |t| + 2 \log_e |t+1| + c \right]_{t=e^{x}}$$
$$= -\underbrace{\log_e e^{x}}_{= x} + 2 \log_e (e^{x}+1) + c$$

- $$\sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{1}{2^n} \operatorname{sen} \frac{1}{n}}_{a_n}$$

a_n è a segni alterni

Studio la conv. assoluta

$$|a_n| = \underbrace{|(-1)^n|}_{=1} \left|\frac{1}{2^n}\right| \left|\operatorname{sen}\frac{1}{n}\right| \quad \text{def. } 0<\frac{1}{n}<\pi$$

$$\left|\frac{1}{2^n}\right| = \frac{1}{2^n} \qquad \left|\operatorname{sen}\frac{1}{n}\right| = \operatorname{sen}\frac{1}{n}$$

def: $|a_n| = \underbrace{\frac{1}{2^n}\operatorname{sen}\frac{1}{n}}_{b_n}$

1° modo $\quad b_n \sim \frac{1}{2^n}\cdot\frac{1}{n} \qquad \sum \frac{1}{2^n n}$ conv.

$$\sqrt[n]{\frac{1}{2^n n}} = \frac{1}{2\sqrt[n]{n}} \to \frac{1}{2} < 1$$

La serie $\sum a_n$ conv. ass, quindi converge.

2° modo

$$b_n \underset{\frac{1}{n}\leq 1}{\leq} \frac{1}{2^n} \qquad \sum \frac{1}{2^n} < +\infty$$

(geometrica)

per confr. $\sum b_n$ converge

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$$b_n = \frac{1}{2^n}\cdot\underbrace{\frac{1}{n}}_{\leq 1} \leq \frac{1}{2^n}\cdot 1$$