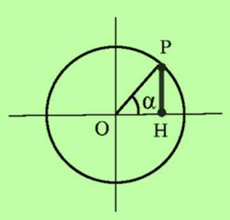
Goniometria (Cenni)

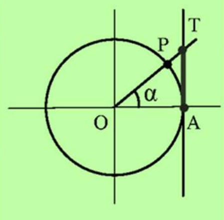
● Definizioni (Cenni)

definizione delle funzioni goniometriche sulla circonferenza goniometrica di centro l'origine degli assi e raggio 1



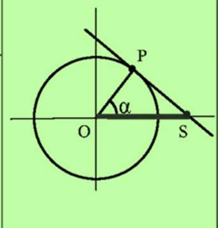
seno α

$$sen\,\alpha = \frac{\overline{PH}}{\overline{OP}} = \overline{PH}$$



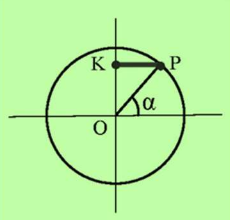
tangente α

$$tg\,\alpha = \frac{\overline{TA}}{\overline{OP}} = \overline{TA}$$



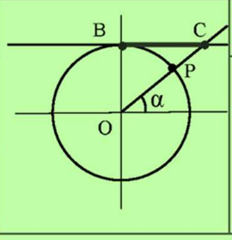
secante α

$$sec\,\alpha = \frac{\overline{OS}}{\overline{OP}} = \overline{OS}$$



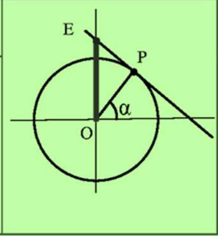
coseno α

$$cos\,\alpha = \frac{\overline{PK}}{\overline{OP}} = \overline{PK}$$



cotangente α

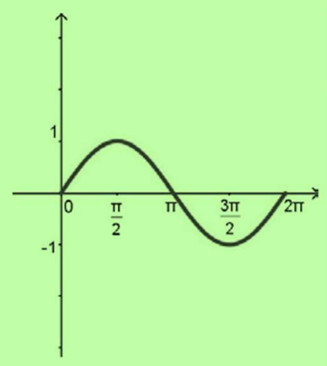
$$ctg\,\alpha = \frac{\overline{BC}}{\overline{OP}} = \overline{BC}$$



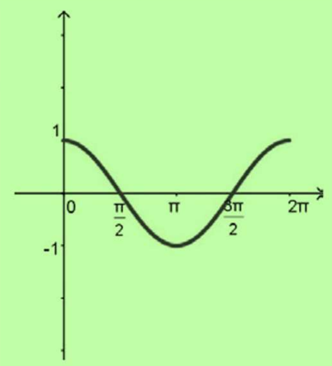
cosecante α

$$cosec\,\alpha = \frac{\overline{OE}}{\overline{OP}} = \overline{OE}$$

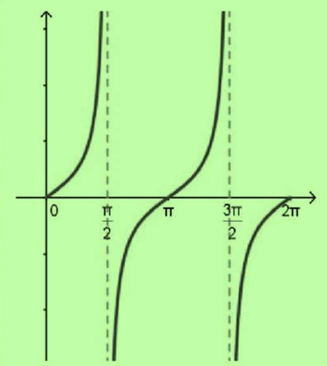
grafici delle funzioni goniometriche



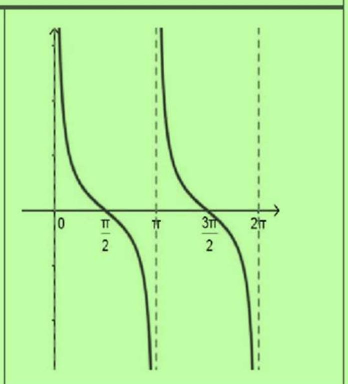
seno



coseno

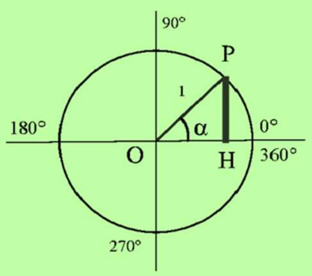


tangente



cotangente

seno

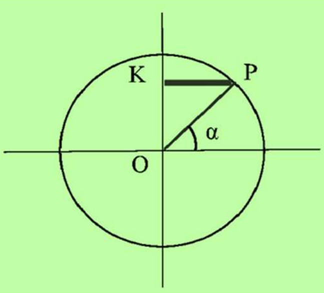


$$sen\,\alpha = \frac{PH}{OP} = \frac{PH}{1} = PH$$

angoli	valori
$\alpha°$	$sen\,\alpha$
0°	0
90°	1
180°	0
270°	−1

segno e crescenza nei quadranti		
quadrante	segno	crescenza
1°	+	↗
2°	+	↘
3°	−	↘
4°	−	↗

coseno



$$cos\,\alpha = \frac{PK}{OP} = \frac{PK}{1} = PK$$

angoli	valori
$\alpha°$	$cos\,\alpha$
0°	1
90°	0
180°	−1
270°	0

segno e crescenza nei quadranti		
quadrante	segno	crescenza
1°	+	↘
2°	−	↘
3°	−	↗
4°	+	↗

● Formule di base

le cinque relazioni fondamentali				
$sen^2\alpha + cos^2\alpha = 1$	$tg\ \alpha = \frac{sen\ \alpha}{cos\ \alpha}$	$ctg\ \alpha = \frac{cos\ \alpha}{sen\ \alpha}$	$sec\ \alpha = \frac{1}{cos\ \alpha}$	$cosec\ \alpha = \frac{1}{sen\ \alpha}$

relazioni che esprimono una funzione goniometrica rispetto alle altre

$sen\ \alpha$ in funzione di ...	$cos\ \alpha$ in funzione di ...	$tg\ \alpha$ in funzione di ...	$ctg\ \alpha$ in funzione di ...
$sen\ \alpha = \pm\sqrt{1 - cos^2\alpha}$	$cos\ \alpha = \pm\sqrt{1 - sen^2\alpha}$	$tg\ \alpha = \pm\frac{sen\ \alpha}{\sqrt{1 - sen^2\alpha}}$	$ctg\ \alpha = \pm\frac{\sqrt{1 - sen^2\alpha}}{sen\ \alpha}$
$sen\ \alpha = \pm\frac{tg\alpha}{\sqrt{1 + tg^2\alpha}}$	$cos\ \alpha = \pm\frac{1}{\sqrt{1 + tg^2\alpha}}$	$tg\ \alpha = \pm\frac{\sqrt{1 - cos^2\alpha}}{cos\ \alpha}$	$ctg\ \alpha = \pm\frac{cos\ \alpha}{\sqrt{1 - cos^2\alpha}}$
$sen\ \alpha = \pm\frac{1}{\sqrt{1 + ctg^2\alpha}}$	$cos\ \alpha = \pm\frac{ctg\alpha}{\sqrt{1 + ctg^2\alpha}}$	$tg\ \alpha = \frac{1}{ctg\ \alpha}$	$ctg\ \alpha = \frac{1}{tg\ \alpha}$

il segno $+$ o $-$ va preso a seconda del segno della funzione nel quadrante in cui si trova l'angolo

addizione e sottrazione	
$sen(\alpha \pm \beta) = sen\alpha \cdot cos\beta \pm sen\beta \cdot cos\alpha$	$tg(\alpha \pm \beta) = \frac{tg\alpha \pm tg\beta}{1 \mp tg\alpha \cdot tg\beta}$
$cos(\alpha \pm \beta) = cos\alpha \cdot cos\beta \mp sen\alpha \cdot sen\beta$	$ctg(\alpha \pm \beta) = \frac{ctg\alpha \cdot ctg\beta \mp 1}{ctg\beta \pm ctg\alpha}$

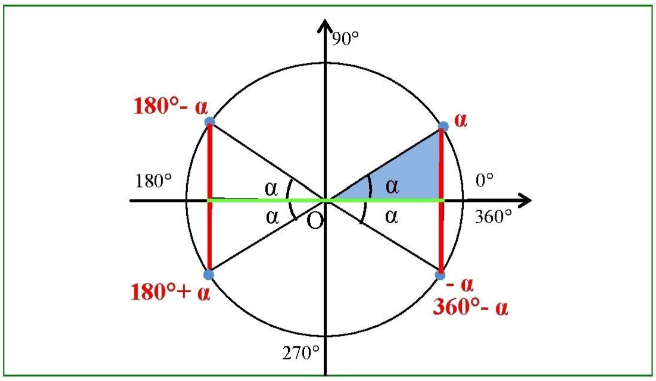
duplicazione		
$sen2\alpha = 2sen\alpha \cdot cos\alpha$		$tg2\alpha = \frac{2tg\alpha}{1 - tg^2\alpha}$
$cos2\alpha = cos^2\alpha - sen^2\alpha$	$cos2\alpha = 1 - 2sen^2\alpha$ $cos2\alpha = 2cos^2\alpha - 1$	$ctg2\alpha = \frac{ctg^2\alpha - 1}{2ctg\alpha}$

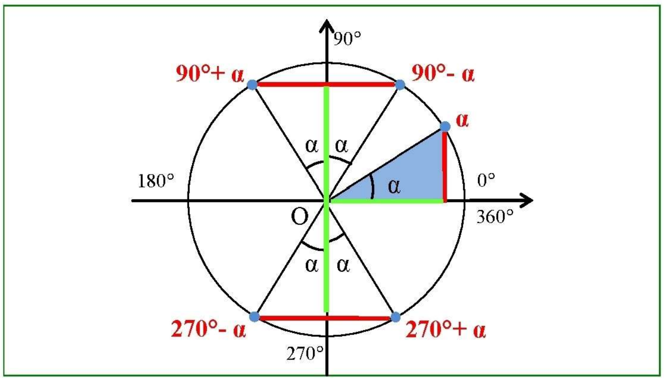
- Angoli notevoli

gradi	radianti	seno	coseno	tangente	cotangente
$0°$	0	0	1	0	∞
$9°$	$\frac{\pi}{20}$	$\frac{\sqrt{3+\sqrt5}-\sqrt{5-\sqrt5}}{4}$	$\frac{\sqrt{3+\sqrt5}+\sqrt{5-\sqrt5}}{4}$	$\frac{4-\sqrt{10+2\sqrt5}}{\sqrt5-1}$	$\frac{\sqrt5-1}{4-\sqrt{10+2\sqrt5}}$
$15°$	$\frac{\pi}{12}$	$\frac{\sqrt6-\sqrt2}{4}$	$\frac{\sqrt6+\sqrt2}{4}$	$2-\sqrt3$	$2+\sqrt3$
$18°$	$\frac{\pi}{10}$	$\frac{\sqrt5-1}{4}$	$\frac{\sqrt{10+2\sqrt5}}{4}$	$\frac{\sqrt{25-10\sqrt5}}{5}$	$\sqrt{5+2\sqrt5}$
$22°30$	$\frac{\pi}{8}$	$\frac{\sqrt{2-\sqrt2}}{2}$	$\frac{\sqrt{2+\sqrt2}}{2}$	$\sqrt2-1$	$\sqrt2+1$
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt3}{2}$	$\frac{\sqrt3}{3}$	$\sqrt3$
$36°$	$\frac{\pi}{5}$	$\frac{\sqrt{10-2\sqrt5}}{4}$	$\frac{\sqrt5+1}{4}$	$\sqrt{5-2\sqrt5}$	$\frac{\sqrt{25+10\sqrt5}}{5}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	1	1
$54°$	$\frac{3}{10}\pi$	$\frac{\sqrt5+1}{4}$	$\frac{\sqrt{10-2\sqrt5}}{4}$	$\frac{\sqrt{25+10\sqrt5}}{5}$	$\sqrt{5-2\sqrt5}$
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt3}{2}$	$\frac{1}{2}$	$\sqrt3$	$\frac{\sqrt3}{3}$
$67°30$	$\frac{3}{8}\pi$	$\frac{\sqrt{2+\sqrt2}}{2}$	$\frac{\sqrt{2-\sqrt2}}{2}$	$\sqrt2+1$	$\sqrt2-1$
$72°$	$\frac{2}{5}\pi$	$\frac{\sqrt{10+2\sqrt5}}{4}$	$\frac{\sqrt5-1}{4}$	$\sqrt{5+2\sqrt5}$	$\frac{\sqrt{25-10\sqrt5}}{5}$
$75°$	$\frac{5}{12}\pi$	$\frac{\sqrt6+\sqrt2}{4}$	$\frac{\sqrt6-\sqrt2}{4}$	$2+\sqrt3$	$2-\sqrt3$
$81°$	$\frac{9}{20}\pi$	$\frac{\sqrt{3+\sqrt5}+\sqrt{5-\sqrt5}}{4}$	$\frac{\sqrt{3+\sqrt5}-\sqrt{5-\sqrt5}}{4}$	$\frac{\sqrt5-1}{4-\sqrt{10+2\sqrt5}}$	$\frac{4-\sqrt{10+2\sqrt5}}{\sqrt5-1}$
$90°$	$\frac{\pi}{2}$	1	0	∞	0
$180°$	π	0	-1	0	∞
$270°$	$\frac{3}{2}\pi$	-1	0	∞	0
$360°$	2π	0	1	0	∞

NB: $\cos(45°)=\frac{\sqrt2}{2}$ $\left(\text{A volte si trova: } \frac{\sqrt2}{2}=\frac{\sqrt2}{\sqrt2\cdot\sqrt2}=\frac{1}{\sqrt2}\right)$; $tg(30°)=\frac{sen(30°)}{\cos(30°)}=\frac{1}{\sqrt3}$ $\left(\text{A volte si trova: } \frac{1}{\sqrt3}=\frac{\sqrt3}{\sqrt3\cdot\sqrt3}=\frac{\sqrt3}{3}\right)$

● Angoli associati





angoli supplementari	
secondo quadrante	
$sen(180^0 - \alpha) = sen\ \alpha$	$sen(\pi - \alpha) = sen\ \alpha$
$cos(180^0 - \alpha) = -\ cos\ \alpha$	$cos(\pi - \alpha) = -\ cos\ \alpha$
$tg(180^0 - \alpha) = -\ tg\ \alpha$	$tg(\pi - \alpha) = -\ tg\ \alpha$
$ctg(180^0 - \alpha) = -\ ctg\ \alpha$	$ctg(\pi - \alpha) = -\ ctg\ \alpha$

angoli complementari	
primo quadrante	
$sen(90^0 - \alpha) = cos\ \alpha$	$sen\left(\frac{\pi}{2} - \alpha\right) = cos\ \alpha$
$cos(90^0 - \alpha) = sen\ \alpha$	$cos\left(\frac{\pi}{2} - \alpha\right) = sen\ \alpha$
$tg(90^0 - \alpha) = ctg\ \alpha$	$tg\left(\frac{\pi}{2} - \alpha\right) = ctg\ \alpha$
$ctg(90^0 - \alpha) = tg\ \alpha$	$ctg\left(\frac{\pi}{2} - \alpha\right) = tg\ \alpha$

angoli che differiscono di un angolo piatto	
terzo quadrante	
$sen(180^0 + \alpha) = -\ sen\ \alpha$	$sen(\pi + \alpha) = -\ sen\ \alpha$
$cos(180^0 + \alpha) = -\ cos\ \alpha$	$cos(\pi + \alpha) = -\ cos\ \alpha$
$tg(180^0 + \alpha) = tg\ \alpha$	$tg(\pi + \alpha) = tg\ \alpha$
$ctg(180^0 + \alpha) = ctg\ \alpha$	$ctg(\pi + \alpha) = ctg\ \alpha$

angoli che differiscono di un angolo retto	
secondo quadrante	
$sen(90^0 + \alpha) = cos\ \alpha$	$sen\left(\frac{\pi}{2} + \alpha\right) = cos\ \alpha$
$cos(90^0 + \alpha) = -\ sen\ \alpha$	$cos\left(\frac{\pi}{2} + \alpha\right) = -\ sen\ \alpha$
$tg(90^0 + \alpha) = -\ ctg\ \alpha$	$tg\left(\frac{\pi}{2} + \alpha\right) = -\ ctg\ \alpha$
$ctg(90^0 + \alpha) = -\ tg\ \alpha$	$ctg\left(\frac{\pi}{2} + \alpha\right) = -\ tg\ \alpha$

angoli esplementari	
quarto quadrante	
$sen(360^0 - \alpha) = -\ sen\ \alpha$	$sen(2\pi - \alpha) = -\ sen\ \alpha$
$cos(360^0 - \alpha) = cos\ \alpha$	$cos(2\pi - \alpha) = cos\ \alpha$
$tg(360^0 - \alpha) = -\ tg\ \alpha$	$tg(2\pi - \alpha) = -\ tg\ \alpha$
$ctg(360^0 - \alpha) = -\ ctg\ \alpha$	$ctg(2\pi - \alpha) = -\ ctg\ \alpha$

angoli la cui somma è 270°	
terzo quadrante	
$sen(270^0 - \alpha) = -\ cos\ \alpha$	$sen\left(\frac{3}{2}\pi - \alpha\right) = -\ cos\ \alpha$
$cos(270^0 - \alpha) = -\ sen\ \alpha$	$cos\left(\frac{3}{2}\pi - \alpha\right) = -\ sen\ \alpha$
$tg(270^0 - \alpha) = ctg\ \alpha$	$tg\left(\frac{3}{2}\pi - \alpha\right) = ctg\ \alpha$
$ctg(270^0 - \alpha) = tg\ \alpha$	$ctg\left(\frac{3}{2}\pi - \alpha\right) = tg\ \alpha$

angoli opposti	
quarto quadrante	
$sen(-\alpha) = -\ sen\ \alpha$	$sen(-\alpha) = -\ sen\ \alpha$
$cos(-\alpha) = cos\ \alpha$	$cos(-\alpha) = cos\ \alpha$
$tg(-\alpha) = -\ tg\ \alpha$	$tg(-\alpha) = -\ tg\ \alpha$
$ctg(-\alpha) = -\ ctg\ \alpha$	$ctg(-\alpha) = -\ ctg\ \alpha$

angoli che differiscono di 270°	
quarto quadrante	
$sen(270^0 + \alpha) = -\ cos\ \alpha$	$sen\left(\frac{3}{2}\pi + \alpha\right) = -\ cos\ \alpha$
$cos(270^0 + \alpha) = sen\ \alpha$	$cos\left(\frac{3}{2}\pi + \alpha\right) = sen\ \alpha$
$tg(270^0 + \alpha) = -\ ctg\ \alpha$	$tg\left(\frac{3}{2}\pi + \alpha\right) = -\ ctg\ \alpha$
$ctg(270^0 + \alpha) = -\ tg\ \alpha$	$ctg\left(\frac{3}{2}\pi + \alpha\right) = -\ tg\ \alpha$

● Formule avanzate

bisezione	
$sen\frac{\alpha}{2} = \pm\sqrt{\frac{1-cos\alpha}{2}}$	$tg\frac{\alpha}{2} = \pm\sqrt{\frac{1-cos\alpha}{1+cos\alpha}} = \frac{sen\alpha}{1+cos\alpha} = \frac{1-cos\alpha}{sen\alpha}$
$cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+cos\alpha}{2}}$	$ctg\frac{\alpha}{2} = \pm\sqrt{\frac{1+cos\alpha}{1-cos\alpha}} = \frac{sen\alpha}{1-cos\alpha} = \frac{1+cos\alpha}{sen\alpha}$

parametriche o razionali $\left(t = tg\frac{\alpha}{2}\right)$	
$sen\alpha = \frac{2t}{1+t^2}$	$tg\alpha = \frac{2t}{1-t^2}$
$cos\alpha = \frac{1-t^2}{1+t^2}$	$ctg\alpha = \frac{1-t^2}{2t}$

prostaferesi	
$sen\,p + sen\,q = 2\,sen\frac{p+q}{2}\cdot cos\frac{p-q}{2}$	$sen\,p - sen\,q = 2\,sen\frac{p-q}{2}\cdot cos\frac{p+q}{2}$
$cos\,p + cos\,q = 2\,cos\frac{p+q}{2}\cdot cos\frac{p-q}{2}$	$cos\,p - cos\,q = -2\,sen\frac{p+q}{2}\cdot sen\frac{p-q}{2}$

Werner	
$sen\alpha\cdot cos\beta = \frac{1}{2}[sen(\alpha+\beta)+sen(\alpha-\beta)]$	$sen\beta\cdot cos\alpha = \frac{1}{2}[sen(\alpha+\beta)-sen(\alpha-\beta)]$
$cos\alpha\cdot cos\beta = \frac{1}{2}[cos(\alpha+\beta)+cos(\alpha-\beta)]$	$sen\alpha\cdot sen\beta = -\frac{1}{2}[cos(\alpha+\beta)-cos(\alpha-\beta)]$

● Esercizi

ES 01)

Calcola: $\sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{3}{4}\pi\right)+\text{tg}\left(-\frac{5}{4}\pi\right)+cotg\left(\frac{3}{2}\pi\right)$

$\left(\text{Trasformo } -\frac{5}{4}\pi\ :\ \text{ So che}\frac{1}{4}\pi=45°\ ,\ \text{ quindi}+\frac{5}{4}\pi=5\cdot 45°=225°\ ,\ \text{ quindi}-\frac{5}{4}\pi=360°-225°=135°=\frac{3}{4}\pi\right)$

$$\left[+\frac{\sqrt{2}}{2}\right]+\left[-\frac{\sqrt{2}}{2}\right]+tg\left(\frac{3}{4}\pi\right)+cotg\left(\pi+\frac{1}{2}\pi\right)=0+\left[\frac{sen\left(\frac{3}{4}\pi\right)}{\cos\left(\frac{3}{4}\pi\right)}\right]+\left[\frac{\cos\left(\frac{3}{2}\pi\right)}{sen\left(\frac{3}{2}\pi\right)}\right]=\frac{\left[+\frac{\sqrt{2}}{2}\right]}{\left[-\frac{\sqrt{2}}{2}\right]}+\frac{[0]}{[-1]}=-1+0=-1$$

ES 02)

Risolvi: $\cos(3x)=\frac{1}{2}$

$\Rightarrow\ 3x=\frac{\pi}{3}+2k\pi\ \vee\ 3x=\frac{5}{6}\cdot 2\pi+2k\pi=\frac{5}{3}\pi+2k\pi\quad\left(\text{Ricorda che }\cos(\alpha)=\frac{1}{2}\ \text{ per }\alpha=60°\vee\alpha=300°\right)\ \Rightarrow$

$\Rightarrow\ x=\frac{\pi}{9}+\frac{2}{3}k\pi\ \vee\ x=\frac{5}{9}\pi+\frac{2}{3}k\pi$

ES 03)

Risolvi: $\cos\left(\frac{\pi}{9}-x\right)=0$

Pongo $t=\frac{\pi}{9}-x$

$\cos(t)=0\quad\Rightarrow\quad t=\frac{\pi}{2}+k\pi\quad\Rightarrow\quad\left[\frac{\pi}{9}-x\right]=\frac{\pi}{2}+k\pi\quad\Rightarrow\quad[\dots]\quad\Rightarrow\quad x=-\frac{7}{18}\pi-k\pi$

ES 04)

Risolvi: $sen(4x)=\cos(2x)$

$sen(2[2x])=\cos(2x)\quad\Rightarrow\quad[2\cdot sen(2x)\cdot\cos(2x)]=\cos(2x)\quad\Rightarrow\quad 2\cdot sen(2x)\cdot\cos(2x)-\cos(2x)=0\quad\Rightarrow$

$\Rightarrow\quad\cos(2x)\cdot(2sen(2x)-1)=0\quad\Rightarrow$

Risolvo F_1: $\cos(2x)=0\quad\Rightarrow\quad 2x=\frac{\pi}{2}+k\pi\quad\Rightarrow\quad x=\frac{\pi}{4}+k\frac{\pi}{2}\quad(x=45°+k\pi)$

Risolvo F_2: $2sen(2x)-1=0\quad\Rightarrow\quad sen(2x)=\frac{1}{2}\quad\Rightarrow$

$\left(\text{Sulla tavola dei valori c'è solo sen}(30°)=\frac{1}{2}\text{ , ma so che sen}(\alpha)=\text{sen}(180°-\alpha)\text{ , quindi anche }\sin(150°)=\frac{1}{2}\right).$

$\Rightarrow\quad 2x=\frac{\pi}{6}+2k\pi\ \vee\ 2x=\frac{5}{6}\pi+2k\pi\quad\Rightarrow\quad x=\frac{\pi}{12}+k\pi\ \vee\ x=\frac{5}{12}\pi+k\pi\quad(x=15°+k\pi\vee x=75°+k\pi)$

Risultato: $x=\frac{\pi}{4}+k\frac{\pi}{2}\ \vee\ x=\frac{\pi}{12}+k\pi\ \vee\ x=\frac{5}{12}\pi+k\pi$

ES 05)

Risolvi: $\sin\left(2x-\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{4}-3x\right)$

(Errore comune: Ricorda che $\sin(\alpha)=\sin(\beta)\;\not\Rightarrow\;\alpha=\beta$; $\sin(\alpha)=\sin(\beta)\;\Rightarrow\;\alpha=\beta+2k\pi$)

$$\Rightarrow\; 2x-\frac{\pi}{3}=\frac{\pi}{4}-3x+2k\pi \quad\Rightarrow\quad 5x=\frac{\pi}{4}+\frac{\pi}{3}+2k\pi \quad\Rightarrow\quad 5x=\frac{7}{12}\pi+2k\pi \quad\Rightarrow\quad x=\frac{7}{60}\pi+\frac{2}{5}k\pi$$

$$\left(\text{Risultati: } \frac{7}{60}\pi;\quad \frac{31}{60}\pi;\quad \frac{55}{60}\pi=\frac{11}{12}\pi;\quad \frac{79}{60}\pi;\quad \frac{103}{60}\pi\right)$$

ES 06)

Risolvi: $\cos\left(\frac{\pi}{2}+x\right)=\cos(x)$

$$\Rightarrow\quad \frac{\pi}{2}+x=x+2k\pi \quad\Rightarrow\quad \frac{\pi}{2}=2k\pi \quad\Rightarrow\quad k=\frac{1}{4} \quad\Rightarrow\quad \text{Non ci interessa, serve risolvere per x} \quad\Rightarrow\quad \text{Cambio metodo} \quad\Rightarrow$$

$$\Rightarrow\quad \left[\cos\left(\frac{\pi}{2}\right)\cdot\cos(x)-sen\left(\frac{\pi}{2}\right)\cdot sen(x)\right]=\cos(x) \quad\Rightarrow\quad [0\cdot\cos(x)-(+1)\cdot sen(x)]=\cos(x) \quad\Rightarrow$$

$$\Rightarrow\quad -sen(x)=\cos(x) \quad\Rightarrow\quad -\frac{sen(x)}{\cos(x)}=\frac{\cos(x)}{\cos(x)} \quad \left(\text{Per}\cos(x)\neq 0\rightarrow x\neq\frac{\pi}{2}+k\pi\right) \quad\Rightarrow\quad -\frac{sen(x)}{\cos(x)}=1 \quad\Rightarrow$$

$$\Rightarrow\quad -tg(x)=1 \quad\Rightarrow\quad x=\frac{3}{4}\pi+k\pi \quad (x=135°\vee x=315°)$$

Perché? $|tg(x)|=1$ agli angoli delle bisettrici (45°, 135°, 225°, 315°).

In particolare, è $tg(x)=-1$ quando il seno è positivo e il coseno negativo, o viceversa.

ES 07)

Riscrivi la seguente funzione con le formule di bisezione: $f(x)=\sin^2(t)$

Ricordo la formula di bisezione per il seno: $\sin\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1-\cos(\alpha)}{2}}$

$$\text{Pongo t}=\frac{\alpha}{2} \quad\Rightarrow\quad \sin(t)=\pm\sqrt{\frac{1-\cos(2t)}{2}} \quad\Rightarrow\quad \sin^2(t)=\left(\pm\sqrt{\frac{1-\cos(2t)}{2}}\right)^2=\frac{1-\cos(2t)}{2}$$