Analisi Matematica – 11.6.2016

1. $f(x) = \dfrac{e^{2x}}{1-x}$

(a) $1-x \neq 0 \qquad x \neq 1 \qquad \text{dom } f = \mathbb{R}\setminus\{1\}$

$f(0) = 1 \qquad (0,1) \in \text{graf } f$

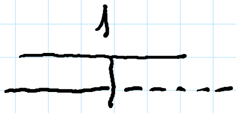
$f(x) = 0 \iff e^{2x} = 0$ non ha soluzione

$f(x) > 0 \iff 1-x > 0 \iff x < 1$

Limiti significativi: $1, \pm\infty$

$x \to 1 \qquad e^{2x} \to e^2 \qquad e^{2x} \sim e^2$

$f(x) \sim \dfrac{e^2}{1-x}$



quindi occorre distinguere

$x \to 1^+ \qquad f(x) \to -\infty$

$x \to 1^- \qquad f(x) \to +\infty$

$x = 1$ asintoto verticale

$x \to +\infty \qquad f(x) \sim \dfrac{e^{2x}}{-x} = -2\,\dfrac{e^{2x}}{2x} \to +\infty$

$x \to -\infty \qquad f(x) \sim \dfrac{e^{2x}}{-x} \to \left\{\dfrac{0}{+\infty}\right\} = 0$

$y = 0$ è asintoto orizzontale per $x \to -\infty$.

Potrebbe esserci un asintoto obliquo a $+\infty$. Ma

$\dfrac{f(x)}{x} \sim \dfrac{e^{2x}}{-x^2} \to +\infty$ per $x \to +\infty$

quindi non esiste.

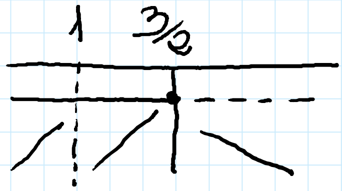
(b) $\forall x \in \text{dom } f$

$$f'(x) = \frac{2e^{2x}(1-x) + e^{2x}}{(1-x)^2} = \frac{e^{2x}(2-2x+1)}{(1-x)^2}$$

$$= \frac{e^{2x}(3-2x)}{(1-x)^2}$$

$$f'(x) \geq 0 \iff 3-2x \geq 0 \iff x \leq 3/2$$

f':



f è crescente in $(-\infty, 1)$ e in $(1, \frac{3}{2})$;
f è decrescente in $(\frac{3}{2}, +\infty)$;
$x = \frac{3}{2}$ è un p.to di massimo relativo di f.

(c) $\forall x \in \operatorname{dom} f$

$$f''(x) = \frac{[2e^{2x}(3-2x) - 2e^{2x}](1-x)^{\cancel{2}} + e^{2x}(3-2x)2\cancel{(1-x)}}{(1-x)^{\cancel{4}\,3}}$$

$$= \frac{e^{2x}[2(3-2x-1)(1-x) + 2(3-2x)]}{(1-x)^3}$$

$$= 2e^{2x}\frac{(2-2x)(1-x) + 3-2x}{(1-x)^3}$$

$$= 2e^{2x}\frac{2-2x-2x+2x^2+3-2x}{(1-x)^3}$$

$$= 2e^{2x}\frac{2x^2-6x+5}{(1-x)^3}$$

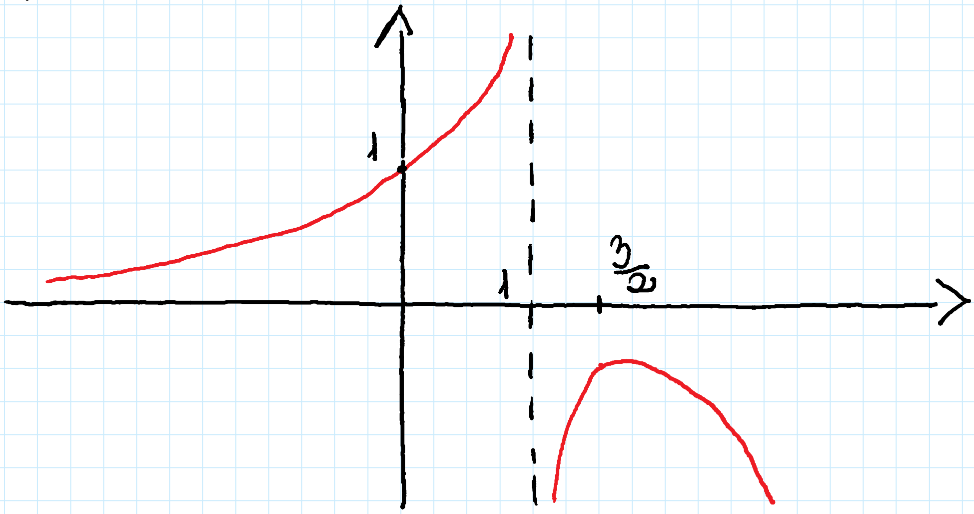
$2x^2-6x+5 > 0 \quad \forall x \in \operatorname{dom} f \quad (\Delta = 36-40 < 0)$
$2e^{2x} > 0 \quad \forall x \in \operatorname{dom} f$

quindi
$$f''(x) > 0 \iff 1-x > 0 \iff x < 1$$

f è convessa in $(-\infty, 1)$;
f è concava in $(1, +\infty)$;

non vi sono punti di flesso.

(d) Grafico di f:



(e) $\text{Im} f = (-\infty, f(\frac{3}{2})] \cup (0, +\infty)$

L'eq. $f(x) = \lambda$ ha

2 soluzioni se $\lambda < f(\frac{3}{2})$;

1 soluzione se $\lambda = f(3/2)$;

0 soluzioni se $f(3/2) < \lambda \leq 0$;

1 soluzione se $\lambda > 0$.

2. $$\lim_{x \to 0} \frac{1 + e^{-x} - 2\cos x}{\log_e(1+x^3)} = ?$$

Si tratta di una forma $\frac{0}{0}$

Occorre scrivere il numeratore come

$$1 + e^{-x} - 2\cos x = (e^{-x} - 1) + (2 - 2\cos x)$$

Per $x \to 0$

$$e^{-x} - 1 \sim -x \quad (-x \to 0)$$

$$2 - 2\cos x = 2(1 - \cos x) \sim 2 \cdot \frac{1}{2} x^2 = x^2$$

$$-x + x^2 \neq 0$$

$$\log_e(1 + x^3) \sim x^3 \quad (x^3 \to 0)$$

$$\log(1+\underset{\substack{\downarrow\\0}}{x^3}) \sim x^3$$

Quindi

$$\ell = \lim_{x\to 0} \frac{-x+x^2}{x^3} = \lim_{x\to 0} \frac{\cancel{x}(-1+x)}{x^{\cancel{3}\,2}}$$

$$= \lim_{x\to 0} \frac{\overset{\to -1}{-1+x}}{\underset{\to 0}{x^2}} = +\infty$$

3.

$$I = \int_0^1 \frac{x+1}{x^2+4x+7}$$

$\to \Delta = 16-28<0$

Occorre trasformare la funzione integranda.

$$\frac{x+1}{x^2+4x+7} = \frac{1}{2}\,\frac{2x+2}{x^2+4x+7} = \frac{1}{2}\,\frac{2x+4-4+2}{x^2+4x+7}$$

$$= \frac{1}{2}\,\frac{2x+4}{x^2+4x+7} - \frac{1}{x^2+4x+7}$$

Inoltre

$$x^2+4x+7 = x^2+4x+4+3 = (x+2)^2+3$$

Quindi

$$I = \frac{1}{2}\int_0^1 \frac{2x+4}{x^2+4x+7}\,dx - \int_0^1 \frac{1}{3+(x+2)^2}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{D(x^2+4x+7)}{x^2+4x+7}\,dx - \frac{1}{3}\int_0^1 \frac{1}{1+\left(\frac{x+2}{\sqrt{3}}\right)^2}\,dx$$

$$= \left[\frac{1}{2}\log(x^2+4x+7) - \frac{1}{\sqrt{3}}\arctan\frac{x+2}{\sqrt{3}}\right]_0^1$$

$$\Big[\;2 \qquad \sqrt{3} \qquad \sqrt{3}\;\Big]_0$$

$$= \frac{1}{2}\log 12 - \frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{3}{\sqrt{3}} - \frac{1}{2}\log 7 + \frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2}{\sqrt{3}} -$$

4. $$\sum_{n=1}^{\infty} \frac{n^2 - \log n}{n^3 + n^5}$$

Si tratta di una serie a termini positivi.
Conviene usare il criterio del confronto asintotico.

$$a_n = \frac{n^2 - \log n}{n^3 + n^5} \sim \frac{n^2}{n^5} = \frac{1}{n^3} \, :$$

$$n^2 - \log n = n^2\Big(1 - \overbrace{\frac{\log n}{n^2}}^{\to 0}\Big) \sim n^2$$

$$n^3 + n^5 = n^5\Big(\underbrace{\frac{1}{n^2}}_{\downarrow\, 0} + 1\Big) \sim n^5$$

Poiché è noto che $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converge anche $\sum_{n=1}^{\infty} a_n$ converge.