Analisi Matematica - 12.4.2019 - prima parte

Derivate e approssimazione di funzioni

Def: Siano $f, g : D \to \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ p.to in cui si possono fare i limiti di f e g.
Si dice che f è un "o-piccolo" di $g(x)$ per $x \to x_0$ e si scrive

$$f(x) = o(g(x)) \quad x \to x_0$$

se esiste $\omega : D \to \mathbb{R}$ tale che

$$f(x) = \omega(x) g(x) \qquad \omega(x) \to 0 \text{ per } x \to x_0.$$

Oss: se $g(x) \neq 0$ def. per $x \to x_0$ allora la def. equivale a richiedere che

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

- $x \to 0 \qquad x^3 = o(x^2)$
$$x^3 = \underbrace{x}_{\omega(x) \to 0 \ \ x \to 0} \cdot x^2$$

- $x \to +\infty \qquad x^2 = o(x^3)$
$$x^2 = \underbrace{\frac{1}{x}}_{\omega(x) \to 0 \ \ x \to +\infty} \cdot x^3$$

- $1 - \cos x = o(x) \qquad x \to 0$
$$1 - \cos x = \underbrace{\frac{1 - \cos x}{x}}_{\omega(x) \to 0} \cdot x$$

Def: $f : (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$. Si dice che f è differenziabile in x_0 se esiste $\alpha \in \mathbb{R}$ tale che $\forall x \in (a,b)$

$$\underbrace{f(x) - f(x_0)}_{\text{incremento di } f} = \underbrace{\alpha(x - x_0)}_{\text{lineare in } x-x_0 \text{ incremento di } x} + o(x - x_0) \qquad x \to x_0$$

- $\displaystyle \lim_{x \to x_0} \frac{o(x - x_0)}{x - x_0} = 0$

L'incremento di f è proporzionale a $x-x_0$ a meno di una funzione in o, a zero per $x \to x_0$ più velocemente di $x-x_0$.

<u>Teorema</u>: f è differenziabile in x_0 se e solo se f è derivabile in x_0 e, in tal caso,

$$\alpha = f'(x_0).$$

<u>Dim</u>: $\Rightarrow$) Ipotesi: f è diff. in x_0 $\Rightarrow$ $\exists \alpha \in \mathbb{R}$:

$$f(x)-f(x_0) = \alpha(x-x_0) + o(x-x_0) \quad x \to x_0 \Rightarrow$$

$$\frac{f(x)-f(x_0)}{x-x_0} = \alpha + \underbrace{\frac{o(x-x_0)}{x-x_0}}_{\to 0}$$

$x \to x_0$

$$\exists \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \alpha$$

$$\Rightarrow \exists f'(x_0) = \alpha$$

$\Leftarrow$) f derivabile in x_0 (Ipotesi)

Tesi: f è diff. in x_0 con $\alpha = f'(x_0)$ cioè

$$f(x) - f(x_0) - f'(x_0)(x-x_0) = o(x-x_0)$$

Quindi calcolo:

$$\frac{f(x) - f(x_0) - f'(x_0)(x-x_0)}{x-x_0} =$$

$$\underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\to f'(x_0)} - f'(x_0) \to 0$$

Oss: Questa equiv. vale solo in una sola variabile.
Se $f: A \to \mathbb{R}$, $A \subset \mathbb{R}^n$, $n > 1$, allora

$$f \text{ diff} \Rightarrow f \text{ è deriv.}$$
$$\not\Leftarrow$$

- Se f è deriv. (o differenziabile) in x_0, detta
$$T(x) = f(x_0) + f'(x_0)(x - x_0) \quad \text{(retta tg in } x_0 \text{ a } f\text{)}$$
per il teo. precedente si ha
- $f(x) - T(x) = o(x - x_0) \quad x \to x_0$

inoltre
- $\underbrace{f(x) - T(x)}_{\text{Errore}} \to 0 \quad x \to x_0$

Ciò significa che l'errore che si commette approssimando f vicino ad x_0 con la retta tangente $T(x)$ tende a 0 più velocemente di $x - x_0$.

Abbiamo visto "$\sim$" asintotico
"o" o piccolo

Altro simbolo: "O" o grande

<u>Def</u>: Nelle stesse ipotesi della def. di o piccolo, si dice che $f(x)$ è un O grande di $g(x)$ per $x \to x_0$ e si scrive
$$f(x) = O(g(x)) \quad x \to x_0$$
se esiste $\omega : D \to \mathbb{R}$ tale che
- $f(x) = \omega(x) g(x)$
- $\omega(x)$ limitata def. per $x \to x_0$

$\exists M \in \mathbb{R} : |\omega(x)| \le M \quad \forall x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$
$\exists \delta > 0$

- Se $g(x) \neq 0$ def. per $x \to x_0$ ciò equivale a richiedere che
$$\frac{f(x)}{g(x)} \text{ è limitata def. per } x \to x_0$$

Che relazioni ci sono tra $\sim$, o, O ?

1) $f(x) = o(g(x)) \quad x \to x_0 \Rightarrow f(x) = O(g(x)) \quad x \to x_0$

$\left[f(x) = \omega(x) g(x) \qquad \omega(x) \to 0 \Rightarrow \omega(x) \text{ limitata def. } x \to x_0 \right]$

$$\left[f(x) = \underset{\substack{\downarrow \\ 0}}{\omega(x)} g(x) \qquad \omega(x) \to 0 \Rightarrow \omega(x) \text{ limitata def. } x \to x_0 \right]$$

2) $f(x) \sim g(x) \quad x \to x_0 \Rightarrow f(x) = O(g(x)) \quad x \to x_0$

[come sopra con $\omega(x) \to 1 \Rightarrow \omega(x)$ lim. def $x \to x_0$]

3) $f(x) \sim g(x) \quad x \to x_0 \Leftrightarrow f(x) = g(x) + o(g(x)) \quad x \to x_0$

$\Rightarrow$)

$f(x) \sim g(x) \quad x \to x_0 \Rightarrow f(x) = \omega(x) g(x) \quad \omega(x) \xrightarrow[x \to x_0]{} 1$

$$f(x) - g(x) = \underbrace{[\overset{\to 1}{\omega(x)} - 1]}_{\omega_1(x) \to 0 \;\; x \to x_0} g(x) = \underset{\substack{\downarrow \\ 0}}{\omega_1(x)} g(x)$$

$\Rightarrow f(x) - g(x) = o(g(x)) \quad x \to x_0$

$\Leftarrow$) Analoga

Dalla 3, posso riscrivere i limiti notevoli in termini di o piccolo

- $\operatorname{sen} x = x + o(x) \quad x \to 0$ $\qquad \operatorname{sen} x \sim x \quad x \to 0$
- $e^x - 1 = x + o(x)$ "
  $e^x = 1 + x + o(x)$ "
- $1 - \cos x = \frac{1}{2} x^2 + o(x) \quad x \to 0$
  $\cos x = 1 - \frac{1}{2} x^2 + o(x)$ $\qquad -o(x) = o(x)$
- $\operatorname{arctg} x = x + o(x)$ "
- $\operatorname{arcsen} x = x + o(x)$ "
- $(1+x)^\alpha - 1 = \alpha x + o(x) \quad x \to 0$
  $(1+x)^\alpha = 1 + \alpha x + o(x) \quad x \to 0$

Proprietà algebriche:

$f_1 = o(g(x)) \qquad f_2 = o(g(x))$

- $f_1(x) \pm f_2(x) = o(g(x))$ $\qquad o(g) \pm o(g) = o(g)$
- $f_1(x) \cdot f_2(x) = o(g^2(x))$ $\qquad o(g) \cdot o(g) = o(g^2)$
- $\forall \alpha \in \mathbb{R} \quad \alpha f_1(x) = o(g(x))$ $\qquad \alpha\, o(g) = o(g)$

$$\left[\alpha f_1(x) = \overbrace{\alpha\, \underset{\substack{\downarrow \\ 0}}{\omega(x)}}^{\omega_1(x)} g(x) = \underset{\substack{\downarrow \\ 0}}{\omega_1(x)}\, g(x)\right]$$

- $f_1(x) = o(\alpha g(x)) \Rightarrow f_1(x) \overset{?}{=} o(g(x))$ $\qquad o(\alpha g) = o(g)$

Funzioni composte :

se $\varepsilon(x) \to 0 \quad x \to x_0$ allora $\operatorname{sen} \varepsilon(x) \sim \varepsilon(x)$

$$\operatorname{sen} \varepsilon(x) = \varepsilon(x) + o(\varepsilon(x)) \quad x \to x_0$$

si può fare per tutti i limiti notevoli.

Calcolo di limiti con o piccolo :

$$\lim_{x \to 0} \frac{\operatorname{tg} 3x + e^{2x} - 1}{\operatorname{arctg} x + \operatorname{sen} 3x} = \lim_{x \to x_0} f(x)$$

$x \to 0 \qquad 3x \to 0$

$$\operatorname{tg} x = 3x + o(3x) = 3x + o(x)$$

$2x \to 0$

$$e^{2x} - 1 = 2x + o(2x) = 2x + o(x)$$

$$\operatorname{arctg} x = x + o(x)$$

$3x \to 0$

$$\operatorname{sen} 3x = 3x + o(3x) = 3x + o(x)$$

$\to 0$

$$f(x) = \frac{5x + o(x)}{4x + o(x)} = \frac{\cancel{x}\left(5 + \frac{o(x)}{x}\right)}{\cancel{x}\left(4 + \underbrace{\frac{o(x)}{x}}_{0}\right)} \longrightarrow \frac{5}{4}$$

Analisi Matematica - 12.4.2019 - seconda parte

- $f(x) = |x|(x+1)$

dom $f = \mathbb{R}$

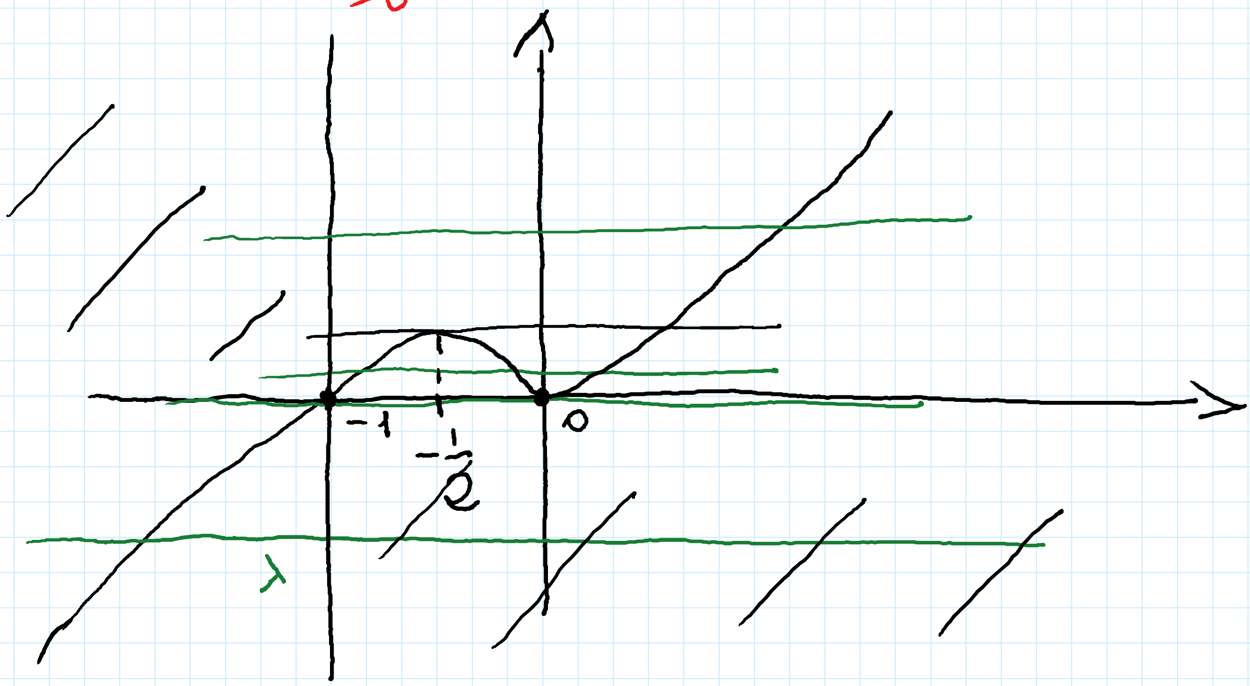
$x=0 \quad f(0) = 0 \cdot 1 = 0$

$f(x)=0 \iff |x|=0$ oppure $x+1=0$

$x=0 \qquad x=-1$

$(0,0) \qquad (-1,0)$

$f(x)>0 \iff \underbrace{|x|}_{\geq 0}(x+1)>0 \iff x\neq 0,\ x>-1$



Limiti: $x \to \pm\infty$

$x \to +\infty \qquad f(x) = |x|(\underbrace{x+1}_{\sim x}) \sim \underbrace{|x|}_{=x} x = x^2 \to +\infty$

$x \to -\infty \qquad f(x) \sim \underbrace{|x|}_{=-x} \cdot x = -x^2 \to -\infty$

Asintoto obliquo? NO perché $\overbrace{\frac{f(x)}{x}}^{\sim x|x|} \sim |x|$

Derivata prima:

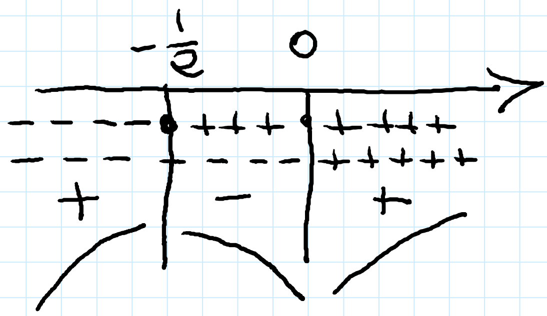
$\forall x \in \mathbb{R} \setminus \{0\}$ Insieme di deriv. è differente da dom f

$$f'(x) = \frac{|x|}{x}(x+1) + |x| \cdot 1$$
$$= |x|\left(\frac{x+1}{x} + 1\right) = |x| \, \frac{x+1+x}{x} = \frac{\overset{\neq 0}{|x|}}{x}(2x+1)$$

$f'(x) \geq 0 \quad 2x+1 \geq 0 \quad 2x \geq -1 \quad x \geq -\frac{1}{2}$

$\quad x > 0$

segno f'



$x = -\frac{1}{2}$ p.to di massimo relativo

$0 \in \text{dom} f$ quindi anche se f non è derivabile in 0 è un p.to di minimo relativo di f

$x \neq 0 \quad f'(x) = \frac{|x|}{x}(2x+1)$

$x \to 0^+ \quad f'(x) = (2x+1) \to 1 \quad \Rightarrow f'_+(0) = 1$

$x \to 0^- \quad f'(x) = -(2x+1) \to -1 \quad \Rightarrow f'_-(0) = -1$

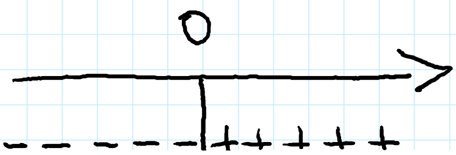
$\Rightarrow x = 0$ è un p.to angoloso

Derivata seconda:

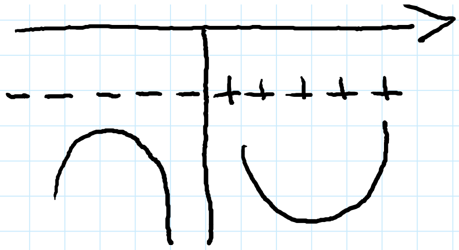
$$f'(x) = \begin{cases} 2x+1 & x > 0 \\ -2x-1 & x < 0 \end{cases} \qquad \nexists \, f'(0)$$

$$f''(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \end{cases}$$

segno f''



segno f''



$x=0$ p.to di flesso

- Al variare di $\lambda \in \mathbb{R}$ det. il numero di sol. dell'eq. $f(x) = \lambda$

Se $\lambda < 0$ 1 soluzione
se $\lambda = 0$ 2 sol.
se $0 < \lambda < f(-\frac{1}{2})$ 3 sol.
se $\lambda = f(-\frac{1}{2})$ 2 sol.
se $\lambda > f(-\frac{1}{2})$ 1 sol.

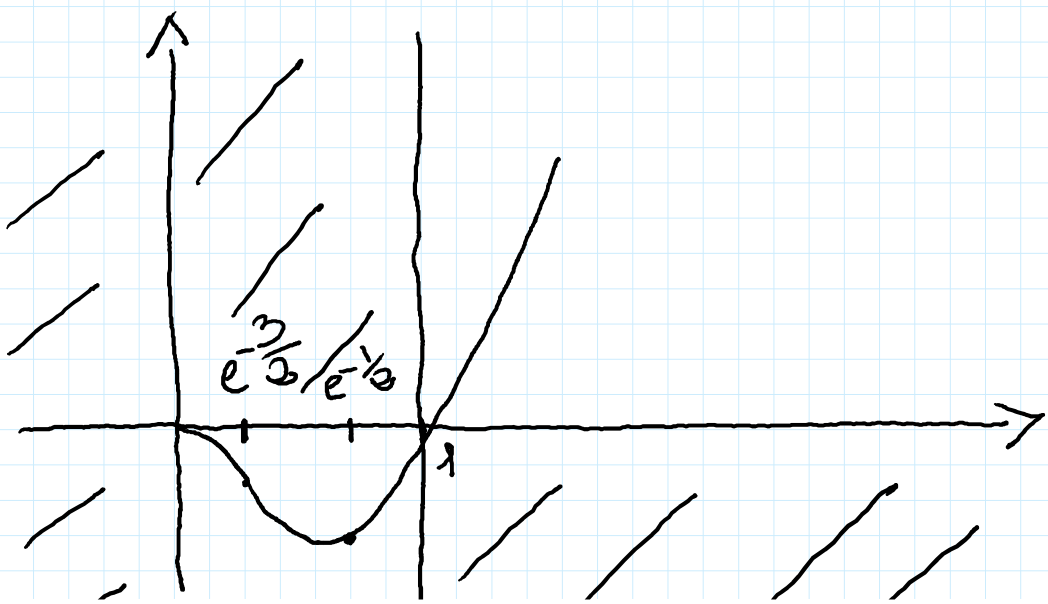
Estremi di f: $\inf_{\mathbb{R}} f = -\infty$, $\sup_{\mathbb{R}} f = +\infty$

- $f(x) = x^2 \log_e x$

$x>0$ $\quad$ dom $f = (0,+\infty)$

$f(x) \geq 0 \underset{x^2 \geq 0}{\iff} \log_e x \geq 0 \iff x \geq 1$

$(1,0) \in$ Graf f



Limiti: $0, +\infty$

$x \to 0 \quad f(x) = x^2 \log_e x \to 0$

$x \to +\infty \quad f(x) \to +\infty \qquad [(+\infty)\cdot(+\infty)]$

Asintoto obliquo? $\quad \dfrac{f(x)}{x} = x \log_e x \to +\infty \quad$ NO

Derivata prima: $\forall x \in (0, +\infty)$

$$f'(x) = 2x \log_e x + x^2 \cdot \frac{1}{x} = 2x \log_e x + x \cancel{\to}$$

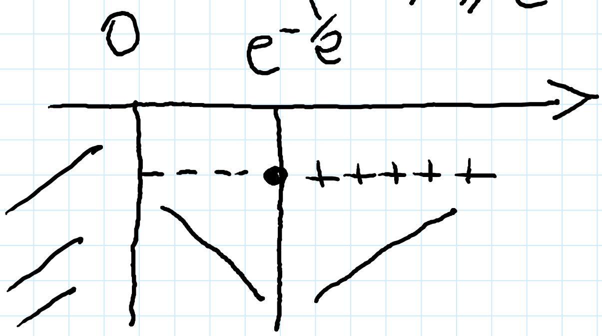
$$= x(2\log_e x + 1) \qquad \underset{x\to 0}{\searrow 0} \quad \searrow 0$$

$f'(x) \geq 0 \qquad x > 0$ vero in dom f

$2\log_e x + 1 \geq 0 \quad 2\log_e x \geq -1 \quad \log_e x \geq -\frac{1}{2}$

$x \geq e^{-1/2}$

f':



$x = e^{-1/2}$ p.to di min. relativo $\quad f(e^{-1/2}) = e^{-1}\left(-\frac{1}{2}\right)$

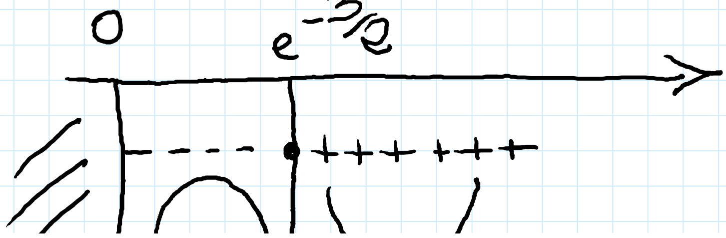
$= -\dfrac{e^{-1}}{2}$

Derivata seconda: $\forall x \in (0, +\infty)$

$$f''(x) = 2\log_e x + 1 + x \cdot \frac{2}{x} = 2\log_e x + 3$$

$f''(x) \geq 0 \qquad \log_e x \geq -\dfrac{3}{2} \qquad x \geq e^{-3/2}$

f'':



$x \to 0 \quad f'(x) \to 0$

- Estremi di f : $\sup_{(0,+\infty)} f = +\infty$

  $\min_{(0,+\infty)} f = f(e^{-\frac{1}{2}})$

- $\operatorname{Im} f = [f(e^{-1/2}), +\infty)$

- $f(x) = \lambda$ ha

  se $\lambda < f(e^{-1/2})$ 0 sol

  se $\lambda = f(e^{-1/2})$ 1 sol.

  se $f(e^{-1/2}) < \lambda < 0$ 2 sol.

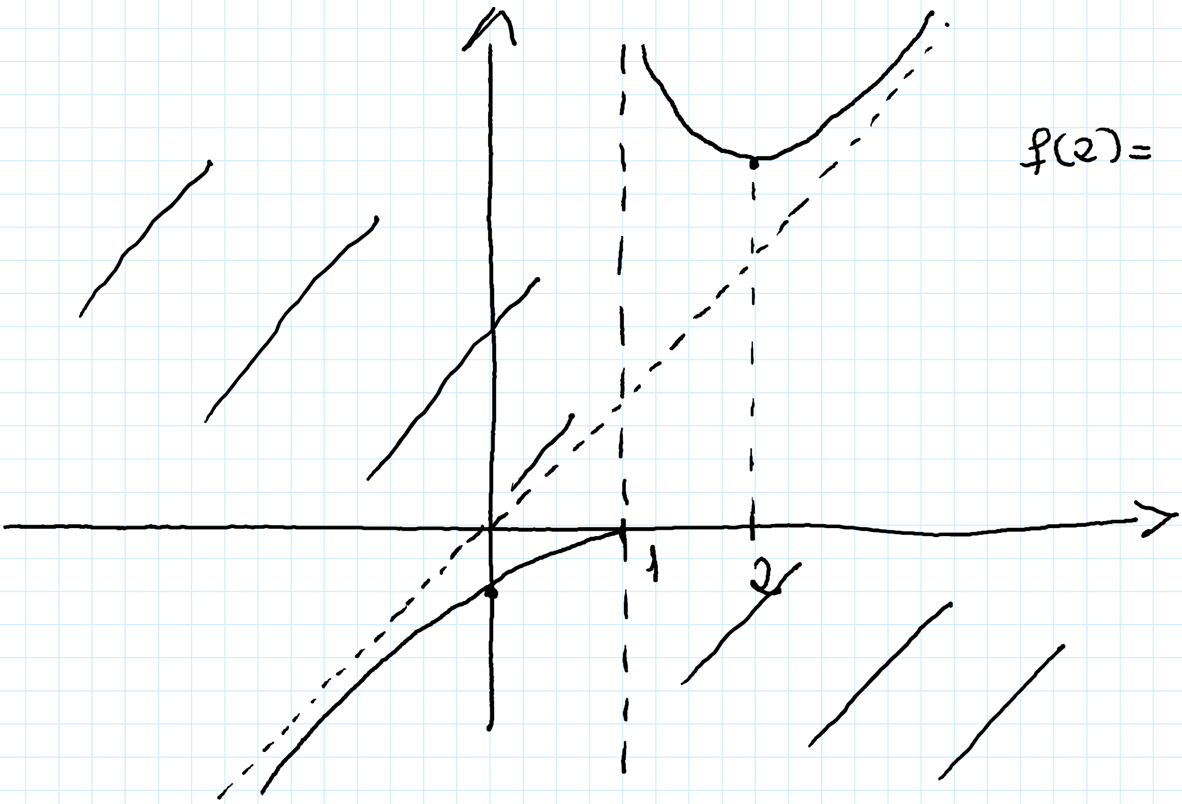
  se $\lambda \geq 0$ 1 sol.

- $f(x) = (x-1)\, e^{\frac{1}{x-1}}$

  $x-1 \neq 0 \quad x \neq 1 \qquad \text{dom} f = \mathbb{R} \setminus \{1\}$

  $f(0) = -e^{-1} \qquad (0, -e^{-1})$

  $f(x) > 0 \iff x-1>0 \quad x>1 \qquad x=1 \notin \text{dom} f$



$f(2) = e$

Limiti: $1 \quad \pm\infty$

$x \to 1 \qquad (x-1) \to 0 \qquad \frac{1}{x-1} \to \left[\frac{1}{0}\right]$

$x \to 1^+ \qquad \frac{1}{x-1} \to +\infty \qquad e^{\frac{1}{x-1}} \to +\infty$

$f(x) \to [0(+\infty)]$?

$$f(x) = \frac{e^{\frac{1}{x-1}}}{\frac{1}{x-1}} \qquad \begin{matrix} +\infty \\ +\infty \end{matrix}$$

$$\lim_{x\to 1^+} f(x) = \lim_{y\to+\infty} \frac{e^y}{y} = +\infty$$

$\to x = 1$ as.

$$\lim_{x \to 1^+} f(x) = \lim_{\substack{y \to +\infty \\ \frac{1}{x-1} = y}} \frac{e^y}{y} = +\infty \Rightarrow x = 1 \text{ as. verticale}$$

$x \to 1^-$ $\quad (x-1) \to 0$

$$\left.\begin{array}{l} (x-1) \to 0 \\ \frac{1}{x-1} \to -\infty \Rightarrow e^{\frac{1}{x-1}} \to 0 \end{array}\right\} \Rightarrow$$

$$f(x) \to 0 \cdot 0 = 0$$

$x \to +\infty$ $\quad e^{\frac{1}{x-1}} \to e^0 = 1$

$$f(x) \sim (x-1) \cdot 1 \to +\infty$$

As. obliquo: $\quad \frac{f(x)}{x} \sim \frac{x-1}{x} \to 1$

$$f(x) - x = (x-1)e^{\frac{1}{x-1}} - x =$$

$$= \underbrace{x\left(e^{\overset{\to 0}{\frac{1}{x-1}}} - 1\right)}_{\sim\, x \cdot \frac{1}{x-1} \to 1} - \underbrace{e^{\frac{1}{x-1}}}_{\to\, e^0 = 1}$$

$$f(x) - x \to 1 - 1 = 0$$

$y = x$ asintoto obliquo a $+\infty$

$x \to -\infty$ $\quad f(x) \to [-\infty \cdot \underset{1}{e^0}] = -\infty$

$$\frac{f(x)}{x} \sim \frac{x-1}{x} \to 1$$

$$f(x) - x \to 0$$

$y = x$ asintoto obliquo a $-\infty$.

Derivata prima: $\forall x \in \operatorname{dom} f$

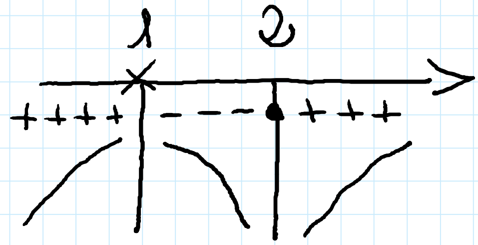
$$f'(x) = 1 \cdot e^{\frac{1}{x-1}} + \cancel{(x-1)} \cdot e^{\frac{1}{x-1}} \cdot \left(-\frac{1}{(x-1)^{\cancel{2}}}\right)$$

$$f'(x) = 1 \cdot e^{\frac{1}{x-1}} + \cancel{(x-1)} \cdot e^{\frac{1}{x-1}} \cdot \left(-\frac{1}{(x-1)^{\cancel{2}}}\right)$$

$$= e^{\frac{1}{x-1}} \left(1 - \frac{1}{x-1}\right) = e^{\frac{1}{x-1}} \; \frac{x-2}{x-1}$$

$f'(x) \geq 0 \qquad \frac{x-2}{x-1} \geq 0 \qquad x < 1 \qquad x \geq 2$

f'



$x = 2$ p.to di minimo relativo

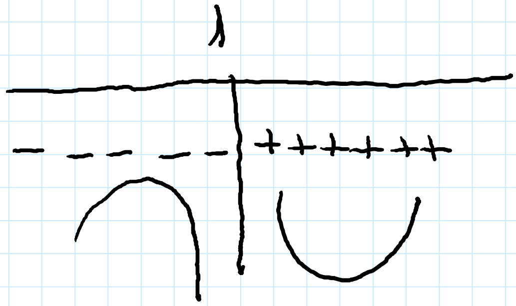
Derivata seconda: $\forall x \in \operatorname{dom} f$

$$f''(x) = e^{\frac{1}{x-1}} \left(-\frac{1}{(x-1)^2}\right) \frac{x-2}{x-1} + e^{\frac{1}{x-1}} \cdot \frac{\cancel{x}-1-\cancel{x}+2}{(x-1)^2}$$

$$= e^{\frac{1}{x-1}} \left[-\frac{x-2}{(x-1)^3} + \frac{1}{(x-1)^2}\right]$$

$$= e^{\frac{1}{x-1}} \; \frac{-\cancel{x}+2+\cancel{x}-1}{(x-1)^3} = e^{\frac{1}{x-1}} \; \frac{1}{(x-1)^3}$$

$f''(x) > 0 \qquad x > 1$



$$\operatorname{Im} f = (-\infty, 0) \cup \left[\overset{\overset{1}{\shortparallel}}{f}(2), +\infty\right)$$

- $f(x) = x - 2\operatorname{arctg} x$

dom $f = \mathbb{R}$

f è dispari $\quad f(-x) = -f(x)$

$\left.\begin{array}{l} f(0)=0 \\ f(x)\geq 0 \end{array}\right\}$ non si possono studiare

Limiti : $x \to \pm\infty$

$x \to +\infty \qquad f(x) \to \left[+\infty - 2\cdot\frac{\pi}{2}\right] = +\infty$

$$\frac{f(x)}{x} = 1 - 2\,\frac{\operatorname{arctg} x}{x} \to 1$$

$$f(x) - x = -2\operatorname{arctg} x \to -2\cdot\frac{\pi}{2} = -\pi$$

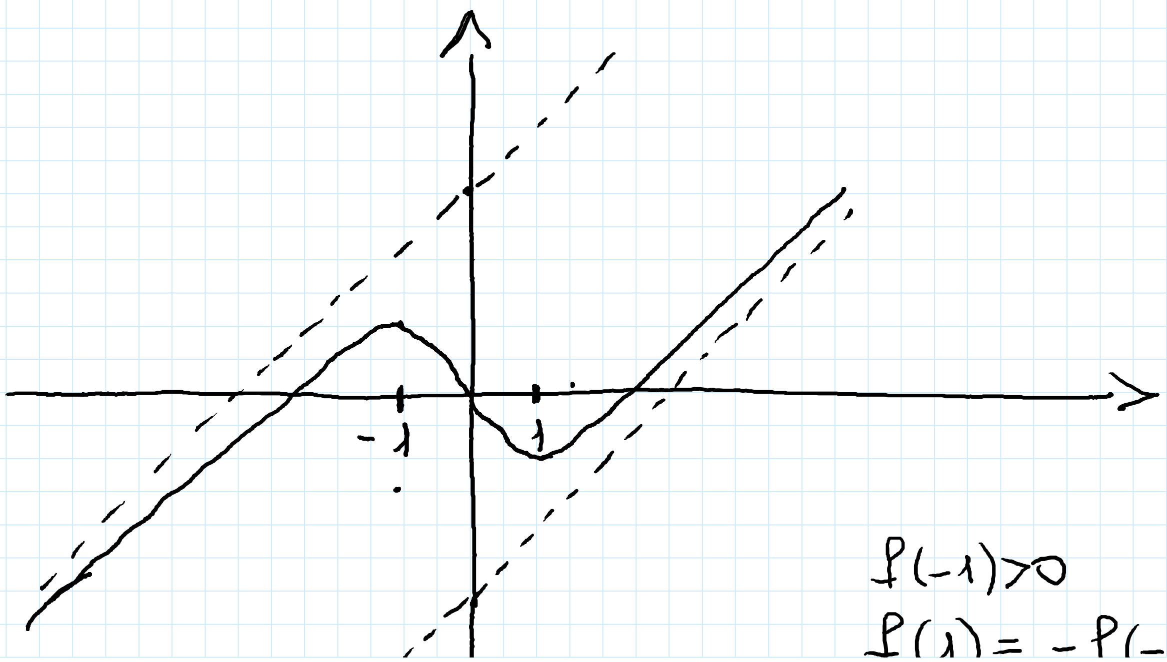
Asintoto obliquo a $+\infty \qquad y = x - \pi$

$x \to -\infty \qquad f(x) \to -\infty \qquad \left[-\infty - 2\left(-\frac{\pi}{2}\right)\right]$

$$\frac{f(x)}{x} = 1 - 2\,\underbrace{\frac{\operatorname{arctg} x}{x}}_{\to 0} \to 1$$

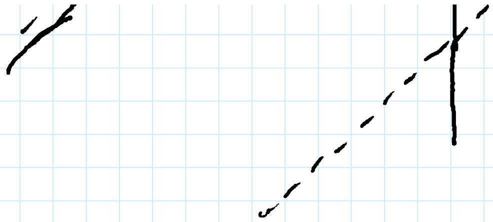
$$f(x) - x = -2\operatorname{arctg} x \to -2\left(-\frac{\pi}{2}\right) = \pi$$

$y = x + \pi$ as. obliquo a $-\infty$



$f(-1) > 0$

$f(1) = -f(-1) < 0$



$f(-1)>0$

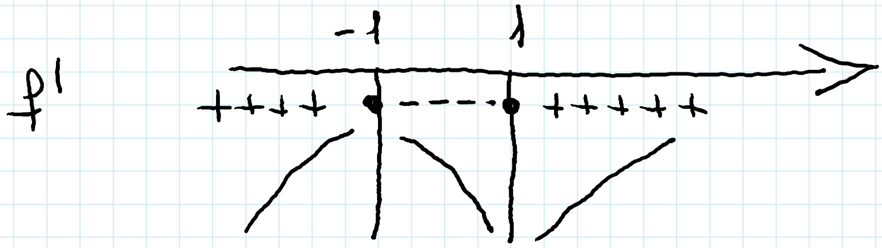
$f(1) = -f(-1) < 0$

Derivata prima: $\forall x \in \mathbb{R}$

$$f'(x) = 1 - 2\,\frac{1}{1+x^2} = \frac{1+x^2-2}{1+x^2} = \frac{x^2-1}{x^2+1}$$

∇ $\quad x^2+1>0$

$f'(x) \geq 0 \qquad x^2-1 \geq 0 \qquad x \leq -1 \qquad x \geq 1$

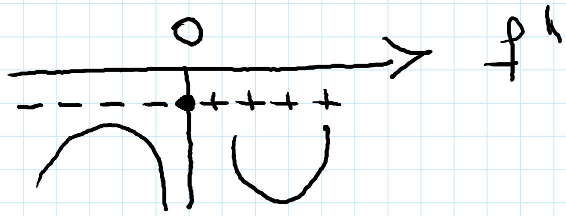


$x = -1$ p.to di max rel.

$x = 1$ p.to di min rel.

$$f''(x) = +2\,\frac{2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

$f''(x) \geq 0 \qquad x \geq 0$



$x=0$ p.to di flesso