Analisi Matematica - 8.3.2019 - prima parte

Le funzioni elementari

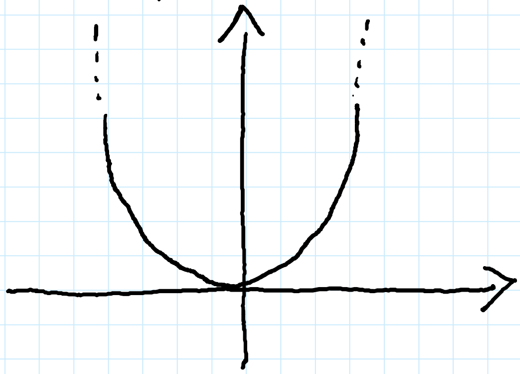
Funzioni potenza

Esponente naturale : sia $n \in \mathbb{N}$, $n \neq 0, 1$ ($n = 1$ già visto)

$f : x \in \mathbb{R} \to x^n \in \mathbb{R}$

Grafici :

n pari



$\text{dom}\, f = \mathbb{R}$

$\text{Im}\, f = [0, +\infty)$

f è pari

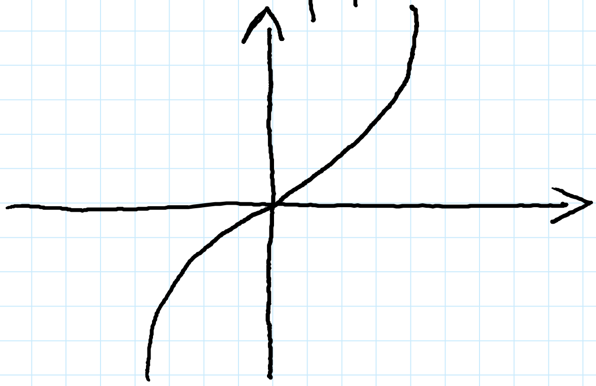
$0 = \min\limits_{\mathbb{R}} f = f(0)$

$\sup\limits_{\mathbb{R}} f = +\infty$

f è strett. decrescente in $(-\infty, 0]$

f è strett. crescente in $[0, +\infty)$

n dispari



$\text{dom}\, f = \mathbb{R}$

$\text{Im}\, f = \mathbb{R}$

f è dispari

$\inf\limits_{\mathbb{R}} f = -\infty$

$\sup\limits_{\mathbb{R}} f = +\infty$

f è strett. crescente

Esponente $\frac{1}{n}$ $n \neq 0, 1$

$x \mapsto x^n$ è strett. crescente se n è dispari

se n è pari in $[0, +\infty)$

In questi casi esiste la funzione inversa :

$y \in \mathbb{R}$ n dispari

$y \in [0, +\infty)$ n pari $(\text{Im}\, f = [0, +\infty))$

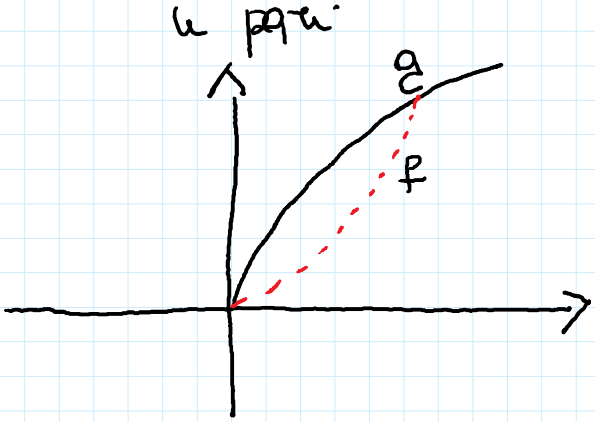
$$f(x) = y$$
$$x^n = y \qquad x = \sqrt[n]{y} = y^{\frac{1}{n}}$$

La funzione potenza con esp. $\frac{1}{n}$ è l'inversa della funz. pot. con esponente n.

$g_c : \mathbb{R} \to \mathbb{R} \quad x \longmapsto x^{\frac{1}{n}} \qquad n$ dispari

$g_c : [0,+\infty) \to [0,+\infty) \quad x \mapsto x^{\frac{1}{n}} \qquad n$ pari

n pari



$\text{dom } g_c = [0,+\infty)$

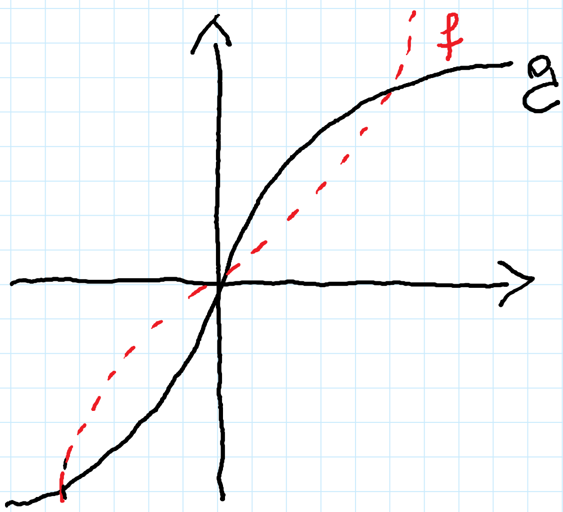
$\text{Im } g_c = [0,+\infty)$

$\min_{[0,+\infty)} g_c = 0 = g_c(0)$

$\sup_{[0,+\infty)} g_c = +\infty$

g_c è strett. crescente

n dispari



$\text{dom } g_c = \mathbb{R} \quad \text{Im } g_c = \mathbb{R}$

$\inf_{\mathbb{R}} g_c = -\infty \qquad \sup_{\mathbb{R}} g_c = +\infty$

g_c è strett. crescente

Eq. e disequazioni

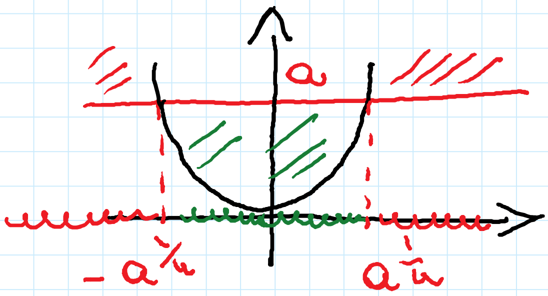
$x^n = a \qquad a \in \mathbb{R}$

n dispari : $x = a^{1/n}$

n pari : $a < 0 \quad \emptyset$

$a = 0 \qquad x = 0$

$a > 0 \qquad x = \pm a^{\frac{1}{n}}$



Disequazioni : $a \in \mathbb{R}$

n dispari

$$x^n \geq a \iff x^n \geq \left(a^{\frac{1}{n}}\right)^n \underset{\substack{x^n \text{ è} \\ \text{strett. crescente}}}{\iff} x \geq a^{\frac{1}{n}}$$

$$x^n > a \iff x > a^{1/n}$$

$$x^n \leq a \iff x \leq a^{1/n}$$

$$x^n < a \iff x < a^{1/n}$$

n pari

Se $a < 0$

$x^n \geq a$	$\mathbb{R}$
$>$	$\mathbb{R}$
$x^n \leq a$	$\emptyset$
$<$	$\emptyset$

Se $a = 0$

$x^n \geq 0$	$\mathbb{R}$
$x^n > 0$	$\mathbb{R} \setminus \{0\}$
$x^n \leq 0$	$\{0\}$
$x^n < 0$	$\emptyset$

Se $a > 0$

$$x^n \geq a \iff x \leq -a^{1/n} \ \vee \ x \geq a^{1/n}$$

$$x^n > a \iff x < -a^{1/n} \ \vee \ x > a^{1/n}$$

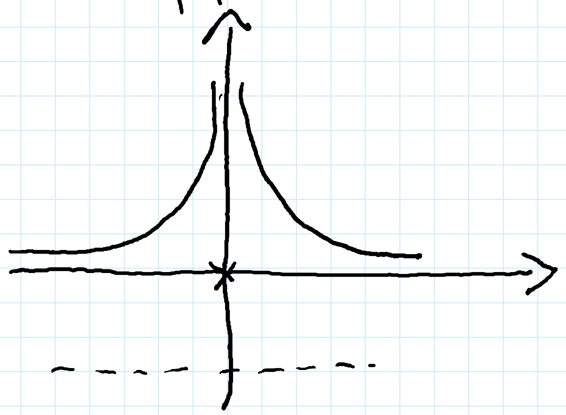
$$x^n \leq a \iff -a^{1/n} \leq x \leq a^{\frac{1}{n}}$$

$$x^n < a \iff -a^{1/n} < x < a^{\frac{1}{n}}$$

Esponente intero negativo : $n \in \mathbb{N}$, $n \neq 0$

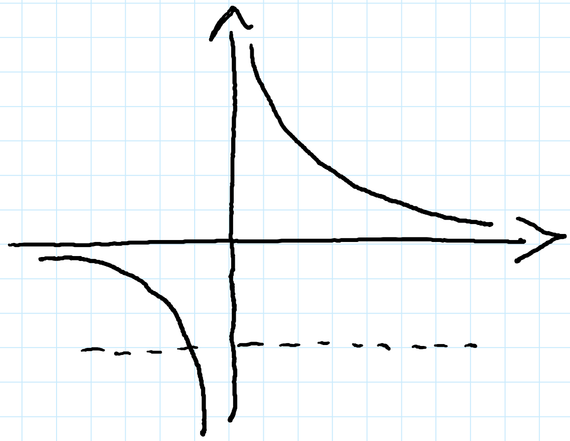
$f: x \to x^{-n}$ $\qquad$ $\text{dom } f = \mathbb{R} \setminus \{0\}$

n pari



$\text{Im } f = (0, +\infty)$

n dispari



$\text{Im } f = \mathbb{R} \setminus \{0\}$

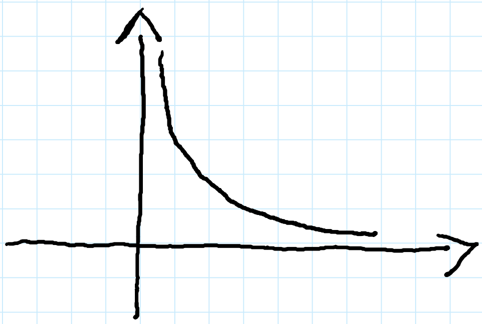
Esponente reale: $\alpha \in \mathbb{R} \setminus \{0\}$

$f(x) = x^{\alpha}$ definita in $[0, +\infty)$ se $\alpha > 0$

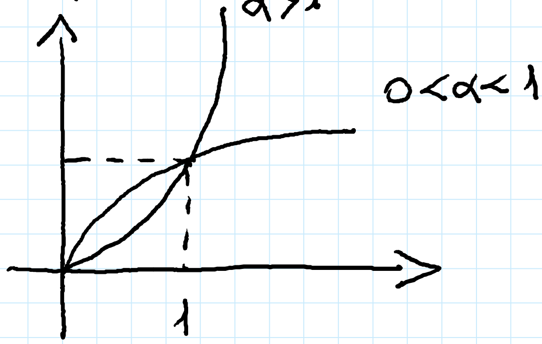
$(0, +\infty)$ se $\alpha < 0$

Grafici:

$\alpha < 0$



$\alpha > 0$



Esempi:

Data $f(x)$, det. il più grande sottoinsieme di $\mathbb{R}$ per cui $f(x)$ è ben definita.

"Det. il dominio di f"

1. $f(x) = \sqrt{x^3 - 1}$

$f(x) = g(h(x)) \qquad h(x) = x^3 - 1$

$g(x) = \sqrt{x} \qquad \text{dom}\, g = [0, +\infty)$

$\Rightarrow \; h(x) \in \text{dom}\, g \iff h(x) \geq 0$

$x^3 - 1 \geq 0$

$x^3 \geq 1 = 1^3$

$x \geq 1 \qquad \text{dom}\, f = [1, +\infty)$

2. $f(x) = \dfrac{1}{x^2 - 1}$

$x^2 - 1 \neq 0$

$x^2 \neq 1$

$x \neq \pm 1 \qquad \text{dom}\, f = \mathbb{R} \setminus \{\pm 1\}$

3. $f(x) = \dfrac{1}{\sqrt{x^2 - 4}}$

$\begin{cases} \sqrt{x^2 - 4} \neq 0 \\ x^2 - 4 \geq 0 \end{cases} \qquad \left(\dfrac{1}{2}\right)$

$\Downarrow$

$x^2 - 4 > 0$

$x^2 > 4 = 2^2$

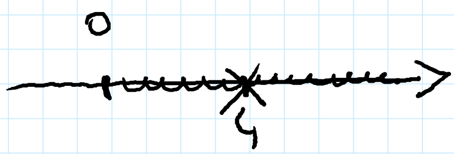
$x < -2$ oppure $x > 2 \qquad \text{dom}\, f = (-\infty, -2) \cup (2, +\infty)$

4. $f(x) = \dfrac{1}{\sqrt{x} - 2}$

$\sqrt{x} - 2 \neq 0$ (denominatore

$x \geq 0$ (per $\sqrt{x}$)

$\begin{cases} \sqrt{x} - 2 \neq 0 \\ x \geq 0 \end{cases} \qquad \begin{cases} x \neq 4 \\ x \geq 0 \end{cases}$



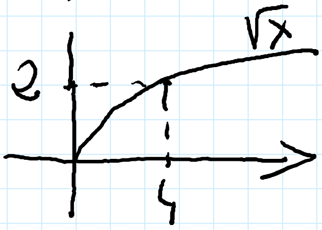
$\sqrt{x}$

$\begin{cases} x \geq 0 \end{cases}$ $\quad$ $\begin{cases} x \geq 0 \end{cases}$ $\quad$ 4

$\sqrt{x} - 2 \neq 0$

$\sqrt{x} \neq 2 = \sqrt{4}$

$x \neq 4$



$\mathrm{dom}\, f = [0,4) \cup (4, +\infty)$

Analisi Matematica - 8.3.2019 - seconda parte

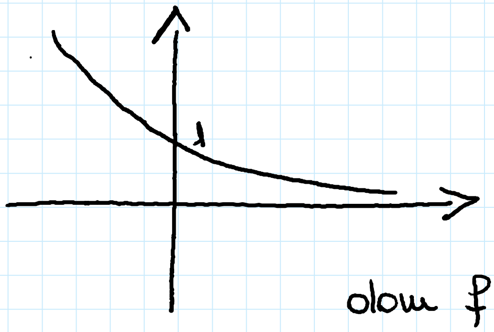
Funzioni esponenziali e logaritmo

Sia $a \in (0,+\infty), a \neq 1$ $\quad (1^x = 1)$

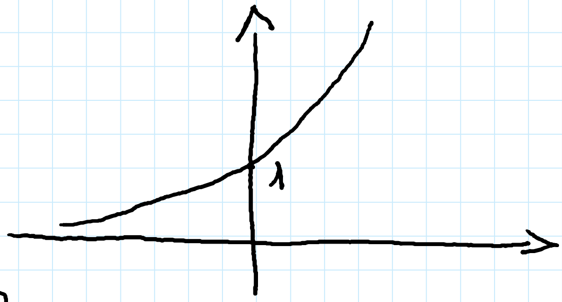
Sia $f : \mathbb{R} \to \mathbb{R} \quad x \longmapsto a^x$

Grafici

$0<a<1$



$a>1$



$$\text{dom } f = \mathbb{R}$$
$$\text{Im } f = (0,+\infty)$$
$$\inf_{\mathbb{R}} f = 0 \qquad \sup_{\mathbb{R}} f = +\infty$$

f è strett. decrescente $\qquad$ f è strett. crescente

In entrambi i casi f è iniettiva, quindi ammette funzione inversa:

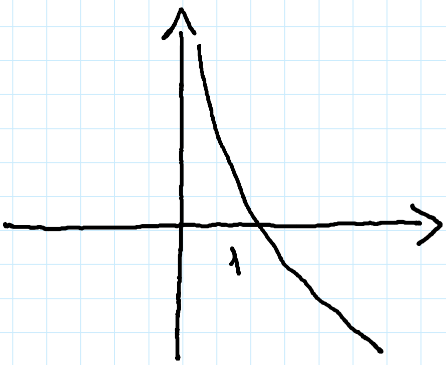
$$a^x = y \qquad y \in (0,+\infty)$$
$$x = \log_a y$$

È la funzione logaritmo.

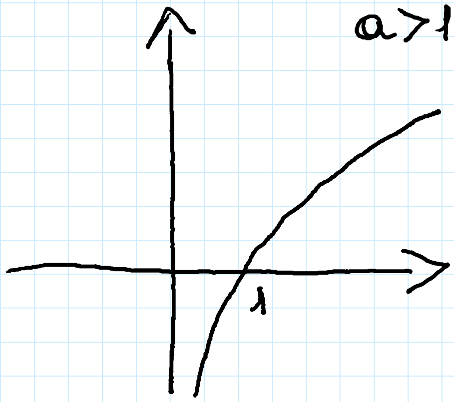
$$g : (0,+\infty) \longrightarrow \mathbb{R}$$
$$x \longmapsto \log_a x$$

Grafici:

$0<a<1$



$a>1$



$\text{dom } g_c = (0, +\infty)$ $\quad$ $\text{Im } g_c = \mathbb{R}$

$\inf_{(0,+\infty)} g_c = -\infty$ $\quad$ $\sup_{(0,+\infty)} g_c = +\infty$

$0 < a < 1$ $\quad$ g_c è strett. decrescente

$a > 1$ $\quad$ " " " crescente

Esempi:

1. $f(x) = \sqrt{\log_c x - 2}$ $\qquad$ $\log_c x = \log_{c_e} x = \ln x$

$e > 1$

$$\begin{cases} \log_c x - 2 \geq 0 \\ x > 0 \end{cases} \qquad \begin{cases} x \geq e^2 \\ x > 0 \end{cases} \qquad x \geq e^2$$

$\log_c x - 2 \geq 0$

$\log_c x \geq 2 = \log_c e^2$ $\quad$ $\overset{\log \text{ strett. crescente}}{\Longrightarrow}$

$x \geq e^2$

$\text{dom } f = [e^2, +\infty)$

2. $f(x) = \log_c \left(e^x + \frac{1}{2}\right)$

$e^x + \frac{1}{2} > 0$

$\underbrace{e^x}_{>0} > \underbrace{-\frac{1}{2}}_{<0} \qquad \forall x \in \mathbb{R}$

$\forall x \in \mathbb{R}$

$\text{dom } f = \mathbb{R}$

3. $f(x) = \log_c \left(e^x - \frac{1}{2}\right)$

$e^x - \frac{1}{2} > 0$

$e^x > \frac{1}{2} = e^{\log_c \frac{1}{2}}$ $\quad$ $\overset{e^x \text{ strett. crescente}}{\Longrightarrow}$

$x > \log_c \frac{1}{2} = \underbrace{\log_c 1}_{=0} - \log_c 2 = -\log_c 2$

$\text{dom } f = (-\log_c 2, +\infty)$

4. $f(x) = \sqrt{1 - \log_c x}$

4. $f(x) = \sqrt{1-\log_e x}$

$$\begin{cases} x>0 \\ 1-\log_e x \geq 0 \end{cases} \qquad \begin{cases} x>0 \\ x \leq e \end{cases}$$

$1-\log_e x \geq 0$

$-1+\log_e x \leq 0$ $\qquad$ $\log_e$ strett. cresc.

$\log_e x \leq 1 = \log_e e$ $\qquad \Rightarrow$

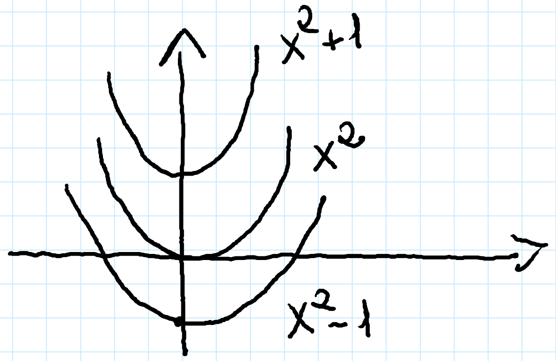
$x \leq e$

$\text{dom} f = (0, e]$

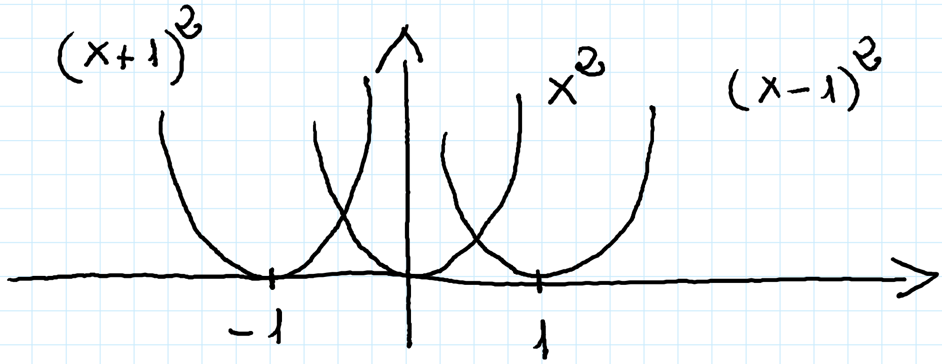
Operazioni con i grafici

Sia dato il grafico di $f(x)$. Come ottengo il grafico di

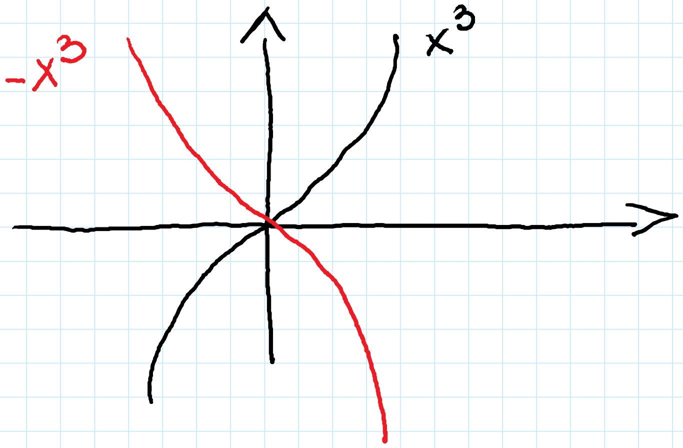
- $g(x) = f(x)+a$ traslazione verticale $\uparrow$ se $a>0$, $\downarrow$ se $a<0$



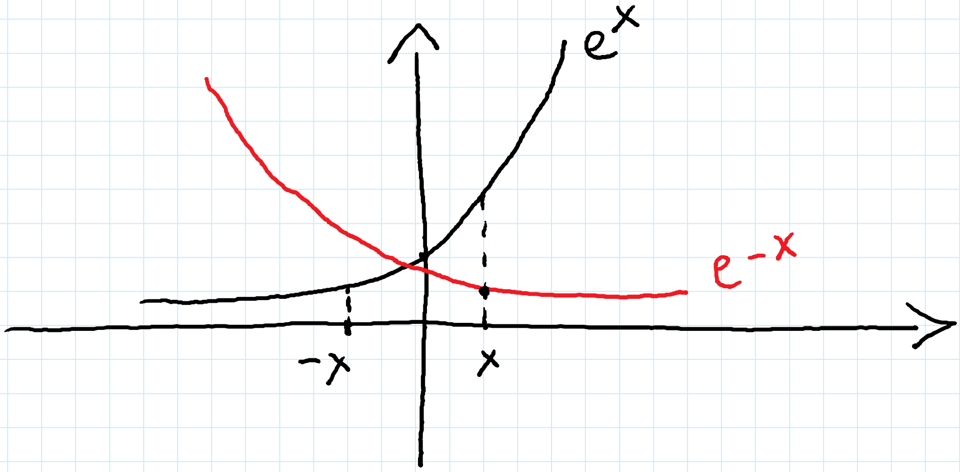
- $g(x) = f(x+a)$ traslazione orizzontale
  $\leftarrow$ se $a>0$
  $\rightarrow$ se $a<0$



• $g(x) = -f(x)$ ribalto rispetto asse x

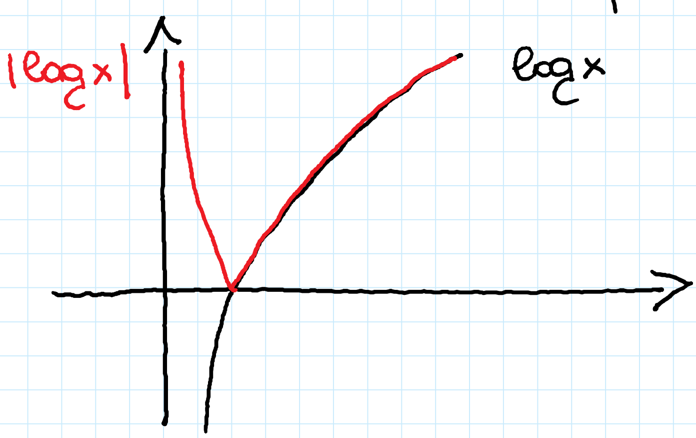


• $g(x) = f(-x)$ ribalto rispetto asse y



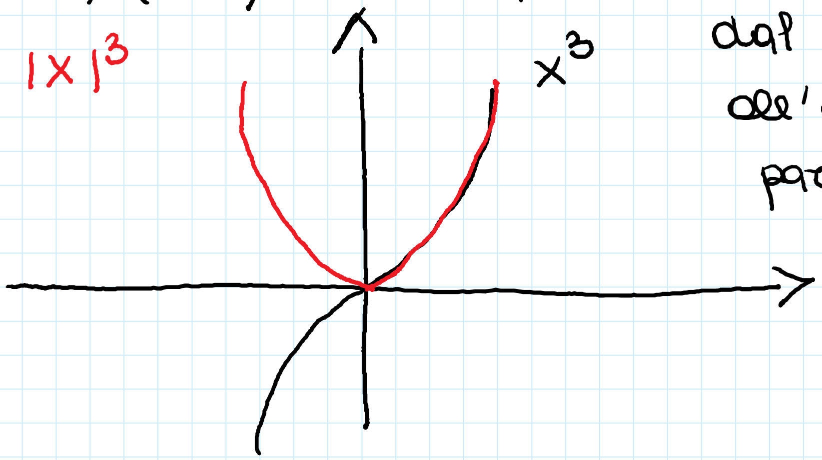
$g(x) = e^{-x}$

• $g(x) = |f(x)|$ ribalto rispetto all'asse x le parti negative

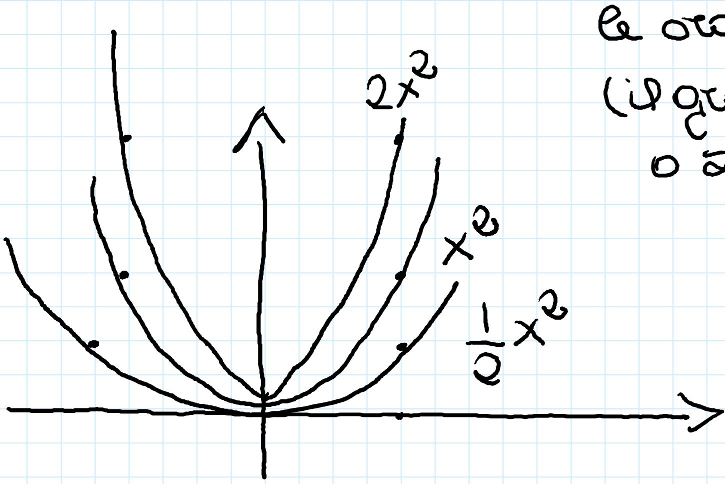


$g(x) = f(|x|)$

- $g(x) = f(|x|)$ sparisce la parte $x \leq 0$ sostituita dal ribaltato rispetto all'asse y della parte con $x \geq 0$



- $g(x) = a\,f(x)$ $a > 0$ moltiplico per a tutte le ordinate (il grafico si restringe o si allarga)



Applicazione: polinomi quadratici

Analisi Matematica - 8.3.2019 - terza parte

Thursday, March 7, 2019 17:16

Polinomi quadratici

Grafico di $g(x) = ax^2+bx+c$ dato il grafico di $f(x) = x^2$

$a, b, c \in \mathbb{R}, \quad a \neq 0$

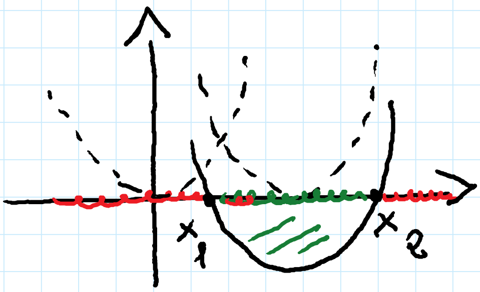
Prendo $a>0$

$$g(x) = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$
$$= a\left[x^2+2\frac{b}{2a}\cdot x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right]$$
$$= a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$$

$\Delta = b^2-4ac$

$$g(x) = a\left[\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right]$$

1. $\Delta>0$



$g(x) = 0 \qquad \left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2} = 0$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2} > 0$$
$$x+\frac{b}{2a} = \pm\frac{\sqrt{\Delta}}{2a}$$
$$x = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x_1 = \frac{-b-\sqrt{\Delta}}{2a} \qquad x_2 = \frac{-b+\sqrt{\Delta}}{2a}$$

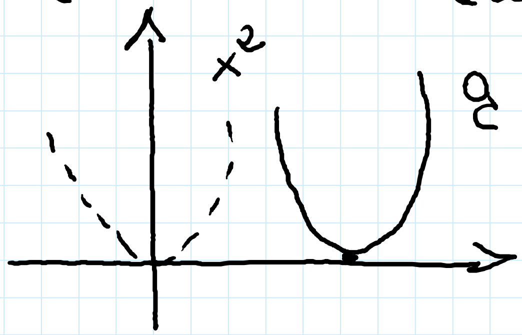
$g(x) \geq 0 \iff x \leq x_1 \text{ o } x \geq x_2$

$g(x) > 0 \iff x < x_1 \text{ o } x > x_2$

$g(x) \leq 0 \iff x_1 \leq x \leq x_2$

$g(x) < 0 \iff x_1 < x < x_2$

2. $\Delta = 0$ $\qquad g(x) = a\left(x + \frac{b}{2a}\right)^2$



$g(x) = 0 \iff x = -\frac{b}{2a}$

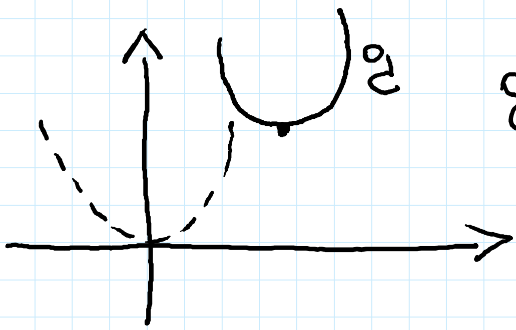
$g(x) \geq 0 \qquad x \in \mathbb{R}$

$g(x) > 0 \qquad x \in \mathbb{R} \setminus \{-b/2a\}$

$g(x) \leq 0 \qquad x = -\frac{b}{2a}$

$g(x) < 0 \qquad \emptyset$

3. $\Delta < 0$



$g(x) > 0 \ \forall x \in \mathbb{R} \Rightarrow$
$g(x) \geq 0$ e $g(x) \neq 0$
$\Rightarrow g(x) \geq 0$

$g(x) = 0 \qquad \emptyset$

$g(x) > 0 \qquad \mathbb{R}$

$g(x) \geq 0 \qquad \mathbb{R}$

$g(x) \leq 0 \qquad \emptyset$

$g(x) < 0 \qquad \emptyset$

Se $a < 0$?

$ax^2+bx+c \geq 0$

$\underbrace{-a}_{\vee \atop 0}x^2-bx-c \leq 0$ → ricado nei casi precedenti

- $f(x)=\sqrt{x^2-x-2}$

$x^2-x-2 \geq 0$

$\Delta = 1+8=9>0 \qquad x=\frac{1\pm\sqrt{9}}{2}=\frac{1\pm3}{2}=\frac{1\pm3}{2}$

$x \leq -1 \quad x \geq 2 \qquad = \begin{cases} -1 \\ 2 \end{cases}$

$\text{dom } f = (-\infty,-1] \cup [2,+\infty)$

- $f(x)=\frac{x}{x^2+2x+1}$

$x^2+2x+1 \neq 0$

$\Delta = 0$

$(x+1)^2 \neq 0$

$x \neq -1$

$\text{dom } f = \mathbb{R} \setminus \{-1\}$

- $f(x) = \log_2 (1+2x-3x^2)$

$1+2x-3x^2>0$

$3x^2-2x-1<0$

$\Delta = 4+12=16 \qquad x=\frac{2\pm\sqrt{16}}{6}=\frac{2\pm4}{6} \begin{cases} -\frac{1}{3} \\ 1 \end{cases}$

$-\frac{1}{3}<x<1$

$\text{dom } f = \left(-\frac{1}{3},1\right)$

- $f(x) = (x^2+2x+3)^{-\pi}$

$x^2+2x+3>0 \qquad x\in\mathbb{R}$

$\Delta = 4-12<0$

$\operatorname{dom} f = \mathbb{R}$

Funzioni periodiche

Def: Sia $A\subseteq\mathbb{R}$, $f: A\to\mathbb{R}$ e $T>0$.
Si dice che f è T-periodica se

- $\forall x\in A \quad x+T\in A$
- $\forall x\in A \quad f(x+T)=f(x)$

Oss: f periodica $\Rightarrow$ f non iniettiva

$x\neq x+T$ ma $f(x)=f(x+T)$

f T-periodica $\Rightarrow$ f KT-periodica $\forall K\in\mathbb{N}\setminus\{0\}$

$$f(x+2T)=f(\underbrace{x+T}+T)=f(x+T)=f(x)$$

$$f(x+KT)=f(x) \quad \forall K\in\mathbb{N}\setminus\{0\}$$

Come è fatto il grafico di una funzione periodica?
Se conosco graf f in un intervallo di ampiezza T, basta traslarlo.

$T=1$