# Scritto di Analisi Matematica - 6.9.2017

1) $f(x) = \operatorname{arctg} \frac{1}{x} + \frac{1}{2} \log(1+x^2)$

(a) Dominio : $x \neq 0$
$1+x^2 > 0 \quad \forall x \in \mathbb{R}$

$\operatorname{dom} f = \mathbb{R} \setminus \{0\}$

Limiti significativi: $0, \pm\infty$

$$x \to 0^+ \quad \left.\begin{array}{l} \frac{1}{x} \to +\infty \Rightarrow \operatorname{arctg} \frac{1}{x} \to \frac{\pi}{2} \\ 1+x^2 \to 1 \Rightarrow \log(1+x^2) \to 0 \end{array}\right\} \Rightarrow$$

$$f(x) \to \frac{\pi}{2}$$

$$x \to 0^- \quad \left.\begin{array}{l} \frac{1}{x} \to -\infty \Rightarrow \operatorname{arctg} \frac{1}{x} \to -\frac{\pi}{2} \\ 1+x^2 \to 1 \Rightarrow \log(1+x^2) \to 0 \end{array}\right\} \Rightarrow$$

$$f(x) \to -\frac{\pi}{2}$$

$$x \to \pm\infty \quad \left.\begin{array}{l} \frac{1}{x} \to 0 \Rightarrow \operatorname{arctg} \frac{1}{x} \to 0 \\ 1+x^2 \to +\infty \Rightarrow \log(1+x^2) \to +\infty \end{array}\right\} \Rightarrow$$

$$f(x) \to +\infty$$

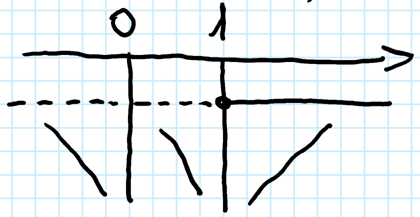
(b) $\forall x \in \operatorname{dom} f$

$$f'(x) = \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{\cancel{2}} \frac{1}{1+x^2} \cdot \cancel{2}x$$

$$= -\frac{\cancel{x^2}}{x^2+1} \cdot \frac{1}{\cancel{x^2}} + \frac{x}{1+x^2}$$

$$= \frac{-1+x}{1+x^2}$$

$f'(x) \geq 0 \iff x-1 \geq 0 \iff x \geq 1$




$f$ è decrescente in $(-\infty, 0)$ e in $(0,1)$,
$f$ è crescente in $(1, +\infty)$.
Il punto $x=1$ è un punto di min. relativo

(c) $\forall x \in \operatorname{dom} f$

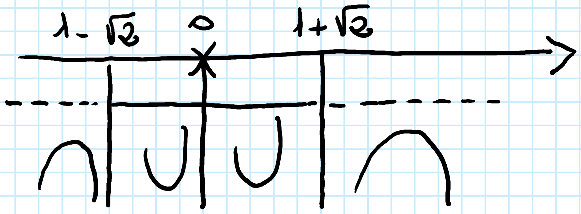
$$f''(x) = \frac{1+x^2-(x-1)2x}{(1-x^2)^2} = \frac{1+x^2-2x^2+2x}{(1+x^2)^2}$$

$$= \frac{-x^2+2x+1}{(1+x^2)^2}$$

$$f''(x) \geq 0 \iff -x^2+2x+1 \geq 0 \iff x^2-2x-1 \leq 0$$

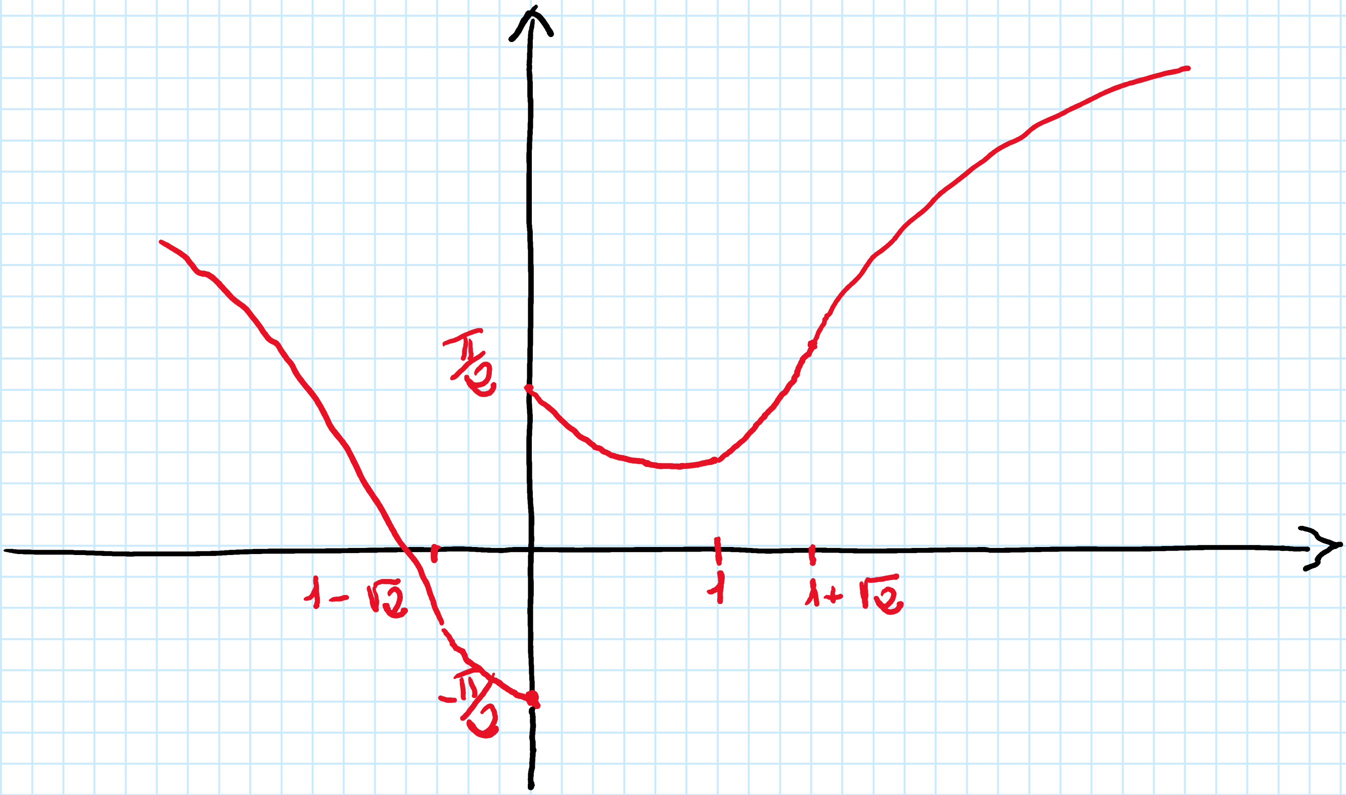
$$\left\{ x = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2} \right\}$$

$$\iff 1-\sqrt{2} \leq x \leq 1+\sqrt{2}, \quad x \neq 0$$




$f$ è convessa in $(1-\sqrt{2}, 0)$ e in $(0, 1+\sqrt{2})$
$f$ è concava in $(-\infty, 1-\sqrt{2})$ e in $(1+\sqrt{2}, +\infty)$
$x = 1 \pm \sqrt{2}$ sono p.ti di flesso

(d) Grafico di $f$




(e) $\operatorname{Im} f = \left(-\frac{\pi}{2}, +\infty\right)$

L'eq. $f(x)=\lambda$ ha:

0 sol. se $\lambda \leq -\pi/2$;

1 sol. se $-\pi/2 < \lambda < f(1)$;

2 sol. se $\lambda = f(1)$;

3 sol. se $f(1) < \lambda < \pi/2$;

2 sol. se $\lambda \geq \pi/2$.

2) $$\lim_{u\to+\infty} \frac{\log u + u^3}{\sqrt{u} + \arctan u} = \ell$$

Per $u \to +\infty$:

$$\log u + u^3 = u^3 \Big( \underbrace{\frac{\log u}{u^3}}_{\to 0} + 1 \Big) \sim u^3$$

$$\sqrt{u} + \arctan u = \sqrt{u} \Big( 1 + \underbrace{\frac{\arctan u}{\sqrt{u}}}_{\to 0} \Big) \sim \sqrt{u}$$

quindi

$$\ell = \lim_{u\to+\infty} \frac{u^3}{\sqrt{u}} = \lim_{u\to+\infty} u^{5/2 > 1} = +\infty .$$

3) $$I = \int_0^1 \frac{e^x+1}{e^{2x}+1}\,dx = \int_0^1 \frac{e^x+1}{(e^{2x}+1)e^x}\, e^x\,dx .$$

Applicando la regola di integrazione per sostituzione con $\varphi(x) = e^x$ e $f(t) = \frac{t+1}{t(t^2+1)}$ si ha

$$I = \int_1^e \frac{t+1}{t(t^2+1)}\,dt .$$

$$\frac{t+1}{t(t^2+1)} = \frac{a}{t} + \frac{bt+c}{t^2+1} = \frac{at^2+a+bt^2+ct}{t(t^2+1)}$$

$$\begin{cases} a+b=0 \\ c=1 \\ a=1 \end{cases} \qquad \begin{cases} b=-a=-1 \\ c=1 \\ a=1 \end{cases}$$

$$I = \int_1^e \left(\frac{1}{t} - \frac{t}{t^2+1} + \frac{1}{t^2+1}\right)dt =$$

$$= \left[\log|t| - \frac{1}{2}\log(t^2+1) + \operatorname{arctg} t\right]_1^e$$

$$= 1 - \frac{1}{2}\log(e^2+1) + \operatorname{arctg} e + \frac{1}{2}\log 2 - \frac{\pi}{4}$$

4) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^5+n^2-n+1}$

$\cos n$ cambia segno, quindi occorre studiare la convergenza assoluta:

$$\left|\frac{\cos n}{n^5+n^2-n+1}\right| \leq \frac{1}{n^5+n^2-n+1} \sim \frac{1}{n^5} \rightarrow \sum \frac{1}{n^5}$$

serie armonica convergente

La serie assegnata risulta quindi assolutamente convergente (si sono usati i criteri del confronto e del confronto asintotico).