INEQUALITES COOKBOOK

A PREPRINT

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ABSTRACT

A collection of inequalities with some proofs.

Contents

Keywords Inequalities

1 Convex Functions

If f is convex, then for all x, y and $t \in [0,1]$

$$f(tx + (1-t)y) \ \leq \ tf(x_1) + (1-t)f(y) \tag{1}$$

If f is differentiable for all x, y, convexity is equivalent to

- The first-order condition
$$f(y) \ \geq \ f(x) + \langle \nabla f(x), y - x \rangle \tag{2}$$
- Montone gradient condition
$$\langle f(x) - \nabla f(y), x - y \rangle \geq 0 \tag{3}$$

2 Strong Convexity

A differentiable function is strongly convex if for some $m > 0$ and all x, y

$$f(y) \ \geq \ f(x) + \langle \nabla f(x), y - x \rangle + \frac{m}{2}\|y - x\|^2. \tag{4}$$

The following conditions are equivalent to strong-convexity

- Using the first-order condition of convexity, g is convex

$$g(x) \ = \ f(x) - \frac{\mu}{2}\|x\|^2 \tag{5}$$

- Using monotone gradient condition of convexity for g,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \ \geq \ \mu\|x - y\|^2 \tag{6}$$

- Using convexity of g, for $t \in [0, 1]$

$$f(tx + (1-t)y) \ \leq \ tf(x) + (1-t)f(y) - \frac{t(1-t)}{2}\|x - y\|^2. \tag{7}$$

The following are implied by strong convexity

- Minimizing the strong convexity condition w.r.t y, gives the Polyack-Lojasiewicz inequality

$$\frac{1}{2}\|\nabla f(x)\|^2 \ \geq \ \mu(f(x) - f(x^*)) \tag{8}$$

- Using Cauchy-Schwartz

$$\|\nabla f(x) - \nabla f(y)\| \ \geq \ \mu\|x - y\| \tag{9}$$

- Using $\phi_x(z) = f(z) - \langle \nabla f(x), z \rangle$ is strongly convex with the same μ,

$$f(y) \ \leq \ f(x) + \langle f(x), y - x \rangle + \frac{1}{2\mu}\|\nabla f(y) - \nabla f(x)\|^2 \tag{10}$$

- Trivally,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \ \leq \ \frac{1}{\mu}\|\nabla f(x) - \nabla f(y)\|^2 \tag{11}$$

If $\|\nabla^2 f\| \geq m$, then f is strongly convex.

3 Lipschitz Gradient

A differential function f has L-Lipschitz continuous gradient if for some $L > 0$

$$\|\nabla f(x) - \nabla f(y)\| \ \leq \ L\|x - y\| \tag{12}$$

This implies

$$g(x) \ = \ \frac{L}{2}\|x\|^2 - f(x) \tag{13}$$

is convex. As such, the following conditions hold

$$f(y) \ \leq \ f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|^2 \tag{14}$$

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \ \leq \ L\|x - y\|^2 \tag{15}$$

$$f(tx + (1-t)y) \ \geq \ tf(x) + (1-t)f(y) - \frac{t(1-t)L}{2}\|x - y\|^2 \tag{16}$$

If f is assumed to be convex, then the above relationships are equivalent. Moreover, the following conditions are also equivalent

$$f(y) \ \geq \ f(x) + \langle \nabla f(x), y - x\rangle + \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|^2 \tag{17}$$

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \ \geq \ \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2 \tag{18}$$

$$f(tx - (1-t)y) \ \leq tf(x) + (1-t)f(y) - \frac{t(1-t)}{2L}\|\nabla f(x) - \nabla f(y)\|^2 \tag{19}$$

4 Asymptotically Convex

4.1 Asymptotically Strongly Convex

A function is asymptotically strongly convex if for all $\|x_0\|, \|x_1\| \geq R$, for some $R > 0$

$$\langle \nabla f(x_0) - \nabla f(x_1), x_0 - x_1 \rangle \ \geq \ m\|x_0 - x_1\|^2. \tag{20}$$

That is, outside of a ball centered at the origin, f is strongly convex.

If f is assumed to have Lipschitz continuous gradient, then there exists some $M > 0$ such that

$$\langle \nabla f(x_0) - \nabla f(x_1), x_0 - x_1 \rangle \ \geq \ m\|x_0 - x_1\|^2 - M, \qquad \forall x, y. \tag{21}$$

Proof. Since ∇f is L-Lipschitz

$$\begin{aligned} |\langle \nabla f(x_0) - \nabla f(x_1), x_0 - x_1 \rangle| \ &\leq \ \|\nabla f(x_0) - \nabla f(x_1)\|\|x_0 - x_1\| && (22)\\ &\leq L\|x_0 - x_1\|^2. && (23) \end{aligned}$$

Then for $\|x_0\|, \|x_1\| \leq R$,

$$|\langle \nabla f(x_0) - \nabla f(x_1), x_0 - x_1 \rangle| \ \leq LR^2, \tag{24}$$

and

$$m\|x_0 - x_1\|^2 \ \leq \ mR^2. \tag{25}$$

Thus, for all x_0, x_1

$$\langle \nabla f(x_0) - \nabla f(x_1), x_0 - x_1 \rangle \ \geq \ m\|x_0 - x_1\|^2 - (m + L)R^2. \tag{26}$$

□

4.2 Asymptotically Stronly Co-coercive

This definition is given in [1]. There exists $R \geq 0$, $K, L_2 > 0$ such that

$$\langle x - y, \nabla f(x) - \nabla f(y) \rangle \ \geq \ K\|x - y\|^2 + \frac{1}{L^2}\|\nabla f(x) - \nabla f(y)\|, \tag{27}$$

for $\|x - y\| \geq R$.

References

[1] Nawaf Bou-Rabee and Katharina Schuh. Nonlinear hamiltonian monte carlo and its particle approximation, 2023.