Rational functions with positive normalised denominator

(no poles)

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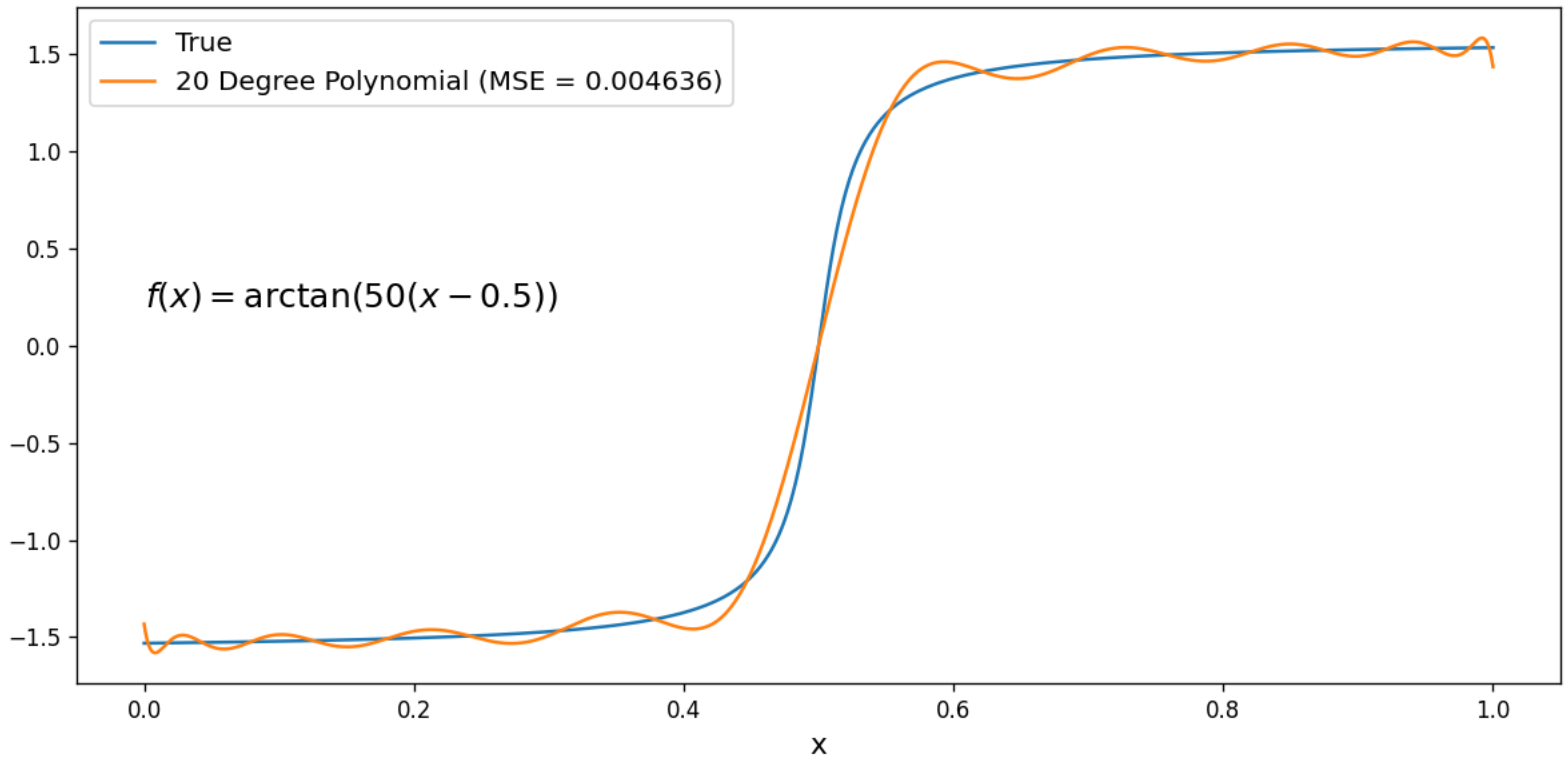
*Thanks to the organisers / LNT

Polynomial Approximation

$$f(x) \approx \sum_{n=0}^{N} a_n P_n(x) = \sum_{n=0}^{N} c_n x^n$$

(e.g. Legendre or Chebyshev)



Rational Approximation

$$f(x) \;\approx\; R_{N,M}(x) = \sum_{n=0}^{N} a_n P_n(x) \Bigg/ \sum_{m=0}^{M} b_m Q_m(x)$$

- Reduces Runge's Phenomena
- Faster convergence than ordinary polynomials

AAA Algorithm

Nakatsukasa, Sète, and Trefethen (2018)

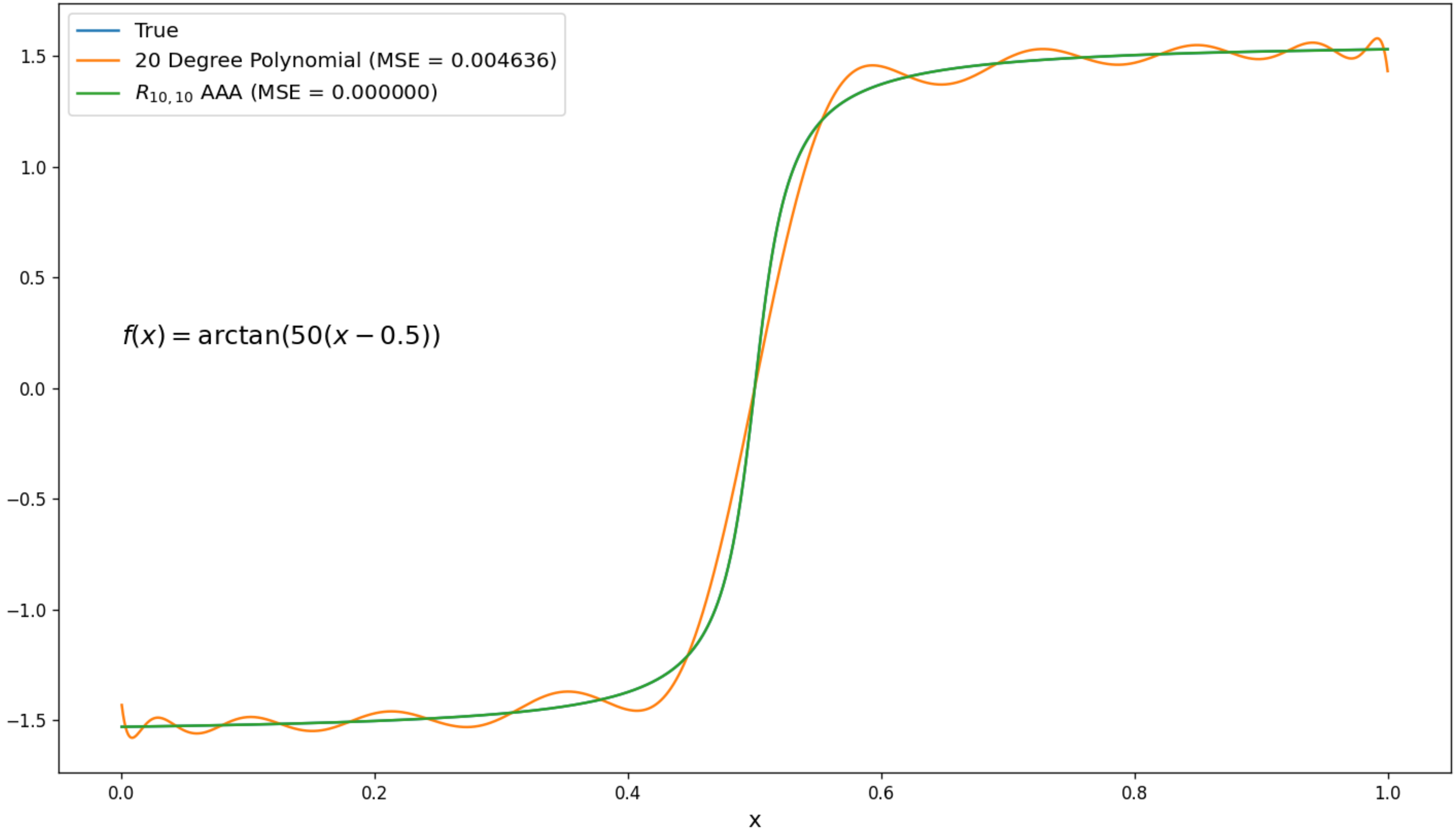
$$f(x) \approx R_{N,N}(x) = \sum_{n=0}^{N} \frac{w_n f_n}{x - x_n} \Bigg/ \sum_{n=0}^{N} \frac{w_n}{x - x_n}$$

$$f_n = f(x_n), \qquad w_n \neq 0$$

$0 < x_0 < x_1 < \ldots < x_N \leq 1$ partitions $[0,1]$. Non-zero denominator at x_i

$$\min_{w} \sum_{i} \left[f_i \left(\sum_{n=0}^{N} \frac{w_n}{x_i - x_n} \right) - \left(\sum_{n=0}^{N} \frac{w_n f_n}{x_i - x_n} \right) \right]^2$$

Normalizing Condition: $\|w\| = 1$



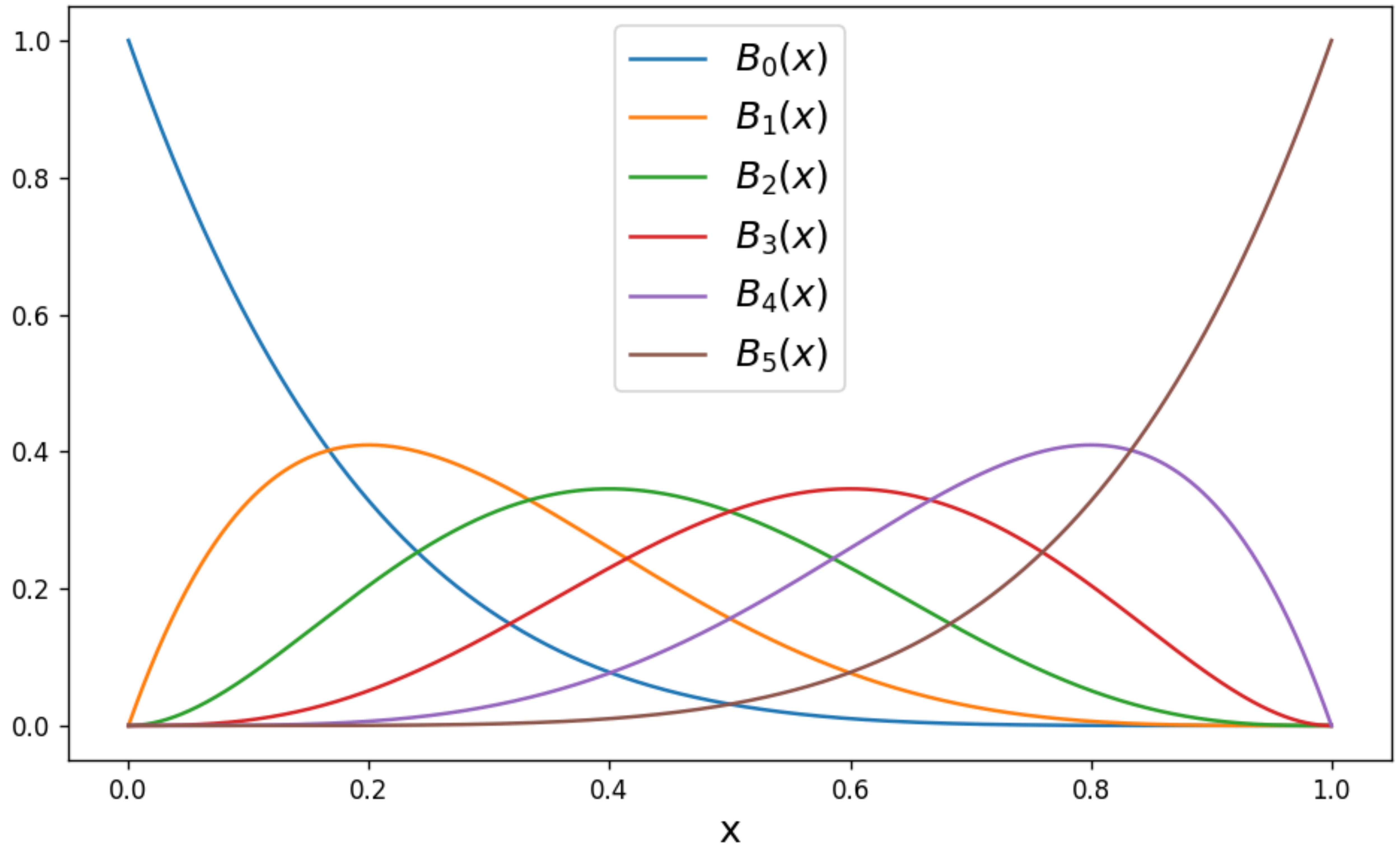
True
20 Degree Polynomial (MSE = 0.004636)
$R_{10,10}$ AAA (MSE = 0.000000)
$f(x) = \arctan(50(x - 0.5))$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.2
0.4
0.6
0.8
1.0
x

New Problem

$$f(x) \;\approx\; R_{N,M}(x) = \frac{P(x)}{Q(x)} \quad \text{with} \quad \underline{Q(x) > 0} \quad \text{for} \quad x \in [0,1]$$

Bernstein Polynomials

$$B_k^{(N)}(x) \;=\; \binom{N}{k} x^k (1-x)^{N-k} \qquad\qquad B_k^{(N)}(x) \;>\; 0 \quad \text{for} \quad x \;\in\; (0,1)$$



Sergei Natanovich Bernstein

Our Proposal

$$f(x) \;\approx\; R_{N,M}(x) = \frac{P(x)}{Q(x)} \quad \text{with} \quad Q(x) > 0 \quad \text{for} \quad x \in [0,1]$$

$$Q(x) \;=\; \sum_{m=0}^{M} w_m B_m(x) \quad \text{where} \quad B_m(x) \;=\; \binom{M}{m} x^m (1-x)^{M-m}$$

Our Proposal

$$f(x) \;\approx\; R_{N,M}(x) = \frac{P(x)}{Q(x)} \quad \text{with} \quad Q(x) > 0 \quad \text{for} \quad x \in [0,1]$$

$$Q(x) \;=\; \sum_{m=0}^{M} w_m B_m(x) \quad \text{where} \quad B_m(x) \;=\; \binom{M}{m} x^m (1-x)^{M-m}$$

Positivity

$$w_m \;\geq\; 0$$

Normalization

$$\sum_m w_m \;=\; 1$$

Our Proposal

$$f(x) \approx R_{N,M}(x) = \frac{P(x)}{Q(x)} \quad \text{with} \quad Q(x) > 0 \quad \text{for} \quad x \in [0,1]$$

$$Q(x) = \sum_{m=0}^{M} w_m B_m(x) \quad \text{where} \quad B_m(x) = \binom{M}{m} x^m (1-x)^{M-m}$$

Positivity

$$w_m \geq 0$$

Normalization

$$\sum_m w_m = 1$$

$$f(x) \approx R_{N,M}(x) = \sum_{n=0}^{N} a_n P_n(x) \Big/ \sum_{m=0}^{M} w_m B_m(x)$$

For some $\{P_n(x)\}_n$

How to Solve

$$f(x) \;\approx\; R_{N,M}(x) = \sum_{n=0}^{N} a_n P_n(x) \Big/ \sum_{m=0}^{M} w_m B_m(x)$$

$$w \;\in\; \Delta^{M+1} \;=\; \left\{ w \in \mathbb{R}^{M+1} \mid w_m \;\geq\; 0 \text{ and } \sum_m w_m \;=\; 0 \right\}$$

Linearized Problem

$$\min_{\substack{a \in \mathbb{R}^{N+1} \\ w \in \Delta^{M+1}}} \left\| f(x) \sum_m w_m B_m(x) \;-\; \sum_n a_n P_n(x) \right\|$$

Very much like Lanczos (1938)

How to Solve

$$f(x) \;\approx\; R_{N,M}(x) = \sum_{n=0}^{N} a_n P_n(x) \Big/ \sum_{m=0}^{M} w_m B_m(x)$$

$$w \;\in\; \Delta^{M+1} \;=\; \left\{ w \in \mathbb{R}^{M+1} \mid w_m \;\geq\; 0 \text{ and } \sum_m w_m \;=\; 0 \right\}$$

Linearized

$$\min_{\substack{a \,\in\, \mathbb{R}^{N+1} \\ w \,\in\, \Delta^{M+1}}} \left\| f(x) \sum_m w_m B_m(x) \;-\; \sum_n a_n P_n(x) \right\|$$

<u>**Non-linearized**</u>

$$\min_{\substack{a \,\in\, \mathbb{R}^{N+1} \\ w \,\in\, \Delta^{M+1}}} \left\| f(x) \;-\; \sum_n a_n P_n(x) \Big/ \sum_m w_m B_m(x) \right\|$$

Linearized Residuals

$$\min_{\substack{a \in \mathbb{R}^{N+1} \\ w \in \Delta^{M+1}}} \int \left(f(x) \sum_m w_m B_m(x) \; - \; \sum_n a_n P_n(x) \right)^2 d\mu$$

$$\approx \min_{a, w} \sum_i \mu_i \left(f(x_i) \sum_m w_m B_m(x_i) \; - \; \sum_n a_n P_n(x_i) \right)^2 \quad \text{for} \quad \mu_i \geq 0$$

with $w_m \geq 0, \qquad \sum_m w_m = 1$

Non-linearized Residuals

$$\min_{\substack{a \in \mathbb{R}^{N+1} \\ w \in \Delta^{M+1}}} \left\| f(x) - \sum_n a_n P_n(x) \Big/ \sum_m w_m B_m(x) \right\|$$

$$\min_{\substack{a \in \mathbb{R}^{N+1} \\ w \in \Delta^{M+1}}} \int \left(f(x) \sum_m w_m B_m(x) - \sum_n a_n P_n(x) \right)^2 \frac{d\mu}{\left(\sum_m w_m B_m(x)\right)^2}$$

Like the linearized part **Like an adjusted measure**

(Like Sanathanan and Koerner. - 1963)

Optimizing Over a Simplex

Chok, and Vasil (2023)

$$\min_{w \in \Delta^M} \; F(w)$$

Enforces positivity

$$\frac{dw_i}{dt} = w_i \left(\nabla_{w_i} F - w \cdot \nabla_w F \right)$$

Enforces unit sum constraint

$$\sum_i \frac{dw_i}{dt} = (w \cdot \nabla_w F) - (w \cdot \nabla_w F) \sum_i w_i$$

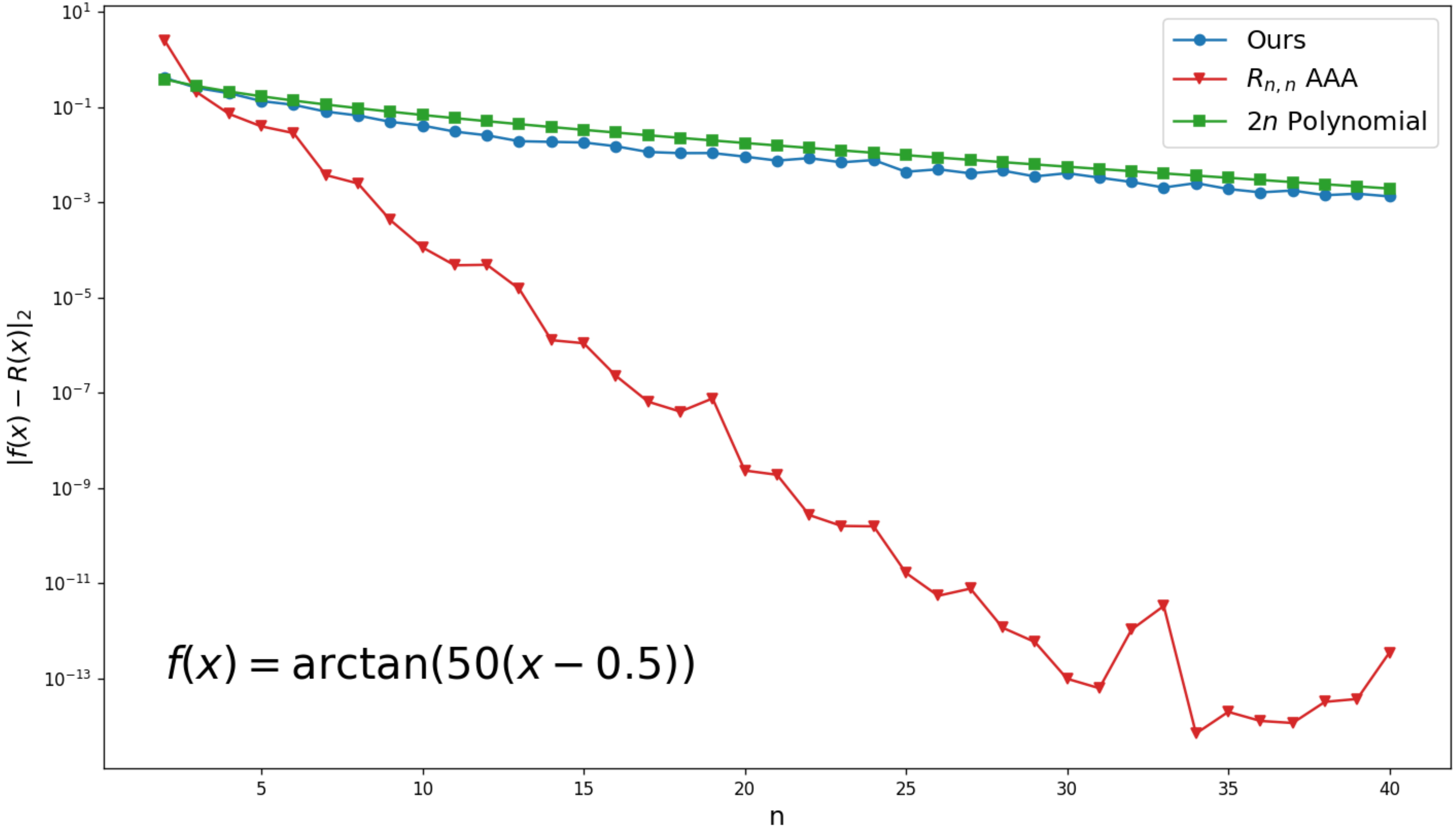
$$F(w) = \left\| f(x) - \sum_n a_n P_n(x) \Big/ \sum_m w_m B_m(x) \right\|$$

How to Solve

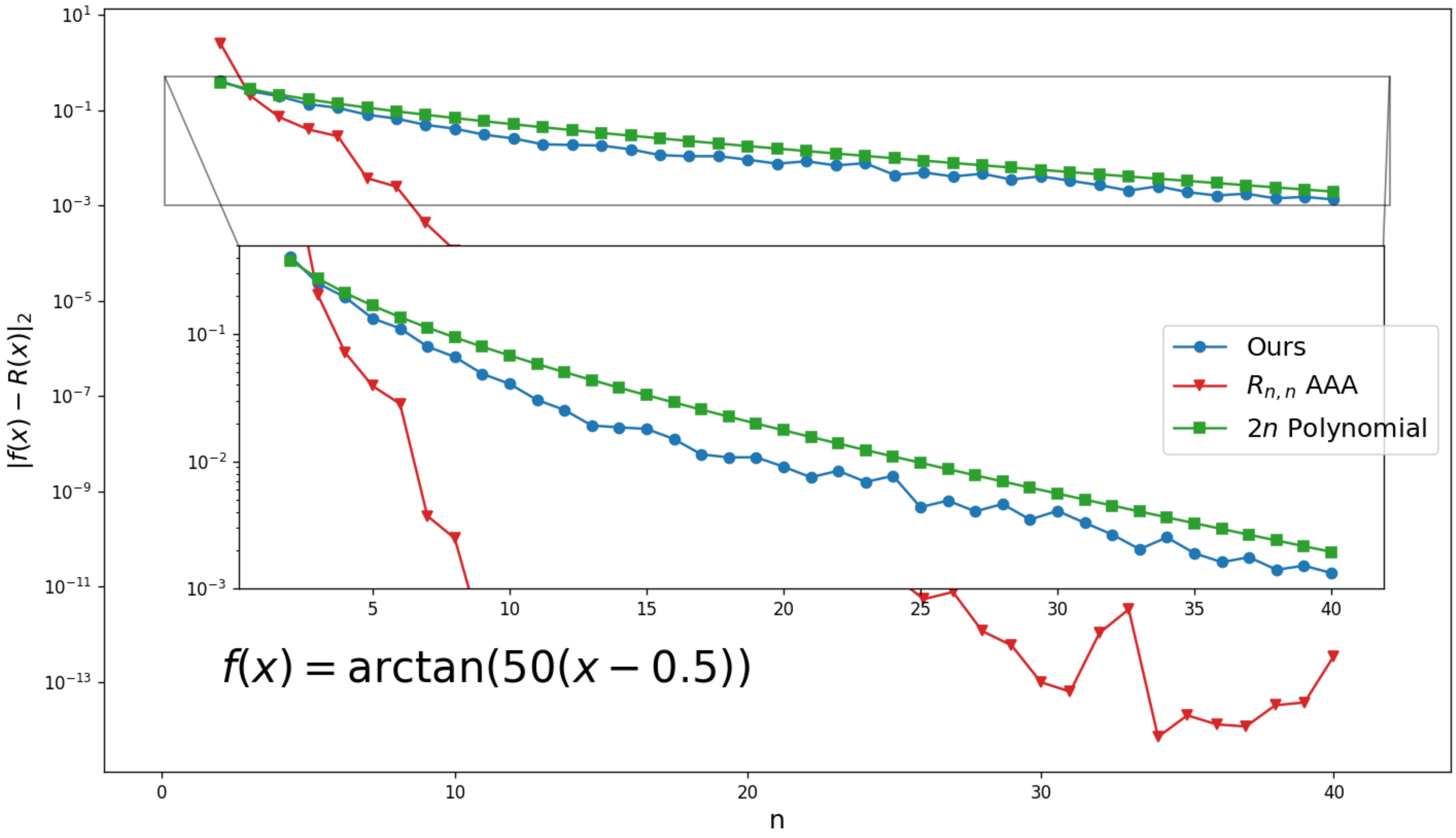
$$F(a, w) = \left\| f(x) \; - \; \sum_n a_n P_n(x) \Big/ \sum_m w_m B_m(x) \right\|$$

1. Fix $a \;\in\; \mathbb{R}^{N+1}$ and a take step(s) in $w \;\in\; \Delta^{M+1}$
2. Fix $w \;\in\; \Delta^{M+1}$ and solve for $a \;\in\; \mathbb{R}^{N+1}$ exactly
3. Repeat

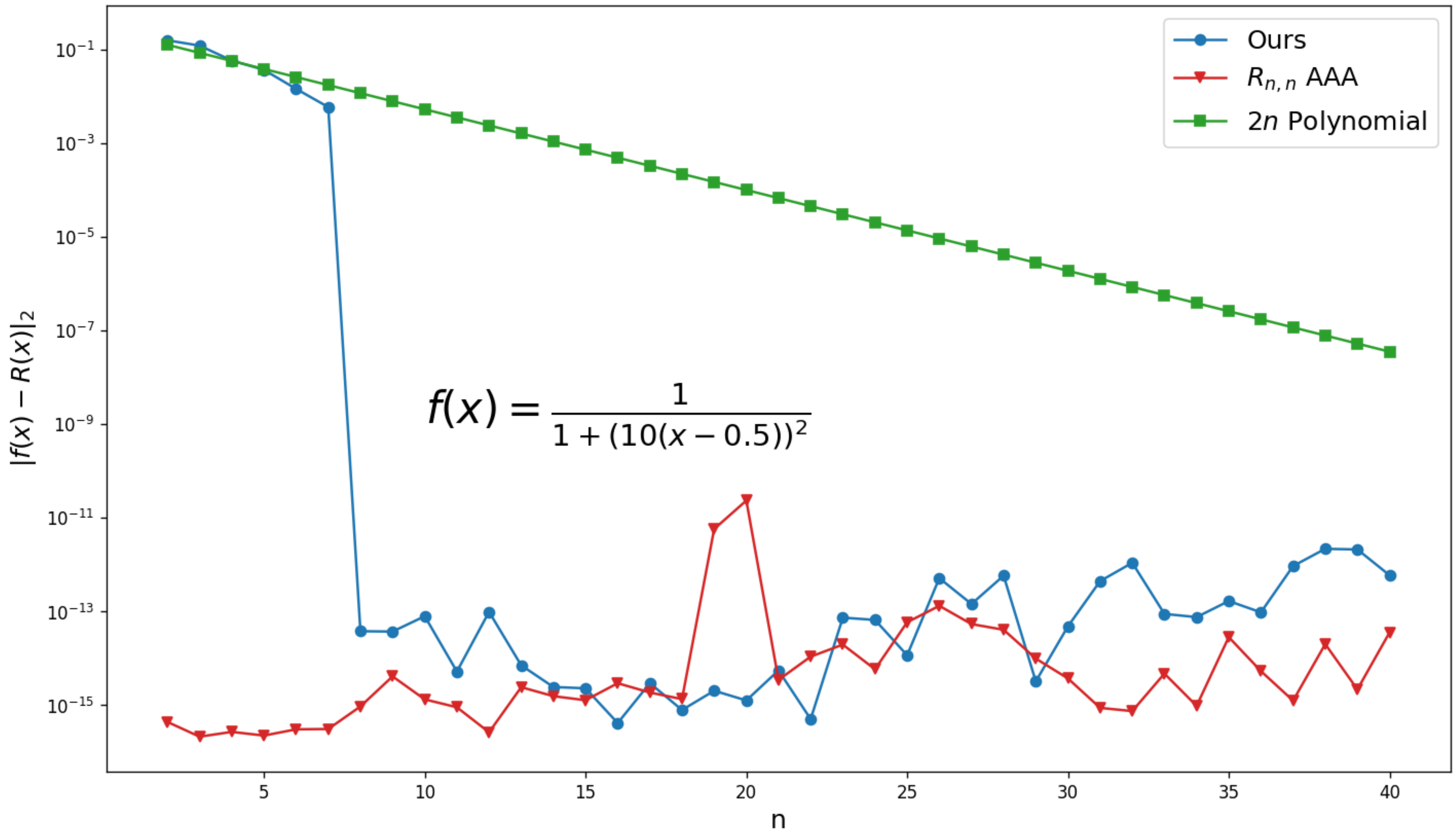
Numerical Convergence



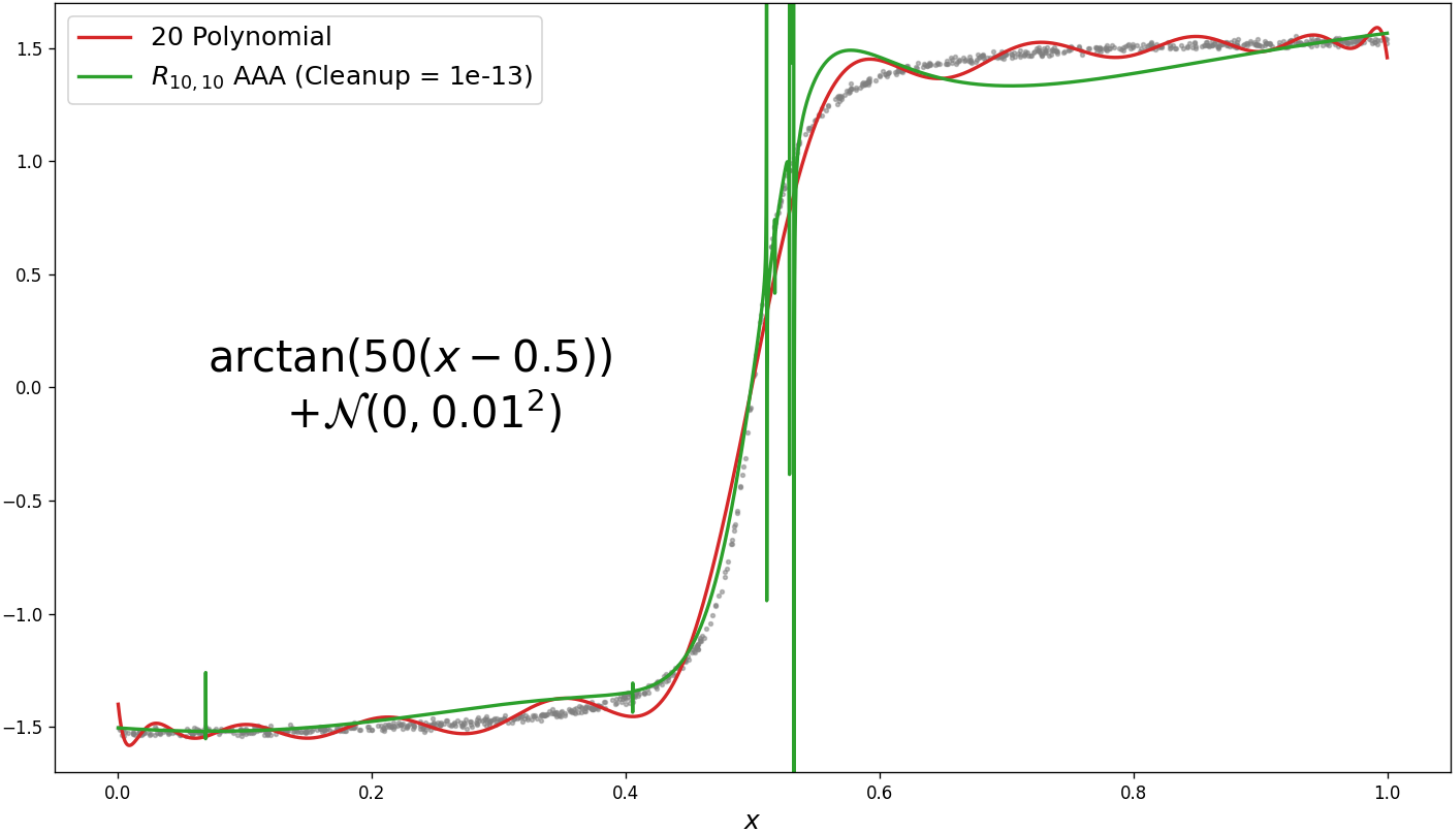
Numerical Convergence



Numerical Convergence (Runge's Example)

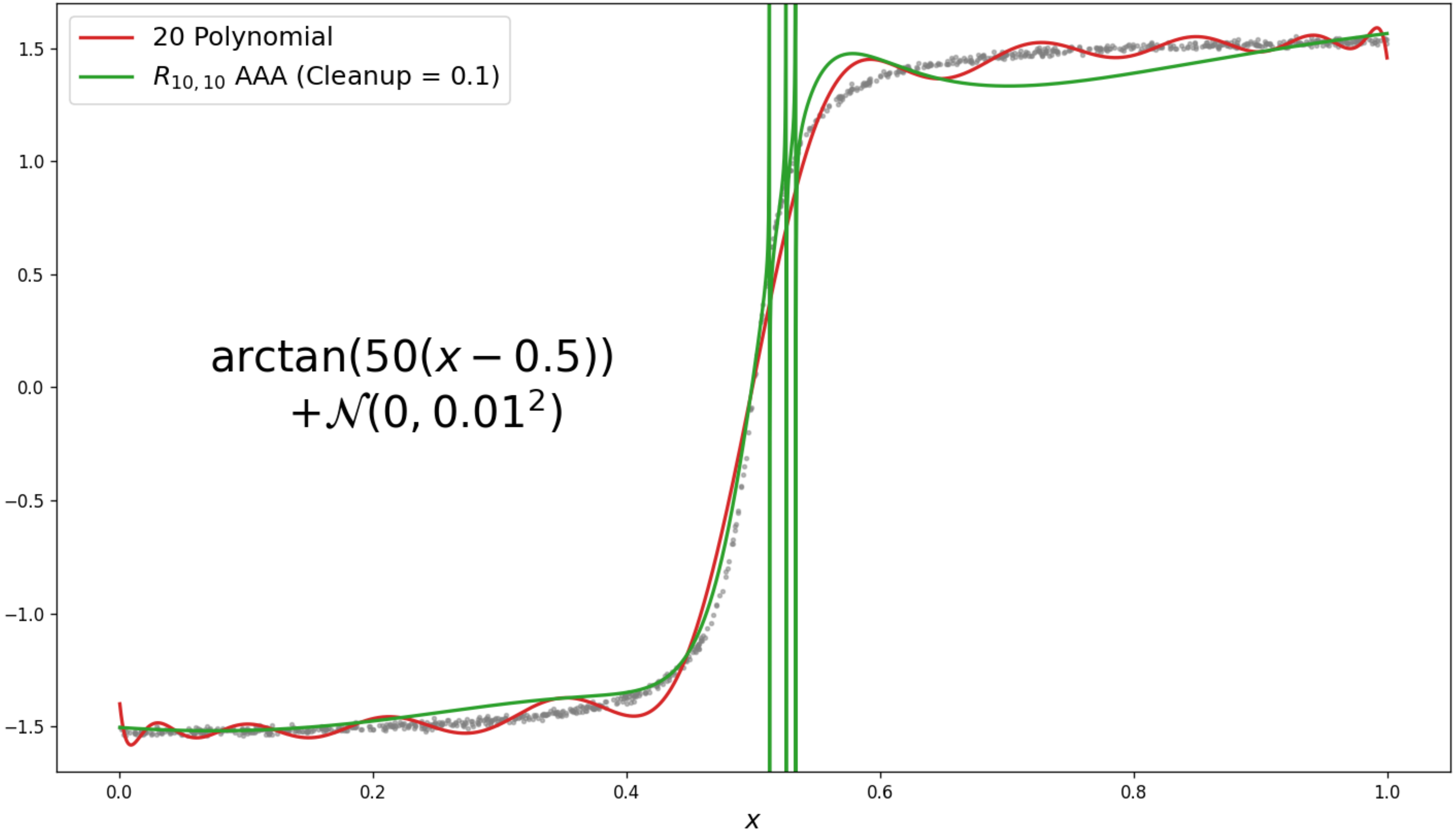


Noisy Data



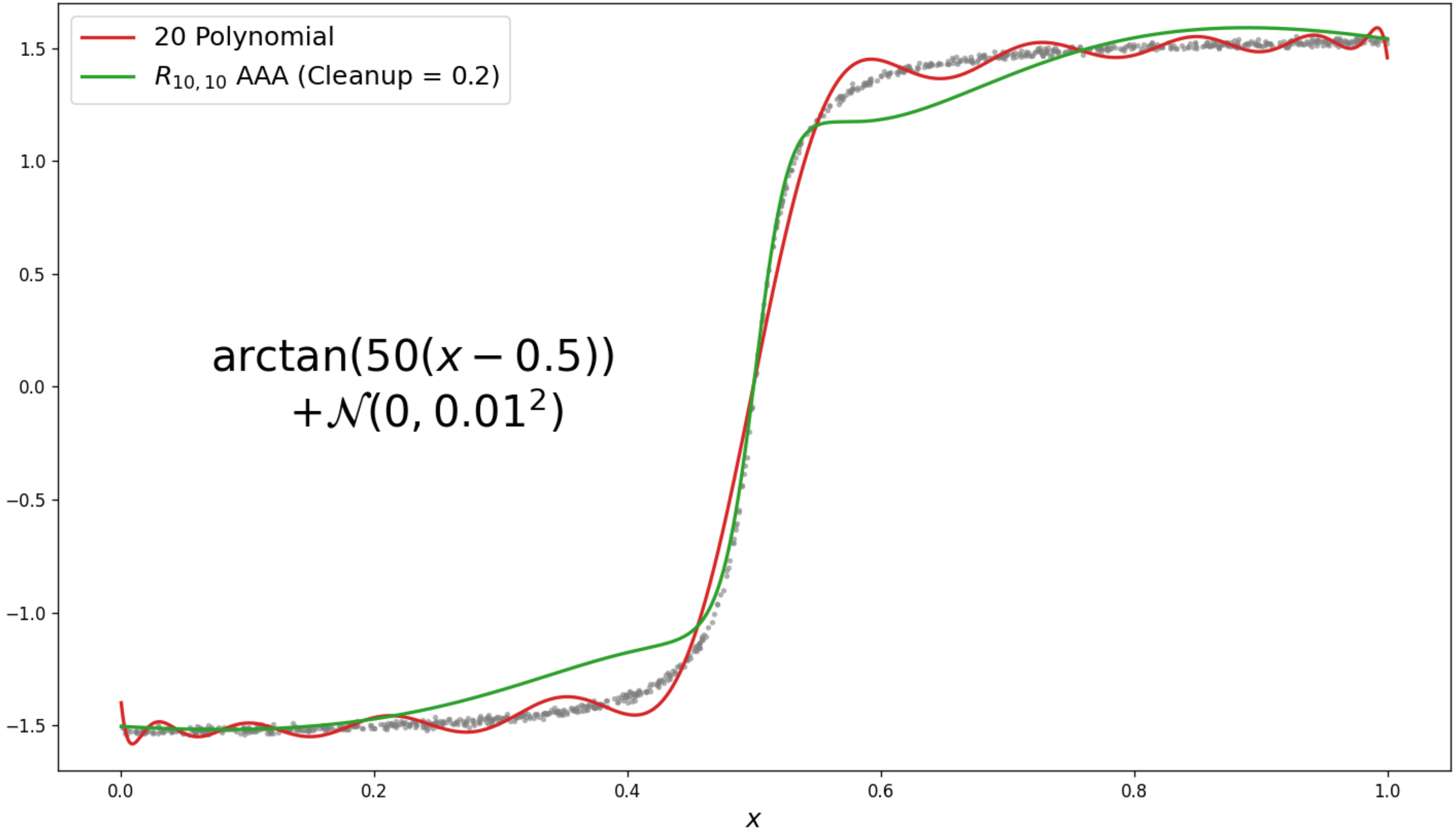
20 Polynomial
$R_{10,10}$ AAA (Cleanup = 1e-13)
$\arctan(50(x-0.5))$
$+\mathcal{N}(0, 0.01^2)$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.2
0.4
0.6
0.8
1.0
x

Noisy Data



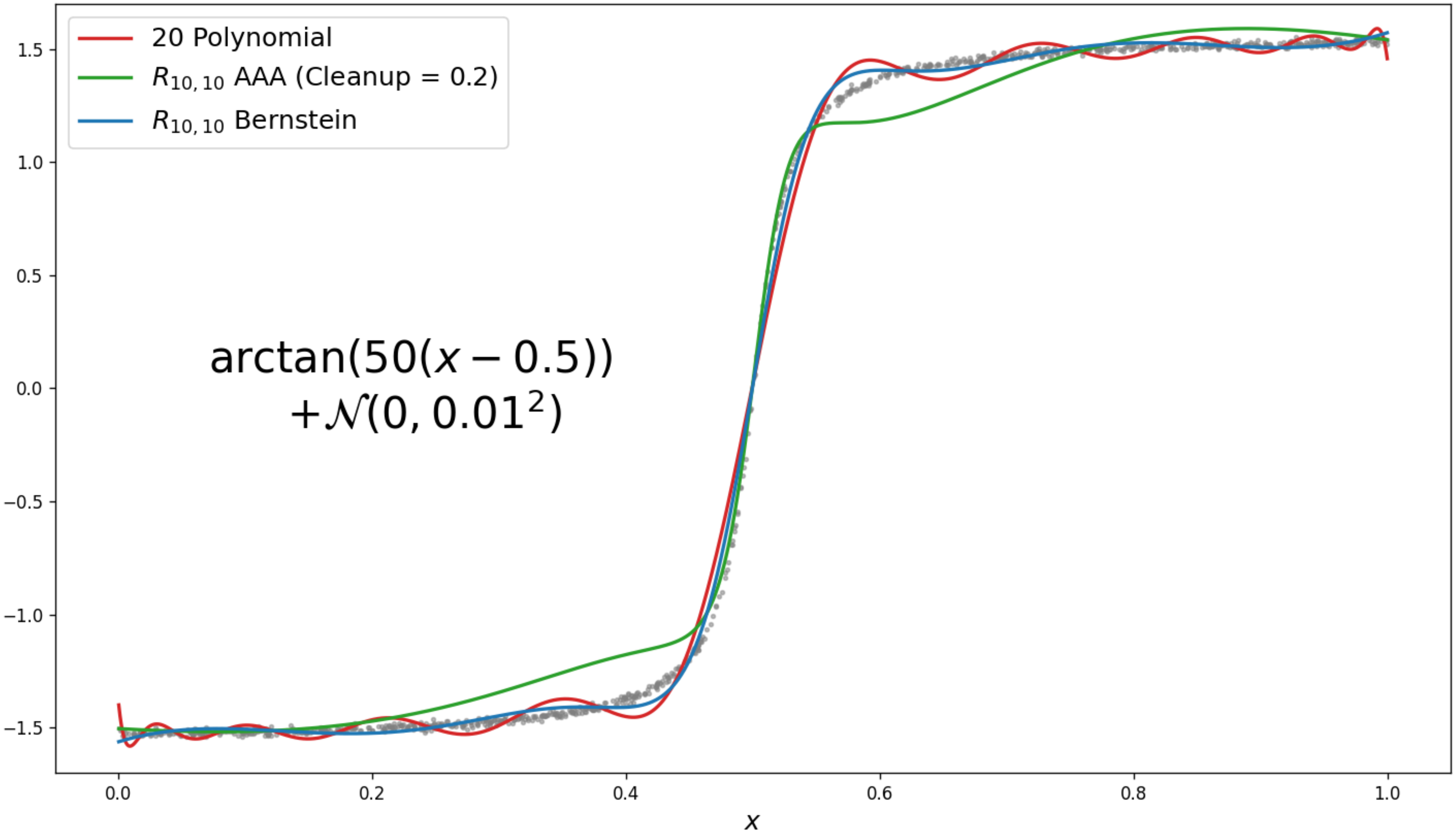
20 Polynomial
$R_{10,10}$ AAA (Cleanup = 0.1)
$\arctan(50(x-0.5))$
$+\mathcal{N}(0, 0.01^2)$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.2
0.4
0.6
0.8
1.0
x

Noisy Data



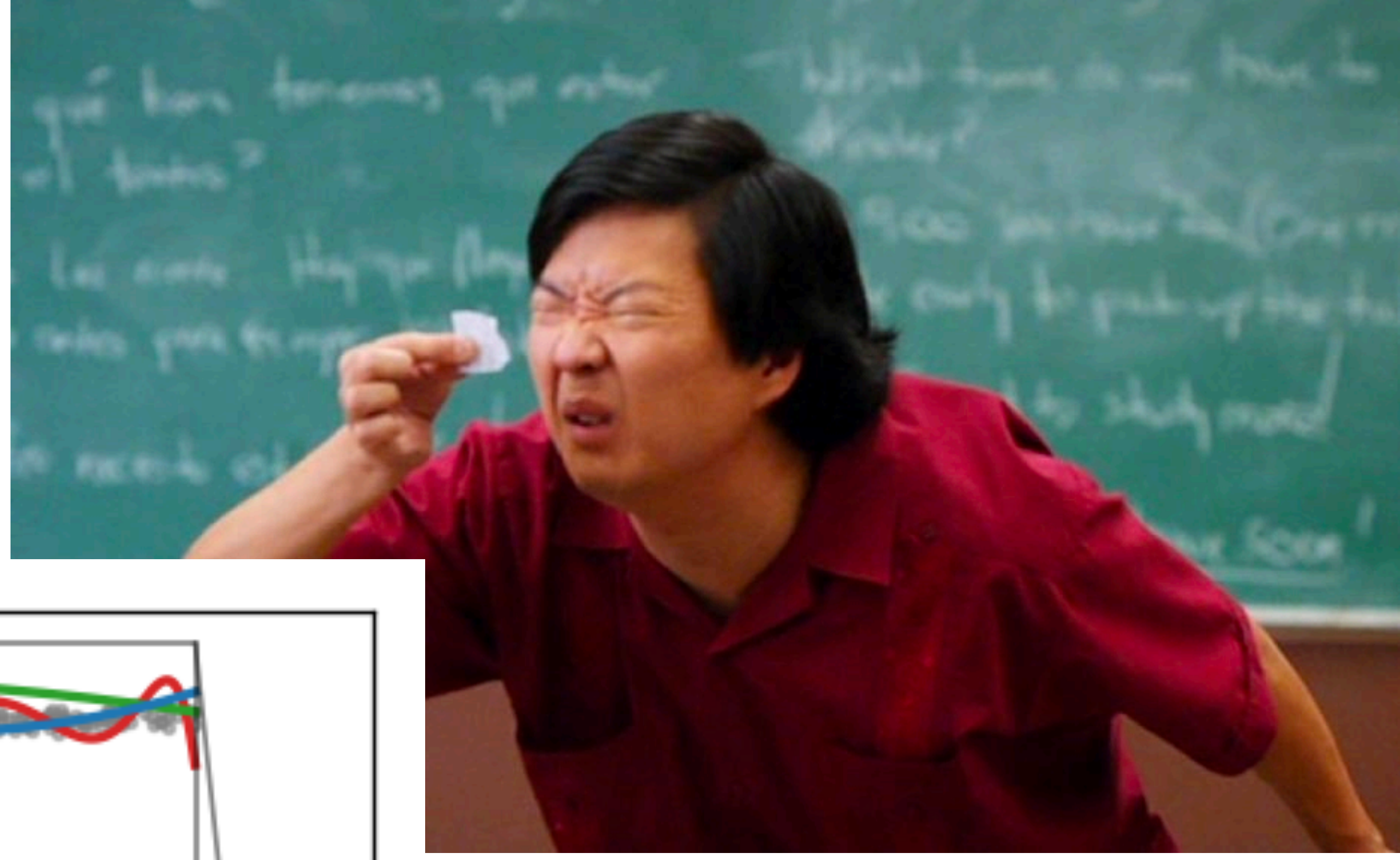
20 Polynomial
$R_{10,10}$ AAA (Cleanup = 0.2)
$\arctan(50(x-0.5))$
$+\mathcal{N}(0, 0.01^2)$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.2
0.4
0.6
0.8
1.0
x

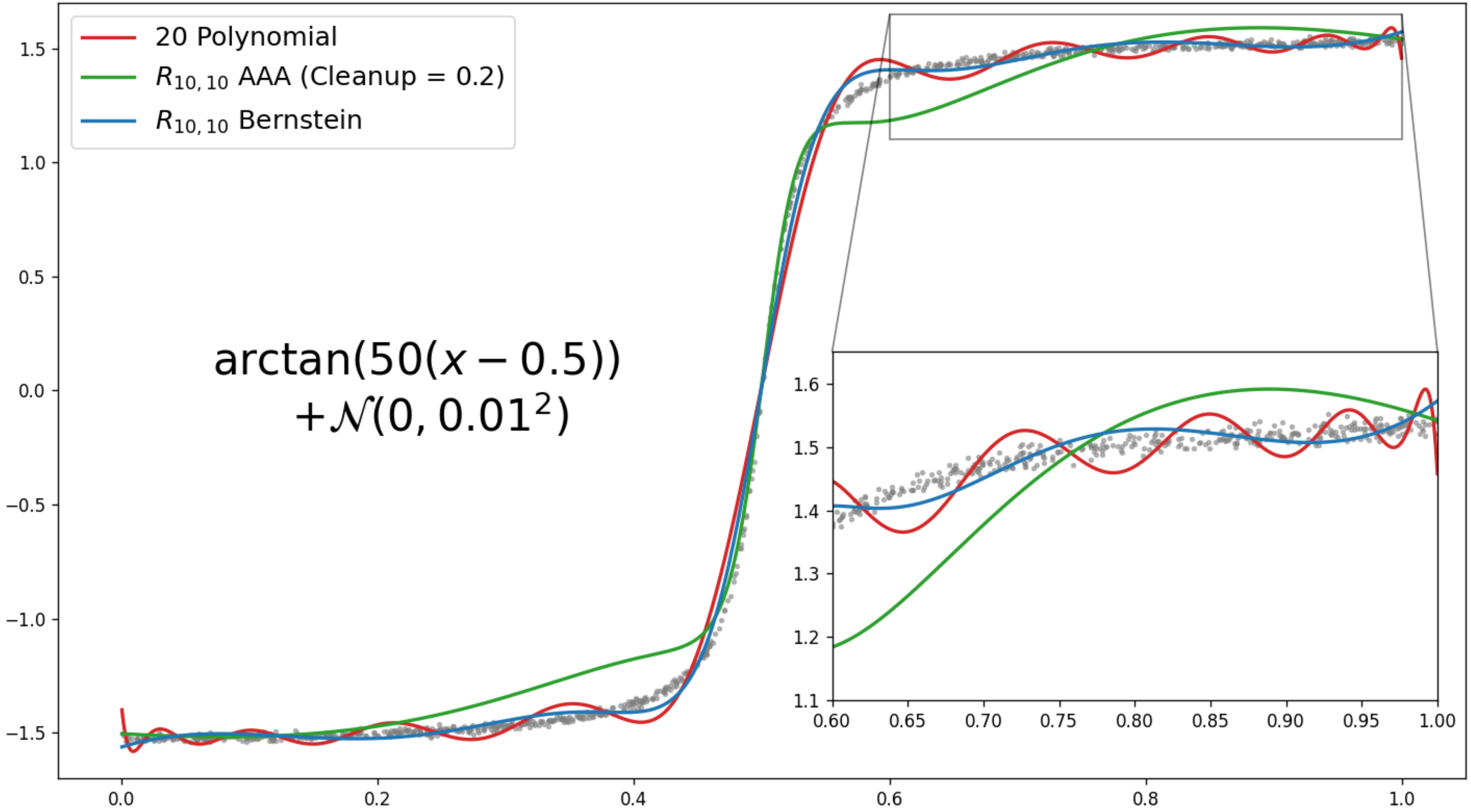
Noisy Data



20 Polynomial
$R_{10,10}$ AAA (Cleanup = 0.2)
$R_{10,10}$ Bernstein
$\arctan(50(x-0.5))$
$+\mathcal{N}(0, 0.01^2)$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.2
0.4
0.6
0.8
1.0
x

Noisy Data




20 Polynomial
$R_{10,10}$ AAA (Cleanup = 0.2)
$R_{10,10}$ Bernstein
$\arctan(50(x-0.5)) + \mathcal{N}(0, 0.01^2)$

Penalization & smoothing

$$f(x) \;\approx\; R_{N,M}(x) \quad \text{and} \quad R_{N,M} \text{ is smooth}$$

$$\approx\; R_{N,M}(x) \quad \text{and} \quad \text{Numerator is smooth}$$

$$\rightarrow \quad \min_{g} \left\| f(x) \;-\; g(x) \right\| \;+\; \sum_{k\geq 0} \lambda_k \int (g^{(k)}(x))^2 \, d\mu_k \quad \text{for} \quad \lambda_k \;\geq\; 0$$

Legendre

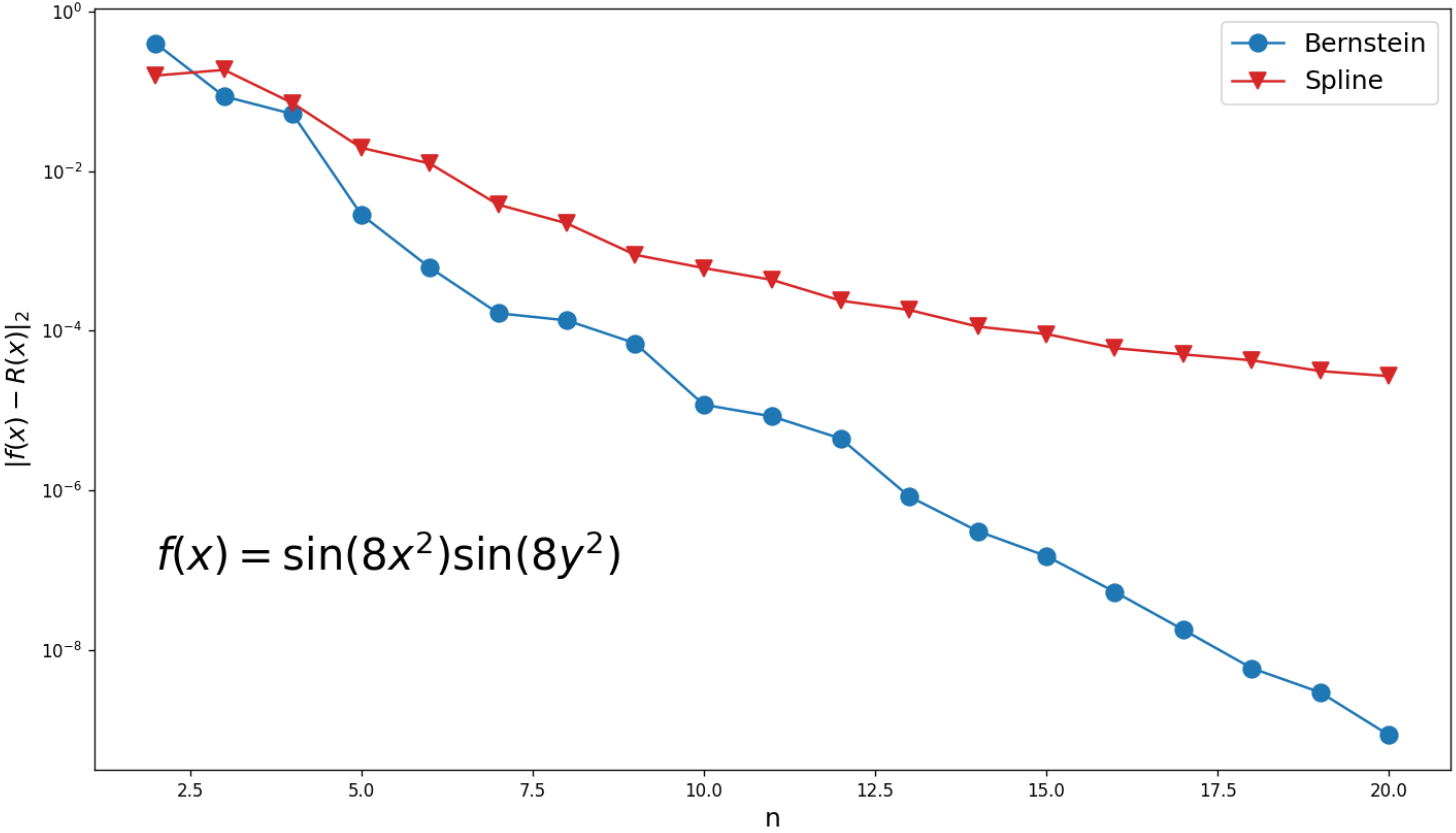
$$\sum_{k\geq 0} \lambda_k \int_0^1 \left(\sum_{n=0}^{N} a_n P_n^{(k)}(x) \right)^2 d\mu_k \;=\; \sum_{n=0}^{N} a_n^2 \, (\lambda_n \, n^{2n})$$

Bivariate

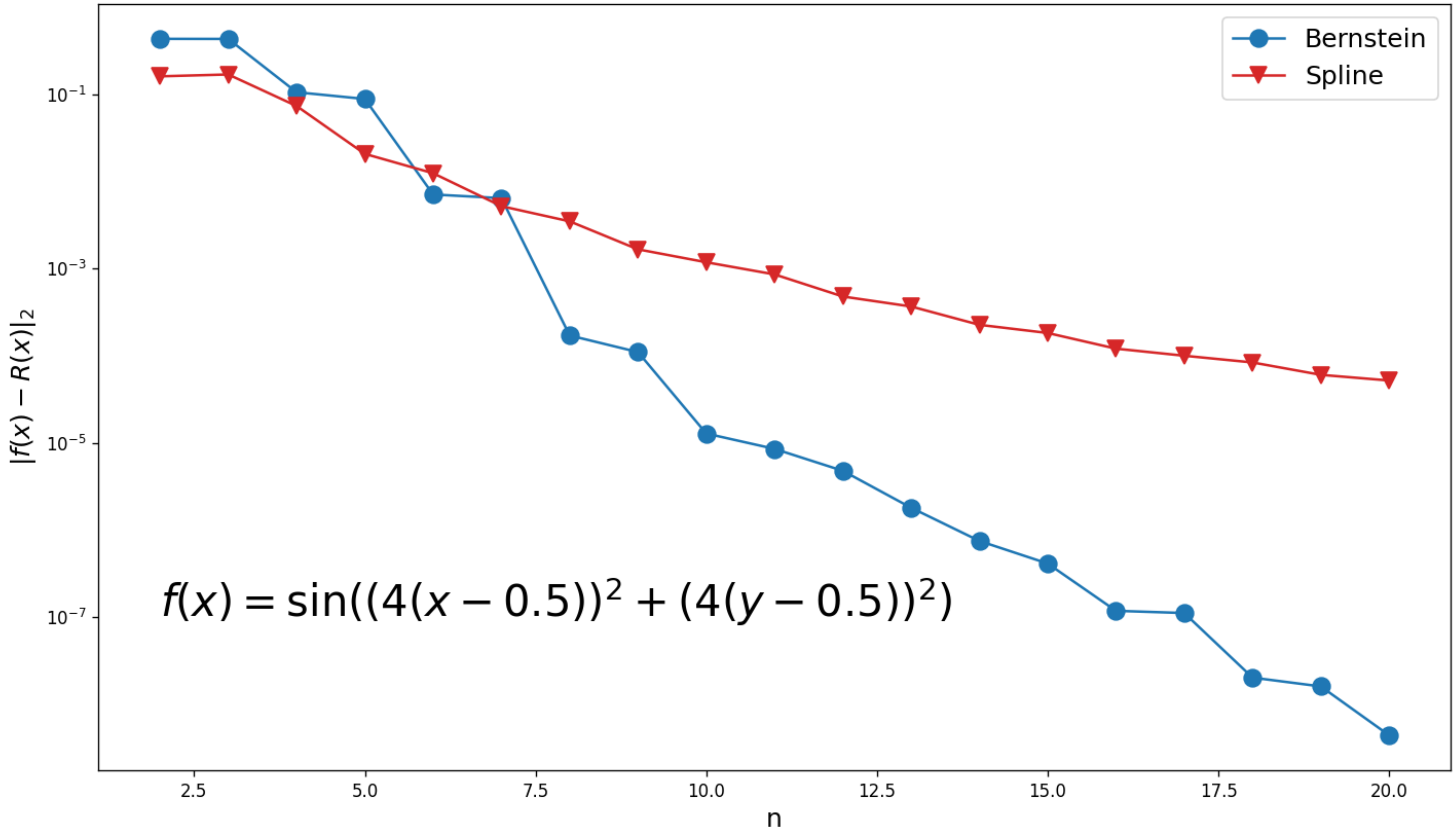
$$R(x, y) = \sum_{n,\, m} a_{n,\, m} P_n(x)\, P_m(y) \Bigg/ \sum_{j,\, k} w_{j,\, k}\, B_j(x)\, B_k(y)$$

$$w_{j,k} \geq 0 \quad \text{and} \quad \sum_{j,k} w_{j,k} = 1$$

Numerical Convergence

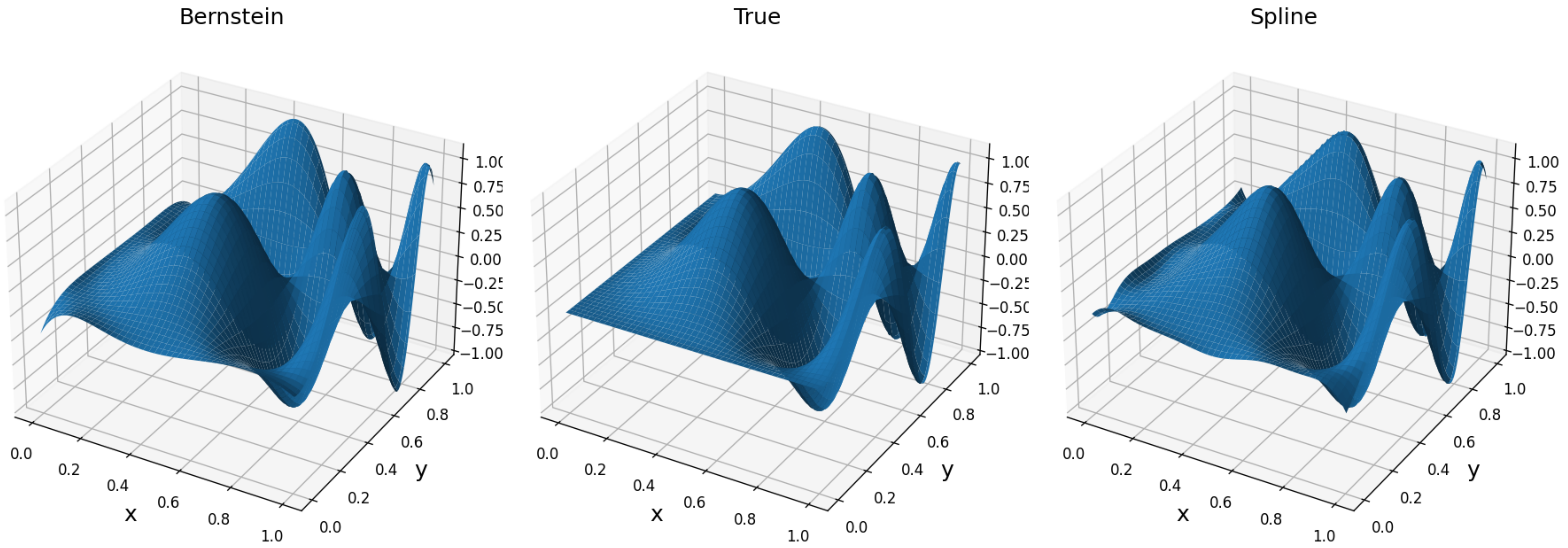


Numerical Convergence



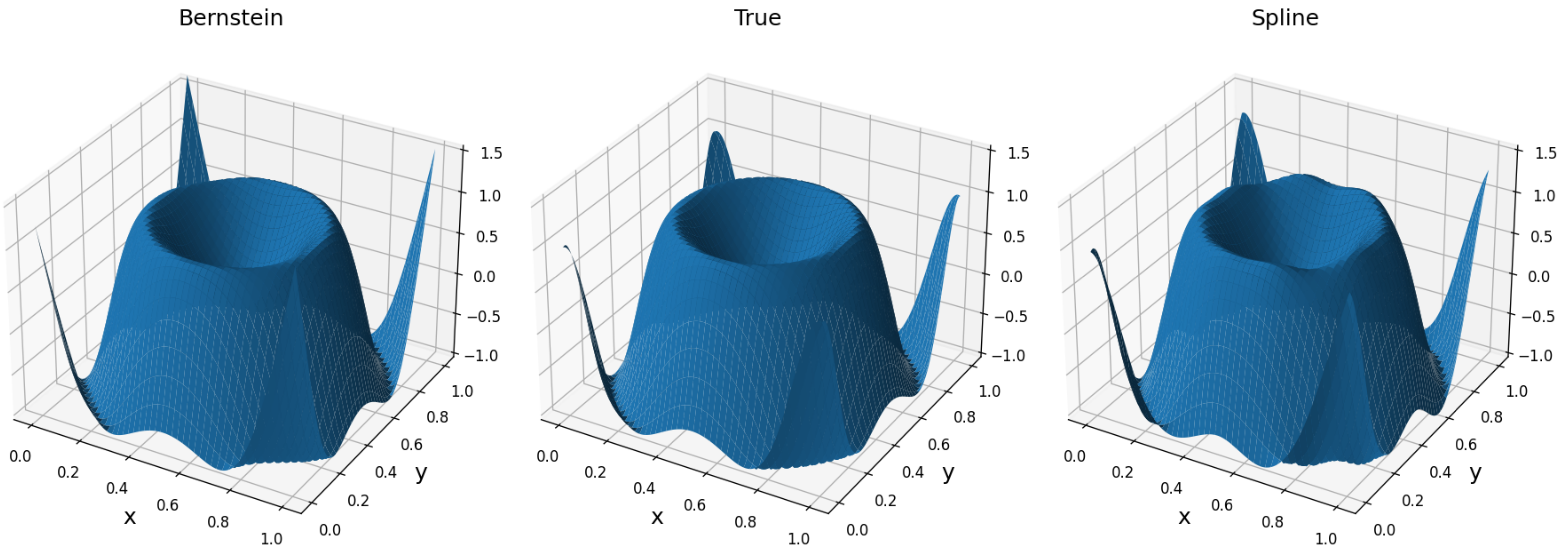
Bernstein vs Spline

$$\sin(8x^2 + 8y^2) + \mathcal{N}(0, 0.1^2)$$

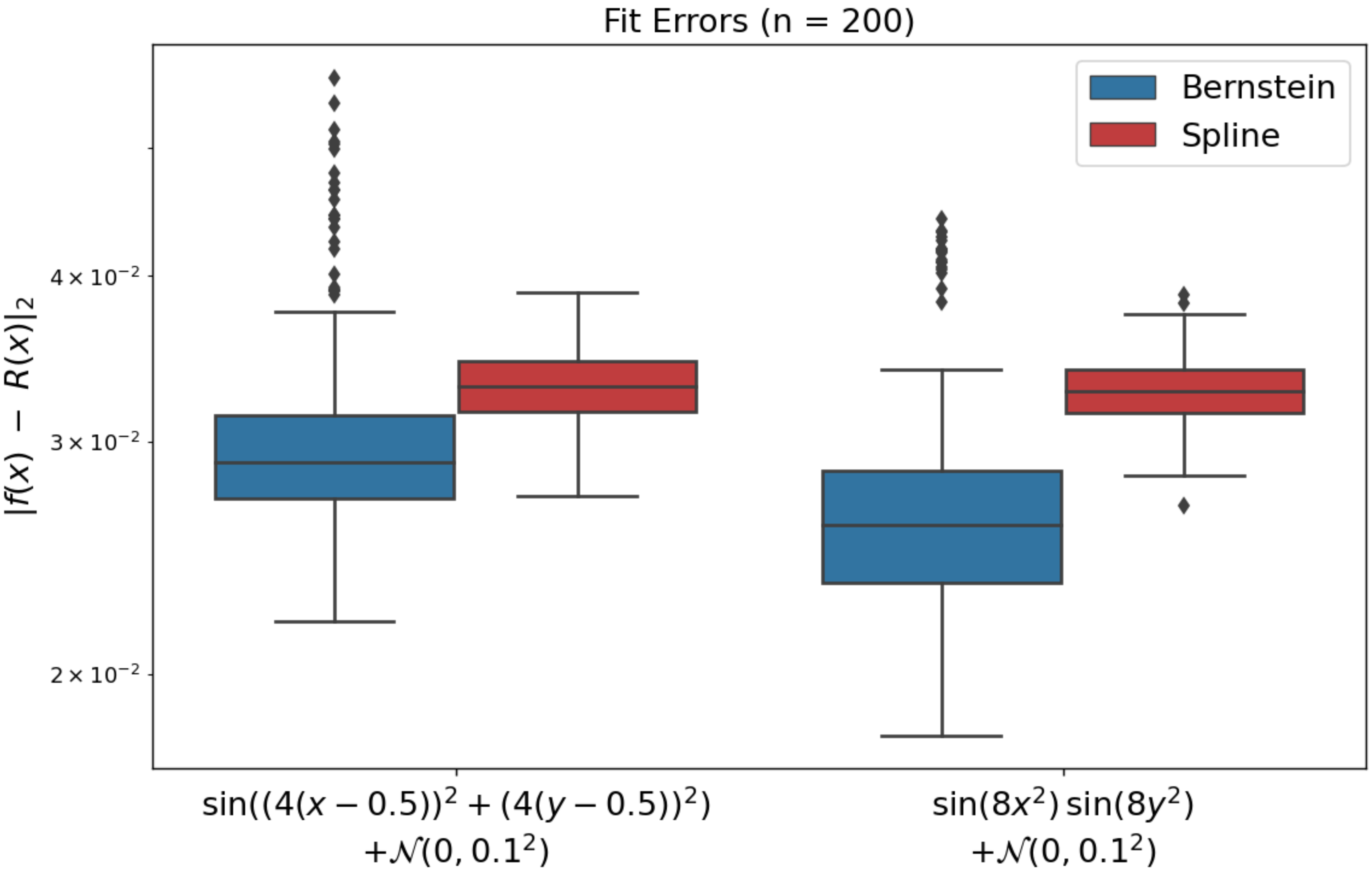


Bernstein vs Spline

$$\sin\left((4(x-0.5))^2 + (4(y-0.5))^2\right) + \mathcal{N}(0, 0.1^2)$$



Bernstein vs Spline



“Which approximation would be more useful in an application?

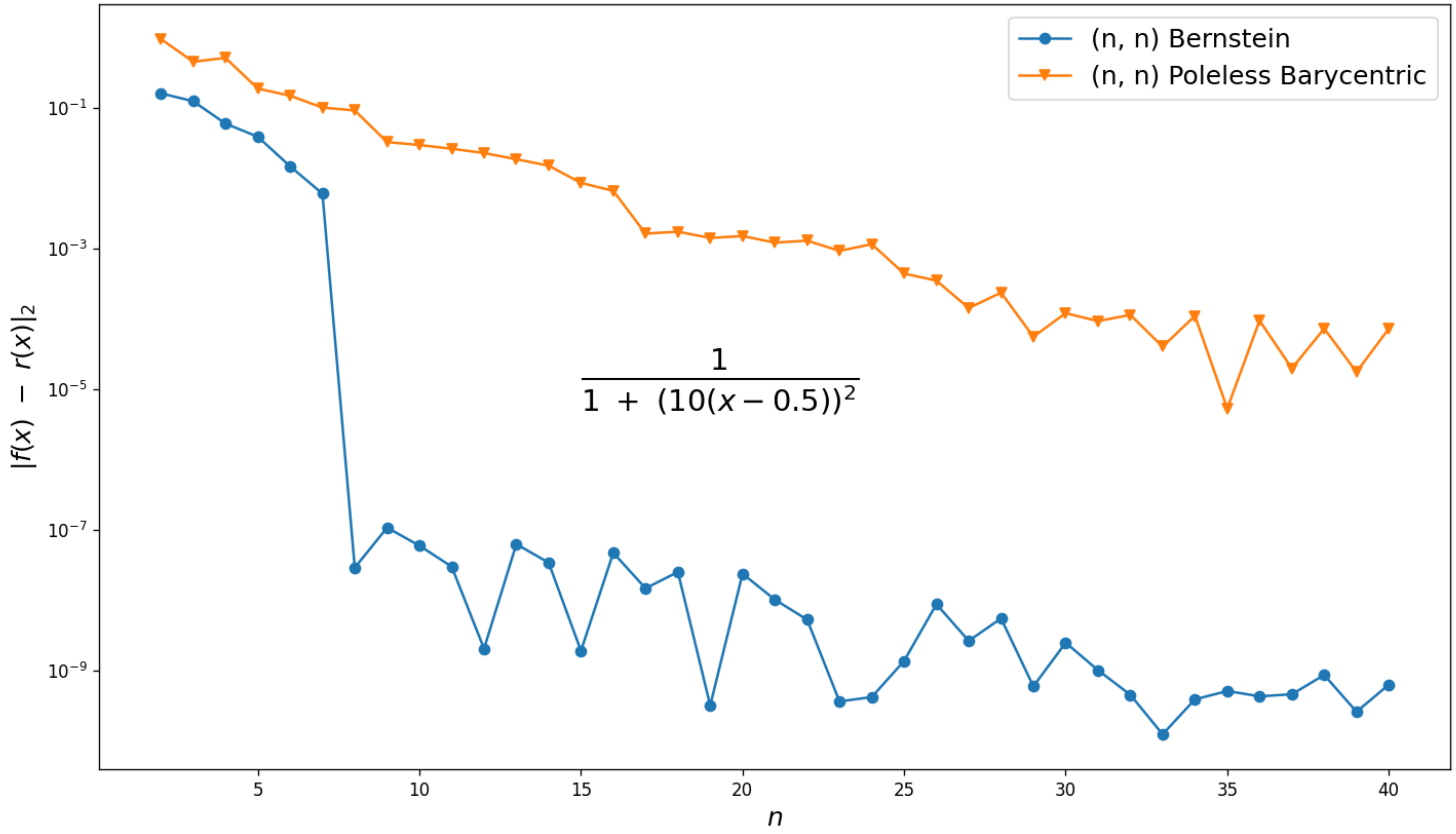
I think the only reasonable answer is, it depends.

Sometimes one really does need a guarantee about worst-case behavior.”

L. N. Trefethen (2012)

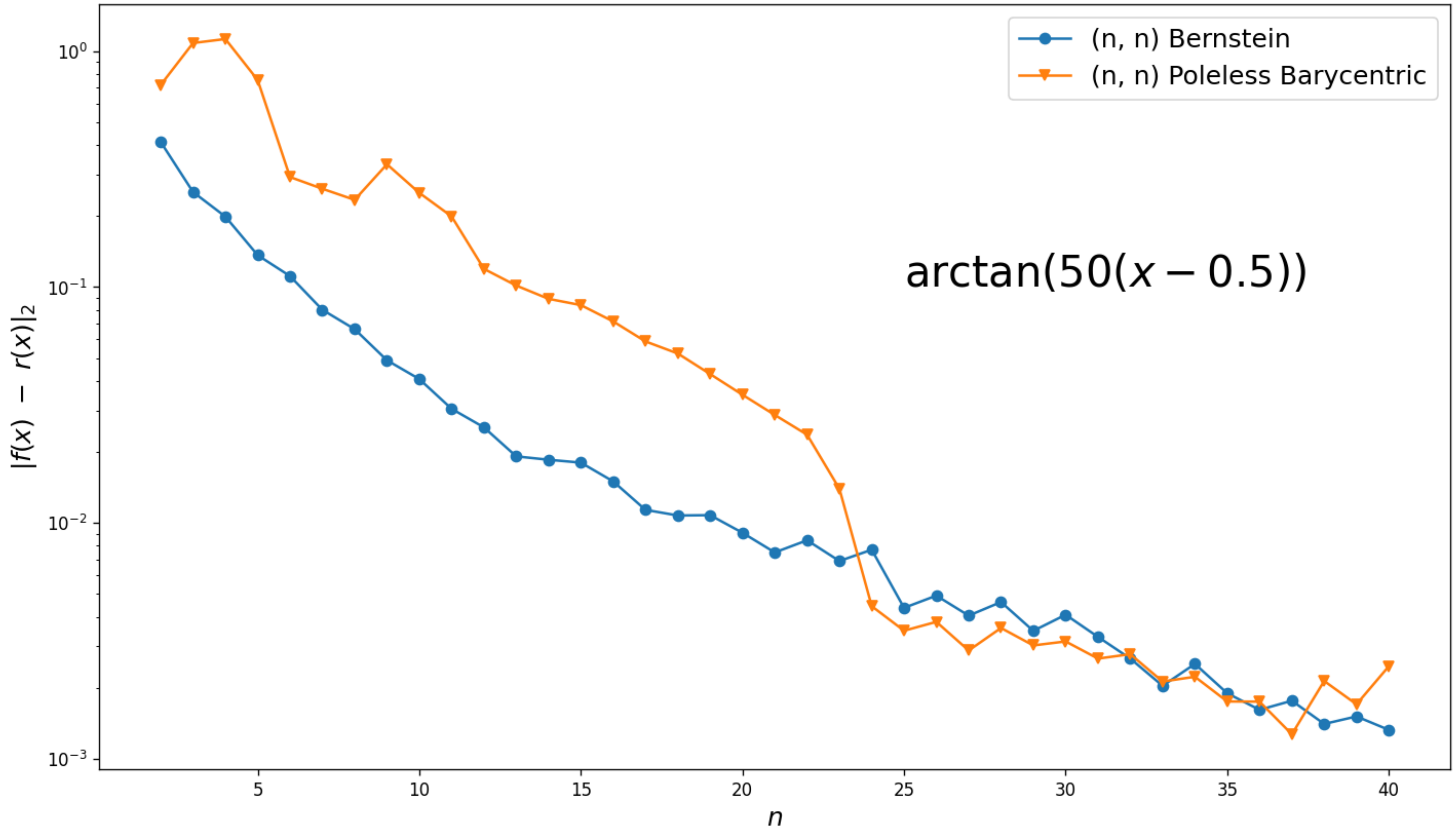
Poleless Barycentric

J. P. Berrut (1988) and M. S. Floater and K. Hormann (2007)



Poleless Barycentric

J. P. Berrut (1988) and M. S. Floater and K. Hormann (2007)



Poleless Barycentric

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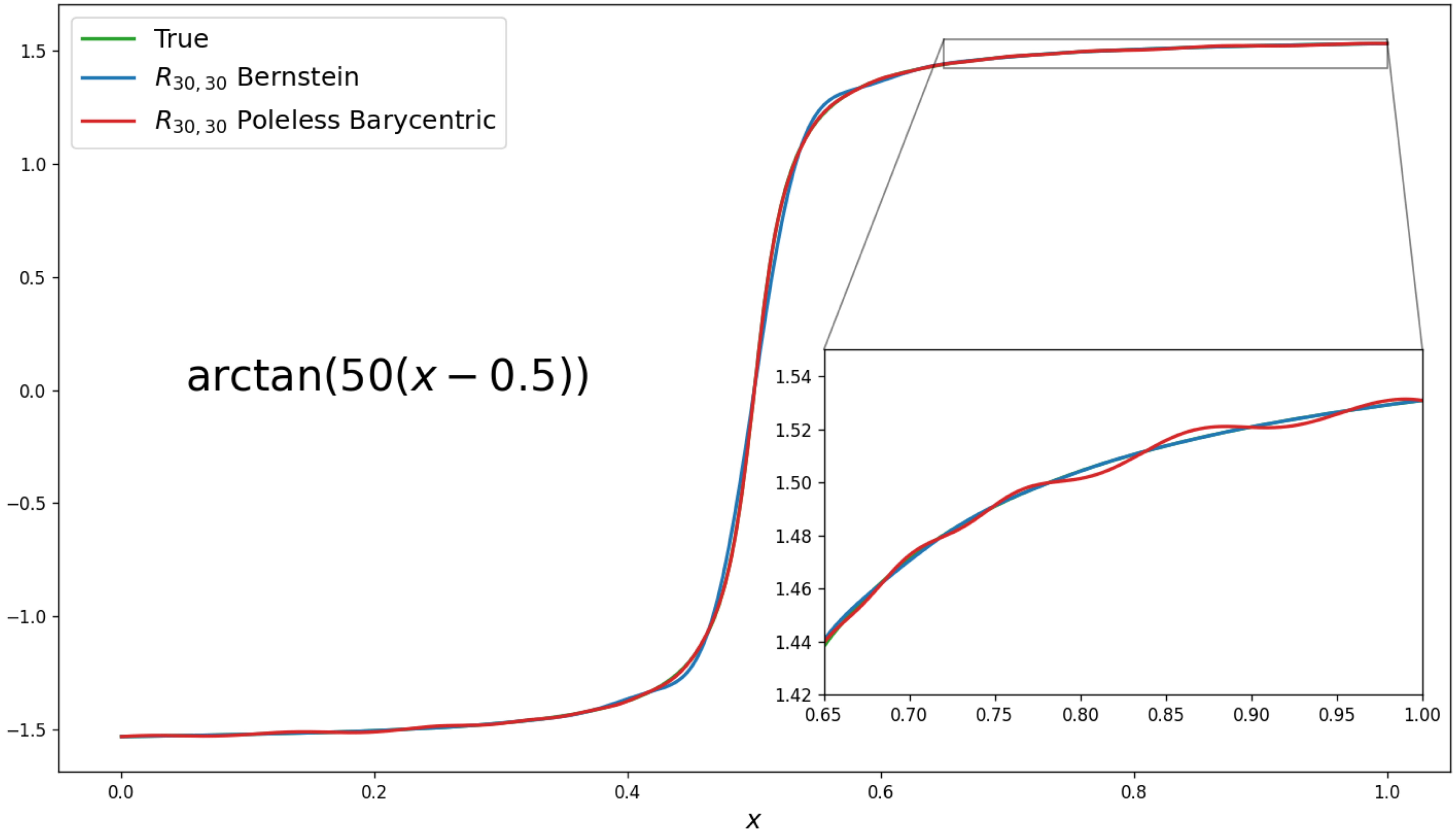
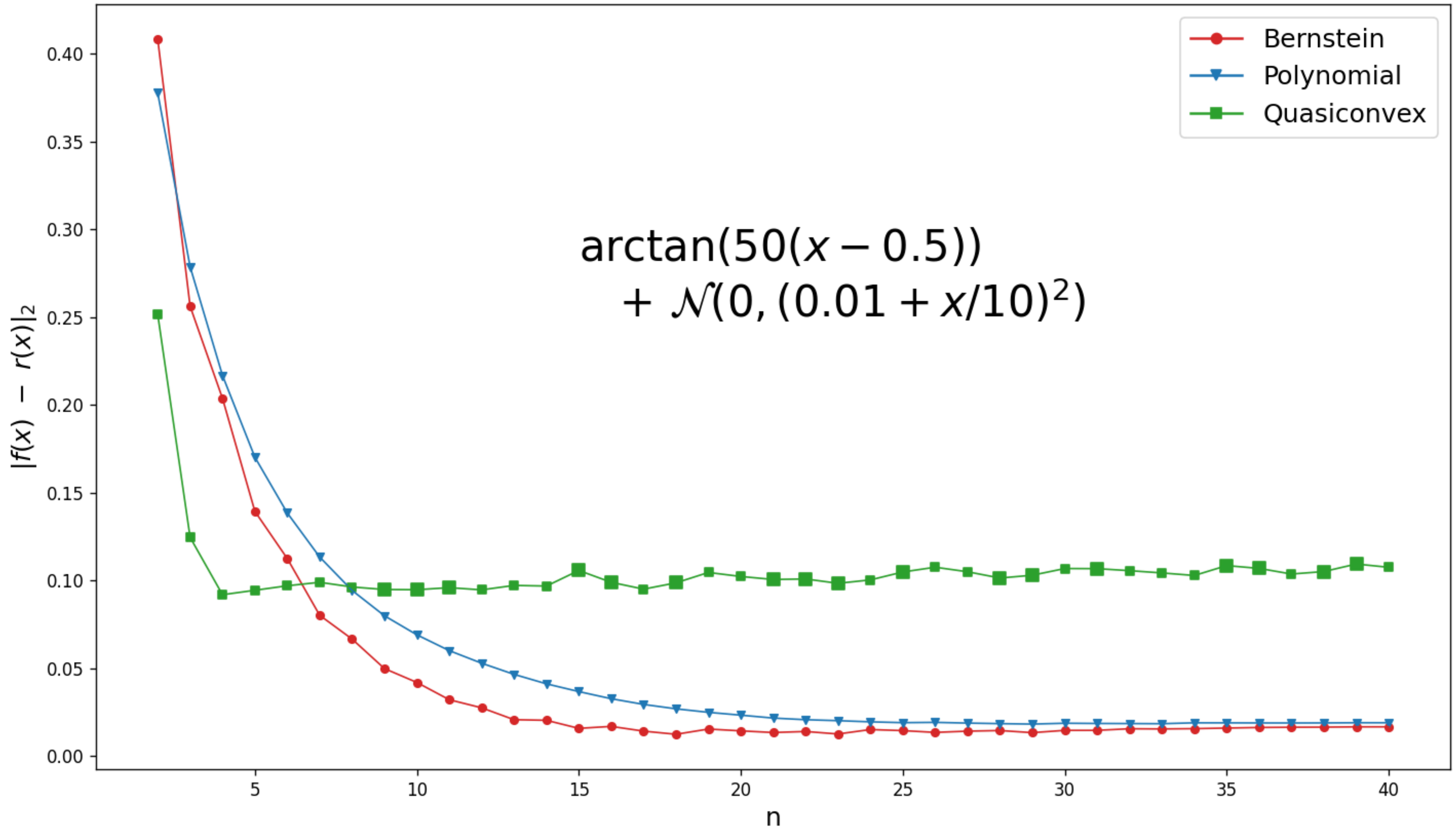


Quasiconvex

V. Peiris, N. Sharon, N. Sukhorukova, and J. Ugon (2021)



Quasiconvex

V. Peiris, N. Sharon, N. Sukhorukova, and J. Ugon (2021)