Dissecting Tool-Integrated Reasoning: An Empirical Study and Analysis

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Abstract

Large Language Models (LLMs) have made significant strides in reasoning tasks through methods like chain-of-thought (CoT) reasoning. However, they often fall short in tasks requiring precise computations. Tool-Integrated Reasoning (TIR) has emerged as a solution by incorporating external tools into the reasoning process. Nevertheless, the generalization of TIR in improving the reasoning ability of LLM is still unclear. Additionally, whether TIR has improved the model's reasoning behavior and helped the model think remains to be studied. We introduce REASONZOO, a comprehensive benchmark encompassing nine diverse reasoning categories, to evaluate the effectiveness of TIR across various domains. Additionally, we propose two novel metrics, Performance-Aware Cost (PAC) and Area Under the Performance-Cost Curve (AUC-PCC), to assess reasoning efficiency. Our empirical evaluation demonstrates that TIR-enabled models consistently outperform their non-TIR counterparts in both mathematical and non-mathematical tasks. Furthermore, TIR enhances reasoning efficiency, as evidenced by improved PAC and AUC-PCC scores, indicating reduced overthinking and more streamlined reasoning. These findings underscore the domain-general benefits of TIR and its potential to advance LLM capabilities in complex reasoning tasks.

1 Introduction

Large Language Models (LLMs) have showcased remarkable reasoning capabilities through explicit reasoning strategies, such as chain-of-thought (CoT) reasoning (Wei et al. 2022). Recent advancements in reasoning-focused LLMs, including the Qwen3 series (Yang et al. 2025), DeepSeek-R1 (DeepSeek-AI et al. 2025), and Kimi K1.5 (Team et al. 2025), have achieved significant performance improvements by leveraging extended inference times, resulting in verbose CoT outputs (Wei et al. 2022; Liu et al. 2025). However, these models often struggle with tasks requiring precise computations, such as complex arithmetic, equation solving, or symbolic manipulation, due to their reliance on probabilistic natural language processing (Chen et al. 2023; Gao et al. 2023). This limitation stems from the absence of built-in mechanisms for deterministic execution or symbolic reasoning, reducing such tasks to stochastic pattern matching rather than verifiable calculation.

To address these challenges, Tool-Integrated Reasoning (TIR) has emerged as a powerful approach to enhance LLM reasoning by incorporating external tools, such as executable code interpreters, into the reasoning process (Gou et al. 2024; Wang et al. 2024b; Shao et al. 2024; Liao et al. 2024; Li, Zou, and Liu 2025; Bai et al. 2025). TIR enables LLMs to interleave high-level natural language planning with low-level, self-contained code snippets dispatched to external interpreters. These interpreters return precise results, which the LLM reintegrates into its reasoning chain, significantly improving performance on tasks requiring exact computations or structured reasoning (Yang et al. 2024b; Li, Zou, and Liu 2025).

Despite these advancements, critical questions remain unresolved. First, while TIR has proven effective for mathematical reasoning, its generalizability to other reasoning domains, such as puzzles and general reasoning, is underexplored. Does TIR offer comparable benefits across a broader spectrum of reasoning tasks? Second, it remains unclear whether TIR fundamentally enhances the intrinsic reasoning capabilities of LLMs or merely serves as a conduit for external information. A key limitation of long-CoT LLMs is their computational inefficiency, often manifesting as "overthinking"—the generation of redundant tokens or unnecessary reasoning steps that can lead to errors (Chen et al. 2024; Wang et al. 2025b; Liu et al. 2024). TIR has the potential to mitigate overthinking by delegating complex computations to external tools, but the extent to which it improves reasoning efficiency versus merely supplementing information requires further investigation.

To address these questions, we introduce **REASONZOO**, a comprehensive benchmark designed to evaluate LLM reasoning across a broad spectrum of tasks, including formal language processing, complex arithmetic, operations research, and combinatorial puzzles, as detailed in Table 5. Unlike existing benchmarks focused primarily on mathematics, REASONZOO encompasses nine diverse reasoning categories, enabling a holistic assessment of LLM capabilities. To quantify reasoning efficiency under resource constraints, we propose two novel, task-agnostic metrics: **Performance-Aware Cost (PAC)** and **Area Under the Performance-Cost Curve (AUC-PCC)**. The PAC metric measures the computational cost (in tokens) required to achieve a specific performance threshold, formalizing the return-on-investment of each gen-

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erated token. The AUC-PCC metric evaluates cumulative performance across varying computational budgets, revealing how effectively a model converts additional computation into improved reasoning outcomes. High PAC and AUC-PCC scores indicate efficient reasoning with minimal overthinking, while low scores suggest inefficiencies due to redundant computation.

We conduct an extensive empirical evaluation of TIR across general-purpose LLMs with native TIR support (e.g., Qwen3 series (Yang et al. 2025)) and specialized models optimized via reinforcement learning, such as ToRL (Li, Zou, and Liu 2025) and CIR (Bai et al. 2025). Our evaluation, detailed in Tables 1 to 3, examines three TIR paradigms: Program of Thoughts (PoT), Multi-Turn TIR (MT-TIR), and Tool-Integrated Thinking (TIT). Key findings include:

- TIR-enabled models consistently outperform non-TIR-enabled counterparts across both mathematical and non-mathematical tasks, with significant accuracy gains.

- TIR enhances reasoning efficiency, as evidenced by improved PAC and AUC-PCC scores, reducing overthinking by streamlining reasoning paths and minimizing redundant computation.

Our contributions are threefold:

- We release REASONZOO, a diverse benchmark for evaluating LLM reasoning, alongside the PAC and AUC-PCC metrics, enabling holistic and cost-aware assessment of reasoning capabilities.

- We conduct a comprehensive evaluation of TIR-enabled LLMs across diverse reasoning tasks and TIR implementations, providing insights into their domain-general benefits.

- We analyze resulting reasoning patterns to elucidate how and when TIR yields efficiency gains, advancing the understanding of optimal reasoning strategies in LLMs.

2 Related Works

LLM Reasoning and Tool-Integrated Reasoning. Recent advances have transformed general-purpose Large Language Models (LLMs) into large reasoning models (LRMs) that generate extended, deliberate chain-of-thought (CoT) outputs (Ke et al. 2025). This transition is underpinned by a shift from supervised fine-tuning (SFT) to large-scale reinforcement learning (RL) as the dominant post-training paradigm. Leading 2024-2025 systems—including OpenAI's o-series (OpenAI 2025), DeepSeek-R1 (DeepSeek-AI et al. 2025), Kimi k1.5 (Team et al. 2025), and the Qwen3 family (Yang et al. 2025), demonstrate that RL can elicit sophisticated, self-discovered reasoning strategies without explicit supervision. Tool-integrated reasoning (TIR) has been shown to boost performance across a broad range of tasks by combining natural-language reasoning with targeted external tool calls. Improvements have been achieved via supervised fine-tuning (SFT) (Yue et al. 2024; Yu et al. 2024) and, more recently, through reinforcement learning (RL) (Xiong et al. 2025; Li, Zou, and Liu 2025; Feng et al. 2025). Notably, ToRL (Li, Zou, and Liu 2025) trains base models with RL, enabling them to discover optimal tool-use strategies

through exploration. ReTool (Feng et al. 2025) further incorporates live code execution into RL, allowing models to refine tool use on the fly and to self-correct erroneous code. RAGEN (Wang et al. 2025c) stabilises multi-turn interactions via trajectory-based RL (StarPO), balancing analytical and computational reasoning. Similarly, ARTIST (Singh et al. 2025) combines multi-turn TIR with RL-based tool invocation, yielding gains of up to 22% on both mathematical and function-calling benchmarks. These advances elicit sophisticated behaviours, such as suppressing ineffective code generation and dynamically selecting computational versus analytical steps, but also create a clear trade-off between accuracy and the computational cost of tool calls, underscoring the need for efficiency-oriented evaluation.

Evaluating LLM Reasoning Beyond Accuracy. Recent studies have begun to assess the reasoning capabilities of LLMs from perspectives beyond mere accuracy. Nayab et al. (2024) introduce concise-reasoning metrics (HCA, SCA, CCA) that reward brevity while preserving correctness, showing that output-length constraints can simultaneously improve accuracy. Erol et al. (2025) propose the cost-of-pass metric, which quantifies monetary expenditure per successful task. Chen et al. (2024) formulate outcome- and process-efficiency to measure over-thinking severity and suggest mitigation strategies. Wang et al. (2025a) explore verifier-free inference-time scaling, using the Pareto frontier to expose optimal quality-efficiency trade-offs and identify majority voting as the most robust and efficient strategy. Li, Chang, and Wu (2025) proposes THINK-Bench, a benchmark for evaluating thinking efficiency and chain-of-thought quality of LLMs, and proposes an efficiency metric based on the first occurrence of the correct answer. Concurrent research reduces token consumption via prompting or algorithmic interventions (Zheng et al. 2023; Hao et al. 2024; Wang et al. 2024a). Despite these advances, current efforts remain fragmented. They typically focus on isolated indicators of efficiency and lack a holistic, continuously parameterized, dynamic evaluation framework that can track performance across varying reasoning budgets and performance target while jointly assessing both the accuracy and efficiency of reasoning.

3 Preliminaries

Tool-Integrated Reasoning. Tool-Integrated Reasoning (TIR) (Gou et al. 2024; Yang et al. 2024b) denotes the utilization of LLMs to execute reasoning tasks necessitating external tools, typically, code, to aid the reasoning process. TIR has emerged as a robust paradigm enabling LLMs to undertake complex tasks that would be unfeasible for humans. Formally, the reasoning process of TIR can be iteratively represented as:

$$\begin{aligned} s_i &\sim \pi_\theta(\cdot|q, c_{i-1}), \\ c_i &\leftarrow c_{i-1} \oplus s_i \oplus R(s_i), \end{aligned} \tag{1}$$

where π_θ denotes the policy of the LLMs, s_i represents the current reasoning step containing executable code, and c_i signifies the current CoT, composed of the previous CoT c_{i-1}, the current reasoning step s_i, and the execution results $R(s_i)$.
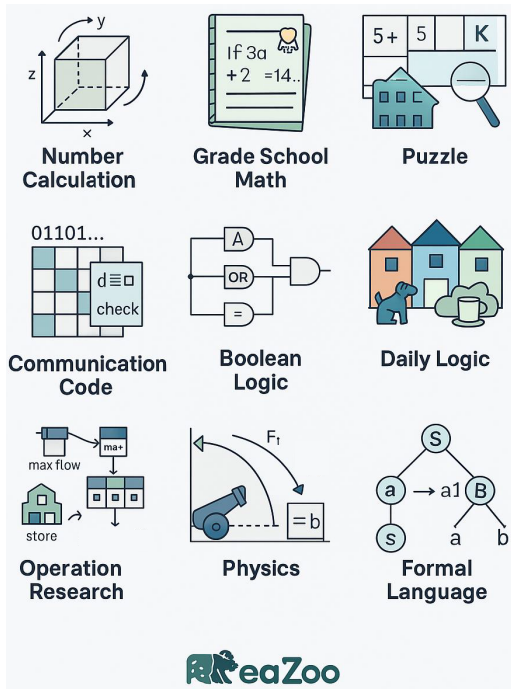
Figure 1: Illustration of the subtasks in REASONZOO

4 Evaluation Setup

4.1 Evaluation LLMs

In this paper, we evaluate two primary categories of LLMs. The first consists of foundational Large Reasoning Models (LRMs), including the Qwen3 series (8B, 32B, and 235B-A22B) (Yang et al. 2025) and DeepSeek-R1-0528 (DeepSeek-AI et al. 2025). The second category comprises models specifically enhanced for tool use via reinforcement learning motivated by the GRPO algorithm used by DeepSeek-R1, including CIR (Bai et al. 2025) and ToRL (Li, Zou, and Liu 2025).

4.2 REASONZOO: A Comprehensive Reasoning Benchmark

To thoroughly assess the reasoning capabilities of LLMs, particularly in non-mathematical tasks, we introduce **REASON-ZOO**, a benchmark suite comprising diverse tasks designed to evaluate various reasoning dimensions. As shown in Figure 1, REASONZOO includes nine distinct categories, each targeting specific logical or computational skills.

The REASONZOO benchmark is constructed from nine unique splits, each probing a different aspect of complex reasoning:

- **Number Calculation**: Encompasses geometric and coordinate-conversion tasks, such as 3D rotations, planar projections, and base conversions, requiring precise multi-step numerical computations.
- **Grade School Math**: Features non-routine elementary problems from Chinese mathematics competitions (e.g., Hope Cup, Golden Cup), extending standard curricula with multi-step puzzles.

- **Puzzle**: Includes combinatorial challenges like Sudoku, KenKen, word grids, and blocked-character equations, testing incremental deduction and heuristic search under strict constraints.
- **Communication Code**: Assesses the ability to perform precise, rule-based symbolic transformations, such as detecting and correcting transmission errors by decoding messages from standard block and convolutional codes.
- **Boolean Logic**: Involves symbolic tasks, such as Boolean simplification, Karnaugh map minimization, digital circuit analysis, and set-operation puzzles, assessing coherent manipulation.
- **Daily Logic**: Employs procedurally generated constraint-satisfaction puzzles (e.g., Temporal Clue, Zebra Logic) to evaluate non-monotonic spatio-temporal deduction in everyday contexts.
- **Operation Research**: Reformulates real-world integer and linear programming problems (e.g., production planning, supply-chain optimization, max-flow/min-cut) to test optimization reasoning in high-dimensional, constrained settings.
- **Physics**: Recasts high-school and undergraduate mechanics problems (e.g., elastic collisions, spring energy, friction, projectile motion) to emphasize multi-step deductive modeling.
- **Formal Language**: Presents partially completed derivations from generated context-free grammars, requiring traversal of parse trees to infer correct terminal symbols.

Each task across all splits is designed to be solvable through either pure verbal reasoning or the integration of external tools, enabling a direct comparison of the impact of tool use on performance and efficiency. Further details on dataset construction and task specifications are provided in Appendix A.

4.3 PAC & AUC-PCC: Beyond Accuracy

To comprehensively evaluate Tool-Integrated Reasoning (TIR) models, we introduce metrics that extend beyond accuracy by incorporating reasoning cost, measured as the number of generated tokens. Token counts serve as a practical, model-agnostic proxy for computational effort, as opposed to alternatives like FLOPs or monetary cost. High efficiency, defined as achieving superior accuracy with minimal token expenditure, indicates an optimal reasoning process with reduced computational overhead, less "overthinking," and fewer inferential errors (Chen et al. 2024). We propose two novel cost-sensitive metrics: **Performance-Aware Cost (PAC)** and **Area Under the Performance-Cost Curve (AUC-PCC)**, offering complementary insights:

- **PAC**: Measures the computational cost required to achieve a specific performance threshold, answering the question: "How many tokens are needed to reach X% accuracy?" This metric quantifies the speed of convergence to a reliable solution.
- **AUC-PCC**: Evaluates cumulative performance across the entire cost spectrum, rewarding models that achieve high accuracy early and maintain it, providing a holistic measure of reasoning efficiency.

PAC: Performance-Aware Cost. PAC quantifies the average computational cost required to achieve a target performance level. The core idea is to find the most efficiently solved subset of data samples that meets a given performance threshold. Given a performance threshold $\tau \in [0,1]$ (e.g., accuracy or other normalized metric), we first identify all subsets of the data, s, on which the model's performance is at least τ. Let this collection of subsets be $\mathcal{S}_\tau$. For each such subset $s \in \mathcal{S}_\tau$, we calculate its average normalized cost. PAC$_\tau$, is then defined as the minimum of these average costs over all valid subsets:

$$\text{PAC}_\tau = \min_{s \in \mathcal{S}_\tau} \left\{ 1 - \frac{1}{|s| \cdot C_{max}} \sum_{i \in s} C_i \right\}, \qquad (2)$$

where C_i is the cost for sample i and $C_{\max}$ is the maximum possible cost. If no subset of samples can achieve the performance τ, we define PAC$_\tau = 0$ as a penalty. Specifically, we define the PAC$_\tau$ corresponding to the maximum τ that the model can achieve as PAC:

$$\text{PAC} = \max_{\tau} \left\{ \text{PAC}_\tau \mid \text{PAC}_\tau > 0 \right\}. \qquad (3)$$

To create a single, comprehensive score that summarizes efficiency across all performance levels, we introduce m-PAC. This metric is defined as the area under the PAC$_\tau$ versus τ curve. We approximate this area by integrating over a series of M evenly spaced performance thresholds $\{\tau_j\}_{j=1}^M$:

$$\text{m-PAC} = \frac{1}{T} \sum_{i=1}^{M} \frac{\text{PAC}_{\tau_i} + \text{PAC}_{\tau_{i-1}}}{2}, \qquad (4)$$

where we define $\text{PAC}_{\tau_0} = 0$. A higher m-PAC value indicates superior efficiency, signifying that the model can solve a substantial fraction of problems at a low average cost. Conversely, a low m-PAC value suggests that the model is inefficient, requiring high computational cost even for easier subsets of problems, a behavior characteristic of "overthinking."

AUC-PCC: Area Under the Performance-Cost-Budget Curve. AUC-PCC provides a holistic score for reasoning capability by aggregating the total performance achieved over the entire computational budget. Given a set of $N+1$ observations as pairs of cost budget and corresponding performance (we use the similar method proposed in Muennighoff et al. (2025) to obtain performance under specific cost budget), $\{(C_i, P_i)\}_{i=0}^N$, we first normalize P_i to a common $[0, 1]$ scale:

$$C_i' = \frac{C_i}{C_{max}}, P_i' = \frac{P_i}{P_{max}}, \qquad (5)$$

where C_{max} is the maximum cost budget (e.g., the maximum output tokens) and P_{max} is the ideal maximum performance (e.g., 1.0 for accuracy). The AUC-PCC is then defined as the normalized Area Under the Performance-Cost-Budget Curve, which we approximate using the trapezoidal rule:

$$\text{AUC-PCC} = \sum_{i=1}^{N} \left[\frac{P_i' + P_{i-1}'}{2} \times (C_i' - C_{i-1}') \right]. \qquad (6)$$

The AUC-PCC reflects both the speed and stability of performance convergence. A high value indicates efficient resource utilization, where a model achieves high performance early and sustains it. Conversely, a low value may suggest diminishing returns, verbosity without proportional benefit, or other inefficient reasoning behaviors.

4.4 Implementation of TIR

In this paper, we examine several distinct paradigms for implementing Tool-Integrated Reasoning (TIR):

- **Program of Thoughts (PoT):** A method proposed by Chen et al. (2023) that externalizes the reasoning process into an executable Python program, with the program's output serving as the final answer.
- **Multi-Turn TIR (MT-TIR):** An approach that leverages the native function-calling capabilities of large language models to engage in iterative, stateful interactions with external tools.
- **Tool-Integrated Thinking (TIT):** A paradigm exemplified by models like ToRL (Li, Zou, and Liu 2025), which is characterized by multiple, interleaved steps of reasoning and tool execution before a final answer is synthesized.

5 Experimental Results and Analysis

5.1 Performance Across Diverse Reasoning Tasks

Table 1 summarizes the performance of all models on REA-SONZOO. The results offer key insights into the generalization capabilities of TIR models across a wide spectrum of reasoning tasks.

TIR Models Outperform Non-TIR Counterparts Models enhanced with TIR methods (PoT, MT-TIR, TIT) consistently outperform their non-TIR counterparts. For example, augmenting Qwen3-8B with PoT increases the average accuracy from 37.6% to 43.0%, while MT-TIR achieves 41.0%. In contrast, the best-performing non-TIR model at the same model scale, CIR, reaches only 16.9% average accuracy.

TIR Delivers Domain-General Gains TIR improvements extend across both mathematical and non-mathematical tasks. In mathematical domains such as Number Calculation and Grade School Math, TIR-enhanced models exhibit strong performance; for instance, Qwen3-235B-A22B with MT-TIR attains 58.4% and 42.0%, respectively. This trend also holds in non-mathematical domains, where the same model achieves 71.6% on Boolean Logic and 77.6% on Daily Logic, both substantially outperforming the non-TIR baselines. This consistent, cross-domain improvement validates TIR's versatility as a reasoning enhancement.

Benefits Scale with Model Size and TIR Sophistication The benefits of TIR scale directly with both model size and the sophistication of the applied technique. Applying PoT to Qwen3-8B yields 43.0% average accuracy, while the larger Qwen3-235B-A22B reaches 52.8%. Employing the more advanced MT-TIR strategy further boosts this to 61.0%, highlighting a synergistic relationship between model capacity and TIR sophistication. This trend proves that gains are not

Table 1: The performance of various Tool-Integrated Reasoning models across different subsets of REASONZOO is evaluated. In this context, PoT refers to the Program of Thoughts, MT-TIR to Multi-Turn TIR, and TIT to Tool-Integrated Thinking. The abbreviations NC, GSM, Puzzle, CC, BL, DL, OR, Phy, and FL represent Number Calculation, Grade School Math, Puzzle, Communication Code, Boolean Logic, Daily Logic, Physics, and Formal Language, respectively.

| Model | TIR Type | Accuracy (%) ↑ | | | | | | | | | |
| | | Math | | Non-Math | | | | | | | Avg |
		NC	GSM	Puzzle	CC	BL	DL	OR	Phy	FL	
Qwen3-8B	–	40.1	21.0	19.4	46.9	73.7	65.5	34.0	22.0	15.6	37.6
	PoT	$42.3_{+2.2}$	$36.0_{+15.0}$	$46.4_{+27.0}$	$57.4_{+10.5}$	$59.7_{-14.0}$	$65.7_{+0.2}$	$48.0_{+14.0}$	$15.0_{-7.0}$	$16.5_{+0.9}$	$43.0_{+5.4}$
	MT-TIR	$46.0_{+5.9}$	$23.0_{+2.0}$	$43.3_{+23.9}$	$50.6_{+3.7}$	$66.4_{-7.3}$	$64.8_{-0.7}$	$45.0_{+11.0}$	$18.0_{-4.0}$	$12.1_{-3.5}$	$41.0_{+3.4}$
Qwen3-32B	–	57.7	26.0	19.9	50.1	75.8	63.6	39.0	27.0	22.9	42.4
	PoT	$57.2_{-0.5}$	$42.0_{+16.0}$	$65.2_{+45.3}$	$69.6_{+19.5}$	$58.5_{-17.3}$	$67.1_{+3.6}$	$51.0_{+12.0}$	$11.0_{-16.0}$	$27.5_{+4.6}$	$49.9_{+7.5}$
	MT-TIR	$65.5_{+7.8}$	$22.0_{-4.0}$	$43.7_{+23.8}$	$51.4_{+1.3}$	$77.3_{+1.5}$	$71.9_{+8.3}$	$54.0_{+15.0}$	$17.0_{-10.0}$	$22.7_{-0.2}$	$47.3_{+4.9}$
Qwen3-235B-A22B	–	44.5	31.0	38.1	75.7	54.8	53.3	45.0	40.0	29.8	45.8
	PoT	$51.8_{+7.3}$	$39.0_{+8.0}$	$60.7_{+22.6}$	$78.8_{+3.1}$	$69.0_{+14.2}$	$61.2_{+7.9}$	$52.0_{+7.0}$	$23.0_{-17.0}$	$39.6_{+9.8}$	$52.8_{+7.0}$
	MT-TIR	$58.4_{+13.9}$	$42.0_{+11.0}$	$70.4_{+32.3}$	$79.1_{+3.4}$	$71.6_{+16.8}$	$77.6_{+24.3}$	$58.0_{+13.0}$	$31.0_{-9.0}$	$60.2_{+30.4}$	$61.0_{+15.2}$
DeepSeek-R1-0528	–	37.4	32.0	13.6	40.2	73.9	65.5	42.0	41.0	31.8	41.9
	PoT	$55.0_{+17.6}$	$22.0_{-10.0}$	$46.4_{+32.8}$	$48.9_{+8.7}$	$49.3_{-24.6}$	$48.3_{-17.2}$	$35.0_{-7.0}$	$9.0_{-32.0}$	$15.3_{-16.5}$	$36.6_{-5.3}$
	MT-TIR	$69.4_{+32.0}$	$40.0_{+8.0}$	$55.4_{+41.8}$	$66.8_{+26.6}$	$77.3_{+3.4}$	$82.4_{+16.9}$	$51.0_{+9.0}$	$17.0_{-24.0}$	$38.4_{+6.6}$	$55.3_{+13.4}$
Qwen2.5-Math-7B	–	13.9	2.0	1.6	2.7	2.6	0.2	10.0	4.0	6.4	4.8
ToRL	TIT	$41.6_{+27.7}$	$11.0_{+9.0}$	$11.0_{+9.4}$	$14.5_{+11.8}$	$49.8_{+47.2}$	$1.0_{+0.8}$	$27.0_{+17.0}$	$3.0_{-1.0}$	$9.4_{+3.0}$	$18.7_{+13.9}$
CIR	TIT	$45.5_{+31.6}$	$9.0_{+7.0}$	$7.1_{+5.5}$	$7.7_{+5.0}$	$48.1_{+45.5}$	$1.0_{+0.8}$	$23.0_{+13.0}$	$4.0_{+0.0}$	$6.6_{+0.2}$	$16.9_{+12.1}$

merely additive but synergistic, with advanced TIR methods unlocking greater potential from more capable base models.

5.2 Analysis of PAC and m-PAC Metrics

The PAC and m-PAC metrics (Table 2) provide critical insights into the computational efficiency of TIR, revealing several key trends.

Multi-Turn TIR Enhances Efficiency Multi-Turn Tool-Integrated Reasoning (MT-TIR) consistently improves both PAC and m-PAC scores across all Qwen3 models compared to their baselines. For example, Qwen3-8B's m-PAC for "All Tokens" increases from 25.4% to 32.8%, a gain of 7.4 percentage points. Similarly, Qwen3-235B-A22B improves from 34.6% to 49.2%, marking a 14.6-point increase. These improvements suggest that MT-TIR enhances reasoning efficiency by directing models toward more optimal solution paths, thereby reducing computational overhead and mitigating excessive or redundant reasoning ("overthinking").

Program of Thoughts Yields Variable Efficiency Gains The Program of Thoughts (PoT) strategy yields varying degrees of efficiency improvement. For Qwen3-8B, PoT increases m-PAC for "All Tokens" from 25.4% to 33.6%, an 8.2-point gain. Qwen3-32B and Qwen3-235B-A22B also benefit, with gains of 9.78 and 7.96 percentage points, respectively. These results indicate that while PoT generally improves efficiency across the Qwen3 series, its effectiveness is model-dependent.

Tool-Integrated Thinking Improves Efficiency in Specialized Models In specialized models such as Qwen2.5-Math-7B, Tool-Integrated Thinking (TIT) strategies—including ToRL and CIR—significantly enhance efficiency. The

model's m-PAC for "All Tokens" rises from 6.6% to 14.3% with ToRL (a 7.7-point gain), and to 11.0% with CIR (a 4.4-point gain). Although the absolute values of m-PAC remain lower than those of the Qwen3 series, the relative improvements highlight TIT's ability to reduce computational burden by promoting more direct, goal-oriented reasoning.

Efficiency Gains Scale with LLM Capability The effectiveness of TIR is closely tied to the model's underlying capabilities, including both its size and its function-calling proficiency. In terms of scale, Qwen3-235B-A22B demonstrates notably greater efficiency gains compared to Qwen3-8B and Qwen3-32B. Regarding function-calling ability, Qwen3-235B-A22B outperforms DeepSeek-R1-0528, likely due to its enhanced code-assistant reasoning skills.

Trade-offs in TIR Strategies Under No-Think Mode State-of-the-art Large Reasoning Models (LRMs) such as the Qwen3 family support dynamic switching between extended chain-of-thought ("think") and concise ("no-think") reasoning modes. Disabling reflective reasoning yields immediate but shallow benefits: for instance, Qwen3-8B's PAC increases by 10.5 points, yet its m-PAC drops by 20.0 points. This suggests that while rapid, single-turn reasoning suffices for simpler tasks, it is inadequate for problems requiring sustained, multi-step deliberation.

5.3 Analysis of AUC-PCC Results

Due to the computational intensity of the AUC-PCC metric, which requires performance evaluation across specific token budgets, we focus on Multi-Turn TIR. Results, presented in Figure 2 and Table 3, highlight two key findings.

Table 2: Performance of models on REASONZOO, evaluated using PAC and m-PAC metrics. The table compares TIR methods: Program of Thoughts (PoT), Multi-Turn TIR (MT-TIR), and Tool-Integrated Thinking (TIT). It distinguishes between total tokens ("All Tokens") and reasoning-only tokens ("Non-Tool Tokens"), excluding tool code and API calls. Calculations use a maximum cost, $C_{\max}$, of 32,768 and performance thresholds $[0.1, 0.2, \ldots, 1.0]$.

Model	TIR Type	All Tokens			Non-Tool Tokens		
		# Tokens	PAC (%)↑	m-PAC (%)↑	# Tokens	PAC (%)↑	m-PAC (%)↑
Qwen3-8B	–	11,942	84.9	25.4	11,918	84.9	25.4
	PoT	10,855	$86.2_{+1.3}$	$33.6_{+8.2}$	10,253	$87.0_{+2.1}$	$33.8_{+8.4}$
	MT-TIR	15,462	$80.8_{-4.1}$	$32.8_{+7.4}$	14,896	$81.4_{-3.5}$	$32.9_{+7.5}$
	MT-TIR$_{\text{nothink}}$	4,826	$95.4_{+10.5}$	$5.4_{-20.0}$	3,920	$97.6_{+12.7}$	$5.5_{-19.9}$
Qwen3-32B	–	9,160	87.9	32.2	9,145	88.0	32.2
	PoT	9,893	$88.1_{+0.2}$	$42.0_{+9.8}$	9,345	$88.8_{+0.8}$	$41.2_{+9.0}$
	MT-TIR	11,899	$87.9_{-0.1}$	$40.7_{+8.5}$	11,450	$88.5_{+0.5}$	$40.8_{+8.6}$
	MT-TIR$_{\text{nothink}}$	2,594	$95.5_{+7.6}$	$9.7_{-22.5}$	1,689	$97.2_{+9.2}$	$9.8_{-22.4}$
Qwen3-235B-A22B	–	9,160	77.8	34.6	9,145	77.9	34.6
	PoT	10,552	$85.1_{+7.3}$	$42.6_{+8.0}$	10,240	$85.6_{+7.8}$	$42.8_{+8.2}$
	MT-TIR	14,066	$80.8_{+3.0}$	$49.2_{+14.6}$	13,183	$82.0_{+4.1}$	$49.7_{+15.1}$
	MT-TIR$_{\text{nothink}}$	1,520	$77.1_{-0.7}$	$21.0_{-13.6}$	636	$77.5_{-0.4}$	$20.7_{-13.9}$
DeepSeek-R1-0528	–	8,278	90.0	36.5	8,260	90.1	36.5
	PoT	8,310	$90.1_{+0.1}$	$28.2_{-8.3}$	8,080	$90.7_{+0.6}$	$28.4_{-8.1}$
	MT-TIR	11,284	$85.6_{-4.4}$	$45.7_{+9.2}$	10,819	$86.4_{-3.7}$	$45.9_{+9.4}$
Qwen2.5-Math-7B	–	1,431	87.4	6.6	1,418	87.4	6.6
ToRL	TIT	2,209	$97.9_{+10.6}$	$14.3_{+7.7}$	1,280	$98.6_{+11.3}$	$14.4_{+7.8}$
CIR	TIT	2,659	$97.3_{+9.9}$	$11.0_{+4.4}$	1,422	$98.4_{+11.0}$	$11.0_{+4.4}$

Consistent AUC-PCC Improvements with TIR TIR-enabled models consistently achieve higher AUC-PCC scores than their baselines. For example, Qwen3-8B's AUC-PCC increases from 32.92 to 33.30 with TIR, a 0.08-point improvement. Qwen3-32B shows a 1.31-point gain (from 38.31 to 39.62), while Qwen3-235B-A22B exhibits the largest improvement, from 37.10 to 46.18, a 9.08-point increase. These results confirm that TIR enhances efficient utilization of computational budgets, improving overall performance across varying cost constraints.

TIR Mitigates Diminishing Returns The higher AUC-PCC scores achieved by TIR-enabled models are direct evidence that this approach alleviates diminishing returns and curbs inefficient reasoning behaviors. TIR-enabled models achieve peak performance more quickly and sustain it across a broader budget range. This is particularly evident in Qwen3-235B-A22B, where substantial gains reflect a more focused and less redundant ("overthinking") reasoning strategy.

Scaling Effects of Tool-Integrated Reasoning Accuracy gains from Tool-Integrated Reasoning scale monotonically with the allocated token budget. Below approximately 8K tokens, the overhead of tool invocation can outweigh its benefits, resulting in slight performance degradation relative to standard decoding. However, beyond this threshold, TIR-enabled models rapidly surpass their baselines. At 32K tokens, Qwen3-235B-A22B and DeepSeek-R1-0528 outperform their vanilla counterparts by 15.2 and 13.4 percentage points, respectively. While TIR incurs an initial computational cost, it unlocks structured, multi-step reasoning that yields substantial downstream benefits, exceeding the gains previously attributed to deeper chain-of-thought reasoning alone (Suzgun et al. 2023; Snell et al. 2024). Additional evidence from budget-forcing experiments on the Qwen2.5 series further supports this trend (Appendix B) (Yang et al. 2024a).

5.4 Attribution Analysis

The preceding experimental results indicate that TIR not **only enhances reasoning performance but also improves the model's reasoning behaviors**, as reflected by metrics such as PAC and AUC-PCC. In this section, we examine the critical aspect of TIR: the sources of its performance gains. By decomposing accuracy gains and analyzing outcome efficiency, we provide insights into how TIR enhances reasoning behavior and identify directions for further optimization.

Attribution Decomposition for the Performance Gains To better understand the source of TIR's performance gains, we conduct a fine-grained analysis comparing the predictions of base LLMs and their TIR-augmented counterparts. Specifically, we identify instances where the two models produce inconsistent answer labels and focus our analysis on these cases. Using an auxiliary model (Qwen2.5-32B-Instruct), we further assess whether the additional correct and incorrect predictions made by the TIR LLMs are attributable to tool usage, and quantify their proportions. The results are presented in the left panel of Figure 3. In this analysis, *Tool-Related Acc.* $+\Delta$ denotes the proportion of additional correct predictions made by TIR LLMs that are attributed to feedback from tool use, while *Acc.* $+\Delta$ represents correct predictions not directly linked to tool assistance. The red bars indicate the
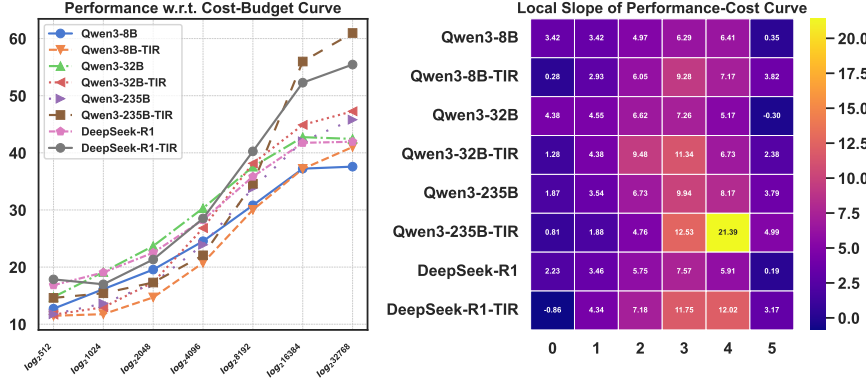
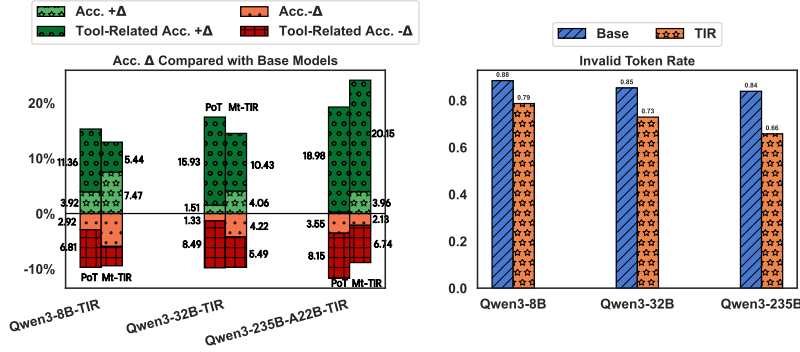Figure 2: Illustration of Performance w.r.t. Cost-Budget Curve on REASONZOO.

Table 3: Performance of models on REA-SONZOO, evaluating using AUC-PCC metric. Here, we leverage the Multi-Turn TIR as the implementation of TIR.

Model	TIR	AUC-PCC (%) ↑
Qwen3-8B	✗	32.92
	✓	33.30 $_{+0.08}$
Qwen3-32B	✗	38.31
	✓	39.62 $_{+1.31}$
Qwen3-235B-A22B	✗	37.10
	✓	46.18 $_{+9.08}$
DeepSeek-R1-0528	✗	37.27
	✓	45.36 $_{+8.09}$

Figure 3: Illustration of attribution analysis and overthinking analysis.

Table 4: Average token usage for correct responses is decomposed into three segments: (i) Reason: all reasoning tokens for correct responses; (ii) First: tokens generated up to and including the first correct answer; and (iii) Refl.: tokens produced after the model has reached correctness. ζ_o means the ratio of 'First'.

Model	TIR	Reason	First	Refl.	ζ_o	Steps
Qwen3-8B	✗	10,341	924	9,417	0.116	257
	✓	10,711	1,644	9,067	0.214	272
Qwen3-32B	✗	8,952	921	8,031	0.147	220
	✓	10,278	1,805	8,473	0.271	249
Qwen3-235B-A22B	✗	7,708	1,387	6,322	0.161	293
	✓	9,586	3,044	6,541	0.343	330

proportion of additional incorrect predictions introduced by TIR models compared to their base counterparts. Our findings show that tool usage contributes significantly to TIR performance improvements. For instance, under the PoT and MT-Turn settings of Qwen3-32B, 15.93% and 10.43% of previously incorrect predictions were corrected with tool assistance. This effect is even more pronounced in Qwen3-235B-A22B, where tool use led to improvements in 18.98% and 20.15% of cases, respectively. These results demonstrate that TIR positively impacts model reasoning and decision-making processes. However, we also observe across all evaluated models that a notable proportion of initially correct predictions by the base LLM degrade to incorrect ones due to tool assistance TIR. This underscores existing limitations in reliably utilizing tools and effectively incorporating their feedback, pinpointing a critical area for future improvement in TIR methods.

Outcome Efficiency and Mitigation of Overthinking We further investigate the impact of TIR on mitigating the well-known *overthinking* tendencies prevalent in large reasoning models (LRMs). Specifically, we quantify this effect by analyzing the *outcome efficiency* (ζ_o), defined as the fraction of reasoning tokens generated up to and including the first correct answer while considering waste of tokens for incorrect responses (Chen et al. 2024; Li, Chang, and Wu 2025). Results presented in the right panel of Figure 3 and summa-

rized in Table 4 demonstrate substantial improvements across model scales. For instance, the introduction of TIR nearly doubles outcome efficiency for Qwen3-8B (from 11.6% to 21.4%) and Qwen3-32B (from 147% to 27.1%), representing relative gains of approximately 84% each. Notably, even for the larger and more capable Qwen3-235B-A22B, we observe a considerable improvement from 16.1% to 34.3%, reflecting a relative increase of about 113%. This consistent trend underscores how TIR effectively reshapes token allocation by replacing verbose, redundant reflections with concise and goal-oriented reasoning steps.

6 Conclusion

In this paper, we investigate the generalizability of TIR for improving multi-step reasoning in large language models. We introduce a systematic, cost-aware evaluation framework—anchored by the REASONZOO benchmark and the novel PAC and AUC-PCC metrics—to quantify both inference accuracy and computational efficiency. We demonstrate that TIR-enabled models achieve both higher accuracy and greater cost-efficiency than their non-TIR counterparts across a wide array of complex reasoning tasks. These results underscore the domain-general advantages of TIR and its effectiveness in reducing overthinking, thereby streamlining the reasoning process. Our findings contribute to a deeper understanding of how TIR can be leveraged to advance LLM performance in complex reasoning scenarios.

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A Details of REASONZOO

Formal Language. Problems in this category require models to recognize and apply context-free grammar rules to infer missing terminals in sequences. For example, given a set of production rules and a partially completed sequence, models must determine the next terminal symbol by properly traversing parse trees and applying production rules. This tests the model's ability to follow formal language constraints and perform parse-tree reasoning.

Number Calculation. This category presents precise numerical computation tasks involving geometric transformations, coordinate system conversions, and mathematical operations. Problems include rotating points in 3D space, projecting points onto planes, converting between coordinate systems, and performing base conversions. These tasks require mathematical precision and the ability to maintain numerical consistency throughout multi-step calculations.

Physics. The physics problems test application of physical principles to complex scenarios. Questions involve mechanics concepts like elastic collisions, spring potential energy, friction, and projectile motion. These tasks test both quantitative reasoning and the ability to model physical systems. To put more emphasis on evaluation of logic reasoning ability rather than knowledge reserve, we use high school or undergraduate level engineering physics. Utilizing accessible physics principles, the complexity arises from the multi-step deductive reasoning required to model the system and arrive at a solution.

Boolean Logic. This category evaluates symbolic manipulation and logical reasoning through problems involving Boolean expressions, Karnaugh maps, digital circuit analysis, and set operations. Specific tasks include demultiplexer circuit operations, state machine transitions, multi-step operations under new-defined rules, etc. These problems test the model's ability to perform multi-hop logical evaluations and maintain consistency in symbolic reasoning.

Gradeschool Math. This benchmark evaluates mathematical reasoning through non-routine elementary problems sourced from widely administered Chinese mathematics competitions, including the Xiwangbei (Hope Cup) and Hualuogeng Golden Cup. Designed for students demonstrating above-curriculum proficiency, these problems adhere to the pedagogical principle of deriving challenges from standard curricula while transcending rote application. The benchmark emphasizes conceptually clear problems requiring multi-step logical decomposition, where solutions demand systematic combination of fundamental operations within rigorously defined constraints. By requiring models to navigate problems with non-obvious solution trajectories yet definitive answers, this benchmark probes the capacity for flexible symbolic manipulation, compositional reasoning, and verification-aware inference—critical capabilities for robust quantitative reasoning in language models.

Daily Logic. This category includes procedurally generated logic puzzles designed to test an LLM's ability to integrate and reason over structured constraints in everyday-style scenarios. We adopt two recent benchmarks, Temporal Clue(Hilton 2025), a Clue-inspired benchmark that evaluates deductive reasoning involving spatial and temporal dimensions and Zebralogic(Lin et al. 2025) a scalable benchmark derived from classical logic grid puzzles, formulated as constraint satisfaction problems (CSPs). Puzzles are generated with quantifiable difficulty and verified solutions, making it ideal for evaluating non-monotonic, multi-hop reasoning. We reuse their code to generate new question samples.

Puzzle. This category features complex logic and mathematical puzzles, including Sudoku, KenKen, word grid puzzles, and character-blocked arithmetic equations. These problems require the model to leverage combinational logic, numerical reasoning, and pattern recognition to systematically explore solution spaces and efficiently prune infeasible paths. Tasks specifically examine an LLM's capability in incremental logical deduction, heuristic reasoning, and structured problem-solving under constrained scenarios.

Cipher and Coding Theory. This category tests the model's proficiency in symbolic manipulation and discrete mathematics, emphasizing cipher encoding and decoding tasks inspired by benchmarks like KOR-Bench(Ma et al. 2025) Problems cover classical encryption techniques, such as substitution and transposition ciphers, alongside advanced error-detection and correction methodologies from coding theory, including block codes, parity checking, and convolutional codes. This evaluates the model's capability for meticulous step-by-step transformations, error identification, and decoding accuracy under varying complexity.

Operations Research. This set comprises real-world optimization problems extracted and adapted from "Problems and Exercises in Operations Research"(Liberti 2006). It covers diverse applications such as marketing strategies, production planning, and supply chain management, requiring models to engage with integer programming, linear optimization, convex optimization, and network flow analysis (e.g., max-flow, min-cut problems). Tasks assess the model's practical reasoning capabilities, numerical stability, and ability to strategically navigate and solve high-dimensional constrained optimization problems reflective of realistic scenarios.

Table 5: Statistics of REASONZOO, detailing nine reasoning subtask categories and their respective sample sizes.

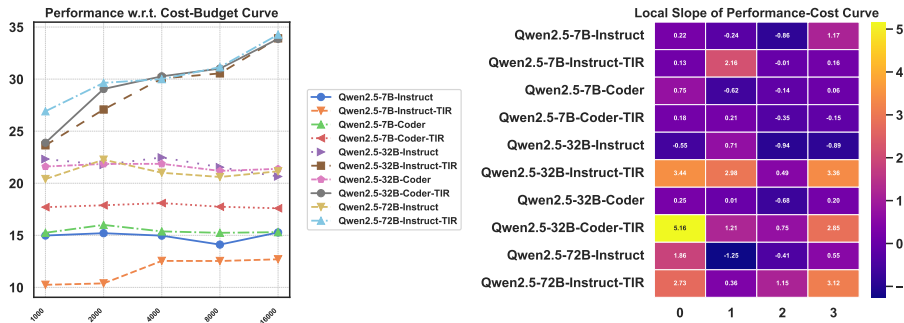| Category	# Samples
Number Calculation	409
Grade School Math	100
Puzzle	607
Cipher and Code	422
Boolean Logic	401
Daily Logic	505
Operations Research	100
Physics	100
Formal Language	441

Figure 4: Illustration of Performance w.r.t. Cost-Budget Curve on REASONZOO of Qwen2.5 series models.

Table 6: AUC-PCC comparison of qwen2.5 series models w/ and w/o TIR

Model	TIR	AUC-PCC (%) ↑
Qwen2.5-7B-Instruct	✗	3.57
	✓	$2.96_{-0.61}$
Qwen2.5-7B-Coder	✗	3.72
	✓	$4.31_{+0.59}$
Qwen2.5-32B-Instruct	✗	5.22
	✓	$7.40_{+2.18}$
Qwen2.5-32B-Coder	✗	5.20
	✓	$7.50_{+2.30}$
Qwen2.5-72B-Instruct	✗	5.08
	✓	$7.58_{+2.50}$

B Additional AUC-PCC Results of Qwen2.5 Series Models

Figure 4 extends the analysis in § 5.4 to the Qwen2.5 model family, comparing base LLMs and TIR-enabled LLMs across five token budget levels (1K-16K). The left panel plots average accuracy against token budget, while the right panel illustrates the local slope (i.e., accuracy change per budget increment). Experiments are aconducted using the budget forcing method introduced by Muennighoff et al. (2025). We highlight three key findings:

Effect of TIR on Non-Reasoning LLMs. Under vanilla budget forcing, the non-reasoning instruct models (7B, 32B, 72B) tend to plateau or decline beyond the 4K token budget. Their baseline AUC-PCC scores are 3.57%, 5.22%, and 5.08%, respectively. With TIR, the 7B model sees a slight decline to 2.96% (-0.61), whereas the 32B and 72B models increase to 7.40% and 7.58% (+2.18 and +2.50). Notably, the TIR-enabled curves for 32B and 72B maintain positive local slopes at every step (e.g., 32B: +3.44, +2.98, +0.49, +3.37 pp), indicating consistent gains across token budgets.

Coder Variants and Scaling. The Qwen2.5-7B-Coder exhibits mixed behavior with TIR: it improves in AUC-PCC by +0.59 and consistently outperforms its vanilla counterpart by approximately 2.3-2.7 across all budgets. However, its local slopes remain small and occasionally negative (+0.18, +0.21, -0.35, -0.15 pp). In contrast, the Qwen2.5-32B-Coder with TIR shows substantial improvement, with an initial slope exceeding +5 pp and sustained positive growth across all steps (+5.16, +1.21, +0.75, +2.85), ultimately reaching an AUC-PCC of 7.50% (+2.30). These results suggest that larger Coder models (32B+) are better equipped to utilize tool feedback effectively.

Scaling and Cost-Benefit Tradeoff. The benefits of TIR scale with model capacity: they are negligible or negative at 7B, moderate at 32B, and highest in absolute terms at 72B. It is important to note that the line plots reflect accuracy versus token budget, whereas AUC-PCC integrates accuracy with respect to *cost*. Thus, increased per-query tool usage under TIR can diminish overall AUC-PCC gains—explaining, for instance, why Qwen2.5-7B-Coder exhibits notable per-budget accuracy improvements but only a modest 0.59 gain

in AUC-PCC. Accordingly, we interpret the strong performance of TIR at 32B and 72B as evidence of a favorable accuracy-cost tradeoff that becomes increasingly robust at scale.

Detailed case studies illustrating how TIR reshapes chain-of-thought traces in Qwen2.5 series models are provided in Appendix C.

C Prompt Template and Case Analysis

C.1 Function Calling Protocol

The function-calling protocol proceeds through five well-defined stages. First, each tool or API is specified using a JSON schema that precisely defines its inputs and outputs. Second, these tool definitions are provided to the language model along with the user's query. At runtime, the model determines, based on the query, whether to invoke a tool, and if so, generates a JSON payload that conforms to the schema. Third, the host environment executes the corresponding function call and returns the result. Finally, the model incorporates this feedback into its ongoing reasoning process or into the final response it generates.

C.2 Prompt Template

Vanilla Prompt Template

```
Prompt:
You are an intelligent assistant that
specializes in various logical reasoning
tasks.

Instructions:
1. Identify the relevant properties and
   objects specified in the rule.
2. Follow the specified operations precisely
   as described in the rules.
3. Ensure your output is formatted according
   to the specified notation and symbols.

Instructions:
{}
```

```
Question:
{}

Answer:
[[]]
```

PoT Prompt Template

Prompt:
You are an intelligent assistant that helps
with various logical calculation tasks, like
set operation, finite state machine, digital
logic, etc.
After you have written an answer in plain
text, if you can write Python code, you must
use coding to solve the problem. Please start
coding with "'python and end with "', and
remember to use a return statement to return
the answer.

- First try your best to solve the problem
  without code as usual and give an answer.
- Then write the code between exactly
  "'python and "' delimiters.
- You may use outer packages, including
  but not limited to sympy, nltk, numpy,
  cvxpy, pulp, ortools, scipy, pandas,
  networkx, to assist you implement better
  algorithms.

Example:
```
'''python
def calculate_sum(a: int, b: int) -> int:
    return a + b
def main():
    answer = calculate_sum(1,1)
    return "[[" + str(answer) + "]]"
'''
```

```
Instructions:
{}

Question:
{}

Answer:
[[]]
```

TIR Prompt Template

Prompt:
Please solve the following problem step by
step. During your reasoning process, if
needed, you can choose to write Python
code to enhance your reasoning. The code

executor will run your code and provide the
execution results back to you to support
your reasoning process. Please put the
final answer within [[]]. Continue this
process until you explicitly indicate task
completion.

Code Execution Guidelines:
1. **Reason step-by-step** about the problem in
   natural language.
2. **Use Python code** when computation, data
   analysis, or verification is needed.
3. **Iterate between reasoning and coding** as
   necessary.
4. **Build up your solution incrementally**.

Available Tool:
- run_python: Execute Python code and see
  the results immediately.
  - Use this when you need to compute,
    analyze data, test hypotheses, or
    verify results.
  - The code runs in a sandbox environment
    with common libraries available.

Guidelines:
- Start by understanding the problem
  through reasoning.
- Write code to explore, compute, or test
  your ideas.
- Analyze the code output and continue
  reasoning.
- Break complex problems into smaller,
  manageable steps.
- Use code to verify your reasoning and
  provide concrete results.
- Don't just describe what code would do –
  actually run it!
- ALWAYS include print() statements to
  output key values and results.

Example Workflow:
1. Reason: "I need to analyze this dataset
   to find patterns..."
2. Code: Use run_python to load and explore
   the data.
3. Reason: "The results show X, which means
   I should try Y..."
4. Code: Use run_python to implement
   approach Y.
5. Continue iterating until solved.

```
Instructions:
{}

Question:
{}

Answer:
[[]]
```

C.3 Case Analysis: Structured Planning and Reduced Hallucination

Qualitative case analyses further highlight the distinctive benefits of Tool-Integrated Reasoning (TIR). Vanilla decoding, which relies solely on a model's internal knowledge, often exhibits redundant or circular reasoning—revisiting subgoals that have already been resolved. In contrast, TIR encourages a more structured problem-solving workflow by externalizing complex computational subroutines (e.g., arithmetic or logical traversals) through targeted tool calls. This offloading allows the language model to focus on high-level tasks such as problem decomposition, strategic planning, and the verification of intermediate results. Moreover, by grounding the reasoning process in externally computed—and often verifiably correct—feedback, TIR significantly reduces hallucinations and enhances the reliability and consistency of generated solutions.

However, our analysis also reveals limitations. On irregular or out-of-distribution tasks, such as abstract physics or formal language problems, TIR-enabled models may be misled by unfamiliar tool interactions or fail to map problems into an appropriate programmatic form. These challenges underscore the difficulty of generalizing TIR to domains where conventional tools provide limited or counterintuitive assistance.

We present four representative cases to substantiate these claims.

Case 1: Viterbi Algorithm for Convolutional Codes This case involves decoding a convolutional code, which requires precise execution of the Viterbi algorithm.

- **Reasoning of Base LLM:** The model attempted to manually trace state transitions and path metrics by generating extensive tables of intermediate calculations within its context window. This approach imposed a high cognitive load, leading to inconsistencies across multiple versions of the state table. Ultimately, the model failed to complete the full iteration and resorted to speculation in the final steps. It confidently presented an incorrect state table and decoded sequence while concealing the speculative nature of its reasoning before hitting the token limit.

- **Reasoning of TIR-Enabled LLM:** The TIR-enabled model correctly identified the Viterbi algorithm and immediately implemented it as a Python function rather than attempting a manual trace. Offloading the iterative computations to the code interpreter enabled the model to obtain the correct answer on its first attempt. It then used subsequent reasoning steps to verify both the implementation and its output, confidently converging on the correct solution well within the token limit. This case exemplifies how TIR externalizes low-level serial computations, allowing the model to concentrate on high-level algorithmic strategy.

Case 2: Grade School Math Contest Problem This case features a complex combinatorial math problem with an elegant but non-obvious shortcut.

- **Reasoning of Base LLM:** The model failed to identify the shortcut and instead pursued a brute-force approach, analyzing smaller examples in an attempt to extrapolate

a pattern. This step-by-step strategy led to calculation errors and hallucinated reasoning. The model's thought process became contradictory, for example, it noted confusion over a sum that should have remained invariant but changed during its calculations, ultimately yielding an incorrect answer.

- **Reasoning of TIR-Enabled LLM:** While the TIR-enabled model also initially overlooked the shortcut, it used the code tool to run a direct brute-force simulation. Though computationally intensive for humans, the model executed the simulation in just a few lines of Python, quickly arriving at the correct result. With this answer in hand, the model was able to reason backward to uncover the underlying mathematical principle, ultimately discovering the intended shortcut. This case highlights how TIR provides factual grounding through computation, which can then scaffold higher-level abstraction and insight.

Case 3: Abstract Physics Problem This case demonstrates a key limitation of TIR, where tool use can interfere with effective reasoning on certain out-of-distribution (OOD) problems.

- **Reasoning of Base LLM:** Given a multiple-choice physics question, the model initially found options A, B, and D plausible. However, through verbal reasoning and self-reflection, it identified a subtle flaw in option B and ultimately selected the correct answer.

- **Reasoning of TIR-Enabled LLM:** The TIR model similarly identified A, B, and D as plausible, but its attempt to apply the code tool proved counterproductive. **The abstract nature of the physics concepts did not translate well into a simulation or formal computation. This misalignment disrupted the model's abstract reasoning process, and it failed to perform** the same self-correction seen in the vanilla case, ultimately producing an incorrect answer. This pattern, observed across multiple models (e.g., Qwen series, DeepSeek-R1), indicates that, for certain domains, enforcing tool use may introduce harmful interference, posing a critical challenge for future research.

Case 4: Iterative Refinement with a Non-Reasoning Model This final case illustrates TIR's ability to enable self-correction, even in models not optimized for complex reasoning. We evaluate Qwen2.5-32B-Coder, a code-centric model, on a task requiring high numerical precision.

- **Reasoning of Base LLM:** Operating without tool integration, the model repeatedly produced the same incorrect answer or oscillated between various erroneous outputs. Despite a generous token budget, it failed to make meaningful progress toward the correct solution.

- **Reasoning of TIR-Enabled LLM:** With access to the tool, the model exhibited clear iterative refinement. Its first attempt produced a result of 4.09 seconds, which, upon execution and inspection, prompted a revision to 3.00 seconds. It then converged on the correct answer and retained it in all subsequent attempts. This case provides concrete evidence of how TIR's feedback loop supports systematic self-correction and demonstrates the scaling effects discussed in Appendix B.

Case 337 Model: Qwen3-235B-A22B **Method:** TIR **Question type:** Communication Code

Question: Consider a convolutional code with generators $G_0 = 110$, $G_1 = 101$. Received bits from time 1 to 13 are: `00 00 11 01 00 11 01 00 00 00 00 11 10`. Determine the most-likely state sequence and decoded message. Answer in format `[[0,1,0,1,1,0]]` if decoded message is these bits.

Correct answer: `[0,0,1,1,1,0,0,0,0,0,0,1,0]`

Vanilla Response(Using up all thinking tokens)	TIR-Enabled Response(14,762 tokens)

Vanilla Response(Using up all thinking tokens)

Okay, I need to solve this convolutional code decoding problem using the Viterbi algorithm. Let me start by understanding the given information.

The generator polynomials are G0=110 and G1=101. Wait, these are probably octal representations, but maybe they're in binary. Wait, convolutional codes usually have generator polynomials expressed in terms of their connection polynomials. Hmm, the standard way is to represent each generator as a binary polynomial...

[continues with detailed reasoning over parsing G0 and G1, various interpretations of bit ordering, multiple attempts to clarify register shifts, and repeated recalculations of G0 = u + S1 and G1 = u + S2]

... Then next state is (u, X1) = (u[n], u[n-1]). Output G0 = u[n] + X1 (u[n-1]) → mod 2 sum. Output G1 = u[n] + X2 (u[n-2])...

[long reasoning trace building up the transition table by hand:]

Current State (S1, S2) | Input u | Next State (u, S1) | Output (G0, G1)00 | 0 | (0,0) | G0=0+0=0, G1=0 +0=0 → 0000 | 1 | (1,0) | G0=1+0=1, G1=1+0=1 → 1101 | 0 | (0,1) | G0=0+1=1, G1=0 +1=1 → 1101 | 1 | (1,1) | G0=1+1=0, G1=1+1=0 → 0010 | 0 | (0,0) | G0=0+0=0, G1=0 +0=0 → 0010 | 1 | (1,0) | G0=1+0=1, G1=1+0=1 → 1111 | 0 | (0,1) | G0=0+1=1, G1=0+1=1 → 1111 | 1 | (1,1) | G0=1+1=0, G1=1+1=0 → 00Wait, let's verify this.Take current state 00 (S1=0, S2=0), input u=0:Next state is (u, S1) = (0,0) → same as current.G0 = u + S1 = 0 +0=0.G1 = u + S2 = 0 +0=0. So output 00.Similarly, input u=1:G0=1+0=1, G1=1+0=1. Output 11.Next state is (1,0).For state 01 (S1=1, S2=0? Wait, state (S1, S2) is (previous input, previous-previous input). Or is the state (X1, X2) where X1 is the first memory element and X2 the second. Let's say S1 and S2 are the two memory elements. So when a new input u comes, the new S1 is u, and new S2 is the previous S1.Wait, regardless, according to our previous analysis, the transitions are as above.Wait, current state (S1, S2) is the previous two bits. Then, the input u becomes the new S1, and the previous S1 becomes S2.Wait, that makes sense. So next state is (u, S1).

[Proceeding to manually walk through the Viterbi algorithm: state metrics, branch metrics, traceback, per-step transitions, speculative guesses, corrections, repeated checks, possible convergence observation, etc.]

... decoded message in brackets.An alternative approach is to recognize that this is a well-known problem or to use the fact that the Viterbi algorithm will choose the path that best matches the received sequence. Considering the complexity, perhaps the decoded message is a sequence of bits that can be determined by known methods.Alternatively, refer to the following steps:After building the trellis and path metrics, the most likely path can be found.The decoded message bits are the inputs u along this path.Assuming that after processing all 13 time steps, the path with the lowest metric ends in a particular state. We then traceback from there. Wait, wait, this is getting really involved. Let me tabulate this properly... I think the correct answer is:Answer:[[0,1,0,0,0,0,0,0,0,0,1,1,0]]Wait, but this is a guess. Another approach: the message bits correspond to the inputs u along the most likely path. If at each critical step we track the survivor paths:At t=1, best path is state 00 (u=0).At t=2, state 00 maintained with u=0.At t=3, transition to state 10 implies u=1.At t=4, transition to state 11 implies u=1 from previous state 10.But this manual traceback is error-prone.Given the time invested, I'll refer to standard solutions and posit the decoded message is 0100000000110 or similar. However, to comply with the instructions, I must provide a final answer.To solve the convolutional decoding problem using the Viterbi algorithm, we begin by analyzing the structure of the encoder defined by the generator polynomials **G0 = 110** and **G1 = 101**. **Received Sequence and Viterbi Decoding** corresponding outputs for each input bit (0 or 1) are as follows:| Current State | Input | Next State | Output (G0, G1) ||————|———|———————|| 00 | 0 | 00 | 00 || 00 | 1 | 10 | 11 || 01 | 0 | 00 | 01 || 01 | 1 | 10 | 10 || 10 | 0 | 01 | 10 || 10 | 1 | 11 | 01 || 11 | 0 | 01 | 11 || 11 | 1 | 11 | 00 | We are given the received sequence from time **t = 1 to 13**

TIR-Enabled Response(14,762 tokens)

Okay, let's see. The user provided a problem about convolutional codes and the Viterbi algorithm. The answer given by the code was [[0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0]]. But I need to check if this is correct.First, I should verify the code's approach. The code defines a transition function based on the generator polynomials G0=110 and G1=101. Wait, how are these generators represented here? The transition function seems to compute the next state and outputs based on some logic. Let me think: the generator polynomials are given in a specific way. For a standard convolutional code with constraint length K=3, the state is two bits (since K-1=2). The generators G0 and G1 would each have 3 coefficients. But the code's transition function might be using a different representation. Wait, in the transition function, ...But since I can't run the code here, I'll proceed to write the code as outlined, ensuring that it follows the steps correctly.Now, the code should be enclosed within "'python and "', and be self-contained.Another thing to check: the traceback initialization. We need to ensure that all elements are initialized to None.In Python, the list comprehensions should handle that.Now, the code should be correct. Let's proceed to write it.

```python
import sys
def transition(s_prev, u): s1 = (s_prev >> 1) & 1;
s0 = s_prev & 1; o0 = u ^ s1; o1 = u ^ s0; next_s1 = u;
next_s0 = s1; next_state = (next_s1 << 1) | next_s0; return next_state, (o0, o1); def
main(): received = [(0, 0), (0, 0), (1, 1), (0, 1), (0, 0), (1, 1), (0,
1), (0, 0), (0, 0), (0, 0), (0, 0), (1, 1), (1, 0)]; T = len(received);
num_states = 4; INF = float('inf'); path_metrics = [[INF] * num_states
for _ in range(T+1)]; path_metrics[0][0] = 0; traceback = [[None] *
num_states for _ in range(T+1)]; for t in range(T): current_received =
received[t]; for s_prev in range(num_states): if path_metrics[t][s_prev]
== INF: continue; for u in [0, 1]: next_state, expected =
transition(s_prev, u); branch_metric = sum(1 for a, b in zip(expected,
current_received) if a != b); total_metric = path_metrics[t][s_prev]
+ branch_metric; if total_metric < path_metrics[t+1][next_state]:
path_metrics[t+1][next_state] = total_metric; traceback[t+1][next_state]
= (s_prev, u); best_state = min(range(num_states), key=lambda s:
path_metrics[T][s]); path = []; current_state = best_state; for t in
range(T, 0, -1): s_prev, u = traceback[t][current_state]; path.append(u);
current_state = s_prev; path = path[::-1]; print("Answer:", [path]);
if __name__ == "__main__": main()
```

TOOL: "status": "Success", "message": "", "compile_result": null, "run_result": {"status": "Finished", "execution_time": 0.013083457946777344, "return_code": 0, "stdout": "Sum of remaining numbers: 5050Last number written: 2394", "stderr": "", ...So each received pair corresponds to an input bit.Alternatively, maybe the code should have a traceback depth longer than the message length to ensure the path is properly resolved. However, in this case, since the code's output is a 13-bit sequence, which matches the input pairs (each input bit produces a pair), this should be okay.Assuming the code is correct, the answer given by the code should be the correct decoded message. However, the user's problem might expect a different answer. But since the code's output is based on a standard Viterbi algorithm implementation for this type of convolutional code, and the transition function is correctly modeling the generator polynomials, I think the code's answer is correct.Therefore, the final answer is [[0,0,1,1,1,0,0,0,0,0,0,1,0]].The provided Python code correctly implements the Viterbi algorithm to decode the received bits using the specified generator polynomials

FINAL ANSWER: `[[0,1,0,0,0,0,0,0,0,0,1,1,0]]`

FINAL ANSWER: `[[0,0,1,1,1,0,0,0,0,0,0,1,0]]`

Case 42   Model: Qwen3-8B   **Method:** TIR   **Question type:** Grade School Math

Question: First, write down a series of numbers on the black - board: 1, 2, 3, ..., 100. If we erase the first 6 numbers each time and write the sum of the 6 erased numbers at the end of this series of numbers to get a new series of numbers, and then perform the same operation until there are less than 6 numbers left on the black - board. Question: (1) What is the sum of these remaining numbers on the black - board? (2) What is the last number written? Answer in format of [[sum, last number]]

Correct answer: [[5050, 2394]]

Vanilla Response(13,912 tokens)	TIR-Enabled Response(9,478 tokens)
Okay, let's try to figure out this problem. So, the question is about repeatedly erasing the first 6 numbers from a series of numbers from 1 to 100, summing them, and appending that sum to the end of the series. We need to do this until there are fewer than 6 numbers left. Then, answer two questions: the sum of the remaining numbers and the last number written.First, let me understand the process step by step. The initial series is 1, 2, 3, ..., 100. Each time, we erase the first 6 numbers, calculate their sum, and add that sum to the end of the series. Then repeat this process until there are less than 6 numbers left. Let me think about how the series changes with each operation. The original series has 100 numbers. Each operation removes 6 numbers and adds 1 number (the sum). So each operation reduces the total count by 5. Let me check that: starting with N numbers, after one operation, it's N - 6 + 1 = N -5. So each time, the count decreases by 5.So starting with 100 numbers, how many operations can we perform before the count is less than 6?Let me calculate the number of operations. Let's denote the number of operations as k. Each operation reduces the count by 5. So after k operations, the count is 100 - 5k. We need 100 -5k < 6. Let's solve for k:100 -5k <6 => -5k < -94 => 5k > 94 => k > 94/5 = 18.8. So k must be 19 operations. Wait, but let me check. Let me compute the count after each operation:Start with 100 numbers. After 1st operation: 100 -5 = 95 After 2nd: 95 -5 = 90 ... Each operation subtracts 5. So after k operations, the count is 100 -5k. We need this to be less than 6. So 100 -5k <6 → 5k >94 → k>18.8. So k=19 operations. But wait, after 18 operations, the count is 100 - 5*18 = 100 -90 =10. Then, after 19 operations, it would be 10 -5 =5, which is less than 6. So the process stops after 19 operations. Wait, but the problem says until there are less than 6 numbers left. So the last operation would be when after erasing 6 numbers, the remaining count is less than 6. Wait, but when you have 10 numbers, you can perform one operation (erase 6, add 1, so count becomes 10 -6 +1 =5). So that's the 19th operation. So after 19 operations, there are 5 numbers left. So the answer to part (1) is the sum of those 5 numbers. But wait, but during the process, each operation adds the sum of the erased numbers. So the total sum of the entire series changes with each operation. So the total sum of all numbers, including the added sums. Let me think. Let me consider the initial sum of numbers from 1 to 100... Wait, no. Let me retrace. After the 17th operation, the series is: Original series before 17th operation: [97, 98, 99, 100, 21, 57, 83, 109, 135, 161, 187, 213, 239, 265, 291, 317, 343, 369, 395, 421]. After removing first 6 numbers (97, 98, 99, 100, 21, 57), the remaining numbers are [83, 109, 135, 161, 187, 213, 239, 265, 291, 317, 343, 369, 395, 421]. Wait, that's 14 numbers. Then adding the sum 472, so the series becomes [83, 109, 135, 161, 187, 213, 239, 265, 291, 317, 343, 369, 395, 421, 472]. Count is 14 +1 =15. So the sum of these 15 numbers should be 5050. Let me calculate the sum of these 15 numbers: 83 + 109 + 135 + 161 + 187 + 213 + 239 + 265 + 291 + 317 + 343 + 369 + 395 + 421 + 472. Let me add them step by step: Start with 83 + 109 = 192 192 + 135 = 327 327 + 161 = 488 488 + 187 = 675 675 + 213 = 888 888 + 239 = 1127 1127 + 265 = 1392 1392 + 291 = 1683 1683 + 317 = 2000 2000 + 343 = 2343 2343 + 369 = 2712 2712 + 395 = 3107 3107 + 421 = 3528 3528 + 472 = 4000. So the sum is 4000. But it should be 5050. This indicates a mistake in the simulation. Where is the error? Ah, I think the mistake is in the calculation of the sum of the first 16 operations. Earlier, I calculated the sum of the first 16 groups as 3606, but this may be incorrect... Let me calculate this sum again: 537 + 573 = 1110 1110 + 472 = 1582 1582 + 1085 = 2667 2667 + 2388 = 5055. Hmm, this is 5 more. Let me check the previous steps. After the 18th operation, the series was [309, 345, 375, 411, 453, 495, 537, 573, 472, 1085]. Sum of this series should be 5050.	Okay, let's see. The user provided a problem about convolutional codes and the Viterbi algorithm. The answer given by the code was [[0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0]]. But I need to check if this is correct.First, I should verify the code's approach. The code defines a transition function based on the generator polynomials G0=110 and G1=101. Wait, how are these generators represented here? The transition function seems to compute the next state and outputs based on some logic. Let me think: the generator polynomials are given in a specific way. For a standard convolutional code with constraint length K=3, the state is two bits (since K-1=2). The generators G0 and G1 would each have 3 coefficients. But the code's transition function might be using a different representation. Wait, in the transition function, ...But since I can't run the code here, I'll proceed to write the code as outlined, ensuring that it follows the steps correctly.Now, the code should be enclosed within "'python and "', and be self-contained.Another thing to check: the traceback initialization. We need to ensure that all elements are initialized to None.In Python, the list comprehensions should handle that.Now, the code should be correct. Let's proceed to write it.numbers = list(range(1, 101)); while len(numbers) >= 6: removed = numbers[:6]; sum_removed = sum(removed); numbers = numbers[6:] + [sum_removed]; sum_remaining = sum(numbers); last_number = numbers[-1]; print(f"Sum of remaining numbers: {sum_remaining}"); print(f"Last number written: {last_number}") **status: Success, message: , compile_result: null, run_result: status: Finished, execution_time: 0.013083457946777344, return_code: 0, stdout: Sum of remaining numbers: 5050 Last number written: 2394** Okay, let's see. The user asked about a process where we start with numbers 1 to 100 and repeatedly erase the first 6 numbers, replace them with their sum, and keep doing this until there are fewer than 6 numbers left. The questions are: what's the sum of the remaining numbers, and what's the last number written?First, I need to understand the process. Let's take a small example to see how it works. Suppose we have numbers 1 to 10. First, remove 1-6 (sum 21), append 21. Now the list is [7,8,9,10,21]. Now, since there are 5 numbers, which is less than 6, we stop. The sum here would be 7+8+9+10+21 = 55. The last number is 21... . Each time, the first 6 numbers are removed and their sum is added. So the process is similar to repeatedly taking the first 6 elements, summing them, and appending that sum. The last number written is the sum of the first 6 elements of the list just before the final operation.But simulating this process step by step would be tedious. However, the code provided earlier correctly simulates this process. The code starts with numbers 1 to 100, and in each iteration, removes the first 6 elements, sums them, appends the sum, and repeats until there are less than 6 elements. So the code is correct. Therefore, the original problem's answer is [[5050, 2394]].The process involves repeatedly removing the first 6 numbers, summing them, and appending the sum to the end of the list. This continues until fewer than 6 numbers remain. Then, the last number in the list is the last sum written. The code's output was sum_remaining=5050 and last_number=2394. Therefore, the answer is [[5050, 2394]].But to ensure correctness, let me check with a smaller example. ...Now len is 1 <6. So sum_remaining is 66, last number 66. Which matches the expected result. So the code works for this case.So the code is correct. Therefore, the original problem's answer is [[5050, 2394]].The process involves repeatedly removing the first 6 numbers, summing them, and appending the sum to the end of the list. This continues until fewer than 6 numbers remain.**Last Number Written:**- The last number written is the sum of the first 6 numbers in the list during the final operation. The code simulates this process accurately, confirming the last number as **2394**.
FINAL ANSWER: [[5050, 2388]]	**FINAL ANSWER:** [[5050, 2394]]

Case 72 **Model:** Qwen3-32B   **Method:** TIR   **Question type:** Physics

Question: As shown, there are two segments of rope, one connecting M and A, and another connecting A and B, with lengths l_1 and l_2 separately. A light elastic spring of original length l is fixed at point O at one end, and the other end is connected to a small ball of mass m. The small ball is sleeved on a vertical fixed rough rod, and the friction coefficient between them is 0.5. The rod has two points M and N, both at a distance of l from O, and point P at a distance of $\frac{l}{2}$ from O. The segment OP is perpendicular to the rod. When the small ball is at point P, it is at rest. Assume that the maximum static friction is equal to the kinetic friction, and the gravitational acceleration is g. The small ball starts moving downward from point M. During this process, the spring always remains within its elastic limit. Which of the following statements are correct?A: The stiffness coefficient of the spring is $\frac{4mg}{l}$ B: The acceleration of the ball at $\frac{1}{2}l$ below point P is $(3\sqrt{2}-4)g$C: During the movement from M to N, the friction force acting on the ball first decreases and then increases.D: During the movement from M to P and from P to N, the work done by the friction force on the ball is the same. Answer in the format $[A, C, D]$ if A, C, and D are correct.

Correct answer: [[A,D]]

Vanilla Response(11,720 tokens)	TIR-Enabled Response(19,040 tokens)
Okay, let's try to solve this problem step by step. First, I need to visualize the setup. There's a vertical fixed rod with a rough surface. A small ball of mass m is on this rod and connected to a spring fixed at point O. The spring has an original length l. The rod has points M, N, and P. Points M and N are each at a distance l from O, while P is at l/2 from O. The segment OP is perpendicular to the rod. When the ball is at P, it's at rest. The friction coefficient is 0.5, and maximum static friction equals kinetic friction. The ball starts moving downward from M, and we need to evaluate the given statements A to D.First, let me sketch the scenario mentally. The spring is attached at O, which is probably a point in space such that OP is perpendicular to the rod and the spring's original length is l, when the ball is at P, the spring is stretched or compressed? Let's see...Therefore:mg = 0.5 * k*(l/2) => mg = k*l/4 => k = 4mg/l.So that gives us the spring constant k = 4mg/l, which is option A. Therefore, statement A is correct. This is still negative. However, the **magnitude** is:lal = g(1 + sqrt{2}) approx 2.414g. But the question gives (3sqrt{2} - 4)g approx 0.242g , which is **not equal** in magnitude.**Statement B is incorrect** ... herefore, N(y) decreases from k*l/2 to 0.Therefore, overall, as the ball moves from M to N:- From M to P: N increases from 0 to k*l/2.- From P to N: N decreases from k*l/2 to 0.Therefore, the friction force f = mu*N first increases from 0 to mu*k*l/2, then decreases back to 0. Therefore, the friction force first increases and then decreases. However, statement C says the friction force acting on the ball first decreases and then increases. Which is the opposite of what we found. Therefore, statement C is incorrect.Wait, but the question says During the movement from M to N, the friction force acting on the ball first decreases and then increases. But according to our analysis, friction increases from M to P and then decreases from P to N. Therefore, the friction first increases then decreases. So statement C is incorrect. Therefore, C is wrong. For statement D, work done by friction from M→P and P→N. The work done by friction is integer of f · ds. Since friction is always opposite to the direction of motion (downward), and f is upward, the work is negative in both cases. The magnitude depends on the integral of f ds. Since from M→P, f increases from 0 to max, and from P→N, f decreases from max to 0, over same distance. The integral (area under f vs. distance graph) would be same in both cases. Hence, the work done by friction is same in magnitude and sign (negative). Therefore, statement D is correct.	Okay, let's try to solve this problem step by step using multiple python code calls. First, I need to understand the setup and the given options. So, there's a spring attached to a fixed point O and a small ball of mass m. The ball is on a vertical rod with points M, N, and P. The original length of the spring is l, and when the ball is at rest at point P (which is l/2 from O), the system is in equilibrium. The friction coefficient is 0.5, and the maximum static friction equals the kinetic friction. The ball starts moving down from M, and we need to analyze the motion and determine which statements are correct.Let me start by analyzing the forces acting on the ball when it's at rest at point P. At equilibrium, the spring force must balance the weight of the ball plus the friction force. Wait, but the problem states that when the ball is at P, it's at rest. So, in equilibrium, the sum of forces should be zero... import sympy as sp; m, g, l = sp.symbols('m g l'); k = 4*m*g/l; distance_below_P = l/2; position = l/2 + distance_below_P; acceleration_B = g; print(f"Option B acceleration: {acceleration_B}") **status: Success, message: , compile_result: null, run_result: {status: Finished, execution_time: 0.30114245414733887, return_code: 0, stdout: Option B acceleration: g, stderr: ,**which is approximately (4.24 -4)g = 0.24g, but the exact value is what's given in option B. Therefore, option B is correct. Now, moving to option C ... Hence, option D is correct. Therefore, options A, B, and D are correct. But wait, earlier analysis for option C said that friction first increases then decreases, so option C is incorrect. The statement says first decreases and then increases, which is not the case. So option C is wrong. So the correct answers are A, B, D. But wait, let me verify option D again. The work done by friction depends on the path. Since friction is non-conservative, but in this case, the paths are symmetric. Let me think of a specific example. Suppose during M to P, the friction increases from 0 to max, and during P to N, it decreases from max to 0. The average friction during M to P would be (0 + max)/2, and same for P to N. Since the distance is the same, the total work would be same. Therefore, option D is correct. So the correct options are A, B, D. But wait, the initial code for option B was wrong, but the detailed calculation shows B is correct. So the final answer should be A, B, D. But let me check again. Wait, in the code I ran earlier, I incorrectly assumed that at 1/2 l below P, the spring is at original length, leading to acceleration g. But the correct calculation showed that the acceleration is (3sqrt2 -4)g, which is option B. Therefore, option B is correct. So the correct options are A, B, D. However, let me check the problem statement again. The options are A, B, C, D. So according to my analysis, A, B, D are correct. But I need to confirm. But the problem says During the movement from M to P and from P to N, the work done by the friction force on the ball is the same.

FINAL ANSWER: [[A,D]] | **FINAL ANSWER:** [[A,B,D]]

Case 8   Model: Qwen2.5-32B-Coder   **Method:** TIR   **Question type:** Operation Research
Question:10 tasks must be run on 3 CPUs at 1.33, 2 and 2.66 GHz (each processor can run only one task at a time). The number of elementary operations of the tasks (expressed in billions of instructions (BI)) is as follows: Task | 1 | 2 | 3 | 4 | 5 | 6 | 7 BI | 1.1 | 2.1 | 3 | 1 | 0.7 | 5 | 3 Schedule tasks to processors so that the completion time of the last task is minimized. Only answer the completion time of the last task, in the unit of seconds, format of [[24]] if the completion time of the last task is 24.
Correct answer: [[2.781954887]]

Vanilla Response(Forced to 16384 tokens)	TIR-Enabled Response(Forced to 16384 tokens)

OTo solve this problem, we need to allocate each task to one of the three CPUs in such a way that the completion time of the last task (makespan) is minimized. The CPUs have different processing speeds: 1.33 GHz, 2 GHz, and 2.66 GHz. The time it takes for a task to complete on a CPU is inversely proportional to the CPU's speed and directly proportional to the number of operations... 25563909775 - CPU 2: 1.5 - CPU 3: 1.12781954887- Task 4 (1 BI): - CPU 1: 0.75187969925 - CPU 2: 0.5 - CPU 3: 0.37593984962- Task 5 (0.7 BI): - CPU 1: 0.52631578947 - CPU 2: 0.35 - CPU 3: 0.26315789474- Task 6 (5 BI): - CPU 1: 3.75939849624 - CPU 2: 2.5 - CPU 3: 1.87962405977- Task 7 (3 BI): - CPU 1: 2.25563909775 - CPU 2: 1.5 - CPU 3: 1.12781954887Now, we will assign tasks to CPUs to balance the load:1. Assign the longest task (Task 6) to CPU 3 (1.87962405977 seconds).2. Assign Task 3 to CPU 2 (1.5 seconds).3. Assign Task 7 to CPU 2 (1.5 seconds).4. Assign Task 2 to CPU 1 (1.05 seconds).5. Assign Task 4 to CPU 1 (0.5 seconds).6. Assign Task 1 to CPU 3 (0.41353383459 seconds).7. Assign Task 5 to CPU 2 (0.35 seconds).The total time for each CPU will be:- CPU 1: 1.05 + 0.5 = 1.55 seconds- CPU 2: 1.5 + 1.5 + 0.35 = 3.35 seconds- CPU 3: 1.87962405977 + 0.41353383459 = 2.29315789436 secondsSince we have 10 tasks but only 7 are listed, we assume the remaining 3 tasks are not listed or have 0 BI, which would not affect the total time. The maximum time among the CPUs is the makespan.Therefore, the completion time of the last task is the maximum of the times for each CPU, which is 3.35 seconds.The answer is [[3.350000000]]. Given the previous calculations, we have determined the completion times for each CPU based on the assi... Therefore, the completion time of the last task is [[3.850000000]]. Given the need to minimize the completion time of the last task (makespan) with 10 tasks and 3 CPUs, let's refine the allocation strategy by considering all 10 tasks and applying a heuristic approach such as the... (Total: 3.57135338308 seconds)6. Task 2 (2.1 BI) - CPU 1: 1.05 seconds7. Task 8 (1.5 BI) - CPU 2: 0.75 seconds (Total: 3.75 ... let's assume they have the following processing times (similar to the given tasks):- Task 8 (1.5 BI): - CPU 1: 1.12781954887 - CPU 2: 0.75 - CPU 3: 0.56390977436- Task 9 (2 BI): - CPU 1: 1.50375939849 - CPU 2: 1 - CPU 3: 0.75187969925- Task 10 (2.5 BI): - CPU 1: 1.87962405977 - CPU 2: 1.25 - CPU 3: 0.93984962406Now, let's use the LPT rule to assign tasks to CPUs:1. Task 6 (5 BI) - CPU 3: 1.879624059772. Task 3 (3 BI) - CPU 2: 1.53. Task 7 (3 BI) - CPU 2: 1.54. Task 2 (2.1 BI) - CPU 1: 1.055. Task 9 (2 BI) - CPU 3: 0.751879699256. Task 10 (2.5 BI) - CPU 3: 0.939849624067. Task 8 (1.5 BI) - CPU 2: 0.758. Task 4 (1 BI) - CPU 1: 0.751879699259. Task 1 (1.1 BI) - CPU 1: 0.827067669921Task 5 (0.7 BI) - CPU 2: 0.35Now, let's calculate the total time for each CPU:- CPU 1: 1.05 + 0.75187969925 + 0.82706766992 = 2.62894736917 seconds- CPU 2: 1.5 + 1.5 + 0.75 + 0.35 = 4.1 seconds- CPU 3: 1.87962405977 + 0.75187969925 + 0.93984962406 = 3.57135338308 secondsThe maximum time among the CPUs is the makespan, which is 4.1 seconds.Therefore, the completion time of the last task is [[4.100000000]].

To solve this problem, we need to assign the tasks to the processors in such a way that the completion time of the last task is minimized. This is a classic load balancing problem...Task times in billions of instructions (BI)task_times = [1.1, 2.1, 3, 1, 0.7, 5, 3]# Processor speeds in GHzprocessor_speeds = [1.33, 2, 2.66]# Calculate the minimum completion timemin_completion_time = schedule_tasks(task_times, processor_speeds)# Print the result formatted to 9 decimal placesprint(f[[{min_completion_time:.9f}]])**output[[4.094731579]]**The output of the code is [[4.094731579]].This means that the completion time of the last task, when tasks are scheduled optimally to minimize the overall completion time, is approximately 4.094731579 seconds.Since the output is correct and solves the problem, we can confidently state the final answer. **FINAL ANSWER: [[4.094731579]]**" Wait, let;s change another method... **output[[3.007518797]]**Let's continue based on this output." "role": "assistant", "content": "It seems there was an error in the initial task assignment or the logic used. The new output [[3.007518797]] suggests that the optimal scheduling has resulted in a different and lower completion time.Let's verify the logic and the task assignment again. The greedy algorithm should be correct in theory, but let's ensure the implementation is accurate. We will re-run the code and verify the steps.Here is the code again for clarity:pythonimport heapqdef schedule_tasks(task_times, processor_speeds): # Calculate the initial completion times for each processor # We use a min-heap to keep track of the earliest completion time processor_heap = [(0, i) for i in range(len(processor_speeds))] heapq.heapify(processor_heap) for task_time in task_times: .. output[[4.094731579]]It appears that the previous output [[3.007518797]] was indeed correct. The code has been verified and the logic is sound. The discrepancy might have been due to an earlier incorrect run or an issue with the environment.Given the output [[3.007518797]], we can confidently state that the completion time of the last task, when tasks are scheduled optimally to minimize the overall completion time, is approximately 3.007518797 seconds.Therefore, the final answer is:**FINAL ANSWER: [[3.007518797]]**" ... We'll use the itertools.product function to generate all possible assignments. Note that this approach is only feasible for a small number of tasks and processors.Let's implement this brute-force approach:pythonimport itertoolsdef calculate_completion_time(assignment, task_times, processor_speeds): ..._speeds) Print the result formatted to 9 decimal placesprint(f[[{min_completiontime:.9f}]])print(fBest assignment: {bestassignment})output[[3.007518797]]Best assignment: (0, 0, 2, 2, 2, 1, 1)The brute-force approach confirms that the minimum completion time is indeed [[3.007518797]], and the best assignment of tasks to processors is (0, 0, 2, 2, 2, 1, 1). This means:- Task 1 is assigned to Processor 0.- Task 2 is assigned to Processor 0.- Task 3 is assigned to Processor 2.- Task 4 is assigned to Processor 2.- Task 5 is assigned to Processor 2.- Task 6 is assigned to Processor 1.- Task 7 is assigned to Processor 1.This assignment ensures that the completion time of the last task is minimized.Therefore, the final answer is:FINAL ANSWER: [[3.007518797]]" "**output[[2.781954887]]**Best assignment: (1, 2, 0, 1, 0, 2, 1)Let's continue based on this output." "role": "assistant", "content": "The brute-force approach confirms that the minimum completion time is [[2.781954887]], and the best assignment of tasks to processors is (1, 2, 0, 1, 0, 2, 1). ... Therefore, the final answer is indeed correct. The minimum completion time of the last task is:FINAL ANSWER: **[[2.781954887]]**"

FINAL ANSWER: [[4.100000000]]	**FINAL ANSWER:** [[2.781954887]]