Monte Carlo study of Whittle estimator in multivariate CCC EGARCH model

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Abstract

The research project investigates asymptotic properties of Whittle estimator in multivariate EGARCH framework using Monte Carlo study. Zaffaroni (2009) establishes consistency and asymptotic normality conditions for Whittle estimator of univariate EGARCH. The aim of this project is extending this study to multivariate case and investigate the asymptotic behavior of additional parameter of correlation between assets.

Evidence from the conducted Monte Carlo simulation suggest that the estimator of correlation between assets' shocks is consistent. This result holds when the estimator is scaled by the square root of sample size. The results for asymptotic distribution are not conclusive.

For the purpose of simulation the methodology of drawing random variables from multivariate standardized generalized error distribution (GED) is developed. Drawing from univariate GED is achieved by applying acceptance-rejection method. Drawing from multivariate GED is based on the results of Solaro (2004).

For the purpose of estimation the relevant moment of GED distribution is derived. This is achieved using a formula from Gradshteyn and Ryzhik (2007).

The relevant tools for simulation are developed in MATLAB. The code is included in the appendices.

Contents

Notation

- T sample size
- n number of assets
- N number of Monte Carlo simulations
- r_t returns of an asset
- z_t i.i.d. shocks of an asset
- σ_t volatility of an asset
- $h_t = log(\sigma_t^2)$ log-volatility of an asset
- ω long run volatility parameter
- $g(z_t)$ innovation function
- θ leverage effect parameter (also exponential distribution parameter in sections 2.1 2.2)
- δ magnitude parameter
- v generalized error distribution tail parameter
- Δ_i AR parameter
- Ψ_i MA parameter
- ϑ vector parameter combining five parameters above
- $\psi(\cdot)$ transfer function (also bigamma function in equation 3.7)
- $\Gamma(\cdot)$ gamma function
- $\Psi(\cdot)$ trigamma function
- $\mathbf{R}$ constant correlation matrix
- ρ_{ij} an element of the above matrix

1. Literature review

Univariate stochastic volatility model is defined as:

$$r_t = z_t \sigma_t, \quad z_t \sim IID(0,1), \quad t \in \mathbb{Z} \tag{1.1}$$

r_t represents returns of an asset and σ_t is the volatility process dependent on past information up to time $t-1$. A popular way to model the volatility σ_t is GARCH(p,q) model of Bollerslev (1986):

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i r_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \tag{1.2}$$

There are two major limitations of the GARCH model 1.2. The first is the necessity to impose restrictions on parameters ω, α_i and β_i in order to avoid possible negative volatility values. The second limitation regards failing to capture the fact that the negative shocks increase volatility more than positive shocks (leverage effect). These issues were the motivation for introducing the Exponential GARCH (EGARCH) model by Nelson (1991):

$$\begin{gathered} h_t = ln(\sigma_t^2) = \omega + \sum_{i=1}^{\infty} \psi_i g(z_{t-i}), \quad \sum_{i=1}^{\infty} \psi_i^2 < \infty \\ g(z_t) = \theta z_t + \delta(|z_t| - \mathbb{E}|z_t|) \end{gathered} \tag{1.3}$$

Modeling logarithm of volatility in 1.3 rather than the volatility itself ensures that $\sigma_t = exp(0.5h_t)$ is always positive, which addresses the first criticism of GARCH model. If θ parameter is negative then negative shocks cause greater increase in volatility, which accounts for the leverage effect and addresses the second criticism. In addition, if δ parameter is positive then shocks greater than average $|z_t|$ increase volatility more than small shocks. In the class of exponential volatility models 1.1 can be rewritten as a linear signal plus noise model:

$$r_t = z_t exp(0.5h_t) \tag{1.4}$$

$$log(r_t^2) = log(z_t^2) + h_t \tag{1.5}$$

Zaffaroni (2009) establishes regularity conditions for consistency and asymptotic normality of Whittle estimator of correlated signal (h_t) plus noise ($log(z_t^2)$) stochastic volatility model. In particular he verifies that the regularity conditions are met by EGARCH 1.3 model. The Multivariate Constant Conditional Correlation (CCC) stochastic volatility model is considered:

$$\mathbf{r}_t = \mathbf{H}_t^{1/2}\mathbf{z}_t \tag{1.6}$$

$\mathbf{r}_t$ is $n \times 1$ returns vector, where n is the number of assets. $\mathbf{z}_t$ is $n \times 1$ returns vector of shocks:

$$\mathbf{z}_t \sim IID(\mathbf{0}, \mathbf{R}) \tag{1.7}$$

R is $n \times n$ constant correlation matrix (hence the name constant conditional correlation) with ones on the diagonal and correlations between assets outside the diagonal. $\mathbf{H}_t$ is $n \times n$ conditional variance matrix dependent on information up to time $t-1$. In EGARCH framework it is defined as follows:

$$\begin{gathered}
\mathbf{H}_t = diag(exp(h_{1t}), ..., exp(h_{nt})) \\
\mathbf{h}_t = \boldsymbol{\omega} + \sum_{i=1}^{\infty} \boldsymbol{\psi}_i \cdot \mathbf{g}(\mathbf{z}_{t-i}) \\
\mathbf{g}(\mathbf{z}_t) = \boldsymbol{\theta} \cdot \mathbf{z}_t + \boldsymbol{\delta} \cdot (|\mathbf{z}_t| - \mathbb{E}|\mathbf{z}_t|) \\
\sum_{i=1}^{\infty} \psi_{ji}^2 < \infty \quad \forall j = 1, ...n
\end{gathered} \tag{1.8}$$

$\cdot$ denotes dot product. Parameters of each diagonal component can be estimated with Whittle estimator. Estimation of $\mathbf{R}$ matrix is the subject of interest of this project. In order to check asymptotic properties of the Whittle estimator Monte Carlo simulation is conducted. First, correlated returns are simulated according to 1.6, 1.7 and 1.8 specification (Chapter 2). Next 1.8 parameters are estimated (Chapter 3). Based on the estimated parameters, shocks $\mathbf{z}_t$ are estimated using 1.6. $\mathbf{R}$ is estimated as correlation matrix between these shocks. The results are discussed in Chapter 4.

2. Simulation methodology

2.1. Generalized Error Distribution

This section deals with simulating correlated shocks 1.7. For the purpose of the simulation standardized (mean 0, variance 1) generalized error distribution (GED) is considered after Nelson's (1991):

$$\begin{gathered} f(z) = \frac{v}{\lambda 2^{1+1/v}\Gamma(1/v)} exp\left(-\frac{1}{2}\left|\frac{z}{\lambda}\right|^{v}\right), \quad z \in \mathbb{R}, \quad v \in \mathbb{R}_{+} \\ \lambda = \left(2^{-\frac{2}{v}}\frac{\Gamma(1/v)}{\Gamma(3/v)}\right)^{1/2} \end{gathered} \tag{2.1}$$

v is tail thickness parameter which allows for fatter ($v < 2$) and thinner ($v > 2$) tails than normal distribution and includes standard normal distribution as special case ($v = 2$). In order to draw from standardized GED the acceptance-rejection method is used. The idea comes from Neumann (1951) and involves drawing from similar but more tractable distribution and then adjusting the draw using uniform distribution. Here, the exponential distribution is used:

$$g(z) = \theta exp(-\theta z), \quad z \in \mathbb{R}_{+} \tag{2.2}$$

In order to draw from the exponential distribution with parameter θ inverse transform method is used. Let X follow $Exp(\theta)$. Inverse cumulative distribution function is equal to:

$$\begin{gathered} F_X(x) = \mathbb{P}(X \leq x) = 1 - \mathbb{P}(X \geq x) = 1 - exp(-\theta x) = y \\ ln(1-y) = -\theta x \\ F_X^{-1}(y) = x = -\frac{ln(1-y)}{\theta} \end{gathered} \tag{2.3}$$

If U is uniformly distributed on [0,1] than $F_X^{-1}(U)$ has the exponential distribution F_X. Proof:

$$\mathbb{P}(F_X^{-1}(U) < x) = \mathbb{P}(U < F_X(x)) = F_X(x) \tag{2.4}$$

So if y is a draw from $U[0,1]$ than $-\frac{ln(1-y)}{\theta}$ is a draw from $Exp(\theta)$. One can simplify this using y rather than $1-y$ because they both follow $U[0,1]$.

To use acceptance-rejection method the domains of GED and the exponential distribution have to be aligned. Since GED is symmetrical around zero, one can use binary random variable with equal probabilities (a fair coin toss) to decide if a draw is positive or negative. Thus the problem reduces to drawing from GED in the positive domain. Then, a constant c is chosen so that $f(z)$ is majorized:

$$cg(z) \geq f(z) \quad \forall z \in \mathbb{R}_+ \tag{2.5}$$

The acceptance-rejection algorithm works as follows:

1. A number $z_0 = -\frac{ln(y)}{\theta}$ is drawn from exponential distribution
2. A number u_0 is drawn from uniform distribution on [0,1]
3. If $u_0 \geq \frac{f(z_0)}{cg(z_0)}$ then z_0 is accepted as a draw from GED, otherwise go to 1

Geometrical rationale of the algorithm is the following: from the area below the majorant $cg(z)$ the area above $f(z)$ is rejected and the area below it is accepted. This yields a random variable which is uniformly distributed under $f(z)$ and is equal in distribution to GED. In order to minimize the number of rejections the ratio $\frac{f(z_0)}{cg(z_0)}$ has to be maximized under the constraint 2.5. This optimization problem is discussed in the next section of this chapter. The acceptance-rejection algorithm is carried out by function "ged.m" (see appendix A).

For the purpose of generating multivariate correlated GED distribution the results of Solaro (2004) are used. If $\mathbf{Y}$ follows $GED(\mathbf{0}, \mathbf{I}, \mathbf{v})$ then $\mathbf{X} = \mathbf{A}^T\mathbf{Y}$ follows $GED(\mathbf{0}, \mathbf{A}^T\mathbf{A}, \mathbf{v})$ [1]. So one needs to decompose the correlation matrix using for example Cholesky algorithm and then use the output matrix $\mathbf{A}^T$ to transform independent GED random vector $\mathbf{Y}$ into correlated $\mathbf{X}$. This is carried out by function "gedsim.m" (see appendix A).

[1] Solaro calls GED multivariate exponential power (MEP) distribution.

2.2. Optimization of GED simulation

Statement 1. *Let $f(z)$ be given by 2.1 and $g(z)$ be given be 2.2. If $\theta = \frac{1}{2\lambda^v}$ and $v > 1$ then $exp(-\theta(z^v - z - v^{\frac{v}{1-v}} + v^{\frac{1}{1-v}})))$ is the optimal value of objective function in the optimization problem:*

$$\max_{c\in\mathbb{R}} \frac{f(z)}{cg(z)} \tag{2.6}$$

under constraint $cg(z) \geq f(z) \quad \forall z \in \mathbb{R}_+$.

Proof. The objective function 2.6 is maximized by minimizing c so the problem reduces to finding minimal c for which $c \geq \frac{f(z)}{g(z)} \ \forall z \in \mathbb{R}_+$. Clearly

$$c_{opt} = \max_{z\in\mathbb{R}_+} \frac{f(z)}{g(z)}. \tag{2.7}$$

Let $d(v) = \frac{v}{\lambda 2^{1+1/v}\Gamma(1/v)}$

$$\begin{aligned} \left(\frac{f(z)}{g(z)}\right)' &= \left(\frac{d(v)exp\left(-\frac{1}{2}\left(\frac{z}{\lambda}\right)^v\right)}{\theta exp(-\theta z)}\right)' = \left(\frac{d(v)}{\theta}exp(-\theta(z^v - z))\right)' = \\ &= -d(v)exp(-\theta(z^v - z))(vz^{v-1} - 1) \end{aligned} \tag{2.8}$$

Setting 2.8 equal to 0 and solving for z yields $z = v^{\frac{1}{1-v}}$. Second derivative is calculated to check the second order condition.

$$\begin{aligned} \left(\frac{f(z)}{g(z)}\right)'' &= (-d(v)exp(-\theta(z^v - z))(vz^{v-1} - 1))' = \\ = d(v)exp(-\theta(z^v - z))&(\theta(vz^{v-1} - 1)^2 - v(v-1)z^{v-2}) = \\ \overset{z=v^{\frac{1}{1-v}}}{=} & -d(v)exp(-\theta(v^{\frac{v}{v-1}} - v^{\frac{1}{v-1}}))v(v-1)v^{\frac{v-2}{v-1}} \end{aligned} \tag{2.9}$$

The last expression in 2.9 is always negative, because $d(v)$ is positive and $v > 1$. So a local maximum of $\frac{f(z)}{g(z)}$ is achieved at $z = v^{\frac{1}{1-v}}$ and is equal to:

$$\frac{f(v^{\frac{1}{1-v}})}{g(v^{\frac{1}{1-v}})} = \frac{d(v)}{\theta}exp(-\theta(v^{\frac{v}{1-v}} - v^{\frac{1}{1-v}})) \tag{2.10}$$

The values at lower boundary is:

$$\frac{f(0)}{g(0)} = \frac{d(v)}{\theta} \tag{2.11}$$

The value in 2.11 is smaller than the value in 2.10 because for $v > 1$ the value of exponential function in 2.10 is greater than 1:

$$\begin{aligned} exp(-\theta(v^{\frac{v}{1-v}} - v^{\frac{1}{1-v}})) > 1 \\ -\theta(v^{\frac{v}{1-v}} - v^{\frac{1}{1-v}}) > 0 \\ v^{\frac{v}{1-v}} - v^{\frac{1}{1-v}} < 0 \\ \frac{v}{1-v} ln(v) < \frac{1}{1-v} ln(v) \\ \frac{v}{1-v} < \frac{1}{1-v} \\ v > 1 \end{aligned} \tag{2.12}$$

The assumption $v > 1$ is used two times in 2.12: $ln(v) > 0$ and $1 - v < 0$. The value in infinity is:

$$\lim_{z\to\infty} \frac{f(z)}{g(z)} = \lim_{z\to\infty} \frac{d(v)}{\theta} exp(-\theta(z^v - z)) = 0 \tag{2.13}$$

From the assumption $v > 1$ it follows that $\lim_{z\to\infty}(z^v - z) = \infty$ which justifies 2.13. So the global maximum is achieved at $z = v^{\frac{1}{1-v}}$ and c_{opt} is equal to 2.10. The value of objective function at c_{opt} is equal to:

$$\frac{f(z)}{c_{opt}g(z)} = \frac{\frac{d(v)}{\theta} exp(-\theta(z^v - z))}{\frac{d(v)}{\theta} exp(-\theta(v^{\frac{v}{1-v}} - v^{\frac{1}{1-v}}))} = exp(-\theta(z^v - z - v^{\frac{v}{1-v}} + v^{\frac{1}{1-v}})) \tag{2.14}$$

which ends the proof.

□

The choice of θ is arbitrary, so it might be subject of optimization too. It is chosen to be equal to $\frac{1}{2\lambda^v}$ for simplicity of calculations. The assumption $v > 1$ is not restraining because $v \in (1, 2)$ still allows for tails heavier than those of normal distribution. For example Nelson's (1991) estimate of v for financial returns equals 1.576. Also, in this paper the estimates of v for SP500 and EuroFirst 300 are found to be bigger than 1 (see Chapter 3 Assumptions section).

2.3. ARMA parametrization

This section deals with calculating conditional volatility process 1.8 based on simulated shocks 1.7. Calculating the values of $\mathbf{g}(\mathbf{z}_t)$ function from 1.8 is straightforward given the simulated shocks and parameters $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$. However, in order to model the $\mathbf{h}_t$ process one needs to finite-parametrize it. Popular way to do it, which was proposed by Nelson (1991), is ARMA (p,q) process. In multivariate setting, assuming p and q are the same for every asset, the process is modeled as follows:

$$(\mathbf{1} - \sum_{i=1}^{p} \boldsymbol{\Delta}_i L^i) \cdot (\mathbf{h}_t - \boldsymbol{\omega}) = (\mathbf{1} + \sum_{i=1}^{q} \boldsymbol{\Psi}_i L^i) \cdot \mathbf{g}(\mathbf{z}_{t-1}) \tag{2.15}$$

L denotes lag operator, $\boldsymbol{\Delta}_i$ and $\boldsymbol{\Psi}_i$ denote AR(p) and MA(q) parameters vectors respectively and $\boldsymbol{\omega}$ is parameter vector which accounts for the long run variance. Because it is constant in time $L\boldsymbol{\omega} = \boldsymbol{\omega}$. It is assumed that the polynomials $(1 - \sum_{i=1}^{p} \Delta_{ij} L^i)$ and $(1 + \sum_{i=1}^{q} \Psi_{ij} L^i)$ have no common roots $\forall j$ and that all roots of AR polynomials $(\mathbf{1} - \sum_{i=1}^{p} \boldsymbol{\Delta}_i L^i)$ lie outside the unit circle. To have explicit formula for $\mathbf{h}_t$ 2.15 can be rewritten:

$$\mathbf{h}_t = (\mathbf{1} - \sum_{i=1}^{p} \boldsymbol{\Delta}_i) \cdot \boldsymbol{\omega} + \sum_{i=1}^{p} \boldsymbol{\Delta}_i \cdot \mathbf{h}_{t-i} + \mathbf{g}(\mathbf{z}_{t-1}) + \sum_{i=1}^{q} \boldsymbol{\Psi}_i \cdot \mathbf{g}(\mathbf{z}_{t-i-1}) \tag{2.16}$$

The volatility vector $\mathbf{h}_t$ is calculated according to 2.16 for every $t = 1, 2, ..., T$, where T denotes the sample size in each simulation. Then the simulated returns are calculated according to 1.6. This is carried out by "simEGARCH.m" function (see appendix A). To summarize, for simulation one needs constant conditional correlation matrix $\mathbf{R}$ with $n(n-1)/2$ different parameters and $n \times (4 + p + q)$ EGARCH parameters. These are $n \times 1$ parameter vectors of:

- long term volatility $\boldsymbol{\omega}$
- leverage effect $\boldsymbol{\theta}$
- magnitude $\boldsymbol{\delta}$
- tail thickness $\mathbf{v}$
- AR (p vectors) $\boldsymbol{\Delta}_i$ and MA (q vectors) $\boldsymbol{\Psi}_i$

3. Estimation methodology

The estimation of the parameters mentioned in previous section is discussed in this chapter. The first section of this chapter deals with Whittle estimation of all parameters except for long term volatility $\boldsymbol{\omega}$ and constant conditional correlation matrix $\mathbf{R}$. Estimation of these is discussed in the second and the third section respectively.

3.1. Whittle estimator

The methodology of Whittle estimation in this section follows Zaffaroni (2009). For simplicity the notation refers to one dimensional case since the procedure for each asset is the same. Let $\vartheta = (\theta^2, \delta, v, \Psi_1, ..., \Psi_q, \Delta_1, ..., \Delta_p)$. The discrete Whittle estimator for a signal-plus-noise model 1.5 is defined as:

$$\begin{gathered} \overset{\wedge}{\vartheta} = \underset{\vartheta \in \Theta}{argmin} Q_T(\vartheta) \\ Q_T(\vartheta) = \frac{1}{T} \sum_{t=1}^{T-1} \left(log(f(\lambda_t; \vartheta)) + \frac{I_T(\lambda_t)}{f(\lambda_t; \vartheta)} \right), \quad \lambda_t = \frac{2\pi t}{T} \end{gathered} \tag{3.1}$$

$I_T(\lambda_t)$ is the sample periodogram based on T observations of $log(r_t^2)$:

$$I_T(\lambda_t) = \frac{1}{2\pi T} \left| \sum_{t=1}^{T} log(r_t^2) exp(i\lambda t) \right|^2, \quad -\pi \leq \lambda < \pi \tag{3.2}$$

$f(\lambda_t; \vartheta)$ is spectral density function of $log(r_t^2)$:

$$\begin{gathered} f(\lambda_t; \vartheta) = \frac{\alpha(\vartheta)}{2\pi} + \frac{\beta(\vartheta)}{2\pi} \left| \psi(e^{i\lambda}; \vartheta) \right|^2 + \\ + \frac{\gamma(\vartheta)}{2\pi} (e^{i\lambda} \psi(e^{i\lambda}; \vartheta) + e^{-i\lambda} \psi(e^{-i\lambda}; \vartheta)) \\ -\pi \leq \lambda < \pi \end{gathered} \tag{3.3}$$

The first term in 3.3 corresponds to the spectrum of $log(z_t^2)$ from 1.5. Under the GED assumption it is a function of the tail thickness parameter:

$$\alpha(\vartheta) = var(log(z_t^2)) = \left(\frac{2}{v}\right)^2 \Psi\left(\frac{1}{v}\right) \tag{3.4}$$

$\Psi(\cdot)$ in 3.4 denotes the trigamma function. The second term in 3.3 corresponds to the spectrum of h_t from 1.5. It consists of two parts: variance of $g(z_t)$, which is function of θ and δ:

$$\begin{aligned} \beta(\vartheta) &= var(g(z_t)) = \theta^2 + \delta^2(1 - \mu_{|z|}^2) \\ \mu_{|z|} &= \frac{\Gamma(2/v)}{\sqrt{\Gamma(1/v)\Gamma(3/v)}} \end{aligned} \tag{3.5}$$

and adjustment for linear filter (ARMA):

$$\psi(z;\vartheta) = \sum_{i=1}^{\infty} \psi_i z^i = \frac{1 + \sum_{i=1}^{q} \Psi_i z^i}{1 - \sum_{i=1}^{q} \Delta_i z^i} \tag{3.6}$$

AR and MA polynomials in 3.6 are assumed to have no common roots and the roots of AR polynomial are assumed to lie outside the unit circle. The third term in 3.3 corresponds to the covariance of h_t and $log(z_t^2)$.

$$\gamma(\vartheta) = \delta cov(log(z_t^2), |z_t|) = \frac{2\delta}{v}\mu_{|z|}\left(\psi\left(\frac{2}{v}\right) - \psi\left(\frac{1}{v}\right)\right) \tag{3.7}$$

$\psi(\cdot)$ in 3.7 denotes bigamma function. The sign of θ is not identifiable trough Whittle estimator, it only yields θ^2, hence the form of ϑ. It is therefore assumed the parameter θ is negative as discussed in the Literature Review chapter. The Whittle estimation is carried out by "whittle.m" (see appendix B). In order to prevent numerical stability issues related with optimization 3.1 the parameter space Θ in "whittle.m" is bounded as follows:

$$\Theta = [0, 0.25] \times [0.01, 0.25] \times [1, 3] \times (-q, q)^q \times (-p, p)^p \tag{3.8}$$

This choice of the particular parameter space is discussed in Discussion section of Chapter 4. "whittle.m" function makes use of bounded optimization function "fminsearchbnd.m", which is provided on Mathworks webside [1].

[1] http://www.mathworks.com/matlabcentral/fileexchange/8277-fminsearchbnd-fminsearchcon

3.2. Long run volatility

The long run volatility parameter ω is not identifiable by Whittle estimator. Its estimation is discussed in this section. Zaffaroni (2009) suggests simple consistent estimator

$$\begin{aligned} \omega &= \mathbb{E}log(r_t^2) - \mathbb{E}log(z_t^2) \\ \hat{\omega} &= \widehat{log(r_t^2)} - \widehat{log(z_t^2)} \end{aligned} \tag{3.9}$$

based on the sample mean:

$$\widehat{log(r_t^2)} = \frac{1}{T}\sum_{t=1}^{T} log(r_t^2) \tag{3.10}$$

and $\widehat{log(z_t^2)}$ which is based on Whittle estimate of tail thickness parameter $\hat{v}$. The relationship between the theoretical moment $\mathbb{E}log(z_t^2)$ and parameter v is derived below.

Statement 2. *Let z follow standardized GED (2.1). Let $\Gamma(\cdot), \psi(\cdot)$ denote gamma and bigamma function respectively. Then:*

$$\mathbb{E}log(z^2) = \frac{2}{v}\psi(1/v) + log\left(\frac{\Gamma(1/v)}{\Gamma(3/v)}\right) \tag{3.11}$$

Proof. As in proof of Statement 1. let $d(v) = \frac{v}{\lambda 2^{1+1/v}\Gamma(1/v)}$

$$\begin{aligned} \mathbb{E}log(z^2) &= \int_{\mathbb{R}} log(z^2) f(z) dz = \\ &= \int_{\mathbb{R}} log(z^2) d(v) exp(-\frac{1}{2}\left|\frac{z}{\lambda}\right|^v) dz = \\ &= 2d(v)\int_0^{\infty} log(z^2) exp(-\frac{1}{2\lambda^v} z^v) dz = \\ &= \frac{4d(v)}{v}\int_0^{\infty} log(z^v) exp(-\frac{1}{2\lambda^v} z^v) dz \end{aligned} \tag{3.12}$$

Substituting for z^v:

$$\begin{aligned} z^v &= t \\ vz^{v-1}dz &= dt \\ dz &= v^{-1}z^{1-v}dt \\ dz &= v^{-1}(z^v)^{1/v-1}dt \\ dz &= v^{-1}t^{1/v-1}dt \end{aligned} \tag{3.13}$$

and using formula 4.352 1 from Gradshteyn, Ryzhik (2007) 3.12 becomes:

$$\begin{aligned} &\frac{4d(v)}{v^2}\int_0^\infty log(t)t^{1/v-1}exp(-\frac{1}{2\lambda^v}t)dt = \\ &= \frac{4d(v)}{v^2}(2\lambda^v)^{1/v}\Gamma(1/v)(\psi(1/v) - log(\frac{1}{2\lambda^v})) = \\ &= \frac{v}{\lambda 2^{1+1/v}\Gamma(1/v)}\frac{4}{v^2}\lambda 2^{1/v}\Gamma(1/v)(\psi(1/v) - log(\frac{1}{2\lambda^v})) = \\ &= \frac{2}{v}(\psi(1/v) - log(\frac{1}{2\lambda^v})) = \\ &= \frac{2}{v}\left(\psi(1/v) - log\left(\frac{1}{2}\left(2^{-\frac{2}{v}}\frac{\Gamma(1/v)}{\Gamma(3/v)}\right)^{-v/2}\right)\right) = \\ &= \frac{2}{v}\left(\psi(1/v) - log\left(\left(\frac{\Gamma(1/v)}{\Gamma(3/v)}\right)^{-\frac{v}{2}}\right)\right) = \\ &= \frac{2}{v}\psi(1/v) + log\left(\frac{\Gamma(1/v)}{\Gamma(3/v)}\right) \end{aligned} \tag{3.14}$$

□

Based on Statement 2. the estimator of $\mathbb{E}log(z_t^2)$ is

$$\widehat{log(z_t^2)} = \frac{2}{\hat{v}}\psi(1/\hat{v}) + log\left(\frac{\Gamma(1/\hat{v})}{\Gamma(3/\hat{v})}\right) \tag{3.15}$$

The long term volatility is estimated according to the above procedure by function "whittleloop.m" (see appendix).

3.3. Constant correlation matrix

Constant correlation matrix remains to be estimated. Having simulated returns $\mathbf{r}_t$, estimated parameters $\hat{\boldsymbol{\vartheta}}$ and $\hat{\boldsymbol{\omega}}$ for all n assets and assuming ARMA(p,q) specification with p and q parameters equal to those used in simulation, the estimated shocks $\hat{\mathbf{z}}_t$ are calculated iteratively:

$$\begin{gathered}\hat{\mathbf{h}}_t = (\mathbf{1} - \sum_{i=1}^{p} \hat{\boldsymbol{\Delta}}_i) \cdot \hat{\boldsymbol{\omega}} + \sum_{i=1}^{p} \hat{\boldsymbol{\Delta}}_i \cdot \hat{\mathbf{h}}_{t-i} + \hat{\mathbf{g}}(\hat{\mathbf{z}}_{t-1}) + \sum_{i=1}^{q} \hat{\boldsymbol{\Psi}}_i \cdot \hat{\mathbf{g}}(\hat{\mathbf{z}}_{t-i-1}) \\ \hat{\mathbf{z}}_t = \mathbf{r}_t \cdot exp(-0.5\hat{\mathbf{h}}_t) \\ \hat{\mathbf{g}}(\hat{\mathbf{z}}_t) = \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{z}}_t + \hat{\boldsymbol{\delta}} \cdot (|\hat{\mathbf{z}}_t| - \mathbb{E}|\hat{\mathbf{z}}_t|)\end{gathered} \tag{3.16}$$

In order to calculate the estimated conditional volatility $\hat{\mathbf{h}}_t$ previous values of $\hat{\mathbf{h}}_{t-l}$ and $\hat{\mathbf{g}}(\hat{\mathbf{z}}_{t-l})$ up to lag $l = max(p, q+1)$ must be known. Therefore, the first observations are set equal to:

$$\hat{\mathbf{h}}_t = (\mathbf{1} - \sum_{i=1}^{p} \hat{\boldsymbol{\Delta}}_i) \cdot \hat{\boldsymbol{\omega}} \quad \forall t < max(p, q+1) \tag{3.17}$$

This iterative procedure yields the $T \times n$ matrix of estimated shocks $\hat{\mathbf{Z}} = [\hat{z}_{it}]$. Then it is used to estimate the constant correlation matrix:

$$\hat{\mathbf{R}} = [\hat{\rho}_{ij}] = \frac{1}{T} \sum_{t=1}^{T} \hat{\mathbf{z}}_t' \hat{\mathbf{z}}_t \tag{3.18}$$

$\hat{\mathbf{z}}_t$ is $n \times 1$ vector of shocks for all assets at time t. The above procedure is carried out by the function "corrmatrix.m" (see appendix B).

4. Monte Carlo study

4.1. Assumptions

This section discusses the assumptions of Monte Carlo simulation. The aim of the Monte Carlo study is to assess the asymptotic behavior of correlation matrix estimator, i.e. the quantity:

$$\sqrt{T}(\hat{\mathbf{R}} - \mathbf{R}) \tag{4.1}$$

In order to do so several sample sizes need to be considered. Since there is a computational trade-off between number of assets and sample sizes in consideration the choice of number of assets is $n = 2$, as it allows to consider more sample sizes. The sample sizes in consideration are:

$$T = [500, 1000, 1500, 2000, 2500, 3000, 3500, 4000] \tag{4.2}$$

The maximal sample size is 4000 because of computer memory constrains. The ARMA(p,q) parameters are assumed in accordance with Nelson (1991) who choses p and q using Bayes information criterion to be equal to 2 and 1 respectively. The values of simulation parameters are set using Chapter 3 methodology to estimate from market data: S&P500 and EuroFirst300 (750 observations: from 24.06.2009 to 28.06.2012). The following table summarizes the results:

Asset/Parameter	ω	θ	δ	v	Ψ	Δ_1	Δ_2
EF300	-9.058	-0.216e-03	0.218	1.913	0.98	0.343	0.599
SP500	-9.102	-0.002e-03	0.128	1.350	0.98	1.132	-0.162

The correlation between the shocks is equal to 0.7349. The procedure of Monte Carlo study is following: for each of $N = 100$ simulations:

1. $T_{max} = 4000$ observations of returns r_t are simulated for both assets according to the simulation methodology from Chapter 2 ("simEGARCH.m")

2. For each sample size from 4.2 the parameters are estimated according to methodology from Chapter 3 ("corrmatrix.m")

3. Estimated AR polynomial $1 - \hat{\Delta}L^1 - \hat{\Delta}_2 L^2$ is checked for the presence of roots within unit circle for both assets. If there are such roots, the simulation is not considered. See appendix C("unitrootcheck.m")

4. After all loop iterations the estimated correlations are standardized according to 4.1 and descriptive statistics are calculated

Theoretically the reason for disregarding cases of AR roots within unit circle is that it violates assumptions of the model, in particular the assumption $C(k_2, l)$ of Zaffaroni (2009) for the asymptotic properties of Whittle estimator. Practically AR roots within unit circle mean explosive behavior of volatility process h_t, which stands in contrast to empirical data (volatility does not explode in time, but reverts to long run average). The way to amend this problem is to increase the sample size as discussed in the Discussion section. The above steps are carried out by "montecarlo.m" function (see appendix C). The results are discussed in the next section.

4.2. Results

The first objective of the study is to investigate consistency of the correlation estimator, i.e. verify if the following holds:

$$\lim_{n\to\infty} \mathbb{P}(|\hat{\rho}-\rho| > \varepsilon) = 0 \quad \forall \varepsilon > 0 \tag{4.3}$$

In order to have one comprehensive measure rather than assess the probability for different ε Markov's inequality is used.

$$\mathbb{P}(|\hat{\rho}-\rho| > \varepsilon) \leq \frac{\mathbb{E}|\hat{\rho}-\rho|}{\varepsilon} \tag{4.4}$$

It follows from 4.4 that if $\mathbb{E}|\hat{\rho}-\rho|$ goes to 0 as $n \to \infty$ then the estimator is consistent. Based on the Monte Carlo simulation one can calculate sample equivalent of $\mathbb{E}|\hat{\rho}-\rho|$: mean absolute deviation from the true parameter.

$$MAD = \frac{1}{N^*} \sum_{i=1}^{N^*} |\hat{\rho}_i - \rho| \tag{4.5}$$

N^* denotes number of simulations $N = 100$ minus the number of rejected instances of AR roots within the unit circle and excluded outlying values [1].

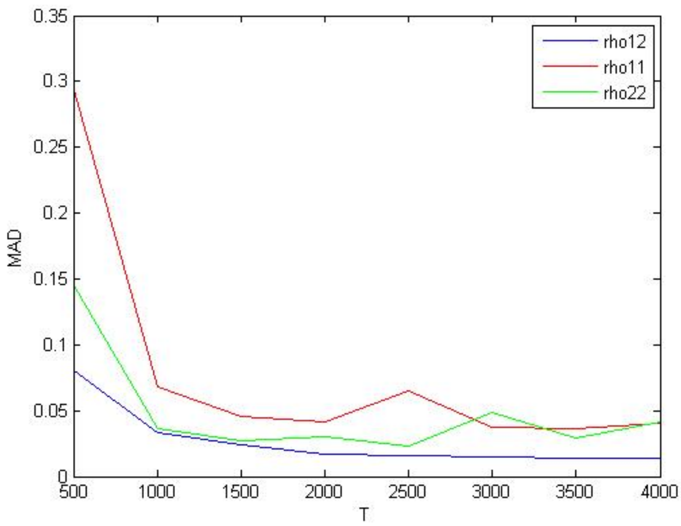

Figure 4.1.: Evolution of MAD of $\hat{\mathbf{R}}$ parameters

[1] 3 observations are excluded in the study. This issue is addressed in Discussion section

MAD calculated separately for different elements of $\hat{\mathbf{R}}$ and different sample sizes is presented in the above plot. The estimator of correlation between assets $\hat{\rho}_{12}$ (the lowest plot) displays consistent improvement in precision as the MAD around the true correlation approaches 0 as sample size increases. This pattern is not so distinct in parameters $\hat{\rho}_{11}$ and $\hat{\rho}_{22}$ as jumps in MAD occur for intermediary sample sizes. MAD can be interpreted as average error of estimator. For example for $T = 4000$ the estimator of correlation errs on average by 0.014 while the estimators $\hat{\rho}_{11}$ and $\hat{\rho}_{22}$ err on average by 0.04. To conclude, there is evidence for the consistency of estimator, particularly strong for the estimator of correlation between assets.

The second objective of the study is to investigate asymptotic distribution of the estimator. To do so the standardized estimators are considered.

$$\sqrt{T}(\hat{\rho} - \rho) \tag{4.6}$$

The evolution of correlation parameter $\hat{\rho}_{12}$ distribution is captured by figure 4.2

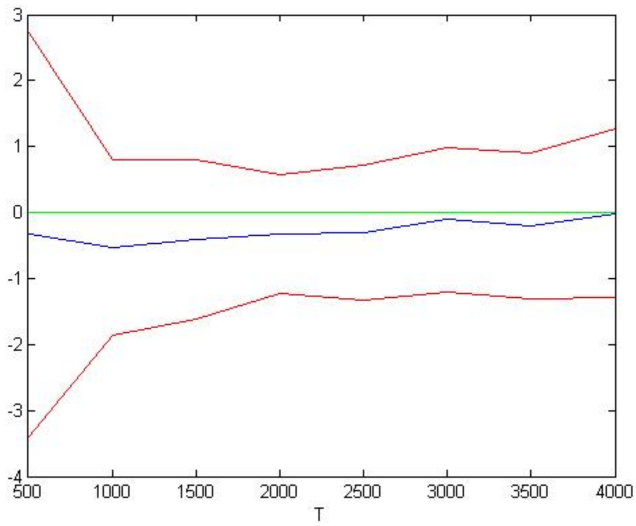

Figure 4.2.: Evolution of mean of $\sqrt{T}(\hat{\rho}_{12} - \rho_{12}) \pm 1$ standard deviation

Figure 4.2 indicates that the mean of the scaled parameter $\sqrt{T}(\hat{\rho}_{12} - \rho_{12})$ converges to 0 as sample size increases. The standard deviation decreases to $T = 2000$ but then slightly increases. So there is good evidence for convergence of the first moment and weaker evidence for the second moment. In order to verify if $\sqrt{T}(\hat{\rho}_{12} - \rho_{12})$ converges in distribution to the normal distribution the table of higher moments along with Jarque -Bera test statistics is presented.

Sample	Skewness	Kurtosis	JB statistic
500	2.3762	15.2425	160.6832
1000	-0.8351	8.7735	47.5954
1500	0.9116	6.0093	19.9786
2000	-0.1248	3.0932	0.1323
2500	1.7854	11.8632	190.2198
3000	1.2790	5.8104	32.9579
3500	1.9799	9.9187	156.6489
4000	2.2227	11.4801	241.5806

Table 4.1.: Empirical moments and Jarque-Bera test statistic of $\sqrt{T}(\hat{\rho}_{12} - \rho_{12})$

The null hypothesis of the Jarque-Bera test is that considered data follows normal distribution. The test statistic is calculated from skewness and kurtosis:

$$JB = \frac{N}{6}(Skewness^2 + \frac{1}{4}(Kurtosis - 3)^2) \tag{4.7}$$

Under the null hypothesis JB statistic follows Chi-squared distribution with two degrees of freedom. The critical value of $\chi^2_{0.005} = 10.597$ and table of test statistics provide strong evidence to reject the null hypothesis of normality for all instances except for $T = 2000$. In this instance there is strong evidence for the contrary: p-value is equal 0.936. Inflated values of skewness and kurtosis result from occurrence of outlying observations. It is not clear, however, to what extent outliers result from statistical properties of the distribution and from issues with numerical optimization used in "whittle.m" function. The occurrence of outlying observations is even more pronounced in the divergence of scaled parameters: $\sqrt{T}(\hat{\rho}_{11} - \rho_{11})$ and $\sqrt{T}(\hat{\rho}_{22} - \rho_{22})$ (see figures on the next page). While the mean of $\sqrt{T}(\hat{\rho}_{11} - \rho_{11})$ still seems to converge to 0, the outliers cause its standard deviation to explode for $T = 500$ and $T = 2500$. Also, the standard deviation seems to increase from $T = 2000$ as in the case of $\hat{\rho}_{12}$. The case of $\sqrt{T}(\hat{\rho}_{22} - \rho_{22}) \pm$ displays divergence not only in the standard deviation, but also the mean. The next section addresses the issue of the outlying observations, which cause this divergence.

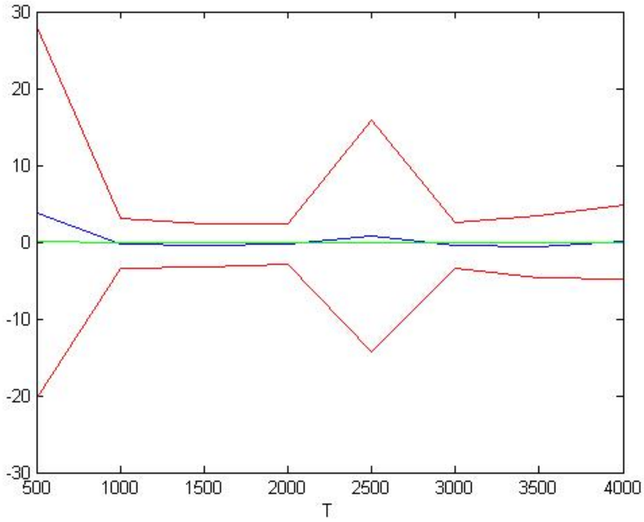

Figure 4.3.: Evolution of mean of $\sqrt{T}(\hat{\rho}_{11} - \rho_{11}) \pm$ 1 standard deviation

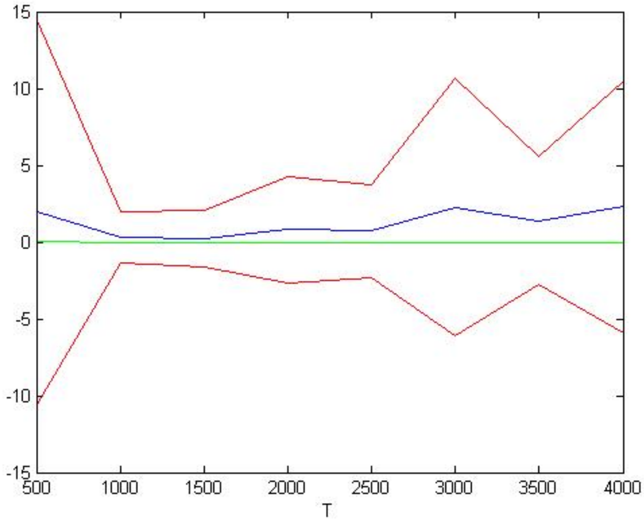

Figure 4.4.: Evolution of mean of $\sqrt{T}(\hat{\rho}_{22} - \rho_{22}) \pm$ 1 standard deviation

4.3. Discussion

The outlying values of the estimator disrupt the conclusions about the consistency and distribution of parameters. The results of the previous section already exclude three most extreme cases of outliers. The first case is correlation estimator $\hat{\rho}_{11}$ for $T = 3500$.

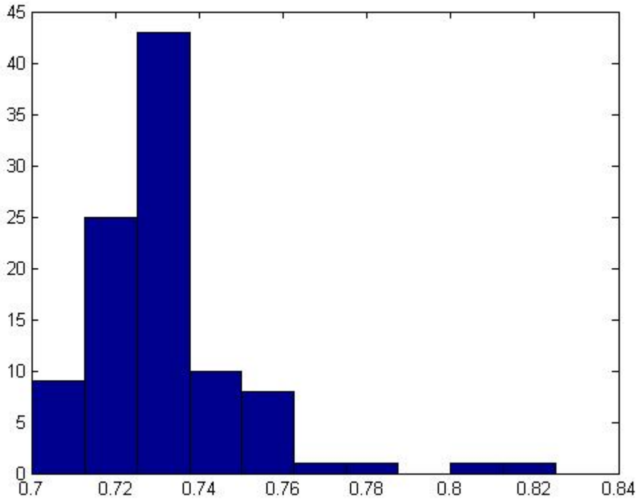

Figure 4.5.: Histogram of $\hat{\rho}_{11}$ excluding one outlying observation

While the histogram 4.3 contains several outlying observations, the excluded observation equals 7.2×10^5, which corresponds to around 3.8×10^7 standard deviations from the mean [2]. This observation corresponds to extremely low value of Whittle estimator for $\hat{\delta} = 0.02$ in one of the assets. The imposed lower boundary for the parameter is 0.01. Similarly, the other two excluded observations of $\hat{\rho}_{11} = 560$ (515 standard deviations from the mean) and $\hat{\rho}_{12} = 6.1$ (144 standard deviations from the mean) correspond to extreme (close to the boundary for given parameter) values of $\hat{\delta}$, $\hat{\theta}$ and $\hat{\upsilon}$. Relaxing the imposed boundaries on these parameters leads to increased number of outlying observations. In order to capture this a similar study is conducted, identical to the initial one except for the parameter space, which allows for more variation:

$$\Theta = [0,1] \times [0,1] \times [0,100] \times [-q,q]^q \times [-p,p]^p \tag{4.8}$$

[2]Standard deviation calculated excluding the outlier

In this study the number of instances where one of the estimates of $\hat{\rho}_{11}$, $\hat{\rho}_{12}$ or $\hat{\rho}_{11}$ lies beyond 100 standard deviations from its mean estimator is eight, compared to three in the initial study. Again these observations are associated with outlying values of parameters $\hat{\vartheta}$ estimated with Whittle estimator. The large scale of deviation from the mean (hundreds of standard deviations) might suggest this phenomenon is attributable to numerical optimization issues rather than statistical distribution of the parameter. So there is a trade-off between more flexible parameter space and the occurrence of outlying values. The parameter space used in the original study (3.8) is chosen so that the number of outliers is minimized, but still enabling the parameters to vary in reasonable intervals: all of them cover the true parameters as well as Nelson's (1991) estimates $\pm$ 3 their standard deviations. Identifying the exact causes of the outliers as well as determining optimal parameter space might be the issue of further study in this area.

The last issue to consider is the rejection of the observations for which the AR roots lie within the unit circle. As previously mentioned this problem in estimation can be solved by increasing the sample size. The following graph shows how the number of rejected observations behaves as the sample size increases:

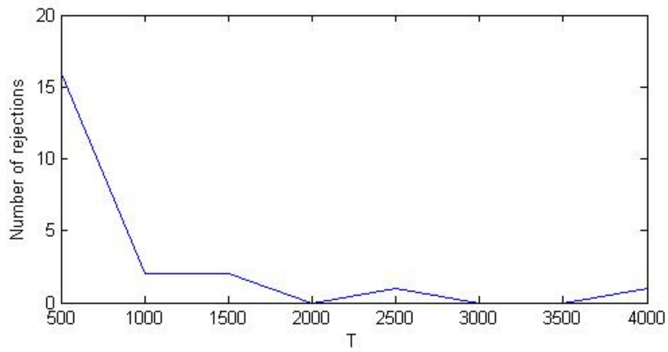

Figure 4.6.: Number of observations with AR roots within the unit circle

The increase in the sample size improves the precision of the estimation and as a result the explosive behavior of volatility (AR roots within the unit circle) is observed less frequently. At $T = 2000$ the number of rejections practically reaches 0.

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Appendices

A. Simulation

ged.m

```
function out=ged3(v)
% lambda defined by Nelson (2.5)
lambda=sqrt(2^(-2/v)*gamma(1/v)/gamma(3/v));
% exponential ditribution parameter
theta=0.5/lambda^v;
const=exp(theta*(v^(v\(1-v))-v^(1\(1-v))));

x=-1;
r=rand;
while(x<r)
   e=-log(rand)/theta; %draw from exp
   x=const*exp(-theta*(e^v-e));
end

if rand>0.5
    e=-e; %symmetry
end

out=e;

end
```

gedsim.m

```
function x = gedsim(v,cor,T)

%Choleski decomposition
U=chol(cor);
n=length(v);
x=zeros(T,n);

for i=1:T
    for j=1:n
       z(j)=ged(v(j));
    end
    x(i,:)=z*U;
end

end
```

simEGARCH.m

```
function returns=simEGARCH(cor,T,omega,theta,delta,thickness,ma,ar)

n=length(cor); % number of assets
p=size(ar,2);
q=size(ma,2);
l=max(p,q+1);

% simulating correlated schocks
z = gedsim(thickness, cor, T+l);
err=zeros(T+l,n);
h=zeros(T+l,n);



% innovation fuction with theoretical moments
```

```
for j=1:n
    err(:,j) = theta(j)*z(:,j) + delta(j)*(abs(z(:,j))
-gamma(1)/sqrt(gamma(3/2)*gamma(1/2)));
    h(:,j)=h(:,j)+omega(j);
    if p>0
        arinv(j,:)=ar(j,end:-1:1);
    else
        arinv(j,:)=zeros(1,0);
    end
    if q>0
        mainv(j,:)=ma(j,end:-1:1);
    else
        mainv(j,:)=zeros(1,0);
    end
    const(j)=(1-sum(arinv(j,:)))*omega(j);

    for i=l+1:T+l % ARMA representation of h function
        h(i,j)=const(j)+arinv(j,:)*h(i-p:i-1,j)
+err(i-1,j)+mainv(j,:)*err(i-1-q:i-2,j);
    end
end

returns = z(l+1:end,:).*exp(0.5*h(l+1:end,:));
end
```

B. Estimation

whittle.m

```
function estimator = whittle(data,initial,ma,ar)

% GED parameters
mi=@(theta) gamma(2/theta(3))/(gamma(3/theta(3))*gamma(1/theta(3)))^0.5;
alpha0=@(theta) (2/theta(3))^2*psi(1,1/theta(3));
beta0=@(theta) theta(1)^2+theta(2)^2*(1-mi(theta)^2);
gamma0=@(theta) 2*theta(2)/theta(3)*mi(theta)*(psi(0,2/theta(3))
-psi(0,1/theta(3)));

% transfer function phi
a=@(z,theta) 1;
for j=1:ma
    a=@(z,theta) a(z,theta)+theta(3+j)*z.^j;
end
b=@(z,theta) 1;
for j=1:ar
    b=@(z,theta) b(z,theta)-theta(3+ma+j)*z.^j;
end
phi =@(z,theta) a(z,theta)./b(z,theta);

% periodogram (T-1x1 vector)
logx2=log(data.^2);
T=length(data);
t=1:T-1;
freq=2*pi*t'/T;
I = (abs(exp(1i*freq*[t,T])*logx2).^2)/(2*pi*T);
```

```
% spectral density function f (T-1x1 vector function)
f =@(theta) 1/2/pi*(alpha0(theta)+beta0(theta)*abs(phi(exp(1i*freq),theta)).^2
+gamma0(theta)*(exp(1i*freq).*phi(exp(1i*freq),theta)
+exp(-1i*freq).*phi(exp(-1i*freq),theta)));

% discrete Whittle function
Q =@(theta) sum(log(f(theta))+I./f(theta))/T;

% minimization
options = optimset('MaxFunEvals',50000,'MaxIter',50000);
epsilon=0.01;
estimator=fminsearchbnd(@(theta) Q(theta),initial,
[0,0,0,-ma*ones(1,ma)+epsilon,-ar*ones(1,ar)+epsilon],
[1,1,100,ma*ones(1,ma)-epsilon,ar*ones(1,ar)-epsilon],options);
```

whittleloop.m

```
function out=whittleloop(data,tol,ma,ar)
%considers GED case
%EGARCH parameters from Nelson (1991), 2 - assumed normal ditribution
par=zeros(1,3+ar+ma);
par(1)=0.0139;
par(2)=0.1559;
par(3)=2;
prev=zeros(1,3+ar+ma);

    while (norm(par-prev,2)>tol)
        prev=par;
        par=whittle(data,par,ma,ar);
    end
    % omega estimation
    v=par(3);
    omega=mean(log(data.^2))-2/v*(psi(0,1/v)-log((gamma(3/v)/gamma(1/v))^(v/2)));
    out=[omega,par];

end
```

corrmatrix.m

```
function [par,c]=corrmatrix(returns,q,p)

T=size(returns,1);
n=size(returns,2);
l=max(p,q+1);
% Whittle estimation
for j=1:n
    par(j,:)=whittleloop(returns(:,j),0.1,q,p);
    arinv(j,:)=par(j,end:-1:end+1-p);
    mainv(j,:)=par(j,end-p:-1:5);
    ss(j)=sum(arinv(j,:));
end

% shocks calculation
h=zeros(T,n);
z=zeros(T,n);
err=zeros(T,n);
for i=1:T
    for j=1:n
        h(i,j)=par(j,1).*(1-ss(j));
        if (i>l)
            h(i,j)=h(i,j)+ arinv(j,:)*h(i-p:i-1,j)+err(i-1,j)
+mainv(j,:)*err(i-1-q:i-2,j);
        end
    end
    z(i,:)=returns(i,:).*exp(-0.5*h(i,:));
    for j=1:n
        err(i,j)=par(j,2)*z(i,j)+par(j,3)*(abs(z(i,j))
-gamma(1)/(gamma(1.5)*gamma(0.5))^0.5);
    end
end

% sample covariance matrix
c=(z'*z)/T;
```

C. Monte Carlo

montecarlo.m

```
function s=montecarlo(N,T)

par =[-9.0582,(0.2163e-03)^2,0.2176,1.9132,0.9800,0.3428,0.5988;-9.1024,
(0.002e-03)^2,0.1277,1.3495,0.9800,1.1317,-0.1621];
c=0.7349;

n=2;
cor=ones(n)*c; % generate correlation matrix
for i=1:n
    cor(i,i)=1;
end

omega=par(:,1);  theta=-sqrt(par(:,2));   delta= par(:,3);  thickness=par(:,4);
ma=par(:,5);  ar=par(:,6:7);

q=size(ma,2);
p=size(ar,2);
s=struct;
l=ones(1,length(T));

for i=1:N
    i
    returns=simEGARCH(cor,T(end),omega,theta,delta,thickness,ma,ar);
    for j=1:length(T)
        [table,corr,z]=corrmatrix2(returns(1:T(j),:),q,p);
        if (unitrootcheck(table(:,end-p+1:end))==0)
            s(j).correlation(l(j),:)=corr(1,2);
            s(j).varz1(l(j),:)=corr(1,1);
```

```
            s(j).varz2(l(j),:)=corr(2,2);
            s(j).shocks=z;
            for k=1:n
                s(j).asset(k).estimator(l(j),:)=table(k,:);
                s(j).asset(k).average=mean(s(j).asset(k).estimator);
                s(j).asset(k).stdev=std(s(j).asset(k).estimator);
            end
            l(j)=l(j)+1;
        end
    end
    save mciteration
end

for j=1:length(T)
    s(j).SampleSize=T(j);
    s(j).AverageCorrelation=nanmean(s(j).correlation);
    s(j).Averagevarz1=nanmean(s(j).varz1);
    s(j).Averagevarz2=nanmean(s(j).varz2);
    s(j).StandardDeviationOfCorrelation=nanstd(s(j).correlation);
    s(j).SkewnessCorrelation=skewness(s(j).correlation);
    s(j).KurtosisCorrelation=kurtosis(s(j).correlation);
    s(j).StandardizedCorrelation=sort(sqrt(T(j))*(s(j).correlation-c));
    s(j).AverageStandardizedCorrelation=nanmean(s(j).StandardizedCorrelation);
    s(j).StandardDeviationOfStandardizedCorrelation
=nanstd(s(j).StandardizedCorrelation);
end
end
```

unitrootcheck.m

```
function ur = unitrootcheck(table)
ur=0;
for i=1:size(table,1)
    if size(table,2)==1
        if table(i,1)>=1
            ur=1;
        end
    else
        % finds the roots of AR polynomial
        [x1,x2]=rootfinder(table(i,1),table(i,2));
        if (abs(x1)<=1 || abs(x2)<=1)
            ur=1; % returns 1 if the process is explosive
        end
    end
end
end
```