Consultație 24 ian

I Enunțați:

1) Teorema lui Lagrange

Fie $f:[a,b]\to\mathbb{R}$. Dacă f e cont pe $[a,b]$ și deriv pe (a,b) atunci $\exists\, c\in(a,b)$ a.î.

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

2) Formula lui Taylor cu rest Lagrange

Fie $a\in\mathbb{R}$ un punct.

$\forall x\in I$, I - interval $\exists\, c$ între a și x

a.î.

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)^1+\dots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{f^{(n+1)}(c)}{(n+1)!}\cdot(x-a)^{n+1}$$

Definiți:

a) distanța Fie $A\neq\emptyset$

O funcție $d:A\times A\to\mathbb{R}$ se numește distanță pe A dacă

1) $d(x,y)=0 \Leftrightarrow x=y$
2) $d(x,y)=d(y,x)$
3) $d(x,y)+d(y,z)\geq d(x,z)$
4) $d(x,y)\geq 0$, $\forall x,y,z\in A$

b) integrala Darboux superioară

Fie $\mathcal{D}([a,b])$ mulțimea diviziunilor intervalului $[a,b]$.

$(d\in\mathcal{D}([a,b]) \Leftrightarrow$ $a\ x_1\ x_2\ x_3\ x_4\ x_5 \dots b)$

Fie $f:[a,b]\to\mathbb{R}$

$M_i=\sup\limits_{x\in[x_{i-1},x_i]} f(x)$ și $S(d)=\sum\limits_{i=1}^{n} M_i(x_i-x_{i-1})$

Numărul $\int_a^b f(x)\,dx=\inf\limits_{d\in\mathcal{D}([a,b])} S(d)$ se numește integrala Darboux superioară

II.

1) Studiați convergența seriei

$$\sum_{n\geq 1} x^n\cdot\frac{a(a+1)\dots(a+n)}{(n+10)!},\ x>0,\ a>0.$$

Fie $x_n=x^n\cdot\frac{a(a+1)\dots(a+n)}{(n+10)!}$

Calculăm $\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n}=$

$$=\lim_{n\to\infty}\frac{x^{n+1}\cdot\frac{a(a+1)\dots(a+n)(a+n+1)}{(n+11)!}}{x^n\cdot\frac{a(a+1)\dots(a+n)}{(n+10)!}}=$$

$$=\lim_{n\to\infty} x\cdot\frac{n+a+1}{n+11}=x$$

Conform criteriului raportului avem că

- pentru $x>1$, seria e divergentă
- pentru $x<1$, seria e convergentă
- pentru $x=1$, criteriul nu decide

Pentru $x=1$, $x_n=\frac{a(a+1)\dots(a+n)}{(n+10)!}$

Calculăm $\lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)=$

$$=\lim_{n\to\infty} n\cdot\left(\frac{n+11}{n+a+1}-1\right)=\lim_{n\to\infty} n\cdot\frac{\cancel{n}+11-\cancel{n}-a-1}{n+a+1}=$$

$$=\lim_{n\to\infty}\frac{n(10-a)}{n+a+1}=10-a$$

Conform criteriului Raabe-Duhamel avem

- pentru $a<9$, seria e convergentă
- pentru $a>9$, seria e divergentă
- pentru $a=9$, criteriul nu decide

Pentru $a=9$, $x_n=\frac{9\cdot 10\cdot\ldots\cdot(n+9)}{(n+10)!}=$

$$=\frac{\frac{(n+9)!}{8!}}{(n+10)!}=\frac{1}{8!\,(n+10)}.$$

Dar $\sum\limits_{n\geq 1}\frac{1}{8!(n+10)}$ divergentă, fiind serie arm gen. $\frac{1}{n^\alpha}$

$$\sum_{n\geq 1}\frac{1}{8!\,(n+10)}=\frac{1}{8!}\sum_{n\geq 1}\frac{1}{n+10}=\frac{1}{8!}\sum_{m\geq 11}\frac{1}{m}$$

$m=n+10$

2. Studiați convergența simplă și uniformă a șirului de funcții $(f_n)_{n\in\mathbb{N}^*}$

$f_n:\mathbb{R}\to\mathbb{R}$, $f_n(x)=\frac{x^7 n^3}{x^{10}+n^{10}}$

Studiem conv s:

$\lim\limits_{n\to\infty} f_n(x)=\lim\limits_{n\to\infty}\frac{x^7n^3}{x^{10}+n^{10}}=0$. Deci

$f_n \xrightarrow[n\to\infty]{s} f$, unde $f:\mathbb{R}\to\mathbb{R}$, $f(x)=0$

Studiem conv unif:

Calculăm $\lim\limits_{n\to\infty}\sup\limits_{x\in\mathbb{R}}|f_n(x)-f(x)|\overset{?}{=}0$

$$f_n'(x)=\frac{7x^6n^3(x^{10}+n^{10})-x^7n^3\cdot 10x^9}{(x^{10}+n^{10})^2}=$$

$$=\frac{7x^{16}n^3+7x^6n^{13}-10x^{16}n^3}{(x^{10}+n^{10})^2}=$$

$$=\frac{x^6n^3(7n^{10}-3x^{10})}{(x^{10}+n^{10})^2}=0 \Leftrightarrow 7n^{10}-3x^{10}=0 \Leftrightarrow$$

$\Leftrightarrow x=\pm\left(\frac{7}{3}\right)^{\frac{1}{10}}\cdot n$

$$f_n\left(n\sqrt[10]{\tfrac{7}{3}}\right)=\frac{\left(\frac{7}{3}\right)^{\frac{7}{10}}\cdot n^{10}}{\frac{7}{3}\cdot n^{10}+n^{10}}=\dots O(n^3)=y$$

x	$-\infty$	$\left(\frac{7}{3}\right)\cdot n$	$+\infty$
$f_n'(x)$	$+++++\,+$	0	$-\,-\,-\,-$
$f_n(x)$	$\nearrow$	y	$\searrow$

Deci $\sup\limits_{x\in\mathbb{R}} f_n(x)=y=O(n^3)\xrightarrow{n\to\infty}\infty$

Deci $f_n \overset{u}{\underset{n\to\infty}{\not\to}} f$

3. Determinați punctele de extrem local ale funcției

$f(x,y)=xy+\frac{1}{x}+\frac{1}{y}$, $f:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}$

$\mathbb{R}^2\setminus\{(0,0)\}$ deschisă și f cont (op cu fct cont)

$\frac{\partial f}{\partial x}=y-\frac{1}{x^2}$, $\frac{\partial f}{\partial y}=x-\frac{1}{y^2}$ cont

$$\begin{cases}\frac{\partial f}{\partial x}=0\\ \frac{\partial f}{\partial y}=0\end{cases}\Leftrightarrow\begin{cases}y-\frac{1}{x^2}=0\\ x-\frac{1}{y^2}=0\end{cases}\Leftrightarrow\begin{cases}y=\frac{1}{x^2}\\ x-\frac{1}{y^2}=0\end{cases}\Leftrightarrow$$

$\Leftrightarrow x-x^4=0 \Leftrightarrow x(1-x^3)=0\Leftrightarrow$

$\Leftrightarrow x(1-x)\underbrace{(1+x+x^2)}_{\Delta=1-4=-3<0}=0 \Rightarrow x\in\{0,1\}$

$\Rightarrow x=1\Rightarrow y=1$

Deci $(x,y)=(1,1)$ singurul punct critic

$\frac{\partial^2 f}{\partial x^2}=\frac{2}{x^3}$, $\frac{\partial^2 f}{\partial x\partial y}=1$

$\frac{\partial^2 f}{\partial y\partial x}=1$, $\frac{\partial^2 f}{\partial y^2}=\frac{2}{y^3}$ cont

$$Hf(x,y)=\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2}\end{pmatrix}=\begin{pmatrix}\frac{2}{x^3} & 1\\ 1 & \frac{2}{y^3}\end{pmatrix}$$

$$Hf(1,1)=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

$\Delta_1=|2|=2>0$
$\Delta_2=\begin{vmatrix}2&1\\1&2\end{vmatrix}=3>0$ $\Big\}\Rightarrow (1,1)$ pct de min local

$<\ >\ <\ >\ <\ >\ \to$ max

$<\ <\ >\ <\ >\ \to$ sau nu decide

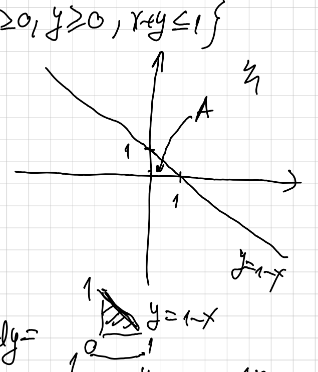
$f(x,y)\leq a\ \forall (x,y)\in\mathbb{R}^2$

$f(x,y)\geq a$

$=\to$ nul

4. Calculați $\iint_A (x^2+y^2)\,dx\,dy$, unde

$A=\{(x,y)\in\mathbb{R}^2 \mid x\geq 0,\ y\geq 0,\ x+y\leq 1\}$



$0\leq y\leq 1-x$

Fie $f:A\to\mathbb{R}$,

$f(x,y)=x^2+y^2$ cont

$A\in J(\mathbb{R}^2)$

Avem $\iint_A f(x,y)\,dx\,dy=$

$$=\int_0^1\int_0^{1-x}(x^2+y^2)\,dy\,dx=\int_0^1\left(x^2 y\Big|_{y=0}^{y=1-x}+\frac{y^3}{3}\Big|_0^{1-x}\right)dx=$$

$$=\int_0^1\left(x^2(1-x)+\frac{(1-x)^3}{3}\right)dx=$$

$$=\int_0^1\left(x^2-x^3+\frac{1}{3}-\frac{3x}{3}+\frac{3x^2}{3}-\frac{x^3}{3}\right)dx=$$

$$=\frac{x^3}{3}\Big|_0^1-\frac{x^4}{4}\Big|_0^1+\frac{x}{3}\Big|_0^1-\frac{x^2}{2}\Big|_0^1+\frac{x^3}{3}\Big|_0^1-\frac{x^4}{12}\Big|_0^1=$$

$=\dots$

* $f(x,y)=\begin{cases}\frac{x^5y^2}{\sqrt{x^{16}+y^8}}, & (x,y)\neq 0\\ 0, & (x,y)=(0,0)\end{cases}$

i) cont
ii) $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$
iii) dif

i) f cont pe $\mathbb{R}^2\setminus\{(0,0)\}$

$\lim\limits_{(x,y)\to(0,0)}\frac{x^5y^2}{\sqrt{x^{16}+y^8}}\overset{?}{=}f(0,0)=0$

$0\leq\left|\frac{x^5y^2}{\sqrt{x^{16}+y^8}}\right|\leq\dots\to 0$

$\left|\frac{x^5y^2}{\sqrt{x^{16}+y^8}}\right|\leq\frac{|x|\cdot x^4\cdot y^2}{\sqrt{x^{16}+y^8}}\leq\frac{|x|\cdot\cancel{x^4 y^2}}{\sqrt{2\cdot x^8 y^4}}=$

$x^{16}+y^8\geq 2\sqrt{x^{16}\cdot y^8}=2x^8\cdot y^4$

$x\to 0$
$y=ax\to 0$
$\forall f(x)=y\to 0$

$=\frac{|x|}{\sqrt{2}}\xrightarrow[(x,y)\to(0,0)]{}0$

Deci din crit cleștelui, $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$

~~iii) f dif pe $\mathbb{R}^2\setminus\{(0,0)\}$ $\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\text{ cont}\right)$~~

~~$df(0,0)(u,v)=\left(\frac{\partial f}{\partial x}\right.$~~

ii) $$\frac{\partial f}{\partial x}=\frac{5x^4y^2\sqrt{x^{16}+y^8}-x^5y^2\cdot\frac{1}{2\sqrt{x^{16}+y^8}}\cdot 16x^{15}}{x^{16}+y^8}$$

$$\frac{\partial f}{\partial y}=\frac{2x^5y\sqrt{x^{16}+y^8}-x^5y^2\cdot\frac{1}{2\sqrt{x^{16}+y^8}}\cdot 8y^7}{x^{16}+y^8}$$

$\frac{\partial f}{\partial x}(0,0)=\lim\limits_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=\lim\limits_{t\to 0}\frac{0-0}{t}=0$

$\frac{\partial f}{\partial y}(0,0)=\lim\limits_{t\to 0}\frac{f(0,t)-f(0,0)}{t}=0$

iii) f e dif pe $\mathbb{R}^2\setminus\{(0,0)\}$

$\exists\, df(0,0)(u,v)=\left(\left(\frac{\partial f}{\partial x}(0,0)\ \ \frac{\partial f}{\partial y}(0,0)\right)\begin{pmatrix}u\\ v\end{pmatrix}\right)^t=0$

$\frac{\partial f}{\partial x}(0,0)u+\frac{\partial f}{\partial y}(0,0)v$

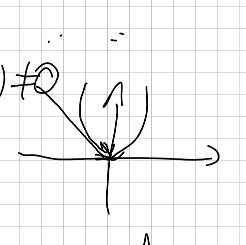
dacă

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|}=0$$

$\lim\limits_{(x,y)\to(0,0)}\frac{x^5y^2}{\sqrt{x^{16}+y^8}\cdot\sqrt{x^2+y^2}}=?$

$0\leq\left|\frac{x^5y^2}{\sqrt{x^{16}+y^8}\sqrt{x^2+y^2}}\right|\overset{?}{\leq}\to 0$

$x=x_n$
$y=y_n$ $\lim\limits_{n\to\infty}(x_n,y_n)\neq 0$



$x_n=\frac{1}{n}\to 0$

$y=y_n=\frac{1}{n^2}\to 0$

$$\lim_{n\to\infty}\frac{\frac{1}{n^5}\cdot\frac{1}{n^4}}{\sqrt{\frac{1}{n^{16}}+\frac{1}{n^{16}}}\sqrt{\frac{1}{n^2}+\frac{1}{n^4}}}=\lim_{n\to\infty}\frac{\frac{1}{n^9}}{\frac{\sqrt{2}}{n^8}\cdot\frac{\sqrt{n^2+1}}{n^2}}=$$

$=\lim\limits_{n\to\infty}\frac{n}{\sqrt{2}\cdot\sqrt{n^2+1}}=\frac{1}{\sqrt{2}}\neq 0 \Rightarrow f$ nu e dif în $(0,0)$

$=0\to$ nu e bine

$f(x)=\begin{cases}x^2, & x\in\mathbb{Q}\setminus\{0,1\}\\ x, & x\in(\mathbb{R}\setminus\mathbb{Q})\cup\{0,1\}\end{cases}$

Cont, deriv

$\mathring{B}$

* $A=[0,4]\setminus\{2\}$ $\mathring{A}$, $\bar{A}$, A', $Fr(A)$, $Izo(A)$?

1) x_0 = pct interior lui A ($x_0\in\mathring{A}$)

$\Leftrightarrow \exists\, r>0$ a.î. $B(x_0,r)\subseteq A$

Dacă $x_0=2$, $\forall r>0$, $\{2\}\notin A$
$x_0=0$
$x_0=4$
$\{2\}\in B(x_0,r)$

Deci $2\notin \mathring{A}$

Dacă $x_0\in\mathbb{R}\setminus[0,4]$, $x_0\notin\mathring{A}$

Dacă $x_0\in A$,

$r=\left|\frac{2-x_0}{2}\right|$

Deci $\mathring{A}=(0,4)\setminus\{2\}$

2) x_0 = pct de acumulare a lui A ($x_0\in A'$)

$\Leftrightarrow \forall r>0$, avem $B(x_0,r)\cap(A\setminus\{x_0\})\neq\emptyset$

$A'=A\cup\{2\}$

3) x_0 = pct aderent ($x_0\in\bar{A}$) $\Leftrightarrow$

$\Leftrightarrow \forall r>0$, $B(x_0,r)\cap A\neq\emptyset$

$\bar{A}=A\cup A'=A\cup\{2\}$

4) x_0 este pct frontieră a lui $A \Leftrightarrow x_0\in\bar{A}\setminus\mathring{A}$

$Fr(A)=\{0,4,2\}$

5) x_0 este pct izolat a lui $A \Leftrightarrow x_0\in\bar{A}\setminus A'$

$Izo(A)=\emptyset$