Brick finiteness of classical Schur algebras

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@ AMSS

§0 Motivation

- To study a new finiteness in wild cases.
- To control derived equivalence of algebras using tilting/silting theory.

§1 Brick finiteness of algebras.

$A = kQ/I$: bound quiver alg over $k = \bar{k}$.

<u>Ex</u> $A = k(1 \longrightarrow 2 \longrightarrow 3)$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "0"'] & k \arrow[r, "1"] \arrow[d, "1"'] & k \arrow[d, "1"] & \simeq & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \arrow[d] \\
k \arrow[r, "1"] & k \arrow[r, "1"] & k & \simeq & \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}
\end{tikzcd}$$

Nakayama funtor $\nu(-) : \mathrm{proj}\, A \longrightarrow \mathrm{inj}\, A$.

Let M be an A-mod. We take a minimal projective presentation of M:

$$\begin{tikzcd}
 & P_1 \arrow[r, "f_1"] \arrow[d, "\nu(-)"] & P_0 \arrow[r, "f_0"] \arrow[d, "\nu(-)"] & M \arrow[r] & 0 \\
0 \arrow[r] \arrow[r] \arrow[d, phantom] & \tau M \arrow[r] & \nu(P_1) \arrow[r, "\nu(f_1)"] & \nu(P_0) &
\end{tikzcd}$$

that is, $\tau M = \ker \nu(f_1)$: the AR-translation of M.

<u>Ex</u> $A = k(1 \longrightarrow 2 \longrightarrow 3)$

- $\tau\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$.

$$\begin{tikzcd}
 & 3 \arrow[r] \arrow[d] & \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \arrow[r] & 0 \\
0 \arrow[r] & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \arrow[r] & 1 &
\end{tikzcd}$$

- $\tau(1) = 2$.

$$\begin{tikzcd}
 & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \arrow[r] \arrow[d] & 1 \arrow[r] & 0 \\
0 \arrow[r] & 2 \arrow[r] & \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \arrow[r] & 1 &
\end{tikzcd}$$

<u>Def [Adachi-Iyama-Reiten, 2014]</u>

(1) M is called τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.

(2) M is called τ-tilting if M is τ-rigid and $|M| = |A|$

(3) (M, P) is called support τ-tilting if M is τ-rigid, $\mathrm{Hom}_A(P, M) = 0$ for some projective A-module P, $|M| + |P| = |A|$.

<u>Ex</u>: $A = k(1 \longrightarrow 2 \longrightarrow 3)$

- $\begin{smallmatrix}1\\2\end{smallmatrix}$, 1. are τ-rigid.
- $\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3$, $\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 1 \oplus 3$, are τ-tilting.
- $(\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3, \begin{smallmatrix}1\\2\\3\end{smallmatrix})$ is support τ-tilting.

We have

$$\text{i}\tau\text{-rigid } A \quad \text{"span"} \quad \tau\text{-tilt } A \subseteq \text{s}\tau\text{-tilt } A \subseteq \tau\text{-rigid } A.$$

Give a support τ-tilting module $M = M_1 \oplus \cdots \oplus M_n$. Take M_j & $\mathrm{Fac}(M/M_j)$ and

ex. $\mathrm{Fac}(\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}) = \oplus\{1, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}, 2, \begin{smallmatrix}2\\3\end{smallmatrix}\}$.

a minimal left $\mathrm{add}(M/M_j)$-approximation π with an exact sequence

$$M_j \xrightarrow{\pi} Z \longrightarrow \mathrm{Coker}\,\pi \longrightarrow 0$$

- (minimality)

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \\ & \searrow^{\pi} & \downarrow h \end{array} \quad \Rightarrow h = id.$$

- (approximation)

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Z \\ & \searrow^{\forall f} & \downarrow \exists g \\ & & Y \end{array} \quad \text{s.t. } g \circ \pi = f.$$

Then, $\mu_j^-(M) = (M/M_j) \oplus \mathrm{coker}\,\pi$ is again a support τ-tilting module.

<u>Def [AIR, 2014]</u>

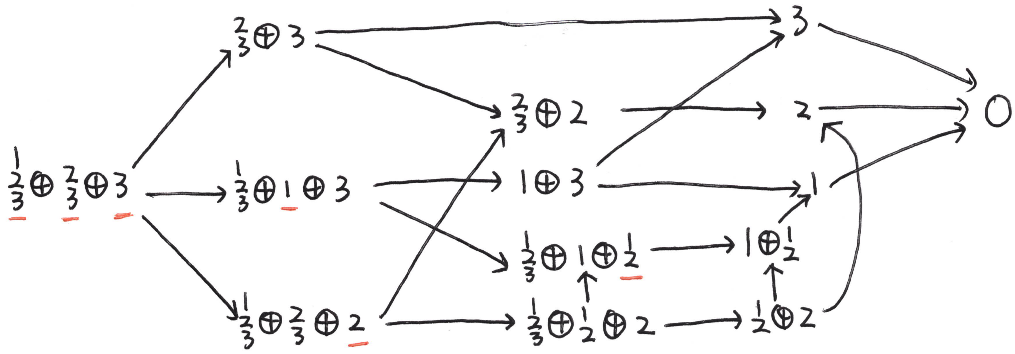
$\mu_j^-(M)$ is the <u>left</u> mutation of M.
right.

We draw an arrow $M \longrightarrow \mu_j^-(M)$. This gives a mutation graph.

<u>Prop [AIR, 2014]</u>

If the mutation graph of A contains a finite connected component Δ, then Δ exhausts all support τ-tilting modules.

<u>Ex</u> $A = k(1 \longrightarrow 2 \longrightarrow 3)$



<u>Remark</u>: $i\tau$-rigid A is easy to find.

An A-module M is called a brick if $\mathrm{End}_A(M) = k$.

Then, A is said to be brick finite if #bricks is finite.

- brick A: the set of bricks
- fbrick A: the set of bricks whose smallest torsion class $T(M)$ is functoriallay fin

<u>Theorem [Demonet — Iyama — Jasso, 2016]</u>

There is a bijection

$$i\tau\text{-rigid } A \longrightarrow \text{fbrick } A$$
$$X \longmapsto X/\mathrm{rad}_B X$$

Where $B := \mathrm{End}_A X$. If $i\tau$-rigid A is finite, fbrick$A$ = brick$A$.

<u>Remark</u>: τ-rigid finite $\Leftrightarrow$ τ-tilting finite $\Leftrightarrow$ brick finite.

<u>Ex</u> $A = k(1 \longrightarrow 2 \longrightarrow 3)$

$$\#\text{brick}A = \#\, i\tau\text{-rigid } A = 6$$

$$\text{brick}A = i\tau\text{-rigid } A = \{ \tfrac{1}{\tfrac{2}{3}}, \tfrac{2}{3}, \tfrac{1}{2}, 1, 2, 3 \}.$$

<u>Prop [DIJ, 2016]</u>

If A is brick-finite, then any A/I and eAe are brick-finite.

Prop The path alg's of extended Dynkin type are brick-infinite.

proof. Let $A=kQ$ with Q being of type $\tilde{A}, \tilde{D}, \tilde{E}$. Then, A admits a preprojective component Δ in its AR-quiver, and Δ contains infinitely many vertices.

Since each vertex in Δ gives a brick, we conclude that A is brick-infinite. $\square$

In this talk, we will use

$$
\begin{tikzcd}
\circ \arrow[r] \arrow[d] & \circ \\
\circ & \circ \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \circ \arrow[d] & \\
\circ \arrow[r] & \circ & \circ \arrow[l] \\
 & \circ \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\circ \arrow[r] & \circ & \circ \arrow[l] \\
\circ & \circ \arrow[l] \arrow[r] \arrow[u] & \circ
\end{tikzcd}
$$

§2 classical schur alg.

Def For any $n \geq 2$, $r \geq 2$,

$$S(n,r) := \mathrm{End}_{kG_r}\left(\bigoplus_{\lambda \in \Omega(n,r)} n_\lambda M^\lambda\right)$$

- G_r: the symmetric group on r symbols.
- $\Omega(n,r)$: the set of partitions of r with at most n parts.
- M^λ: the permutation module of kG_r.
- n_λ: the number of compositions of r which is rearrangement of λ.

Remark If $n \geq r$, then

$$kG_r \simeq \mathrm{End}_{S(n,r)}\left(\bigoplus_{\lambda \in \Omega(n,r)} n_\lambda M^\lambda\right)$$

Let S^λ be the specht module of kG_r w.r.t. λ.

- If char $k=0$, $\{S^\lambda \mid \lambda$: partition of $r\}$ is the set of simples.
- If char $k=p>0$, $\{D^\lambda \mid \lambda$: p-regular partition of $r\}$, where $D^\lambda = S^\lambda/(\mathrm{rad}\, S^\lambda)$, is the set of simples.

Def Let $M^\lambda = \bigoplus_{i=1} Y_i$. There is a unique i such that $S^\lambda \subseteq Y_i$, and this Y_i is unique. We call Y_i the Young module for λ, and denote it by Y^λ.

Prop [Martin, 1993]

The set $\{Y^\lambda \mid \lambda \in \Lambda(n,r)\}$ is a complete set of indecomposable direct summand of M^λ, for all $\lambda \in \Lambda(n,r)$.

We have that the basic alg of $S(n,r)$ is given as

$$\overline{S(n,r)} \simeq \mathrm{End}_{kG_r}\Big(\bigoplus_{\lambda \in \Lambda(n,r)} Y^\lambda\Big)$$

To study the quiver with relations of $\overline{S(n,r)}$, we only need to know

- the filtration multiplicities $[Y^\lambda ; S^\mu]$
- the decomposition matrix $[S^\lambda : D^\mu]$.

Ex $n=2, r=11 \overset{p=2}{\vee}$. kG_{11} has only two blocks, and the decomposition matrices are

$$\begin{array}{c} (11) \\ (9,2) \\ (7,4) \end{array} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & 0 & 1 \end{pmatrix} \qquad \begin{array}{c} (10,1) \\ (8,3) \\ (6,5) \end{array} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{pmatrix}$$

principal block B_1 $\qquad$ 2-core $(2,1)$ B_2

This gives

$S^{(10,1)} = D^{(10,1)}$

$S^{(8,3)} = \begin{array}{c} D^{(8,3)} \\ D^{(10,1)} \end{array}$

$S^{(6,5)} = \begin{array}{c} D^{(6,5)} \\ D^{(8,3)} \end{array}$

By [Henke, 1999], whe have

$$[Y^\lambda : S^\mu] = \begin{array}{c} (10,1) \\ (8,3) \\ (6,5) \end{array} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{pmatrix}$$

Then, $Y^{(10,1)} = S^{(10,1)} = D^{(10,1)}$

$$Y^{(8,3)} = \begin{array}{cl} \boxed{D^{(10,1)}} & = S^{(10,1)} \\ \boxed{\begin{array}{c} D^{(8,3)} \\ D^{(10,1)} \end{array}} & = S^{(8,3)} \end{array}$$

$$
Y^{(6.5)} = \begin{tikzcd}[column sep=small]
 & D^{(8.3)} \arrow[dl, dash] \arrow[dr, dash] & & D^{(10.1)} \arrow[dl, dash] \\
D^{(10.1)} & & D^{(6.5)} \arrow[r, dash] & D^{(8.3)}
\end{tikzcd}
$$

$= S^{(8.3)}$, $= S^{(10.1)}$, $= S^{(6.5)}$

By direct calculation, $\mathrm{End}_{kG_{11}}(Y^{(10.1)} \oplus Y^{(8.3)} \oplus Y^{(6.5)})$ is isomorphic to

$$
\begin{tikzcd}
(10,1) \arrow[r, shift left, "\alpha_1"] & (6.5) \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & (8.3) \arrow[l, shift left, "\beta_2"]
\end{tikzcd}
\Big/ \begin{array}{l} \alpha_1\beta_1,\ \beta_2\alpha_2, \\ \alpha_1\alpha_2\beta_2,\ \alpha_2\beta_2\beta_1. \end{array}
$$

similarly. $\mathrm{End}_{kG_{11}}(Y^{(11)} \oplus Y^{(9.2)} \oplus Y^{(7.4)})$ is isomorphic to

$$
\begin{tikzcd}
(11) \arrow[r, shift left, "\alpha_1"] & (7.4) \arrow[l, shift left, "\beta_1"]
\end{tikzcd}
\quad (9.2) \Big/ \alpha_1\beta_1
$$

Ex ① $p=2$. the quiver of $\overline{S(2.10)}$ is displayed as

$$
\begin{tikzcd}
 & & (6.4) \arrow[r, shift left] \arrow[d, shift left] & (10) \arrow[l, shift left] \arrow[d, shift left] \\
(8.2) \arrow[r, shift left] & (7.3) \arrow[l, shift left] \arrow[r, shift left] & (5^2) \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & (9.1) \arrow[l, shift left] \arrow[u, shift left]
\end{tikzcd}
$$

② $p=3$. the quiver of $\overline{S(3.10)}$ is displayed as

$$
\begin{tikzcd}
\circ \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift left] & \circ \arrow[l, shift left] \\
\circ \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[r, shift left] \arrow[u, shift left] & \circ \arrow[l, shift left]
\end{tikzcd}
$$

Lemma If $S(n.r)$ is brick-infinite, then so are $S(n.n+r)$ and $S(N,r)$, for $N>n$.

Theorem [W. 2020]

A complete list of minimal brick infinite Schur alg's is given as

(1) $p=2$, $n=2$, $r=10, 21$.
$n=3$. $r=6.7.8$.
$n=4$. $r=4$
$n=5$. $r=5$.

(2) $p=3$. $n=2$. $r=12.13$
$n=3$. $r=6,10.11$.
$n=4$. $r=7.8$.

(3) $p\geq 5$. $n=2$. $r=p^2+p$, p^2+p+1
$n=3$. $r=2p$. $2p+1$. $2p+2$.

Remark: $p=3$. $S(n.r)$ is brick infinite
if and only if it is wild