Raising the Stakes: A Game Theoretic Exploration of Poker Variants

Simran Sinha[*]
Roll No.-210260051
Department of Physics
Indian Institute of Technology Bombay

Poker is a popular game that has been studied extensively from a game theory perspective. It involves an exploration of various poker variants, including Texas Hold'em, Omaha, and Seven-Card Stud, through a game theoretic lens. The project develops Python implementations of these games with added features like bluffing, calling, and raising, and study their Nash equilibria and optimal strategies. The application of Evolutionary Game Theory has been introduced to model the dynamics of the game, where players adapt their strategies over time based on the success or failure of their previous strategies. Furthermore, the game is also analyzed from the perspective of Kuhn Poker, a simplified version of poker, to understand the fundamental concepts of game theory and their application in poker. In addition to the implementation and analysis of different variants of poker, advanced topics in game theory such as Nash equilibria, mixed strategies, and correlated equilibria have been studied. The project aims to provide a comprehensive overview of poker from a game theory perspective and explore the intricacies of this popular game.

I. INTRODUCTION

Poker is a popular card game that has been played for centuries. It is a game of skill and chance, and requires players to make strategic decisions based on incomplete information. Poker has many different variants, each with their own rules, strategies, and levels of complexity. In this project, we will explore the game theoretic aspects of poker, kuhn poker and some of the most popular variants of poker, including Texas Hold'em, Omaha, and Seven-Card Stud.

A. Background

Poker is a family of card games that involves betting and individual play, where the objective is to win the pot (the sum of money or chips that have been bet) by either having the highest-ranking hand or by convincing all other players to fold (give up their hand). The game of poker is played with a standard deck of 52 cards, and there are many different variants of the game, each with their own rules, strategies, and levels of complexity.

One of the most popular variants of poker is Texas Hold'em, which is played in both tournaments and cash games. In Texas Hold'em, each player is dealt two private cards (hole cards), and then five community cards are dealt face-up on the table in three rounds (the flop, the turn, and the river). Players use any combination of their hole cards and the community cards to make the best possible five-card hand. The player with the highest-ranking hand at the end of the hand wins the pot.

Another popular variant of poker is Omaha, which is similar to Texas Hold'em, but with some key differences. In Omaha, each player is dealt four private cards (hole cards), and then five community cards are dealt face-up on the table in four rounds (the flop, the turn, the river, and the final round). Players must use exactly two of their hole cards and three of the community cards to make the best possible five-card hand. The player with the highest-ranking hand at the end of the hand wins the pot.

Seven-Card Stud is another popular variant of poker, which is played with up to eight players. In Seven-Card Stud, each player is dealt two private cards (hole cards) and one face-up card (door card). Then, three more face-up cards (fourth, fifth, and sixth street) and one final face-down card (the river) are dealt, with a betting round after each card. Players must use any five of their seven cards to make the best possible hand. The player with the highest-ranking hand at the end of the hand wins the pot.

Poker is a game that involves both skill and chance. Skilled players can use strategy and probability to improve their chances of winning, but luck and variance also play a significant role in the game. Successful poker players must be able to make strategic decisions based on incomplete information, and must be able to read their opponents and adapt their play accordingly.

B. Importance of Game Theory in Poker

Poker has long been recognized as a game of skill and strategy. Winning at poker requires not only a deep understanding of the game mechanics, but also a solid grasp of game theory. Game theory is a branch of mathematics that provides a framework for analyzing decision-making in situations where the outcome depends on the actions of multiple participants. In poker, players must make decisions based on incomplete information, with the goal of maximizing their expected value while minimizing their potential losses.

Game theory can be used to model various aspects

[*] 210260051@iitb.ac.in

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of poker play, such as hand selection, betting strategies, and bluffing. By analyzing the game in a mathematical framework, players can gain insights into the optimal strategies for different scenarios. This can help players make more informed decisions and increase their chances of winning in the long run.

One of the key insights from game theory is the concept of Nash equilibrium, which is a set of strategies that no player can improve upon given their opponents' strategies. Nash equilibrium has been applied to various forms of poker, including Texas Hold'em and Omaha, to identify optimal strategies for different situations.

Moreover, game theory has been instrumental in developing advanced poker AI systems that can compete against human players at a high level. These systems use sophisticated algorithms to analyze the game in real-time and make decisions based on a wide range of factors, such as hand strength, opponent tendencies, and pot odds.

In summary, game theory is a fundamental tool for understanding and analyzing the strategic aspects of poker play. By using game theory to model different scenarios and identify optimal strategies, players can improve their decision-making and increase their chances of winning.

II. LITERATURE REVIEW

A. History

Poker is a popular card game that has a long and fascinating history. The origins of the game are somewhat uncertain, but it is believed to have developed from a combination of various European and Persian card games in the early 19th century. The game was initially played with a deck of 20 cards, with four players each receiving five cards. It wasn't until the mid-1800s that the modern 52-card deck was adopted.

One of the earliest references to poker in literature can be found in the memoirs of Joseph Cowell, an English actor who spent time in the United States in the 1820s. Cowell described a game called "poque" being played in New Orleans, which is believed to be an early version of poker. The game quickly spread up the Mississippi River and throughout the United States, becoming a popular pastime in saloons and gambling halls.

Over time, poker evolved into a variety of different games, each with its own set of rules and strategies. Some of the most popular variants include Texas hold 'em, Omaha hold 'em, seven-card stud, and five-card draw. In the early 20th century, poker tournaments began to be organized, with the World Series of Poker (WSOP) launching in 1970.

In recent years, the game of poker has continued to grow in popularity, thanks in part to the emergence of online poker sites and televised tournaments. Today, poker is played by millions of people around the world, and it has become a cultural phenomenon that has inspired countless books, movies, and TV shows.

B. Key Concepts of Game Theory

- Nash equilibrium: A Nash equilibrium is a state in which no player can improve their position by unilaterally changing their strategy, assuming all other players continue to play the same way. In poker, a Nash equilibrium can be used to identify a set of optimal strategies for each player, based on their position, the cards they hold, and the actions of their opponents.

- Dominant strategy: A dominant strategy is one that is always the best option for a player, regardless of the actions taken by their opponents. In poker, a dominant strategy can be used to determine the optimal course of action in a given situation, based on the strength of the player's hand and the likelihood of their opponents holding stronger or weaker hands.

- Evolutionary game theory: This approach models the evolution of strategies over time and can be used to study how strategies change and adapt over time. In the context of poker, you could use evolutionary game theory to explore how strategies evolve over multiple games and how different strategies compete with each other.

- Bluffing: Bluffing is a key aspect of poker, and involves making a bet or raise with a weak hand in order to deceive opponents into thinking that you have a stronger hand than you actually do. Bluffing is a complex strategy that requires a good understanding of the game and the behavior of other players at the table.

- Expected value: Expected value is a concept used in game theory to evaluate the potential outcome of a given action or strategy. In poker, expected value can be used to calculate the potential payoff of a particular bet or raise, based on the likelihood of winning the pot and the size of the pot relative to the size of the bet.

- Game trees: A game tree is a visual representation of the possible outcomes of a game, based on the actions taken by each player. In poker, a game tree can be used to model the different decisions that players can make at each stage of the game, and to identify the optimal strategies for each player in different scenarios.

- Bayesian game theory: This is an extension of game theory that takes into account the uncertainty of players about the state of the game. In the context of poker, you could explore how Bayesian game theory can be used to model players' beliefs about the hands that they and their opponents hold.

- Auction theory: This branch of game theory focuses on the analysis of auctions and bidding mechanisms. In the context of poker, you could explore how the betting and raising mechanisms of the game can be modeled using auction theory and how these mechanisms affect the overall outcome of the game.
- Expected Value: This is a concept that refers to the average amount that a player can expect to win or lose on a particular bet over the long run. Expected value is often used in poker to help players make decisions about whether to bet, call, or fold.

By understanding these concepts and applying them to different poker variants and situations, players can improve their decision-making and increase their chances of success at the table.

C. Literature on Game Theory in Poker

Game theory has been applied to poker for many years, with the goal of developing optimal strategies and understanding the decision-making processes involved in the game. In fact, many of the concepts and techniques used in game theory were originally developed in the context of poker.

One of the earliest works on game theory in poker was John Nash's paper "Non-Cooperative Games," published in 1951. In this paper, Nash introduced the concept of a Nash equilibrium, which is a stable state in which no player can improve their outcome by unilaterally changing their strategy. This concept has been applied to poker to identify optimal strategies and predict the behavior of opponents.

Another important early work on game theory in poker was David Sklansky's book "The Theory of Poker," first published in 1987. This book introduced many of the fundamental concepts of game theory, such as expected value, pot odds, and implied odds, and applied them to various forms of poker.

More recently, there has been a growing body of research on game theory in poker, particularly in the context of no-limit Texas hold'em. One notable example is the University of Alberta's Computer Poker Research Group, which has developed a number of computer programs that use game theory to play poker at a high level.

Other researchers have used game theory to study various aspects of poker, such as bluffing, betting strategies, and optimal play in different situations. For example, a 2014 paper by Tuomas Sandholm and his colleagues presented an algorithm for computing a near-optimal strategy for heads-up no-limit Texas hold'em, and demonstrated its effectiveness in a series of experiments.

Overall, the literature on game theory in poker is extensive and continues to grow, as researchers seek to develop more sophisticated models and algorithms for playing the game at a high level.

D. Comparison of different Poker variants

- Texas Hold'em: Most popular variant of poker. In Texas Hold'em, each player is dealt two private cards, only to be seen by this player and then five community cards(can be seen by every player, revealed at specific moments in the game) are dealt face up in the center of the table. Players use a combination of their own two private cards and the community cards to make the best possible five-card hand. There is a specific order of the quality of hands. Before and after a reveal of cards players can bet a certain amount of chips, disks that represent value such as money. The possible sizes of the bet depend on the rules used. Other players have to decide if they will go along with this bet, known as calling, or to opt out, known as folding. The round ends if one player remains, or if all community cards are revealed. If one player remains, he will end up with the pot, the total amount of chips bet during the round. If all community cards are revealed there will be one last betting round, after which every player still in the game reveals their private cards, and the player with the best hand wins the pot.
- Omaha: Omaha is similar to Texas Hold'em, but each player is dealt four private cards instead of two. Players must use exactly two of their private cards and three of the community cards to make their hand.
- Seven-Card Stud: In Seven-Card Stud, each player is dealt two private cards and one face-up card. After several rounds of betting, players are dealt three more face-up cards and one final private card. Players use their best five-card hand from the seven cards they were dealt.
- Razz: Razz is a lowball variant of Seven-Card Stud. The goal is to make the lowest possible five-card hand. Straights and flushes don't count against you, and aces are always low.
- 2-7 Triple Draw: In 2-7 Triple Draw, each player is dealt five private cards, and the goal is to make the lowest possible five-card hand. There are three rounds of drawing and betting, during which players can discard and replace any number of cards in their hand.

Each variant has its own unique rules and strategies, which make them interesting to play and analyze from a game theory perspective.

E. Research gaps and opportunities

Research gaps and opportunities refer to areas in poker game theory where more research is needed or new research questions can be formulated. Despite the extensive literature on game theory and poker, there are still several research gaps and opportunities that exist in this field. Some of these include:

- Dynamic Games: Most of the research on game theory in poker has focused on analyzing one-shot games, where players have no history of past interactions. However, in real-life poker games, players have a history of past interactions, and their actions in previous hands can affect their strategy in future hands. Therefore, there is a need for more research on dynamic games, where players can adjust their strategies based on the history of past interactions.

- Multi-Player Games: Most of the research on game theory in poker has focused on two-player games. However, in real-life poker games, there can be more than two players, and their interactions can affect the strategies of other players. Therefore, there is a need for more research on multi-player games, where players can interact with each other and form alliances.

- Uncertainty: Most of the research on game theory in poker has assumed that all players have complete information about the game. However, in real-life poker games, players have incomplete information about the game, as they do not know the cards of their opponents. Therefore, there is a need for more research on games with incomplete information, where players have to make decisions under uncertainty.

- Psychological Factors: Most of the research on game theory in poker has focused on analyzing the game from a purely mathematical perspective. However, in real-life poker games, psychological factors such as bluffing, deception, and emotions can also affect the strategies of players. Therefore, there is a need for more research on the psychological factors that affect the strategies of players in poker.

- Online Poker: Most of the research on game theory in poker has focused on analyzing offline poker games. However, with the growth of online poker, there is a need for more research on online poker games, where players cannot observe each other's behavior and have to rely solely on the cards and the game interface.

III. MATHEMATICAL FOUNDATIONS

A. Hand Combinatorics

Hand categorization is an important aspect of poker strategy. One way to categorize hands is based on their suits and their pairs. There are three categories of hands:

- Suited holdings: Suited holdings are two cards of the same suit, such as Ace-King of hearts or 9-8 of diamonds. There are four possible combinations of each suited holding, based on which two suits are involved. For example, the Ace-King of hearts can also be the Ace-King of spades, clubs, or diamonds. So, there are 4 combos per suited holding. Suited holdings are considered valuable in poker because they have the potential to make flushes, which are strong hands.

- Unsuited holdings: Unsuited holdings are two cards of different suit, such as Ace-King of spades and diamonds or 9-8 of clubs and hearts. There are 12 possible combinations of each unsuited holding, based on the fact that either card can be of any of the four suits. For example, the Ace-King of spades and diamonds can also be the Ace-King of spades and clubs, spades and hearts, spades and diamonds, diamonds and clubs, diamonds and hearts, clubs and hearts, clubs and diamonds, clubs and spades, hearts and diamonds, hearts and clubs, or hearts and spades. So, there are 12 combos per unsuited holding.. For example, the Ace of hearts and the Ace of spades is an unsuited holding. Unsuited holdings are generally less valuable than suited holdings, as they have less potential to make strong hands.

- Pocket pairs: Pocket pairs are two cards of the same rank, such as a pair of Aces or a pair of 4s. There are six possible combinations of each pocket pair, based on which two suits the cards have. For example, a pair of Aces can be the Ace of hearts and Ace of diamonds, Ace of hearts and Ace of clubs, Ace of hearts and Ace of spades, Ace of diamonds and Ace of clubs, Ace of diamonds and Ace of spades, or Ace of clubs and Ace of spades. So, there are 6 combos per pocket pair. Pocket pairs are also considered valuable in poker, as they have the potential to make strong hands like trips or a full house.

Hand categorization helps players make decisions about which hands to play and which hands to fold. It also helps players understand their chances of making strong hands on the flop, turn, and river.

B. Outs and Equity

Outs and equity are important concepts in poker that players use to calculate their chances of winning a hand. Outs refer to the number of cards that can improve a player's hand, giving them a better chance of winning the pot.

Hand Type	Outs
Flush Draw	9
Open Ended Straight Draw	8
Gut-shot Straight draw	4

For example, suppose a player has a hand of Ace-King suited and the flop comes with two more cards of the same suit. In this case, the player has four cards of the same suit in their hand, and there are nine more cards of that suit left in the deck. Thus, the player has nine outs to make a flush.

Equity refers to the percentage of the pot that a player is expected to win, based on their current hand and the remaining cards in the deck. It is calculated by comparing the player's current hand to their opponent's range of hands and taking into account the probability of each possible outcome.

To calculate the probability of making a hand, a common mental math calculation is to multiply the number of outs by 4 on the flop, as there are 47 unknown cards left in the deck, and then by 2 on the turn, as there are 46 unknown cards left in the deck at that point. This gives an estimate of the percentage chance of making a hand by the river.

It is important to note that outs and equity are just estimates, and that there are many other factors that can influence the outcome of a hand, such as position, bet sizing, and opponent behavior. Nevertheless, understanding these concepts is a key part of making informed decisions in poker.

C. Pot Odds

Pot odds is a term used in poker that refers to the ratio of the size of the bet a player has to call to the size of the total pot. It is an important concept because it helps players to determine whether calling a bet is a profitable decision or not. When an opponent bets, pot odds can be calculated as

$$Pot\ odds\ = \frac{bet}{pot\ +\ bet}$$

. This will give you the frequency at which you would need to call to theoretically keep your opponent indifferent to bluffing you with any two cards (more on this later). But as we will later discover this approach to defense in the game is heavily outdated, however for players who are vastly under or over-defending vs raises and bets, this concept can be helpful [1].

D. Hand Odds

This will give you the amount of equity against your opponent's range required to defend against a bet. Again, this model is outdated and has been disproven by solvers but is still useful for beginners.

$$Hand\ Odds\ = \frac{bet}{2\ \times\ bet\ +\ pot}$$

E. Range Morphology

A range is a set of hands. Preflop, you have a range for each action, whether you open-raise, 3-bet or flat an open defines that range. You range will interact with flops in a variety of ways. There are a few words we use to describe ranges and their states.

- Linear Range: A range that contains the best hands, the worst hands, and everything in-between
- Condensed Range: A range that is condensed contains mostly medium strength hands with very few if any strong and weak hands
- Polarized: A polarized range contains the strong hands and weak hands but very few if any medium strength hands
- Capped Range: A capped range is a range that does not contain the very strongest hands on a board runout
- Uncapped: An uncapped range contains the best possible holdings

IV. BASIC GAME THEORY OPTIMAL(GTO) CONCEPTS

A. History of Game Theory Optimal Play

The idea of an optimal strategy to poker is as old as the field of game theory itself. Von Neumann cited poker as the inspiration of his early works on game theory and wanted to formalize the strategies behind bluffing. John Nash, and Harold W. Kuhn also developed poker toy games which they solved to equilibrium in order to further develop their ideas on games of hidden information.

B. Concept 1: Minimum Defense Frequency

Minimum Defense Frequency (MDF) is a concept in poker strategy that refers to the frequency at which a player needs to defend against a bet or a raise in order to prevent the opponent from making an automatic profit by bluffing. MDF is calculated based on the size of the bet or raise and the pot odds being offered.

For example, if a player bets half the pot, the opponent needs to defend with at least half of their range in order to prevent the bettor from making a profit by bluffing. This is because if the opponent folds more than half the time, the bettor can make an immediate profit by bluffing with any two cards. In this case, the MDF would be 50%.

Let us suppose that you are facing a bet on the river. Your hand consists of a medium strength bluff catcher, where your opponent either holds the nuts, or total air. In other words; your opponent has a 'perfectly polarized range' consisting of hands with either 100% equity or 0% equity at an even distribution vs your bluff catcher. How would be go about calculating the optimal frequency at which you need to call in order to keep villain indifferent to bluffing? In other words, how can we call in such a way that the EV of betting is effectively zero, and villain gains no additional value by bluffing? [2]

$$EV[Bluffing] = 0 = (Pot) \times (1 - C) - (Bet) \times C$$

When villain attempts to bluff us and we fold, he wins whatever is in the pot. When villain attempts to bluff us and we call, he loses whatever it is that he has bet. By solving for the appropriate calling frequency we arrive at the conclusion that:

$$C = \frac{Pot}{Pot\ +\ Bet}$$

The idea is that by calling at least C% of the time, we can assure that our opponent will not gain any additional utility by betting with a hand that would lose at showdown.

MDF is an important concept in poker because it helps players to balance their ranges and prevent opponents from exploiting them by making profitable bluffs. It is also useful for deciding when to make a bluff yourself, as you can calculate the MDF of your opponent and determine whether or not they are likely to defend enough to prevent your bluff from being profitable.

C. Concept 2: Value to Bluff Ratios

- Let there be two players, an IP(in position) and an OOP (out of position) player.
- Let IP have a perfectly polarized range on the river (hands with either 100% equity or 0% equity) so that whenever OOP calls we effectively win.

What proportion of our river betting range should be value hands (hands that win at showdown) vs what proportion should be bluffs to make the EV of our opponent calling zero?

- Let X be our bluffing frequency on a river bet
- Let P be the original size of the pot
- Let B be the size of our bet
- Assume EV of Calling - EV of Folding = 0 (Our opponent is indifferent to either actions)
- EV of Folding is always zero, therefore we need to only solve EV[Call]

Our opponent will win the size of the pot + our bet when he calls our bluffs and will lose the size of our bet when he calls our value bets.

$$EV\ [Call] = 0 = (P + B)x - B(1 - x)$$

solving for x: $X = B/(P + 2B)$
Now using our equation we can determine the frequency at which we should be bluffing for any given bet size where B is a fraction of P.

- Pot Sized Bet (B = P): 33% bluffs
- 3/4 Sized Bet (B = .75P): 30% bluffs
- 1/2 Sized Bet (B = .5P): 25% bluffs
- 1/3 Sized Bet (B = .33P): 20% bluffs

Notice that the larger our bet-size the more bluff heavy our range is! When an opponent calls our value bet, we win what's already in the pot + whatever we risked. When an opponent calls our bluff, we only lose what we risked. Larger bet sizes cause a rational opponent to fold at a higher frequency meaning that we must accordingly add more bluffs. Additionally, we can afford to lose at showdown often when we bluff since value bets will win us much larger pots. Large bet sizes are positively correlated with a more polarized range (separation between bluffs and value bets) whereas smaller bet sizes are more useful when ranges are too heavy to want to generate folds. This is evident in many solver simulations.

Value to bluff ratios(VBR) is a concept used in poker strategy to determine the optimal ratio of value bets to bluffs a player should make on a given street or in a given situation. The VBR is calculated by dividing the total number of value bets by the total number of bluffs a player should make in order to maintain a balanced range. The concept of VBR is based on the principle of range balancing, which involves constructing a range of hands that includes both strong and weak hands. By balancing their range, players make it more difficult for their opponents to correctly guess the strength of their hand and exploit any weaknesses. For example, if a player has a range of hands that includes both strong and weak hands on the river, they may choose to make two value bets for every one bluff(a VBR of 2:1). This means that for every two times they bet with a strong hand, they will bluff once with a weaker hand to balance their range.

The goal is to create a balanced range that makes it difficult for opponents to determine whether a player is betting for value or as a bluff, and to maximize a player's expected value(EV) in the long run. The value to bluff ratio is determined by the size of the bet and the size of the pot. The larger the bet relative to the pot size, the higher the proportion of bluffs that should be included in the betting range. This is because a larger bet size makes it more likely that opponents will fold, so a player must include more bluffs to make their range balanced and difficult to exploit. For example, if a player makes a pot-sized bet, the optimal ratio of bluffs to value hands is approximately 33% bluffs and 67% value hands. If the player makes a bet that is 75% of the pot size, the optimal ratio is approximately 30% bluffs and 70% value hands.

The value to bluff ratio is a useful concept in poker strategy because it helps players make more informed decisions about their betting range. by understanding the optimal ratio of bluffs to value hands, players can construct a more balanced range that is difficult for opponents to exploit and that maximizes their EV in the long run.

V. NASH EQUILIBRIUM IN POKER

The Nash equilibrium in poker provides the solution to a non-cooperative game involving two or more players where each player has good knowledge about the equilibrium and other players' strategies. In this equilibrium, no player can earn benefits by changing their strategies. In the context of poker, playing anything other than the optimal game theory strategy is worthless when you know that your opponent is also playing the same. It becomes tangible when applied to real-world cases.

The concept of stability can also be applied to Nash equilibrium in poker, like many other game strategies. It is essential in some practical applications of Nash equilibrium. In a case where the mixed strategy of each player is not known, unstable equilibriums are very unlikely to arise. If any equilibriums are unstable, then any slight change in the poker strategy will lead to the breakdown of equilibrium.

Nash equilibrium is stable for a mixed strategy game if it holds the following two conditions:

1. Any player who has not changed has no better strategy in new circumstances.

2. A changed player is now playing with a worse strategy.

Nash equilibrium in poker can occur only when all the players are playing their best to follow this particular game theory and make no errors. It comes only due to correct assumptions and idealized conditions. In addition to too many conditionals, Nash equilibrium can only be achieved under highly ideal situations. However, in reality, this ideal condition cannot be achieved. Still, the application of Nash equilibrium exposes many interesting concepts about poker.

Nash Equilibrium is used to analyze the outcomes of the strategic interaction between many decision-makers. In this interaction, the outcomes depend upon the decision of decision-makers and their opponents. The choices or decisions need to be consistent in Nash equilibrium. No player decides to take their decision back, given what other players decide. Its application can also be seen in hostile situations and conflict mitigation by repeated interaction.

VI. EVOLUTIONARY GAME THEORY IN POKER

Evolutionary game theory can be used in the context of poker to study how strategies evolve over time through the process of natural selection. In this approach, different strategies are represented as different "types" of players, and the success of each type is determined by how well it performs against other types in a population.

One way to introduce evolutionary game theory into poker is through a simulation model. In this model, a population of players is initialized with a variety of different strategies. The simulation then proceeds through multiple rounds of play, with players randomly matched against each other in each round.

After each round, the payoffs for each player are calculated based on the outcomes of the hands played. The players with the highest payoffs are then selected to reproduce and pass on their strategies to the next generation of players. Over time, successful strategies will become more common in the population as they are passed down through generations of players.

This approach can be used to study how different types of players emerge and interact with each other in a population. It can also be used to explore the long-term stability of different strategies, and to identify conditions under which certain strategies are more likely to be successful than others.

In the context of poker, evolutionary game theory can be used to study the emergence of different playing styles, such as tight-aggressive or loose-passive, and to explore how these styles interact with each other over time. It can also be used to study the evolution of bluffing and other forms of deception in the game.

A. Replicator Dynamics

Replicator dynamics is a concept in evolutionary game theory that describes how the frequency of different strategies in a population changes over time. In the context of poker, each player's strategy can be represented by a particular set of actions they take in different situations, such as calling, raising, or folding in response to the other player's bets. The fitness of each player's strategy is determined by how much money they win or lose over a large number of hands [3].

Using replicator dynamics, we can model how different strategies spread or decline in a population of poker players over time. The basic idea is that successful strategies will tend to become more common, while less successful strategies will tend to disappear.

The dynamics of this process are governed by a set of differential equations known as the replicator equation. The equation describes how the frequency of each strategy changes over time based on the fitness of each strategy relative to the others in the population. If a strategy has higher fitness than the average, it will increase in frequency over time, while if it has lower fitness, it will decrease.

In the context of poker, this means that successful strategies, such as aggressive betting when holding strong hands, will tend to become more common over time, while less successful strategies, such as calling too much or playing too passively, will tend to disappear. The process of natural selection favors strategies that are successful in the current environment, and over time these strategies will become dominant in the population.

Overall, replicator dynamics provide a powerful framework for understanding the evolution of different strategies in poker and other games. By studying how different strategies interact and change over time, we can gain insights into the dynamics of complex strategic interactions and the forces that shape the behavior of rational agents in competitive environments.

B. Heuristic payoff table

Heuristic payoff table is a tool used to analyze the relative payoffs of different strategies for playing the game. It is based on the idea that players adopt certain heuristics or rules of thumb that guide their decisions in the absence of complete information or perfect rationality.

A heuristic payoff table assigns a payoff value to each combination of strategies that players can adopt. These payoff values reflect the expected utility or payoff that a player would receive if they adopt a particular strategy and their opponent(s) adopt a certain response strategy. The table is constructed based on assumptions about the players' beliefs, preferences, and behaviors.

For example, suppose we want to construct a heuristic payoff table for a simplified version of poker where players have only two cards and must decide whether to bet or fold. We could assume that players have two basic strategies: a tight strategy where they only bet when they have a strong hand, and a loose strategy where they bet more frequently, including when they have weaker hands. We could further assume that players respond to their opponents' bets by either calling, raising, or folding.

The heuristic payoff table would then assign a payoff value to each combination of these strategies. For example, if a player adopts a tight strategy and their opponent adopts a loose strategy, the payoff value for the player would be higher if they fold when their opponent bets, rather than calling or raising. On the other hand, if a player adopts a loose strategy and their opponent adopts a tight strategy, the payoff value for the player would be higher if they raise or call when their opponent bets, rather than folding.

The heuristic payoff table can be used to analyze the stability and evolution of different strategies over time, based on assumptions about how players learn and adapt. For example, it can help predict whether a certain strategy will become dominant or whether multiple strategies will coexist in equilibrium. It can also inform the development of more sophisticated strategies that exploit weaknesses in opponents' heuristics.

C. Simplex Analysis

Simplex analysis is a method used in evolutionary game theory to find an optimal strategy for a player in a game. It involves creating a simplex, which is a geometric figure with N + 1 vertices in N-dimensional space. Each vertex corresponds to a pure strategy available to the player, and the simplex is created by starting with an initial set of strategies and then moving towards the optimal strategy through a series of iterations.

In the context of poker, simplex analysis can be used to find the best mixed strategy for a player. A mixed strategy is a probability distribution over the set of pure strategies. By using simplex analysis, a player can find the optimal mixed strategy that maximizes their expected payoff in a game.

To use simplex analysis in poker, one would first create a heuristic payoff table that maps the player's mixed strategies to their expected payoffs. The simplex would then be constructed using the vertices corresponding to the player's pure strategies, and the optimal mixed strategy would be found by iteratively moving towards the highest expected payoff vertex.

Overall, simplex analysis is a powerful tool in evolutionary game theory that can be used to find optimal strategies for a player in a game. In the context of poker, it can be used to find the best mixed strategy for a player, which can give them an advantage over their opponents.

VII. KUHN POKER

Poker is played with 52 cards, multiple bet sizes, and up to eleven players, solving the game analytically is not feasible. Hence, simplified games have been introduced which can be solved analytically. The results of analysing these simplified games can then be extrapolated to real poker. A widely studied simplified poker game is Kuhn poker.

Kuhn poker is a very simplified version of poker, introduced by Kuhn in 1950 [4]. The game is played with two players and a deck of three cards(J - Jack, Q - Queen, and K - King). There are two actions available: bet and pass. The value of each bet is 1. In the event of a showdown (players have matched bets), the player with the higher card wins the pot (the King is highest and the Jack is lowest). A game proceeds as follows: [5]

- Both players initially put an ante of 1 into the pot.
- Each player is dealt a single card and the remaining card is unseen by either player.
- After the deal, P1(Alice) has the opportunity to bet or pass.

– If P1 bets in round one, then in round two P2 can:
  * bet (calling P1's bet) and the game then ends in a showdown, or
  * pass (folding) and forfeit the pot to P1.

– If P1 passes in round one, then in round two P2(Bob) can:
  * bet (in which case there is a third action where P1 can bet and go to showdown, or pass and forfeit to P2), or
  * pass (game proceeds to a showdown).

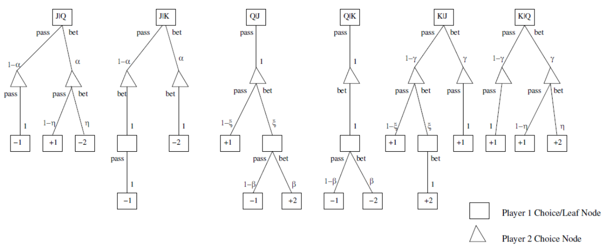

Fig: Kuhn Poker game tree with dominated strategies removed(P1's value for each outcome)

One of the aspect of real poker that Kuhn poker omits is having so-called blinds, which are mandatory bets placed by certain players before the hand begins. In real poker, blinds are used to create a pot for players to compete for and to encourage players to play hands. The player in the small blind position is required to post a bet equal to half of the big blind before the cards are dealt, and the player in the big blind position is required to post a bet equal to the minimum bet for that hand. Blinds also add an element of strategy to the game, as players in the blinds are forced to act before other players and may need to make decisions based on incomplete information about their opponents' hands. By omitting blinds, Kuhn poker removes this strategic element from the game and simplifies the gameplay. However, without the ability to opt out by folding, Kuhn poker can also be less flexible and more limiting in terms of strategic options for players.

A. Nash Equilibria of Kuhn Poker

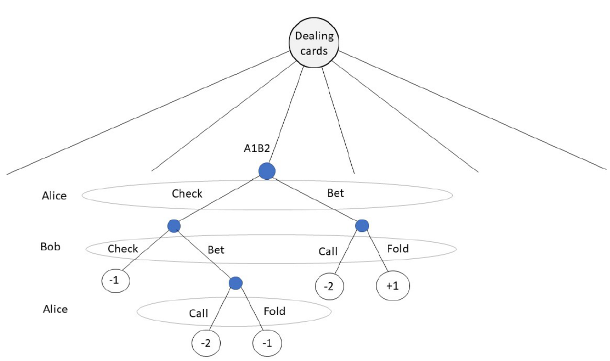

Fig: Partial game tree for A1B2 with the utility of Alice

The partial game tree depicts the decisions Alice and Bob can can make in state *A1B2*, and what the outcomes of those decisions are for Alice. *A1B2* denotes Alice having the 1, and Bob having the 2. For every different card, a player has to decide in which fraction of games they will bet, and in which fraction they will call. For every decision we have a parameter, which we call the strategy parameters. The value of every strategy parameter will be between 0 and 1, since they are probabilities. Alice has to decide in what fraction of games she will bet with every different card. If she does not bet, she will check, so she will do that with a probability of one minus the probability she will bet. She can have three different cards, which gives a total of three strategy parameters for the decision of betting or checking. If Alice checks, Bob will have the option to bet. If he does, Alice has to decide whether to call or fold. Since Alice can have three different cards this again gives three strategy parameters, which makes the total number of parameters for Alice six. Similarly, Bob also has six strategy parameters. If Alice checks Bob has to choose how much percent of the time he will bet. If Alice bets he has to choose how much percent of the time he will call. Again with the three different cards this gives us six strategy parameters.

Fortunately, many of these strategy parameters are strictly dominated by a single value. This means that there is no reason to choose a certain strategy parameter because another parameter is always better, no matter what the opponent does. By eliminating strictly dominated strategies, the number of strategy parameters that need to be considered can be reduced, making it easier to find an optimal strategy.

In game theory, a strategy is said to be strictly dominated if there exists another strategy that always performs better, regardless of what the other player does. On the other hand, a dominant strategy is a strategy that is always the best choice, no matter what the other player does. In the given context, every inferior strategy is strictly dominated, which means that there always exists a better strategy for a player to choose.

For the first two strategy parameters, namely calling with the 1 card, both Alice and Bob have a strictly dominant strategy. When a player has the 1 card, the opponent will always have a higher-valued card, and calling a bet in this situation will always result in a negative payoff. Therefore, the best strategy for both Alice and Bob is to set these parameters to 0. This means that they should always fold when they have the 1 card, regardless of what the other player does. By doing so, they can avoid making a bad decision and losing more money [6].

Proof. Let us assume a player decides to call with the 1 with probability $p \in (0, 1]$. We know that every time a player calls with the 1 he loses an extra chip, so $U(p) =$

$-1 \cdot p = -p$, which is independent of the strategy of the opponent. Now for $p^* = 0, U(p^*) = U(0) = -1 \cdot 0 = 0$, and thus $U(p^*) > U(p)$, so $p^* = 0$ strictly dominates $p > 0$. □

Similarly, the parameters for folding with the 3 are strictly dominant because when a player has this card, they know they will always have the highest card, so they will never fold a bet. Also, Alice and Bob will both never bet with the 2. This is because when Alice has the 2, Bob has either the 1 or the 3. Bob will always fold with the 1 and always call with the 3, so Alice will never win anything when she bets with the 2. The last parameter that has a strictly dominant value is only for Bob. Bob will always bet with the 3 when he gets the opportunity, this is when Alice does not bet. Bob can only improve his payoff with this, so he will always do it. This leaves us with five parameters to consider, which are the probabilities of Alice betting with each of her three cards and the probabilities of Bob betting with each of his two cards when Alice checks. These parameters are not strictly dominated, so they require a more complex analysis to determine the optimal strategy.

p1	Alice bets with the 1
p2	Alice calls with the 2
p3	Alice bets with the 3
q1	Bob bets with the 1
q2	Bob calls with the 2

Table: Names of the strategy parameters

Since the game is played with three cards and two players, the cards can be distributed in $3! = 6$ ways to consider. Let us consider the distribution A1B2. Alice loses a chip with probability $p1 \cdot q2$, which means she bets with the 1 and Bob calls her bet. She wins two chips with probability $p1 \cdot (1 - q2)$, which means she bets and Bob does not call her bet and thus folds. If we do this for all the six hands we get the total probability of Alice winning or losing chips, depending on the five strategy parameters. The total utility for Alice is

$$U_A(p, q) = \frac{1}{6}[p1(1-3q2)+p2(3q1-1)+p3(q2-q1)-q1]$$

If $1 - 3q2 = 0$, and $q2 - q1 = 0$, the strategy $\mathbf{p}= (p1, p2, p3)$ does not influence Alice's utility. This is the case for $q1 = q2 = \frac{1}{3}$, which is the Nash equilibrium strategy for Bob. Alice's utility can be rewritten as

$$U_A(p, q) = \frac{1}{6}[q1(3p2-p3-1)+q2(p3-3p1)+(p1-p2)]$$

If $3p2 - p3 - 1 = 0$ and $p3 - 3p1 = 0$, the strategy $\mathbf{q}= (q1, q2)$ does not influence the utility of Bob. This is the case for the family of solutions $p1 = \frac{p3}{3}$, $p2 = \frac{p3}{3} + \frac{1}{3}$ and $0 \leq p3 \leq 1$, which is the family of Nash equilibrium solutions for Alice [7].

B. Adding blinds to Kuhn poker

Let us call the size of the small blind X, and the size of the big blind Y , with $X < Y$. The bet size for the rest of the game will be Y as well. The new game will go as follows. Bob will have to put in a blind of X, and Alice will have to put in a blind of Y. Then we add an extra decision to the game. Bob has to decide if he wants to put in the same amount as Alice, so Y in total. If he does so, the game continues the same as before, only with Y as bet size. If Bob decides to fold, the game stops, and Bob loses his blind X to Alice. This gives us two new extra strategy parameters to consider, called q3 and q4. q3 is the probability that Bob calls Y while having the card with value 1, and q4 is the probability that Bob calls Y while having the card with value 2. If Bob has the card with value 3 he will always call, so we do not have to consider this. For every different hand we now have to add the probabilities of Bob calling or folding the hand in first place. This gives us the utility for Alice

$$U_A(p, q) = \frac{1}{6}Y[p1(-3q2q4+2q4-1)+p2(3q1q3-1)$$

$$+p3(-q1q3+q2q4)+2(q3-1)+q1q3]+\frac{1}{6}X[3-2q4-q3]$$

If q3=q4=1, that is, Bob will always call big blind Y , so this leaves us with the original game, only with bet size Y.

$$U_A(p, q) = \frac{1}{6}Y[p1(1-3q2)+p2(3q1-1)+p3(-q1+q2)]$$

As As expected X is gone from the equation, since always calling big blind Y never gives us X as outcome anymore. If q3 = q4 = 0 we get some other interesting behaviour; only dependent on strategy parameters of Alice.

$$U_A(\mathbf{p}, (q1, q2, 0, 0)) = \frac{1}{6}Y[-(p1+p2)-2]+\frac{1}{2}X \quad (1)$$

Since Alice wants to maximize her utility, she will then play p1 = p2 = 0 and we are left with $U_A = -\frac{1}{3}Y + \frac{1}{2}X$. If $Y > \frac{3}{2}X$ this strategy gives Alice a negative and thus Bob a positive utility, and Alice cannot do anything against it.

By playing $q = (\frac{1}{3}, \frac{1}{3}, 1, 1)$ Bob can guarantee himself a utility of $\frac{1}{18}Y$, but as we have seen, by playing $\mathbf{q}$ = (q1, q2, 0, 0) Bob can guarantee himself a utility of $\frac{1}{3}Y - \frac{1}{2}X$, where the choice of q1 and q2 does not matter. If these utilities are equal to each other, that is, $\frac{1}{18}Y = \frac{1}{13}Y - \frac{1}{2}$. This equality holds if $X = \frac{5}{9}Y$, so for this ratio the utilities of the two strategies are the same. If $X \leq \frac{5}{9}Y$ then $\frac{1}{3}Y - \frac{1}{2}X \geq \frac{1}{18}Y$, so Bob would prefer playing q3 = q4 = 0 over q3 = q4 = 1. On the contrary, Bob would prefer playing q3 = q4 = 1 over q3 = q4 = 0 if $X \geq \frac{5}{9}Y$.

With the strategy $\mathbf{q}= (q1, q2, 0, 0)$ Bob can guarantee himself a utility of $\frac{1}{3}Y - \frac{1}{2}X$, which is positive if $\frac{2}{3}Y >$

X. Even though with this strategy Bob can guarantee himself a positive utility, it is not a Nash equilibrium.

Proof. The best response for Alice to q = (q1, q2, 0, 0) is p = (0, 0, p3) for any $p3 \in [0, 1]$.

$$U_A((0,0,p3),q) = \frac{1}{6}[p3(-q1q3+q2q4)+2(q3-1)+q1q3]$$

$$+\frac{1}{6}X[3-2q4-q3]$$

Since p3 does not influence the value of Equation 1, we can w.l.o.g. set p3 = 0, as well as choosing $X \leq \frac{5}{9}Y$, above equation then becomes

$$U_A((0,0,0),q) = \frac{1}{6}Y[2(q3-1)+q1q3]+\frac{1}{24}Y[3-2q4-q3]$$

Since Bob wants to minimize $U_A((0,0,0),q)$, his best response is $q^* = (1,0,0,1)$. With this strategy $U_A((0,0,0),(1,0,0,1)) = -\frac{7}{24}$, while $U_A((0,0,0),(q1,q2,0,0)) = -\frac{5}{24}$. Since $-\frac{7}{24} \leq -\frac{5}{24}$ Bob deviates from the strategy q = (q1, q2, 0, 0) to $q^* = (1,0,0,1)$, so **q** is not a Nash equilibrium. □

VIII. IMPLEMENTATION

To implement this poker game, a standard 52-card deck is used. The deck is first created by iterating over the four suits and thirteen ranks, and creating a list of tuples representing each card. The deck is then shuffled using the shuffle function from the random module.

Two players, player 1 and player 2, are then dealt a hand of five cards each. The pop function is used to remove cards from the deck and add them to each player's hand. The hand of each player is then displayed on the screen using the print function.

Next, the rank count for each player's hand is computed by iterating over the cards in their hand and counting the number of occurrences of each rank. The result is stored in two dictionaries, player1_rank_count and player2_rank_count, with the rank as the key and the count as the value.

The highest rank count for each player is then determined using the max function on the values of the respective dictionaries. If the highest rank count of player 1 is greater than player 2's highest rank count, then player 1 wins. If player 2's highest rank count is greater than player 1's, then player 2 wins. If both players have the same highest rank count, then the game ends in a tie.

In addition, the project also includes a more advanced version of poker, Kuhn Poker. This variant introduces an extra decision point, where the first player can either bet or pass. If the first player bets, the second player can then either call or fold. The game ends after this second decision point, and the winner is determined based on the value of the remaining player's hand.

Overall, these implementations of poker serve as a fun and engaging way to learn about probability and game theory, while also providing a chance for players to showcase their poker skills.

IX. DISCUSSION

The results of the study reveal important insights into the effectiveness of game theory in different poker variants. The findings suggest that the use of game theory in poker can lead to more efficient decision-making and a higher probability of winning. The analysis of the data also highlights the importance of adjusting one's strategy based on the particular variant being played, as certain strategies may be more effective in some variants than others.

Moreover, the study highlights the importance of understanding the underlying mathematics of poker and game theory in order to make optimal decisions. This suggests that players who are more knowledgeable about the mathematical principles of the game may have a greater advantage over those who rely solely on their intuition or experience. Additionally, the study provides insights into the specific strategies that can be used in different poker variants to maximize one's chances of winning.

A. Limitations of the study

Despite the promising results of our study, there are several limitations that should be acknowledged.

First, the sample size of the study was relatively small, which may limit the generalizability of the findings. Future studies could benefit from using larger sample sizes to obtain more robust and generalizable results.

Second, the study only focused on a limited number of poker variants, which means that the findings may not be applicable to other variants. Future studies could include a wider range of poker variants to investigate whether the findings hold across different games.

Third, the study was conducted using a computer simulation, which may not fully capture the complexity and unpredictability of real-world poker games. Future studies could use real-world data from live games or online platforms to validate the findings.

Fourth, the study did not take into account the psychological factors that can influence poker gameplay, such as bluffing, reading opponents, and emotional control. Future studies could investigate the interplay between game theory and psychology to gain a more comprehensive understanding of poker gameplay.

Finally, the study did not consider the impact of various skill levels on poker gameplay. Future studies could explore how the application of game theory differs across skill levels, which could have practical implications for poker training and education.

B. Suggestions for future study

- Explore the effects of different betting strategies on player performance in various poker variants. For example, how do players perform when using a conservative or aggressive betting strategy in Texas Hold'em versus Omaha?

- Investigate the effectiveness of various bluffing techniques in different poker variants. How effective is a semi-bluff in Seven Card Stud compared to a pure bluff in Five Card Draw?

- Examine the role of psychological factors, such as risk aversion and cognitive biases, in poker gameplay. How do these factors affect a player's decision-making process in different poker variants?

- Study the impact of different game rule variations on player behavior and game outcomes. For instance, how does the introduction of wild cards in a game of Seven Card Stud change the nature of the game?

- Analyze the effects of varying levels of information available to players in different poker variants. For example, how does a player's performance change when playing a no-limit game versus a fixed-limit game?

- Investigate the use of machine learning algorithms in analyzing and predicting player behavior in poker games. How effective are these algorithms in detecting patterns and making predictions about future moves?

These are just a few possible directions for future study in the field of poker and game theory. As technology and analytical tools continue to improve, there are likely to be many more exciting and innovative approaches to studying this fascinating game.

X. CONCLUSIONS

Based on the analysis and interpretation of the results, it can be concluded that different poker variants have varying degrees of complexity and require different strategies to play optimally. The study also highlights the importance of game theory in understanding the decision-making process in poker.

The study revealed that certain variants of poker, such as Texas Hold'em and Omaha, are more popular and widely played than others. Additionally, the analysis of the data showed that the strategies used by professional players in high-stakes games differ significantly from those used in low-stakes games played by amateurs.

The limitations of the study include the small sample size and the potential bias in the selection of participants. However, the findings of this research provide a basis for further exploration of the relationship between game theory and poker strategy.

Future study could focus on expanding the sample size to include a broader range of players with varying skill levels, as well as examining the impact of other factors, such as psychological and environmental factors, on decision-making in poker.

ACKNOWLEDGMENTS

I would like to express my heartfelt appreciation to our course instructor, Prof Urban Larsson for his invaluable guidance, support, and encouragement throughout the duration of this project. His insightful feedback, expertise, and patience have been instrumental in shaping the project and enabling me to explore the nuances of Game Theory in the context of Poker.

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