Tutorial 1

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PH 421: Photonics
Due: 12:30 Thursday, August 10, 2023

Problem 1.1. Revision of lasers

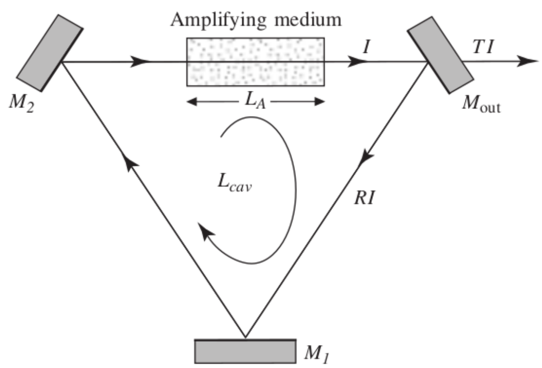


Figure 1: A schematic of a laser

Consider a ring cavity laser where the gain in intensity for a single pass across the amplifying medium is $G^{(0)} = e^{g^{(0)}L_A}$, such that $I_{\text{out}} = G^{(0)} I_{\text{in}}$. The mirrors M_1, M_2 are perfectly reflecting whereas the mirror M_{out} has a finite intensity transmission coefficient of T. Derive the condition for laser oscillation to commence.

Problem 1.2. Population inversion in two level systems Consider a two level system (levels e and g) pumped at a rate w. The lifetime of the upper state is τ_e. Write down the rate equations and derive the steady state population inversion: $(N_e - N_g)/N$. Is it feasible to use such a system for optical amplification?

Problem 1.3. Nd:YAG laser

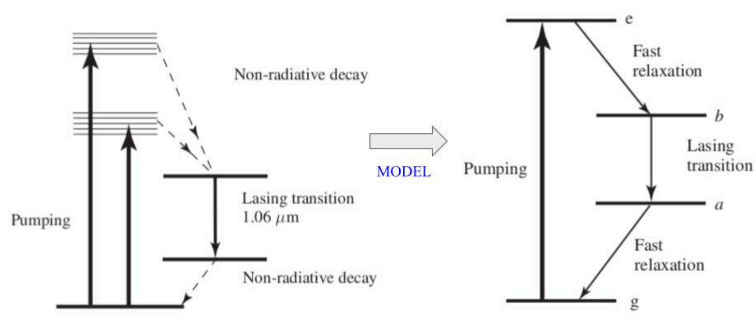


Figure 2: Left: Nd:YAG laser energy levels. The laser can be pumped optically using wavelengths 500 – 800nm. Right: A four level model of the laser. The pumping rate is w and the de-excitation times of the levels are τ_e, τ_b, τ_a related by $\tau_e, \tau_a \ll \tau_b$.

This is a common type of laser where Nd^{3+} (neodymium) ion is embedded at low concentration in glass (neodymium-doped glass) or in crystalline YAG (neodymium-doped yttrium aluminium garnet). Write down the rate equations for the populations in all the levels and under suitable approximations derive an expression for the population inversion: $(N_b - N_a)/N$ in the steady state.

Problem 1.4. Linear absorption versus nonlinearity
Linear optical absorption is a phenomenon where some of the incident photons "disappear", typically into heating of the material through phonon modes.
(a) Derive an expression for the (linear) absorption coefficient in terms of the linear optical susceptibility. *Hint:* You may use $(n + \imath\kappa)^2 = \epsilon = 1 + \chi^{(1)}$.
(b) Within the framework of the quantum mechanical picture we discussed in Lecture 2, please discuss whether high absorption in a material will have any impact on the efficiency of nonlinear optical phenomena such as second harmonic generation.

Problem 1.5. Third order susceptibility in a centrosymmetric crystal using a classical model
(a) Using a classical model of the interaction between a weak electric field and an electron, please show that the second order atomic displacement is zero.
(b) Using the same model, please provide an expression for the third order optical susceptibility.

Problem 1.6. Talking numbers: optical susceptibilities
(a) List the values (in SI units) of first order, second order and third order optical susceptibilities of common nonlinear materials.
(b) Using a classical model of the interaction between a weak electric field and an electron, please provide estimates for the second and third order optical susceptibilities and compare them with the numbers in part (a).

Problem 1.7. Gradient refractive index (GRIN) lenses
GRIN lenses are commonly used use collimate or reimage the light from a fiber. For the GRIN lens shown

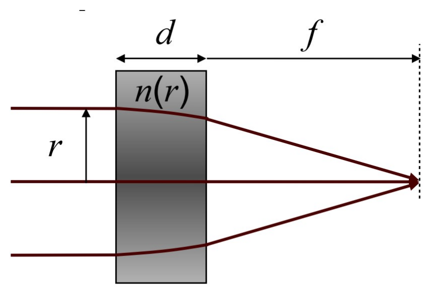


Figure 3: A GRIN lens with a parabolic refractive index profile $n(r) = n_{\max}(1 - \alpha r^2/2)$

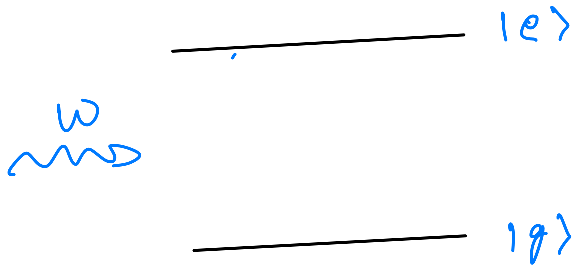
in the figure, find the focal length f in the paraxial limit. Please make reasonable approximations and state them.

Q1)

$$\underbrace{I(1-T) \times e^{g(0) \ell_A}}_{\text{Intensity after a complete loop}} \geq I$$

$$\therefore \quad e^{g(0) L_A}(1-T) \geq 1$$

Q2)



$$\frac{dN_e}{dt} = \underbrace{-\omega(N_e - N_g)}_{\text{Stimulated emission}} - \underbrace{\frac{N_e}{\tau_e}}_{\text{spontaneous emission}}$$

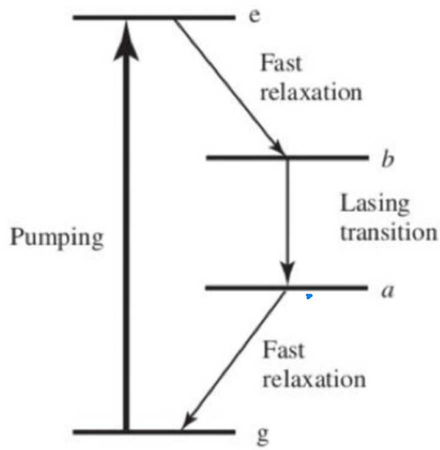
$$\frac{dN_g}{dt} = -\frac{dN_e}{dt}$$

At steady state $\frac{dN_e}{dt} = 0 \quad \therefore \; N_e - N_g = \frac{-N_e}{\omega \tau_e} < 0$

$\therefore$ It's not feasible

$\therefore$ Two state population inversion is not possible.

Q3)



$$\frac{dN_e}{dt} = -W(N_e - N_g) - \frac{N_e}{\tau_e}$$

$$\frac{dN_b}{dt} = \frac{N_e}{\tau_e} - \frac{N_b}{\tau_b}$$

$$\frac{dN_a}{dt} = \frac{N_b}{\tau_b} - \frac{N_a}{\tau_a}$$

$$\frac{dN_g}{dt} = \frac{N_a}{\tau_a} + W(N_e - N_g)$$

$$\left[\frac{dN_e}{dt} = \underbrace{-WN_e}_{\text{stimulated emission}} + \underbrace{WN_g}_{\text{stimulated absorption}} - \underbrace{\frac{N_e}{\tau_e}}_{\text{spontaneous emission}} \right]$$

At steady state, $\frac{dN_e}{dt} = \frac{dN_b}{dt} = \frac{dN_a}{dt} = \frac{dN_g}{dt} = 0$

$$\therefore N_e = \frac{\tau_e N_b}{\tau_b}, \quad N_b = \frac{\tau_b N_a}{\tau_a} \quad \ldots (i)$$

$$\Rightarrow N_e = \frac{\tau_e}{\tau_a} N_a \quad \ldots (ii)$$

$$N_g = \frac{N_e}{W\tau_e} + N_e = \left[\frac{1}{W\tau_e} + 1\right] \frac{\tau_e}{\tau_a} N_a$$

$$\frac{N_b - N_a}{N} = \frac{\left[\frac{\tau_b}{\tau_a} - 1\right] \cancel{N_a}}{\cancel{N_a}\left[1 + \frac{\tau_b}{\tau_a} + \frac{\tau_c}{\tau_a} + \frac{\tau_c}{\tau_a} + \frac{1}{\omega \tau_a}\right]}$$

$$\because \tau_b \gtrsim \tau_c, \tau_a$$

$$\approx \frac{\frac{\tau_b}{\tau_a}}{\left[\frac{\tau_b}{\tau_a} + \frac{1}{\omega \tau_a}\right]} = \frac{\tau_b}{\tau_b + \frac{1}{\omega}}$$

Q 4)

a) $E = E_0 e^{i[kx - \omega t]}$

But in different refractive index

$$k \longrightarrow \eta K$$

$$\therefore E_{med^m} = E_0 e^{i[\eta K x - \omega t]}$$

$$\eta = \eta + i\kappa$$

$$= E_0 \left[e^{i\eta K x - \omega t}\right] e^{-\kappa K x}$$

$$= \underbrace{\left(E_0 e^{-\kappa K x}\right)}_{\text{decaying amplitude}} \left(e^{i\eta K x - \omega t}\right)$$

$$\text{Absorption} = K\,K \Rightarrow K\left[\underbrace{\sqrt{1+\chi^{(1)}} - n}_{i}\right]$$

b) Nothing can be said in general. It might be possible as $K\uparrow$, dissipation also starts (apart from absorption) which in turn has unpredictable consequences of second harmonic generation.

Q^n 5)

a) Using Lorenz model,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0 x + a x^3 = \frac{\lambda \tilde{E}(t) q}{m}$$

$$x = \lambda x^{(1)} + \lambda^2 x^{(2)} + \lambda^3 x^{(3)} + \cdots$$

$$\lambda^2\left(\ddot{x}^{(2)} + 2\gamma\dot{x}^{(2)} + \omega_0 x^{(2)}\right) = 0$$

$$\Rightarrow \quad \ddot{x}^{(2)} + 2\gamma\dot{x}^{(2)} + \omega_0 x^{(2)} = 0$$

$\therefore$ Steady state $sol^n \longrightarrow 0$

$\therefore$ we take $x^{(2)} = 0$

b) We's solve for $x^{(1)}$ first

$$\ddot{x}^{(1)} + 2\gamma\dot{x}^{(1)} + \omega_0 x^{(1)} = \frac{\tilde{E}(t) q}{m}$$

$$\therefore \quad x^{(1)}(\omega_j) = \frac{E_j q}{m D(\omega_j)} \quad \left[D(\omega_j) = \omega_j^2 - \omega_0^2 - 2\gamma i \omega_j\right]$$

$\therefore$ For λ^3 term

$$\ddot{x}^{(3)} + 2\gamma\dot{x}^{(3)} + \omega_0 x^{(3)} + a[x^{(1)}]^3 = 0$$

(Rest in Boyde sec 1.4.3)

Problem 1.6)

2nd order approx^m :

$\Rightarrow$ linear and non-linear contribution becomes comparable when displacement $\tilde{x} \sim$ size of atom, i.e. of the order of lattice constant d.

$\Rightarrow$ order of magnitude gives $m\omega_0^2 d = mad^2$

$\therefore \; a = \dfrac{\omega_0^2}{d}$ [$\because$ ω_0, d are roughly same for most solids.]

Now, under highly non-resonant cond^n,

$D(\omega) \longrightarrow \omega_0^2$, $\quad N = \dfrac{1}{d^3}$, $\; a = \dfrac{\omega_0^2}{d}$

$$\chi^{(2)} = \frac{e^3}{\varepsilon_0 m^2 \omega_0^4 d^4},$$

using $\omega_0 = 10^{16}$ rad/s $\quad d = 3\,\text{Å}$, $\; e = 1.6\times10^{-19}$ C

$m = 9.1\times10^{-31}$ Kg

$$\chi^{(2)} \simeq 6.9\times10^{-12}\ \text{m/V}$$

3rd order :

Similarly for 3rd order permutation,

$m\omega_0^2 d = mbd^3 \quad \Rightarrow \quad b = \dfrac{\omega_0^2}{d^2}$

Again in highly non-resonant cond^n

[i.e $D(\omega) \simeq \omega_0^2$]

$$\chi^{(3)} \simeq \frac{N b e^4}{\varepsilon_0 m^2 \omega_0^8} = \frac{e^4}{\varepsilon_0 m^3 \omega_0^6 d^5} \simeq 344\ \mathrm{pm}^2/\mathrm{V}^2$$

[Actual
$$\chi^{(3)}_{ijkl}(\omega_q, \omega_m, \omega_n, \omega_p) = \frac{N b e^4 \delta_{jk}\delta_{il}}{\epsilon_0 m^3 D(\omega_q) D(\omega_m) D(\omega_n) D(\omega_p)}.$$
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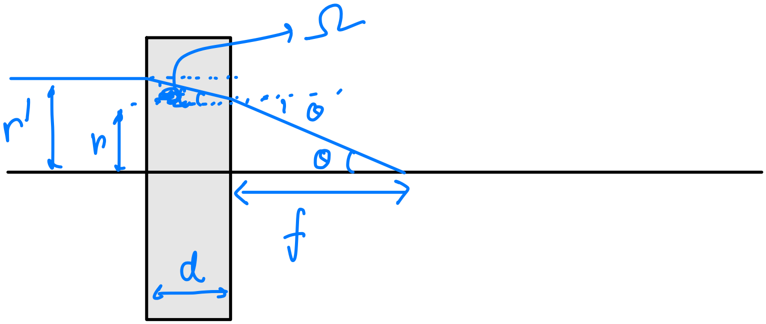
Problem 1.7



Equating optical path lengths

$$f + d n_{max} = \sqrt{f^2 + r^2} + n_{max}\left[1 - \frac{\alpha r^2}{2}\right] \underbrace{\frac{d}{\cos\Omega}}$$

second order term

∴ we can neglect.

$$f + d n_{max} \approx f\left[1 + \frac{r^2}{2f^2}\right] + n_{max} - n_{max}\frac{\alpha r^2}{2}$$

$$\therefore \quad f = \frac{1}{\alpha n_{max} d}$$