Date: / /

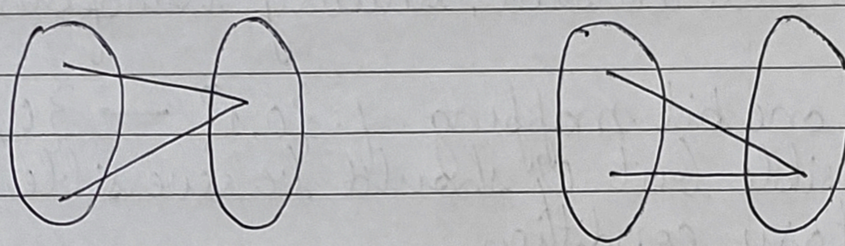
THE DEUTSCH ALGORITHM

The Deutsch problem

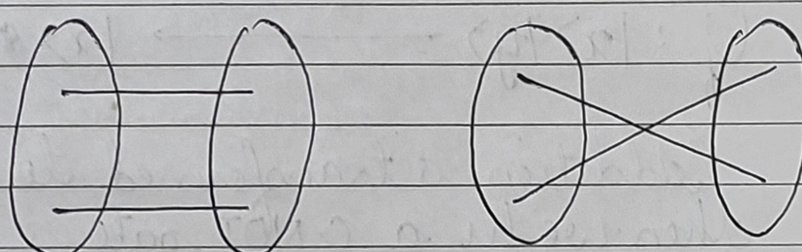
$f : \{0,1\} \longrightarrow \{0,1\}$

Decide if f is balanced or constant

f is constant



f is balanced (one-to-one correspondence)



calculate $f(0)$, $f(1)$

If they are equal, then the function must be constant, if they are unequal then the function must be balanced.

Or, $(f(0) + f(1)) \begin{cases} = 0 \text{ ; function } f \text{ must be a constant} \\ = 1 \text{ ; function } f \text{ must be balanced} \end{cases}$

$\boxed{\text{Answer}: f(0) + f(1)}$

Knowing either of them doesn't help, have to know both of them, which means, to solve this decision problem, we have to compute the function f twice. That's the complexity or the difficulty of solving the problem.

Deutsch has shown that in a quantum way, this can be

Date: / /

reduced to single.

To approach this problem in a quantum way, we have to find the quantum reallocation of the function f
We have to consider the transformation U_f; a transf. from state to state, unitary transformation

The one bit mapping $f: \{0,1\} \longrightarrow \{0,1\}$ may not be reversible but U_f should be reversible because of the unitary condition

$$U_f : |x\rangle|y\rangle \xrightarrow[\text{(C-NOT means addition)}]{\text{C-NOT gate}} |x\rangle|y+f(x)\rangle$$ {quantum counterpart}

Addition is transformed to unitary transformation then we hv a C-NOT gate

<u>Phase Kickback</u>

{$|-\rangle$ is exploited to hv a phase vector}

$$U_f|0\rangle|-\rangle = U_f \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |0\rangle|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle|f(0)\rangle - |0\rangle|1+f(0)\rangle\right)$$

$$= \begin{cases} f(0)=0 & |0\rangle|-\rangle \\ f(0)=1 & -|0\rangle|-\rangle \end{cases}$$

$|1+1\rangle = |0\rangle$

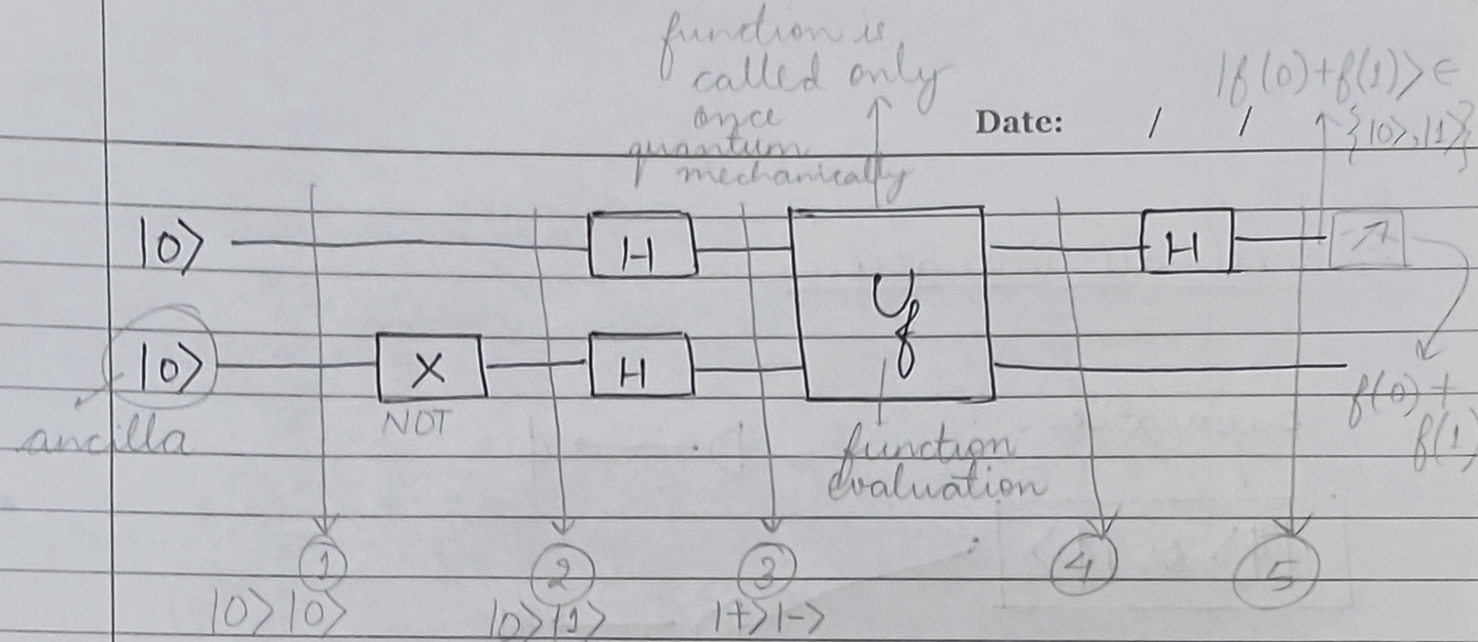
$$= (-1)^{f(0)}\,|0\rangle|-\rangle$$

qubit in the first register

$$U_f|x\rangle|-\rangle = \underbrace{(-1)}_{\text{phase vector}}{}^{f(x)}\,|x\rangle|-\rangle$$

which is $|-\rangle$

(appears due to the interaction b/w the first and second qubit)



(4.) $U_f \, |+\rangle|-\rangle = \frac{1}{\sqrt{2}}\left(U_f \, |0\rangle|-\rangle + U_f \, |1\rangle|-\rangle\right)$

$= \frac{1}{\sqrt{2}}\left((-1)^{f(0)} |0\rangle|-\rangle + (-1)^{f(1)} |1\rangle|-\rangle\right)$

$= (-1)^{f(0)} \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{f(0)+f(1)} |1\rangle\right)}|-\rangle$

$= \begin{cases} \text{if } f(0)+f(1) = 0 & |+\rangle \\ \text{if } f(0)+f(1) = 1 & |-\rangle \end{cases}$

(5) $|+\rangle \longrightarrow |0\rangle \,, \; f(0)+f(1) = 0$ $\quad f = \text{const}$

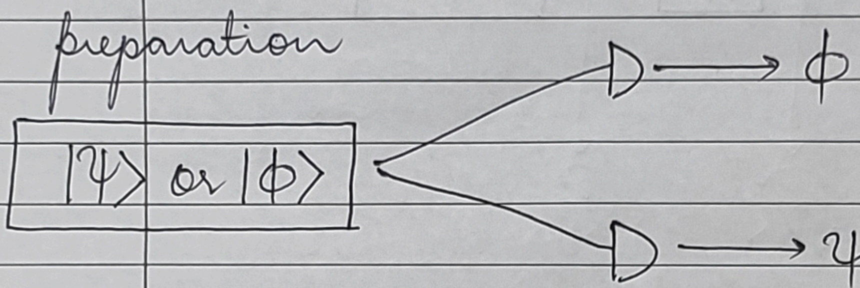
$|-\rangle \longrightarrow |1\rangle \,, \; f(0)+f(1) = 1$ $\quad f = \text{balanced}$

$U_f |x\rangle|y\rangle = |x\rangle|y+f(x)\rangle$

$U_f = \sum_{x,y} |x\rangle\langle x| \otimes |y+f(x)\rangle\langle y|$

Date: / /

quantum channel discrimination



Suppose there's a preparation that give you a state ψ or ϕ with some probability and we prepare two detectors and the detection event of first one conclude state ϕ, detection event ~~of~~ in the second detector conclude state ψ. This defines the problem of state discrimination and how difficult it is

success probability $P_{success} \le \frac{1}{2} + \frac{1}{4} \| |\psi\rangle - |\phi\rangle \|_1$

(max. possible probability in the discrimination between two quantum states)

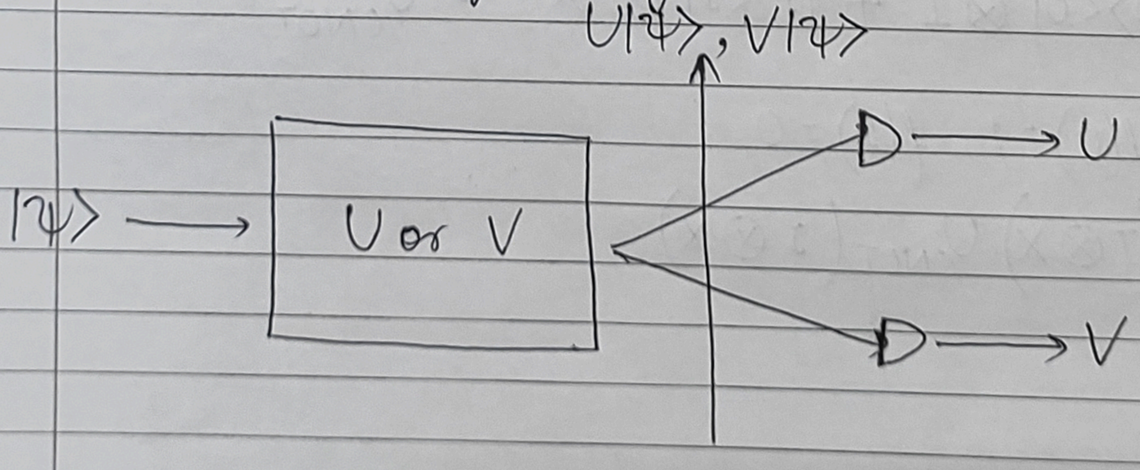
We can translate this to the scenario of discriminating between two unitary transformations

In QM, we have a state, dynamics and measurement and there are two possibilities in dynamics

Date: / /

there is a box and
Suppose, we have a state which is centered to this box and there are two possible unitary transformations U or V. Then we prepare two detectors & detector event in the first detector conclude U has been prepared in the box and detection event in the second one concludes V has been applied.
This is the scenario of discriminating between two unitary transformations.

After the transformation (i.e, the box) there are two possibilities: first possibility is $U|\psi\rangle$ or the second is $V|\psi\rangle$. (this is precisely state combinations)
So we want to discriminate between two states and we hope that these two states are possibly best differentiable.
So we are looking for the best and optimal input state s.t the two states are optimally discriminated.



maximising over all possible input states

$$\max_{|\psi\rangle} \| U|\psi\rangle - V|\psi\rangle \|_1$$

Date: / /

Goal was to find if the function is constant or balanced
Or this is equivalent to saying "discriminate between the two cases"

(1.) f = constant

$$U_f = |0\rangle\langle 0| \otimes \sum_y |y+f(0)\rangle\langle y| + |1\rangle\langle 1| \otimes \sum_y |y+f(1)\rangle\langle y|$$

$$U_f \in \{\underbrace{I \otimes I}_{\text{if } f=0},\ \underbrace{I \otimes X}_{\text{if } f=1}\}$$

X-operation (flipping)

which means U_f corresponds to local unit of transformation (LU), i.e, which cannot generate internal state

$$\begin{cases} I = |+\rangle\langle +| + |-\rangle\langle -| \\ X = |+\rangle\langle +| - |-\rangle\langle -| \end{cases}$$

(2.) f = balanced

assume: $f(0)=0$, $f(1)=1$

$$U_f = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X = U_{CNOT}$$

assume $f(0)=1$, $f(1)=0$

$$U_f = (I \otimes X)\, U_{CNOT}\, (I \otimes X)$$

when f is balanced unitaries must be entangled unitaries

Date: / /

In the context of the Deutsch algorithm, channel discrimination refers to the task of distinguishing between two different channels or paths through which a quantum state can evolve. The Deutsch algorithm uses channel discrimination to determine whether a given Boolean function is constant or balanced.

In the Deutsch algorithm, the input to the function is represented by a single qubit, and the output is also represented by a single qubit. The function is applied to the input qubit using a quantum gate, and the output qubit is then measured to determine the result of the function.

To determine whether the function is constant or balanced, the Deutsch algorithm uses channel discrimination to distinguish between two different channels through which the input qubit can evolve: the constant channel and the balanced channel. The constant channel corresponds to a function that is constant (i.e., always returns the same output), while the balanced channel corresponds to a function that is balanced (i.e., returns different outputs for different inputs).

To perform channel discrimination, the Deutsch algorithm uses a quantum circuit that includes a Hadamard gate, which puts the input qubit into a superposition of the 0 and 1 states. The function is then applied to the input qubit and the output qubit is measured. Depending on the result of the measurement, the Deutsch algorithm can

Date: / /

determine whether the function is constant or balanced.

Overall, the channel discrimination is an important concept in the Deutsch algorithm, as it follows the algorithm to distinguish between different channels through which the input qubit can evolve and to determine whether a given Boolean function is constant or balanced.