$\boxed{\text{协方差: } cov(X,Y) = (E(X) - \mu)(E(Y) - \nu)}$ 白噪声序列.

2. $X_t = Z_t + \theta_1 Z_{t-1}$ $Z_t \sim WN(0, \sigma^2)$ $t = 0, 1, 2, \ldots, n$

$X_t = \phi X_{t-1} + Y_t$ 错误拟合. 推导 $\{Y_t\}$ 的自相关函数 ACF.

① 计算 $\{X_t\}$ 的方差、协方差: 独立.

$$\gamma_X(0) = Var(X_t) = Var(Z_t + \theta_1 Z_{t-1}) = Var(Z_t) + \theta_1^2 Var(Z_{t-1})$$

$$= \sigma^2 + \theta_1^2 \sigma^2 = (1 + \theta_1^2)\sigma^2$$

$$\gamma_X(1) = cov(X_t, X_{t-1}) = cov(Z_t + \theta_1 Z_{t-1}, Z_{t-1} + \theta_1 Z_{t-2})$$

$$= cov(Z_t, Z_{t-1}) + cov(\theta_1 Z_{t-1}, Z_{t-1}) = \theta_1 Var(Z_{t-1}) = \theta_1 \sigma^2$$

当 $h > 1$ 时, $\gamma_X(h) = 0$. $\because Z_t$ 和 Z_{t-h} 对于 $h > 1$ 是独立的.

② 拟合 AR(1) 模型

$$\gamma_X(h) = \phi \gamma_X(h-1)$$

对于 $h = 1$: $\gamma_X(1) = \phi \gamma_X(0)$, 即 $\phi = \dfrac{\gamma_X(1)}{\gamma_X(0)} = \dfrac{\theta_1 \sigma^2}{(1+\theta_1^2)\sigma^2} = \dfrac{\theta_1}{1+\theta_1^2}$

③ 计算 $\{Y_t\}$ 的 ACF:

$\{Y_t\}$ 是 AR(1) 模型中的误差项.

$\{Y_t\}$ 的方差: ~~$X_t = \phi X_{t-1} + Y_t$~~, $Y_t = X_t - \phi X_{t-1}$ $\gamma_X(0) = \phi^2 \gamma_X(0) + Var(Y_t)$

$\Rightarrow (1 - \phi^2) \gamma_X(0) = Var(Y_t)$

$\Rightarrow Var(Y_t) = \left(1 - \dfrac{\theta_1^2}{(1+\theta_1^2)^2}\right)(1+\theta_1^2)\sigma^2$

$\therefore$

同乘 Y_t 取期望 $\gamma_Y(0) = EX_t^2 + \phi^2 EX_{t-1}^2 - 2\phi EX_t X_{t-1} = (1+\phi^2)(1+\theta_1^2)\sigma^2 - 2\phi\theta_1\sigma^2$

同乘 Y_{t-1} 取期望 $\gamma_Y(1) = EX_t X_{t-1} - \phi EX_t X_{t-2} - \phi EX_{t-1}^2 + \phi^2 EX_{t-1}X_{t-2}$

$$= \gamma_X(1) - \gamma_X(2) - \phi \gamma_X(0) + \phi^2 \gamma_X(1)$$

$$= \theta_1 \sigma^2 - \phi(1+\theta_1^2)\sigma^2 + \phi^2 \theta_1 \sigma^2$$

$$= (1 + \phi^2)(\theta_1 \sigma^2) - \phi(1+\theta_1^2)\sigma^2$$

同乘 Y_{t-2} 取期望 $\gamma_Y(2) = (X_t - \phi X_{t-1})(X_{t-2} - \phi X_{t-3}) = -\phi EX_{t-1}X_{t-2} = -\phi\theta_1\sigma^2$

当 $k \geq 3$ 时, $\gamma_Y(k) = 0$.