利用直接差分法分析微分方程边值问题

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目录

1 题目

$$\begin{cases} Lu = -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) + \dfrac{\mathrm{d}u}{\mathrm{d}x} + u = x\cos x + \cos x, x \in [0,\pi] \\ u(0) = 1; u(\pi) = -1 \end{cases} \tag{1.1}$$

简单推导可以得到

$$u - x\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = x\cos x + \cos x \tag{1.2}$$

2 离散形式

利用直接差分法可以得到:

$$\begin{aligned} & u_i - x_i\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = x_i\cos x_i + \cos x_i, \quad i = 1,2,\cdots,n-1 \\ & u_0 = 1, u_n = -1, h = \frac{\pi}{n} \end{aligned} \tag{2.1}$$

再整理得到

$$\begin{aligned} & -\frac{x_i}{h^2}u_{i+1} + (1+\frac{2x_i}{h^2})u_i - \frac{x_i}{h^2}u_{i-1} = x_i\cos x_i + \cos x_i, \quad i = 1,2,\cdots,n-1 \\ & u_0 = 1, u_n = -1, h = \frac{\pi}{n} \end{aligned} \tag{2.2}$$

写成矩阵形式为

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{x_1}{h^2} & (1+\frac{2x_1}{h^2}) & -\frac{x_1}{h^2} & \cdots & 0 & 0 & 0 \\ 0 & -\frac{x_2}{h^2} & (1+\frac{2x_2}{h^2}) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (1+\frac{2x_{n-2}}{h^2}) & -\frac{x_{n-1}}{h^2} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{x_{n-1}}{h^2} & (1+\frac{2x_{n-1}}{h^2}) & -\frac{x_{n-1}}{h^2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} (x_0+1)\cos x_0 \\ (x_1+1)\cos x_1 \\ (x_2+1)\cos x_2 \\ \vdots \\ (x_{n-2}+1)\cos x_{n-2} \\ (x_{n-1}+1)\cos x_{n-1} \\ (x_n+1)\cos x_n \end{bmatrix} \tag{2.3}$$

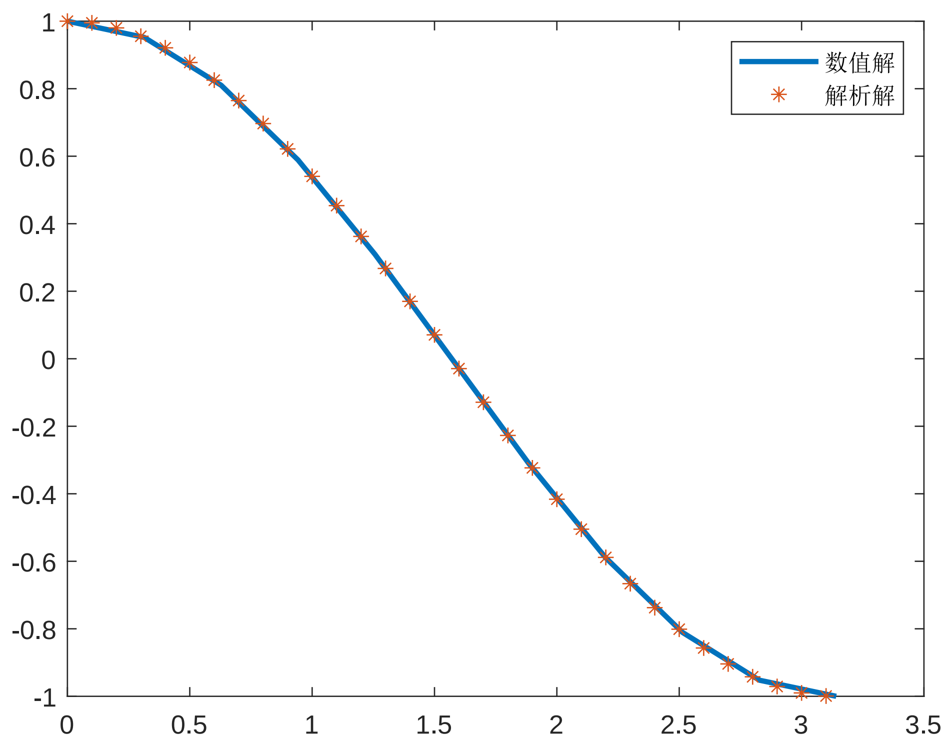
3 代码

Listing 1: **main.m**

```
clc,clear,close all
a = 0;
b = pi;
n = 10;
h = (b-a)/n;
x = a:h:b;
```

```
A1 = diag(-ones(1,n-1).*(x(2:end-1))/h^2);
A2 = diag(ones(1,n-1).*(1+2*x(2:end-1)/h^2));
A3 = A1;
Atop = [1,zeros(1,n)];
Amid = [A1,zeros(n-1,2)] + [zeros(n-1,1),A2,zeros(n-1,1)] + [zeros(n-1,2),A3];
Abottom = [zeros(1,n),1];

A = [Atop;Amid;Abottom];
b = [1,(x(2:end-1)+1).*cos(x(2:end-1)),-1]';
u = A\b;
plot(x,u,'linewidth',2)
hold on
x = a:0.1:pi;
plot(x,cos(x),'*')
legend('数值解','解析解')
```



4 误差分析

有限差分法的潜在的误差来源是中心差分公式的截断误差，以及在求解方程组时带来的误差。此处我们调用的是 MATLAB 内置的求解方程组的算法，精度较高。因此，截断误差占优，误差是 $O(h^2)$，因而我们期望随着子区间 $n+1$ 升高，误差降低为 $O(h^2)$.

我们对于此问题测试了这种方差,图显示了最大误差对于不同 n 取值对应的解的误差 E 的取值。在 log-log 图中,$\lg(E) - \lg(n)$ 是一条斜率为 -2 的直线,意味着,$\lg E \approx\ + b \lg n$, 其中 b = -2;换句话说,与我们预期一致,误差为 $E \approx Kn^{-2}$