Infinity-jam session 1

November 3, 2022

Chapter 1

It's your box!

On a piece of paper, draw three rows of boxes. Each row can contain any number of boxes. Each player in turn chooses any number of boxes from one (and only one!) of the rows to erase. The winner is the one who erases the last box. Find a partner, and play this game with them. Let n_k denote the number of boxes in row k. For what values of (n_1, n_2, n_3) does the first player (the one who makes the first move) have a winning strategy? When does the second player have a winning strategy?

Chapter 2

When one door closes...

Suppose we have 1000 doors in a row marked 1,2,3,...,999,1000. All doors are closed. Now, we pick a number n lying between 1 and 1000, and we change the condition of the doors (opening it if it is closed, and closing it if it is open), whose numbers are divisible by n. To elaborate, initially all the doors were closed. We pick 1, and as every number is divisible by 1, we open all the doors. Then we pick 2, and as only even numbers are divisible by 2, we close the even numbered doors, leaving the odd numbered doors open. We continue this process for 3,4,5,...,1000. Now the question is, how many doors are open and how many are closed at the end of this process?

Chapter 3

Red Blue Red

Suppose we colour all the integers with two colours, red and blue. Verify that regardless of how we choose the colouring, we always find three integers which are the same colour and are evenly spaced? (By “evenly spaced”, we mean that the integers should be of the form a, a+d and a+2d where d is a positive integer.

Chapter 4

Jumping Frog

1. Suppose n points are arranged in a circle, in a circular order, each the same distance from the previous one. A frog sitting on one of these points makes a jump of distance d every minute. The direction in which the frog jumps and the distance d are constant. You get to make one guess every minute where the frog is (knowing neither d, the direction, or the initial position), and you win if you can guess correctly. Can you give a strategy so that you eventually win?
2. Instead of n points in a circle, suppose there are infinite points arranged uniformly in order (think about integers on the number line). Can you still device such a strategy? What if we have points with integer co-ordinates in the whole plane or the wholespace (our naive frog still jumps same distance every minute, and in the same direction)? 3. What if we begin with the the rational numbers instead of integers on the number line? What if we have all the points on the real line?

Chapter 5

A Taste of dynamics

Definition: Given a function $f : X \to X$, the forward orbit of $x \in X$ (with respect to f) is the set of points $\{x, f(x), f(f(x)), f(f(f(x))), ...\}$ (basically iteratively applying f on x). $f^{(n)}(x)$ refers to $f(f...f(x)..)$ (taken n times).

A Guided Tour

Define $F : R \to R$ such that $F(x) = 5x(1 - x)$
(0) Can you come up with a pictorial algorithm to find the forward orbit of a point $a \in R$ (with respect to F) ? (Hint: instead of thinking about what F does to a point numerically, you might want to look at the graph of F) Find the orbits of a few points using this algorithm.

(1) Can you approximately "find" the set Λ (contained in $[0, 1]$) such that the orbit never goes out of $[0, 1]$? (What popular math object does this remind you of?)

(2) If we exclude the part of $[0, 1]$ whose image is outside of $[0, 1]$ we obtain two intervals. Call the one on the left "I_0" and the one on the right "I_1." To each point in Λ, associate a sequence of 1s and 0s such that the nth digit of the sequence corresponds to 0 if $F^{(n)}(x) \in I_0$ and 1 if $F^{(n)}(x) \in I_1$. Call the set of sequences obtained, Σ_2. Can you define a map $\sigma : \Sigma_2 \to \Sigma_2$ such that σ applied to the sequence associated with x is equal to the sequence associated with $f(x)$? Or, if $S : \Lambda \to \Sigma_2$ is the map defined (as the association of each x to the sequence of 1s and 0s), find $\sigma : \Sigma_2 \to \Sigma_2$ such that $\sigma \circ S = S \circ F$.

(3) Can you define a distance function (basically a function that takes a pair of sequences and outputs a numerical estimate of how "close" they are, you need to define what closeness here means) on Σ_2 such that S is continuous. (Meaning

for each input x, S takes points close to the input to points close to the output $S(x)$.? Is S^{-1} also continuous here?

Following the steps above leads us to an example of a topological conjugacy (whatever that means).