

§ 1. Simplicial Sets

1.1. Simplicial Sets

Let Δ be the category of finite ordinal numbers.

- **Objects:** Ordered sets $\mathbf{n} = \{0 \rightarrow 1 \rightarrow \dots \rightarrow n\}$ for $n \geq 0$.
- **Morphisms:** Order-preserving maps $\theta : \mathbf{m} \rightarrow \mathbf{n}$.

A simplicial set is a contravariant functor:

$$X : \Delta^{op} \rightarrow \mathbf{Sets}$$

It consists of sets X_n (n -simplices) and structure maps induced by morphisms in Δ .

1.2. Coface and Codegeneracies

Special morphisms in Δ generate all morphisms:

- **Coface maps** $d^i : \mathbf{n-1} \rightarrow \mathbf{n}$ ($0 \leq i \leq n$): The injective map missing i .
- **Codegeneracy maps** $s^j : \mathbf{n+1} \rightarrow \mathbf{n}$ ($0 \leq j \leq n$): The surjective map covering j twice.

Cosimplicial Identities (in Δ):

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && \text{if } i < j \\ s^j d^i &= d^i s^{j-1} && \text{if } i < j \\ s^j d^j &= \text{id} = s^j d^{j+1} \\ s^j d^i &= d^{i-1} s^j && \text{if } i > j + 1 \\ s^j s^i &= s^i s^{j+1} && \text{if } i \leq j \end{aligned}$$

Simplicial Identities (Structure maps of X): For a simplicial set X , let $d_i = X(d^i)$ (faces) and $s_j = X(s^j)$ (degeneracies):

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j \\ d_i s_j &= s_{j-1} d_i && \text{if } i < j \\ d_j s_j &= \text{id} = d_{j+1} s_j \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j + 1 \\ s_i s_j &= s_{j+1} s_i && \text{if } i \leq j \end{aligned}$$

1.3. Simplicial Abelian Groups

A simplicial abelian group is a functor:

$$A : \Delta^{op} \rightarrow \mathbf{Ab}$$

where \mathbf{Ab} is the category of abelian groups.

- Let $\mathbb{Z}Y$ be the free abelian group on a simplicial set Y .
- **Moore Complex:** A chain complex defined by:

$$(\mathbb{Z}Y)_n \xrightarrow{\partial} (\mathbb{Z}Y)_{n-1}$$

with boundary $\partial = \sum_{i=0}^n (-1)^i d_i$.

1.4. Classifying Space (Nerve)

Let \mathcal{C} be a small category. The **classifying space** (or nerve) $B\mathcal{C}$ is a simplicial set defined by:

$$B\mathcal{C}_n = \hom_{\mathbf{Cat}}(\mathbf{n}, \mathcal{C})$$

An n -simplex is a string of composable arrows of length n :

$$a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_n$$

1.5. Standard n -Simplex & Yoneda Lemma

The **standard n -simplex** Δ^n is the representable functor:

$$\Delta^n = \hom_{\Delta}(-, \mathbf{n})$$

Yoneda Lemma: There is a natural bijection between n -simplices of X and simplicial maps from Δ^n to X :

$$\hom_S(\Delta^n, X) \cong X_n$$

Let $\iota_n = \text{id}_{\mathbf{n}} \in (\Delta^n)_n$. A map $\phi : \Delta^n \rightarrow X$ corresponds to the simplex $x = \phi(\iota_n)$. Conversely, $x \in X_n$ defines $\iota_x : \Delta^n \rightarrow X$.

1.6. Boundary ($\partial\Delta^n$) and Horns (Λ_k^n)

Boundary: $\partial\Delta^n \subset \Delta^n$ is the smallest subcomplex containing all faces $d_j(\iota_n)$.

Construction of $\partial\Delta^n$: The set of j -simplices $(\partial\Delta^n)_j$ is defined as:

$$(\partial\Delta^n)_j = \begin{cases} (\Delta^n)_j & \text{if } 0 \leq j \leq n-1 \\ \deg_j & \text{if } j \geq n \end{cases}$$

where \deg_j denotes the set of iterated degeneracies of elements in $(\Delta^n)_{n-1}$. Essentially, in dimensions $\geq n$, it contains only degenerate elements originating from lower dimensions.

Example: $\partial\Delta^3$ (Boundary of the 3-simplex).

- It contains all proper faces of Δ^3 but not the non-degenerate 3-simplex ι_3 (or its degeneracies).
- **Non-degenerate elements:**

- 4 vertices (0-simplices).
- 6 edges (1-simplices).
- 4 faces (2-simplices): $d_0(\iota_3), d_1(\iota_3), d_2(\iota_3), d_3(\iota_3)$.

- Geometrically, it forms the surface of a tetrahedron.

k -th Horn: $\Lambda_k^n \subset \Delta^n$ ($n \geq 1, 0 \leq k \leq n$) is the subcomplex generated by all faces $d_j(\iota_n)$ **except** the k -th face $d_k(\iota_n)$.

Example: Λ_2^3 (The 2nd horn of the 3-simplex) is the subcomplex of Δ^3 generated by the faces d_0, d_1 , and d_3 (omitting d_2).

§ 2. Geometric Realization

2.1. The Simplex Category

Let X be a simplicial set. The **simplex category** of X , denoted $\Delta \downarrow X$, is defined as follows:

- **Objects:** The simplices of X , represented as simplicial maps $\sigma : \Delta^n \rightarrow X$.
- **Morphisms:** A morphism from $(\sigma : \Delta^n \rightarrow X)$ to $(\tau : \Delta^m \rightarrow X)$ is a commutative diagram in \mathbf{S} :

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\theta^*} & \Delta^m \\ \sigma \searrow & & \swarrow \tau \\ & X & \end{array}$$

where $\theta^* : \Delta^n \rightarrow \Delta^m$ is induced by an ordinal map $\theta : m \rightarrow n$ in Δ .

2.2. Decomposition Lemma

Any simplicial set X is a colimit of its simplices:

$$X \cong \underset{(\Delta^n \rightarrow X) \in \Delta \downarrow X}{\text{colim}} \Delta^n$$

2.3. Geometric Realization

The **geometric realization** of a simplicial set X , denoted $|X|$, is the topological space defined by the colimit:

$$|X| = \underset{(\Delta^n \rightarrow X) \in \Delta \downarrow X}{\text{colim}} |\Delta^n|_{\mathbf{Top}}$$

where $|\Delta^n|_{\mathbf{Top}} \subset \mathbb{R}^{n+1}$ is the standard topological n -simplex $\{(t_0, \dots, t_n) \mid \sum t_i = 1, t_i \geq 0\}$.

2.4. Adjunction

The realization functor $|\cdot| : \mathbf{S} \rightarrow \mathbf{Top}$ is left adjoint to the singular functor $S(\cdot) : \mathbf{Top} \rightarrow \mathbf{S}$.

- **Singular Set:** For $Y \in \mathbf{Top}$, $S(Y)_n = \hom_{\mathbf{Top}}(|\Delta^n|, Y)$.
- **Adjunction Isomorphism:** For $X \in \mathbf{S}, Y \in \mathbf{Top}$:

$$\hom_{\mathbf{Top}}(|X|, Y) \cong \hom_{\mathbf{S}}(X, S(Y))$$

Properties:

- $|\cdot|$ preserves all colimits (as a left adjoint).
- The adjunction is natural in X and Y .

2.5. CW Complex Structure and Skeletons ($sk_n X$)

$|X|$ is a CW-complex. Its structure is defined via the filtration of X by skeletons.

- The n -skeleton $sk_n X \subseteq X$ is the subcomplex generated by all simplices of X of degree $\leq n$.
- $X = \bigcup_{n \geq 0} sk_n X$.

2.6. Non-degenerate Simplices

Let NX_n be the set of **non-degenerate** n -simplices of X :

$$NX_n = \{x \in X_n \mid x \notin \bigcup_{i=0}^{n-1} s_i(X_{n-1})\}$$

2.7. Pushout Construction

$|sk_n X|$ is obtained from $|sk_{n-1} X|$ by attaching n -cells corresponding to non-degenerate simplices. There is a pushout diagram in \mathbf{S} (and consequently in \mathbf{Top} after realization):

$$\begin{array}{ccc} \coprod_{x \in NX_n} \partial \Delta^n & \longrightarrow & sk_{n-1} X \\ \downarrow & & \downarrow \\ \coprod_{x \in NX_n} \Delta^n & \longrightarrow & sk_n X \end{array}$$

where the maps are induced by the characteristic maps of the simplices in NX_n .

2.8. Standard Simplex Presentation

The boundary of the standard simplex, $\partial \Delta^n$, and by extension any realization involving boundaries, is governed by the **cosimplicial identities**.

2.9. Coequalizer Presentation

The realization of the boundary $|\partial \Delta^n|$ is the image of the map induced by face operators. It can be described via the coequalizer:

$$\coprod_{0 \leq i < j \leq n} |\Delta^{n-2}| \rightrightarrows \coprod_{k=0}^n |\Delta^{n-1}| \rightarrow |\partial \Delta^n|$$

2.10. Generating Relations

The coequalizer identifies faces along their common boundaries using the relation:

$$\text{For } 0 \leq i < j \leq n : d^j d^i = d^i d^{j-1}$$

This ensures that the $(n-1)$ -faces of $|\Delta^n|$ are glued correctly along their $(n-2)$ -faces to form the boundary S^{n-1} .

2.11. Kelley Spaces (\mathbf{CGHaus})

The category of topological spaces \mathbf{Top} is often inconvenient for homotopy theory regarding products. The realization functor is best viewed as taking values in \mathbf{CGHaus} (Compactly Generated Hausdorff spaces).

2.12. The Product Problem

In general \mathbf{Top} , the natural map is a continuous bijection but not necessarily a homeomorphism:

$$|X \times Y| \not\cong |X| \times |Y|_{\mathbf{Top}}$$

2.13. The Kelley Product

Let \times_{Ke} denote the product in the category \mathbf{CGHaus} (the k -ification of the standard product topology).

- **Theorem:** The realization functor preserves finite products if the target is considered as \mathbf{CGHaus} :

$$|X \times Y| \cong |X| \times_{Ke} |Y|$$

- **Consequence:** The functor $|\cdot| : \mathbf{S} \rightarrow \mathbf{CGHaus}$ preserves all finite limits.