

Notes on the Tensor Product of Local Rings

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1 The Isomorphism Formula of Localized Tensor Products

Let A and B be commutative rings over a base ring k . Let $S \subset A$ and $T \subset B$ be multiplicative subsets.

Theorem 1.1 (Localization Commutes with Tensor Product). *The tensor product of two localized rings is isomorphic to the localization of their tensor product with respect to the product of their multiplicative sets:*

$$(S^{-1}A) \otimes_k (T^{-1}B) \cong (S \otimes T)^{-1}(A \otimes_k B) \quad (1)$$

where $S \otimes T = \{s \otimes t \mid s \in S, t \in T\}$ is the multiplicative set generated by simple tensors of the denominators.

Interpretation: The "Separable Denominator" Constraint

The elements of the ring $(S^{-1}A) \otimes_k (T^{-1}B)$ can be viewed as fractions:

$$\frac{\alpha}{s \otimes t}$$

where $\alpha \in A \otimes_k B$. The crucial observation is that **allowable denominators must be separable**. That is, a denominator must be strictly of the form $f(a) \cdot g(b)$. A "mixed" element (e.g., $1 - (x + y)$) is generally **not** a valid denominator, even if it is non-zero at the focal point.

2 Why Tensor Products of Local Rings are (Often) Non-Local

Although the formula in Section 1 is a "localization," the resulting ring is generally **not a local ring** (i.e., it may have more than one maximal ideal).

Example 2.1 (The Standard Counterexample). Consider the localization of polynomial rings at the origin:

$$A = k[x]_{(x)}, \quad B = k[y]_{(y)}$$

Their tensor product is:

$$R = k[x]_{(x)} \otimes_k k[y]_{(y)} \cong S^{-1}(k[x, y])$$

where $S = \{f(x)g(y) \mid f(0) \neq 0, g(0) \neq 0\}$.

Proof that R is not local: Consider the element $z = 1 - (x + y) \in R$.

1. At the origin $(0, 0)$, z evaluates to $1 \neq 0$. In the full local ring $k[x, y]_{(x, y)}$, z would be a unit.
2. However, in R , the inverse z^{-1} exists if and only if $1 - (x + y)$ divides some element in S (separable polynomials).

3. Since $1 - (x + y)$ is an irreducible polynomial that mixes variables, it cannot divide any $f(x)g(y)$.
4. Thus, z is **not invertible** in R .
5. Since $z \notin (x, y)$ (the ideal generated by the origin), z must belong to some **other** maximal ideal \mathfrak{m}' .

Therefore, R has at least two maximal ideals: $\mathfrak{m}_0 = (x, y)$ and \mathfrak{m}' .

3 The Artinian Case: When Locality is Preserved

If the rings involved are Artinian, the "nilpotency" property forces the result to be local (assuming trivial residue field extension).

Proposition 3.1. *Let A, B be Artinian local k -algebras with residue fields isomorphic to k (i.e., $A/\mathfrak{m}_A \cong k, B/\mathfrak{m}_B \cong k$). Then $A \otimes_k B$ is an Artinian local ring.*

Sketch of Proof. 1. Since A, B are Artinian, their maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$ consist entirely of **nilpotent elements**.

2. Any element in $\mathfrak{m}_A \otimes B + A \otimes \mathfrak{m}_B$ is a sum of nilpotents, hence nilpotent.
3. Consider the element $u = 1 - (x + y)$ from the previous example. In the Artinian case, x and y are nilpotent. Thus $x + y$ is nilpotent.
4. **Key Algebraic Fact:** If η is nilpotent, then $1 - \eta$ is a unit (inverse is $\sum \eta^i$).
5. Therefore, any potential "ghost" element like $1 - (x + y)$ becomes invertible and cannot generate a new maximal ideal.
6. The only non-invertible elements are those contained in $\mathfrak{m}_{total} = (\mathfrak{m}_A, \mathfrak{m}_B)$, making it the unique maximal ideal.

□

4 Geometric Interpretation: Generic Points

Why does $k[x]_{(x)} \otimes k[y]_{(y)}$ have extra maximal ideals? This can be understood via the spectrum of the ring and **generic points**.

- **The Setup:** $\text{Spec}(A \otimes B)$ roughly corresponds to the product space. The localization process removes all closed points (a, b) where $a \neq 0$ or $b \neq 0$.
- **The Survivor:** The origin $(0, 0)$ survives. This corresponds to the maximal ideal $\mathfrak{m} = (x, y)$.
- **The Ghosts:** Consider a curve C passing near but not through the origin, e.g., the line $L : x + y = 1$.
 - In the full plane \mathbb{A}^2 , this line is defined by a prime ideal $\mathfrak{p} = (x + y - 1)$. It is a **generic point** of the line.
 - In the localized ring R , all specific *closed points* on this line (like $(1, 0), (0, 1)$) have been removed (turned into units) because their coordinates are non-zero.
 - However, the generic point itself (the "soul" of the line) remains.

- Since there are no closed points "below" it in the partial order of inclusion (they were all removed), this generic point \mathfrak{p} becomes a **maximal ideal** in R .

Remark 4.1. In summary: The tensor product of non-Artinian local rings is a **semi-local ring**. It contains one "rational" maximal ideal (the origin) and infinitely many "generic" maximal ideals corresponding to curves that do not pass through the origin but intersect the domain of definition.