Group Representation

January 5, 2026

1 Atiyah-Segal Completion Theorem

Definition 1.1 (Representation Ring)**.** Let $\{V_1, V_2, \ldots, V_k\}$ be the complete set of non-isomorphic irreducible complex representations of a finite group G. The representation ring $R(G)$ is defined as the set of all formal linear combinations of these irreducible representations with **integer coefficients**:

$$R(G) = \left\{ \sum_{i=1}^{k} n_i V_i \mid n_i \in \mathbb{Z} \right\}$$

The ring structure is defined by:

1. **Addition:** Component-wise addition of integers.

$$\left(\sum n_i V_i\right) + \left(\sum m_i V_i\right) = \sum (n_i + m_i) V_i$$

2. **Multiplication:** Defined by the tensor product, extended linearly. If $V_i \otimes V_j = \bigoplus_l c_{ij}^l V_l$, then:

$$V_i \cdot V_j = \sum_{l=1}^{k} c_{ij}^l V_l$$

Elements with negative coefficients (where some $n_i < 0$) are called **virtual representations**.

Definition 1.2 (Augmentation Ideal $I(G)$)**.** The augmentation ideal $I(G)$ is the kernel of the dimension homomorphism $\varepsilon : R(G) \to \mathbb{Z}$. It consists of all virtual representations of dimension zero:

$$I(G) = \{x \in R(G) \mid \dim(x) = 0\}$$

It is generated by elements of the form $[V] - \dim(V) \cdot 1$, where 1 denotes the trivial representation.

Construction 1.3 (EG by Excising Fixed Points)**.** Let $\mathcal{V} \cong \mathbb{R}^\infty$ be an infinite-dimensional vector space equipped with a faithful linear action of the finite group G (e.g., the infinite direct sum of the regular representation).

We define the **singular set** Z as the union of the fixed-point subspaces of all non-identity elements:

$$Z := \bigcup_{g \in G, g \neq e} \mathcal{V}^g = \{v \in \mathcal{V} \mid \exists g \neq e, g \cdot v = v\}$$

The universal bundle EG is constructed as the complement of this set:

$$EG := \mathcal{V} \setminus Z$$

The classifying space is the quotient $BG = EG/G$.

Construction 1.4 (The Classifying Space BG)**.** Given the universal space $EG = \mathcal{V} \setminus Z$ constructed above, where the action of G is free. We define the **classifying space** BG as the orbit space of this action:

$$BG := EG/G = \{G \cdot v \mid v \in EG\}$$

Construction 1.5 (The Natural Map $\alpha : R(G) \to K(BG)$)**.** Let $G \to EG \to BG$ be the universal principal bundle constructed previously. We define the map α by converting algebraic representations into geometric vector bundles via the **associated bundle construction**.

Given a finite-dimensional complex representation V of G:

1. **The Mixing (Balanced Product):** Consider the product space $EG \times V$. We define the associated vector bundle $\mathcal{E}_V$ over BG as the quotient by the diagonal G-action:

$$\mathcal{E}_V := EG \times_G V = (EG \times V)/\sim$$

   where the equivalence relation is defined by $(g \cdot e, g \cdot v) \sim (e, v)$ for all $g \in G$.

2. **The Mapping:** The map α assigns to each representation class $[V]$ the homotopy class of its associated bundle $[\mathcal{E}_V]$:

$$\alpha : R(G) \longrightarrow K(BG)$$
$$[V] \longmapsto [EG \times_G V]$$

This map α is a ring homomorphism, translating the algebraic data of G-representations into the topological data of flat vector bundles over BG.

Theorem 1.6 (Atiyah-Segal Completion Theorem)**.** Let G be a finite group. The natural map $\alpha : R(G) \to K(BG)$ induces an isomorphism of rings between the I-adic completion of the representation ring and the topological K-theory of the classifying space:

$$\widehat{R(G)}_I \xrightarrow{\cong} K(BG)$$

where $I \subset R(G)$ is the augmentation ideal and $\widehat{R(G)}_I = \varprojlim_n R(G)/I^n$.

Remark 1.7 (Flatness and Torsion)**.** Since a finite group G has the discrete topology, the map $EG \to BG$ is a covering map. Consequently, every associated bundle $\mathcal{E}_V$ is canonically **flat** (possessing zero curvature).

This implies that all **rational** Chern classes vanish via Chern-Weil theory:

$$c_k(\mathcal{E}_V) \otimes \mathbb{Q} = 0$$

Therefore, unlike the case of Lie groups, the Atiyah-Segal theorem for finite groups describes an isomorphism that is purely about the **torsion** information in the K-theory of BG.