# A Study in Advanced Talks

Mathematical Notes Collection

Notes & Expositions

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# Perfectoid Spaces I: The Weight-Monodromy Conjecture

*Speaker: Peter Scholze*

## 1.1 Introduction

These notes reconstruct the motivations leading to the theory of Perfectoid Spaces, rooted in Peter Scholze's undergraduate years in Bonn (circa 2007)[cite: 2]. The central motivation discussed here is the **Weight-Monodromy Conjecture** (WMC) by Deligne.

## 1.2 The Setup and The Monodromy Operator

### 1.2.1 Geometric Setup

Let $K = \mathbb{Q}_p$ be the $p$-adic field. Let $X$ be a smooth projective scheme over $\mathbb{Q}_p$[cite: 2]. Fix a prime $\ell \neq p$[cite: 3]. We are interested in the $\ell$-adic cohomology of $X$:

$$V := H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell).$$

This vector space admits a continuous action of the absolute Galois group $G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$[cite: 4].

### 1.2.2 The Monodromy Theorem

To state the conjecture, we must first define the structure of this Galois representation. The inertia group $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ acts on $V$. Even if $X$ does not have good reduction, Grothendieck's $\ell$-adic Monodromy Theorem describes this action.

**Theorem 1.1** (Grothendieck's $\ell$-adic Monodromy)**.** *There exists a nilpotent operator $N : V \to V(-1)$, called the **Monodromy Operator**, such that on an open subgroup of the inertia $I_{\mathbb{Q}_p}$, the action of $\sigma \in I_{\mathbb{Q}_p}$ is given by*

$$\rho(\sigma) = \exp(t_\ell(\sigma)N),$$

*where $t_\ell : I_{\mathbb{Q}_p} \to \mathbb{Z}_\ell$ is the $\ell$-adic tame character[cite: 10, 11].*

**Definition 1.2** (Weight Decomposition)**.** The representation $V$ admits a decomposition based on the eigenvalues of the Geometric Frobenius $\Phi$. We say $V$ has a **weight decomposition**[cite: 8]:

$$V = \bigoplus_{j=0}^{2i} V_j,$$

where the eigenvalues of $\Phi$ acting on $V_j$ are Weil numbers of weight $j$.

The monodromy operator $N$ respects this structure in a specific way. Since $N$ maps $V$ to $V(-1)$ (a Tate twist), it lowers the weight by 2[cite: 13]:

$$N : V_j \longrightarrow V_{j-2}(-1).$$

## 1.3 The Weight-Monodromy Conjecture

Deligne proposed the following conjecture regarding the interplay between the filtration by weights and the monodromy operator.

**Conjecture 1.3** (Deligne's Weight-Monodromy Conjecture)**.** *For any $j \geq 0$, the power of the monodromy operator induces an isomorphism[cite: 17, 25]:*

$$N^j : V_{i+j} \xrightarrow{\sim} V_{i-j}(-j).$$

Intuitively, this suggests that the monodromy operator acts like the hard Lefschetz operator, reflecting the weight filtration across the center $i$.

## 1.4 Examples and Evidence

### 1.4.1 Case 1: Good Reduction

Assume $X$ has good reduction, meaning there exists a smooth projective model $\mathcal{X}$ over $\mathbb{Z}_p$ such that $\mathcal{X}_{\mathbb{Q}_p} \cong X$[cite: 19].

**Lemma 1.4.** *If $X$ has good reduction, the action of the inertia group $I_{\mathbb{Q}_p}$ is trivial[cite: 22].*

Consequently, the monodromy operator is trivial ($N = 0$)[cite: 23]. In this case, Conjecture 1.3 implies that $V_j = 0$ for all $j \neq i$. Thus $V = V_i$. This reduces to the **Weil Conjectures** for the reduction $X_{\mathbb{F}_p}$, which are known[cite: 26].

### 1.4.2 Case 2: The Tate Curve

Consider an elliptic curve $E$ over $\mathbb{Q}_p$ with split multiplicative reduction. As a rigid analytic space, $E$ can be uniformized as the **Tate Curve**[cite: 30]:

$$E(\overline{\mathbb{Q}}_p) \cong \mathbb{G}_m(\overline{\mathbb{Q}}_p)/q^{\mathbb{Z}}, \quad \text{with } |q| < 1.$$

We compute $H^1(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$. Using the Hochschild-Serre spectral sequence (or the Kummer sequence for $\mathbb{G}_m$), we have an exact sequence[cite: 39]:

$$0 \longrightarrow \overline{\mathbb{Q}}_\ell(0) \longrightarrow H^1(E, \overline{\mathbb{Q}}_\ell) \longrightarrow \overline{\mathbb{Q}}_\ell(-1) \longrightarrow 0.$$

Here:

- The subspace $V_0 = \overline{\mathbb{Q}}_\ell(0)$ has weight 0.
- The quotient $V_2 = \overline{\mathbb{Q}}_\ell(-1)$ has weight 2.
- The center weight is $i = 1$.

The Monodromy operator $N$ maps the weight 2 part to the weight 0 part:

$$N : V_2 \xrightarrow{\sim} V_0(-1).$$

This verifies the conjecture for $j = 1$: $N^1 : V_{1+1} \to V_{1-1}(-1)$ is an isomorphism[cite: 40].

## 1.5 Strategy: Reduction to Equal Characteristic

### 1.5.1 Known Results

The conjecture is known in the following cases:

1. Dimension of $X$ is $\leq 2$[cite: 47].

2. **Equal Characteristic:** If $X$ is defined over a function field $F \cong \mathbb{F}_p((t))$ rather than $\mathbb{Q}_p$, the conjecture was proved by Deligne using $L$-functions[cite: 52, 53].

### 1.5.2 Rapoport's Suggestion

The strategy to prove WMC in mixed characteristic ($\mathbb{Q}_p$) is to reduce it to the equal characteristic case ($\mathbb{F}_p((t))$).

- **Idea:** Base change to a highly ramified extension $K/\mathbb{Q}_p$[cite: 60].

- Let $e$ be the ramification index. The ring of integers behaves like:

$$\mathcal{O}_K/p \cong \mathbb{F}_q[t]/t^e.$$

- As $e \to \infty$, this ring approximates $\mathbb{F}_q[[t]]$[cite: 63, 65].

### 1.5.3 The Obstruction and Perfectoid Spaces

However, for any finite $e$, the approximation is insufficient. There are algebraic obstructions to deforming $X_{\mathcal{O}_K/p}$ to a scheme over $\mathbb{F}_q[[t]]$[cite: 72, 76].

This failure suggests that we must pass to the limit $e = \infty$. This leads to the definition of **Perfectoid Fields** (fields with infinite ramification). In the world of perfectoid spaces, we can construct a rigorous isomorphism (Tilting) between geometric objects in mixed characteristic and equal characteristic:

$$X_{\text{perfectoid}} \longleftrightarrow X^{\flat}_{\text{equal char}}.$$

This correspondence allows the transfer of cohomological results (like Deligne's theorem) back to $\mathbb{Q}_p$, as realized in Scholze's work [cite: 78-81].

# Algebraic Geometry in Mixed Characteristic

*Speaker: Bhargav Bhatt*

## 2.1 Motivation: The Defect of Classical Cohomology

The motivating goal for modern $p$-adic geometry is to establish a geometric understanding of integral cohomology classes, particularly torsion, in an algebraic setting.

**Theorem 2.1** (de Rham 1931, Hodge 1941)**.** *Let $X$ be a compact complex Kähler manifold (e.g., $X \subset \mathbb{CP}^m$). The integration of differential forms over cycles yields the following isomorphisms:*

$$H^n(X;\mathbb{C}) \simeq H^n_{\mathrm{dR}}(X;\mathbb{C}) \simeq \bigoplus_{i+j=n} H^{i,j}(X).$$

This classical result provides a bidirectional bridge between topology and geometry:

- **Geometry to Topology:** The symmetry $H^{i,j} \simeq \overline{H^{j,i}}$ implies constraints on the topological Betti numbers (e.g., if $n$ is odd, $\dim H^n(X;\mathbb{C})$ must be even).
- **Topology to Geometry:** Topological constraints, such as $\pi_1(X) = 0$, force the vanishing of holomorphic 1-forms ($H^{1,0}(X) \subset H^1(X;\mathbb{C}) = 0$).

*Remark* 2.2 (The Defect). The comparison above relies on coefficients in $\mathbb{C}$. Consequently, it completely misses the torsion information in $H^*(X;\mathbb{Z})$! The goal of mixed characteristic geometry is to understand $H^*(X;\mathbb{Z}/p^n)$ geometrically for algebraic varieties.

## 2.2 The Geometric Backdrop: Hensel's $p$-adic World

To bridge characteristic 0 and characteristic $p$, we utilize the ring of $p$-adic integers $\mathbb{Z}_p$.

**Definition 2.3** (Mixed Characteristic Setting)**.** We define the following base rings and fields:

- $\mathbb{Z}_p = \left\{\sum_{i\geq 0} a_i p^i \mid a_i \in \{0,\dots,p-1\}\right\}$: The bridge between characteristics.
- $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$: Characteristic 0.
- $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$: The $p$-adic analog of $\mathbb{C}$.
- $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$: Characteristic $p$.

For a smooth projective variety $X$ over $\mathbb{Z}_p$, we obtain a correspondence between the generic fiber (analytic/topological side) and the special fiber (algebraic side):

$$\begin{tikzcd}
X_{\mathbb{C}_p} \arrow[d, "\text{generic fiber}"'] & X \arrow[d] & X_{\mathbb{F}_p} \arrow[d, "\text{special fiber}"] \\
\mathrm{Spec}(\mathbb{C}_p) \arrow[r] & \mathrm{Spec}(\mathbb{Z}_p) & \mathrm{Spec}(\mathbb{F}_p) \arrow[l]
\end{tikzcd}$$

Historically, the comparison principle (Tate 1966, Grothendieck 1970, Fontaine 1978) suggested that the mod $p^n$ topology of $X_{\mathbb{C}_p}$ should be recoverable from the algebraic geometry of $X_{\mathbb{F}_p}$ augmented with specific linear algebra data (e.g., Frobenius actions).

## 2.3 Prismatic Cohomology

Prismatic cohomology serves as a 21st-century upgrade to this principle, unifying various cohomology theories.

**Theorem 2.4** (Bhatt-Morrow-Scholze, 2016, 2018)**.** *Let $X$ be a smooth formal scheme over $\mathbb{Z}_p$. There exists a cohomology theory $H^*_{\triangle}(X;\mathbb{F}_p) \in \mathrm{Mod}^{fg}(\mathbb{F}_p[[T]])$ satisfying the following comparisons:*

1. ***Topological Comparison (Generic Fiber):***

$$H^*_{\triangle}(X;\mathbb{F}_p)[1/T] \simeq H^*(X_{\mathbb{C}_p};\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p((T))$$

*This recovers the cohomology of the generic fiber.*

2. ***Differential Comparison (Special Fiber):***

$$H^*_{\triangle}(X;\mathbb{F}_p)/T \simeq H^*_{\mathrm{dR}}(X_{\mathbb{F}_p})$$

*This recovers the de Rham cohomology of the reduction mod $p$.*

There is also an integral variant $H^*_{\triangle}(X;\mathbb{Z}_p) \in \mathrm{Mod}^{fg}(\mathbb{Z}_p[[T]])$.

## 2.4 The Shape of Prismatic Cohomology

One of the most powerful aspects of prismatic cohomology is how it organizes different cohomology theories into a single geometric family over the parameter space $\mathrm{Spec}(\mathbb{Z}_p[[T]])$.

The following table illustrates the correspondence between the specializations of the "prism" and classical cohomology theories:

Table 2.1: Specializations of $H^*_{\triangle}(X;\mathbb{Z}_p)$ (Fibers over $\mathrm{Spec}(\mathbb{Z}_p[[T]])$)

| **Locus** | **Condition** | **Recovered Theory** |
|---|---|---|
| $p$-axis | $T = 0$ | **Crystalline / de Rham**<br>(Geometry of $X_{\mathbb{F}_p}$) |
| $T$-axis | $p = 0$ | **Mod $p$ Étale**<br>(Char $p$ topology) |
| Hodge-Tate Locus | $T = p$ (approx.) | **Hodge-Tate** |
| Generic Locus | $T$ inverted | **Étale Cohomology ($p$-adic)**<br>(Topology of $X_{\mathbb{C}_p}$) |

This structure effectively upgrades the classical Hodge filtration to a deformation over the variable $T$.

## 2.5 Detailed Applications

The algebraic construction of prismatic cohomology has provided new tools to address long-standing questions where torsion classes were previously problematic.

### 2.5.1 Algebraic K-Theory

Classical results by Atiyah, Hirzebruch, and Bott established a spectral sequence relating the topological K-theory $K(X)$ to singular cohomology for nice spaces. A major open question (Beilinson 1982) was to find an analog for the algebraic K-theory of rings.

**Theorem 2.5** (Clausen-Mathew-Morrow 2018, B-Morrow-Scholze 2018)**.** *For a $p$-complete commutative ring $R$, there exists a natural "motivic" filtration on the $p$-adic K-theory $K_{\mathrm{et}}(R;\mathbb{Z}_p)$ where the graded pieces are given by* ***syntomic cohomology****:*

$$\mathrm{gr}^i K_{\mathrm{et}}(R;\mathbb{Z}_p) \simeq H^*_{\mathrm{syn}}(R, \mathbb{Z}_p(i))[2i].$$

*Syntomic cohomology is a new object determined entirely by prismatic cohomology.*

This structural result has led to concrete calculations that were previously out of reach:

1. **Odd Vanishing (B-Scholze 2019):** For many "large" rings, such as $\mathcal{O}_{\mathbb{C}_p}/p^n$, the odd homotopy groups vanish: $\pi_{\mathrm{odd}}K(\mathcal{O}_{\mathbb{C}_p}/p^n) = 0$. This relies on $q$-de Rham complexes and André's flatness lemma.

2. **Even Vanishing (Antieau-Krause-Nikolaus 2022):** Using absolute prismatic cohomology, it is shown that:

$$\pi_{2k}K(\mathbb{Z}/p^n) = 0 \quad \text{for all } k \gg 0.$$

### 2.5.2 Kodaira Vanishing in Mixed Characteristic

The classic Kodaira vanishing theorem states that for $X \subset \mathbb{P}^n_{\mathbb{C}}$ smooth projective of dimension $d$, $H^{<d}(X, \mathcal{O}(-1)) = 0$. This is known to fail in positive characteristic (Raynaud 1978, Totaro 2021).

**Theorem 2.6** (Global KV up to Finite Covers, Bhatt 2020)**.** *For $X \subset \mathbb{P}^n_{\mathbb{Z}_p}$ projective of relative dimension $d$, there exists a finite cover $\pi : Y \to X$ such that the torsion part of the cohomology is annihilated by the pullback map:*

$$\mathrm{Image}\left(H^{<d}(X, \mathcal{O}(-1))_{\mathrm{tors}} \xrightarrow{\pi^*} H^{<d}(Y, \pi^*\mathcal{O}(-1))_{\mathrm{tors}}\right) = 0.$$

This theorem is established using prismatic cohomology and Riemann-Hilbert constructions for perverse $\mathbb{F}_p$-sheaves. It also has a purely local commutative algebra formulation:

**Theorem 2.7** (Local KV / Splinter Property, Bhatt 2020)**.** *Fix a finite extension $\mathbb{Z}_p[x_1, \ldots, x_n] \subset R$. Then there exists a finite extension $R \subset S$ such that any relation $\sum a_i x_i = 0$ in $R/p$ becomes trivial in $S/p$. Conceptually, if $R^+$ is the integral closure of $R$ in* $\mathrm{Frac}(R)$*, then $R^+/p$ is Cohen-Macaulay over $R/p$.*

*Remark* 2.8 (Application to Minimal Model Program)*.* These vanishing results were key ingredients in establishing the Minimal Model Program for arithmetic 3-folds with residue characteristic $p > 5$ (B-Ma-Patakfalvi-Schwede-Tucker-Waldron-Witaszek 2020).

### 2.5.3 Other Recent Developments

Prismatic cohomology has also been instrumental in several other major results:

- **Tate Conjecture in Char 2:** Proved for K3 surfaces in characteristic 2 by Madapusi Pera (2016) and Ito-Ito-Koshikawa (2018).
- **$p$-adic Upper Half Space:** Colmez-Dospinescu-Nizioł (2019) proved that $H^*(\Omega; \mathbb{Z}_p)$ is $p$-torsionfree for Drinfeld's $p$-adic upper half space.
- **Essential Dimension:** Farb-Kisin-Wolfson (2021) proved that for an abelian variety $A/\mathbb{C}$, the multiplication map $[p]$ has essential dimension $\dim(A)$ for $p \gg 0$.
- **Poincaré Duality:** Established for $\mathbb{Z}_p$-étale cohomology in $p$-adic analytic geometry by Zavyalov (2021).

# Lie Algebras and Homotopy Theory

*Speaker: Based on Notes*

## 3.1 Group Structures and Loop Spaces

Recall that if $G$ is a group, we have the commutator map $G \times G \to G$ defined by $(x, y) \mapsto xyx^{-1}y^{-1}$. If $G$ is a Lie group, differentiation of the commutator map yields the Lie bracket on the tangent space at the identity:

$$[\cdot, \cdot] : T_eG \times T_eG \to T_eG.$$

We wish to apply this intuition to topological spaces. Let $X$ be a topological space with a base point $x_0 \in X$.

**Definition 3.1** (Loop Space)**.** The loop space $\Omega X$ is defined as the space of based paths:

$$\Omega X = \{p : [0, 1] \to X \mid p(0) = p(1) = x_0\}.$$

There is a well-known isomorphism relating the fundamental group of the loop space to the homotopy groups of the base space:

$$\pi_1(\Omega X) \cong \pi_2(X), \quad \text{and generally} \quad \pi_k(\Omega X) \cong \pi_{k+1}(X).$$

The space $\Omega X$ admits a binary operation via concatenation of loops. Let $p, q \in \Omega X$. We define the product $p * q$ as:

$$(p * q)(t) = \begin{cases} p(2t) & 0 \le t \le \frac{1}{2} \\ q(2t - 1) & \frac{1}{2} \le t \le 1 \end{cases}$$

This operation is not strictly associative, but it is *associative up to homotopy.* Similarly, inversion is defined up to homotopy by $\bar{p}(t) = p(1 - t)$.

We can define a map $\Omega X \times \Omega X \to \Omega X$ analogous to the group commutator:

$$(p, q) \mapsto (p * q) * (\bar{p} * \bar{q}).$$

This construction leads to algebraic structures on homotopy groups similar to Lie algebras.

## 3.2 The Whitehead Bracket and Graded Lie Algebras

The commutator in the loop space induces the **Whitehead bracket**.

**Definition 3.2** (Whitehead Bracket)**.** The Whitehead bracket is a bilinear map:

$$[\cdot, \cdot] : \pi_{a+1}(X) \times \pi_{b+1}(X) \to \pi_{a+b+1}(X).$$

We can organize the homotopy groups into a graded object. Let us define the shifted graded vector space $L_*$ by:

$$L_n := \pi_{n+1}(X).$$

Then the total space $L = \bigoplus_{n \ge 0} L_n = \bigoplus_{n \ge 0} \pi_{n+1}(X)$ carries the structure of a **Graded Lie Algebra (GLA)**.

**Theorem 3.3** (Structure of Homotopy Groups)**.** *The bracket defined above satisfies:*

1. ***Graded Skew-symmetry:*** *For* $x \in L_a, y \in L_b$,

$$[x, y] + (-1)^{|x||y|}[y, x] = 0.$$

2. ***Graded Jacobi Identity:***

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]].$$

*Remark* 3.4 (Homotopy Operations)*.* A homotopy operation of $n$-variables is a map:

$$\pi_{a_1}(X) \times \cdots \times \pi_{a_n}(X) \to \pi_b(X).$$

The Whitehead bracket is a 2-variable operation. The **Hilton-Milnor Theorem** essentially states that all such homotopy operations can be built from 1-variable operations and the Whitehead bracket.

## 3.3 Rational Homotopy Theory

We now transition to **Rational Homotopy Theory**, where we ignore torsion by tensoring with $\mathbb{Q}$.

$$\pi_*(X) \otimes \mathbb{Q}.$$

This simplifies calculations significantly (Type II algebraic topology—computable and concrete).

### 3.3.1 Quillen's Theorem

Quillen established an equivalence between the homotopy category of simply connected rational spaces and the category of Differential Graded Lie Algebras (DGLA).

**Definition 3.5** (Differential Graded Lie Algebra)**.** A DGLA is a pair $(L_*, d)$ where $L_*$ is a graded Lie algebra and $d : L_* \to L_{*-1}$ is a differential satisfying:

1. $d^2 = 0$.
2. **Leibniz Rule:** $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$.

Taking the homology of a DGLA, $H_*(L_*, d)$, yields a graded Lie algebra. In Quillen's model for a space $X$, we have:

$$H_*(L_*(X)) \cong \pi_{*+1}(X) \otimes \mathbb{Q}.$$

**Definition 3.6** (Rational Homotopy Equivalence)**.** A map $f : X \to Y$ between simply connected spaces is a *rational homotopy equivalence* if it induces an isomorphism on rational homotopy groups:

$$f_* : \pi_*(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_*(Y) \otimes \mathbb{Q}.$$

In the DGLA setting, this corresponds to a **quasi-isomorphism** (a map inducing an isomorphism on homology) $L_*(X) \to L_*(Y)$.

**Theorem 3.7** (Quillen)**.** *The construction $X \mapsto L_*(X)$ defines an equivalence of categories:*

$$\left\{ \begin{matrix} \textit{Simply connected} \\ \textit{rational homotopy types} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{Reduced DGLAs over } \mathbb{Q} \\ \textit{up to quasi-isomorphism} \end{matrix} \right\}.$$

## 3.4 Lie Algebras in General Categories

We can generalize the notion of a Lie algebra to an arbitrary category $\mathcal{A}$, provided $\mathcal{A}$ has sufficient structure.

**Definition 3.8** (Requirements for $\mathcal{A}$)**.** Let $\mathcal{A}$ be a category that is:

1. Cocomplete (has all colimits).

2. Additive.

3. Symmetric Monoidal: There exists a tensor product $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ that is commutative and associative up to isomorphism, preserving colimits in each variable.

A **Lie algebra object** $L \in \mathcal{A}$ is equipped with a bracket morphism $br : L \otimes L \to L$ satisfying skew-symmetry and the Jacobi identity.

**Example 3.9.**

- If $\mathcal{A} = \mathbf{Ab}$ (Abelian groups), $\mathcal{L}ie(\mathcal{A})$ is the category of standard Lie algebras.

- If $\mathcal{A} = \mathbf{GrAb}$ (Graded Abelian groups), $\mathcal{L}ie(\mathcal{A})$ is the category of Graded Lie Algebras.

- If $\mathcal{A} = \mathbf{Ch}$ (Chain complexes), $\mathcal{L}ie(\mathcal{A})$ is the category of DGLAs.

## 3.5 The Universal Category for Lie Algebras

We can formalize the category of Lie algebras using a universal construction.

**Claim 3.10.** There exists a **universal category** $\mathcal{E}$ equipped with a generic Lie algebra object $L_u$.

This allows us to define the category of Lie algebras in any symmetric monoidal category $\mathcal{A}$ via functors.

**Definition 3.11** (Functorial Definition of Lie Algebras)**.** The category of Lie algebras in $\mathcal{A}$, denoted as $\mathcal{L}ie(\mathcal{A})$, is defined as the category of tensor functors from $\mathcal{E}$ to $\mathcal{A}$ that preserve colimits:

$$\mathcal{L}ie(\mathcal{A}) \triangleq \{F : \mathcal{E} \to \mathcal{A} \mid F \text{ is a } \otimes\text{-functor preserving colimits}\}.$$

Under this correspondence, the value of the functor on the universal object recovers the Lie algebra: $F(L_u) \in \mathcal{L}ie(\mathcal{A})$.

### 3.5.1 Structure of the Universal Category

The universal category $\mathcal{E}$ is constructed such that its objects are generated by tensor powers of the universal Lie algebra object $L_u$.

- Objects in $\mathcal{E}$ are of the form of colimits of direct sums:

$$\bigoplus_{\alpha} L_u^{\otimes n_\alpha} \to \bigoplus_{\beta} L_u^{\otimes m_\beta}.$$

- $\mathcal{E}$ serves as an abelian category containing $L_u$.

*Remark* 3.12 (Homological Context). Recall that if $\mathcal{A}$ is an abelian category, the category of chain complexes $\mathbf{Ch}(\mathcal{A})$ admits a projective generator. We can consider the derived category:

$$D(\mathcal{A}) = \mathbf{Ch}(\mathcal{A})[\text{quasi-iso}^{-1}].$$

There is a construction of the Lie functor in the derived setting:

$$\mathcal{L}ie : D(\mathcal{E}) \to D(\mathbf{Ab}).$$

### 3.5.2 Connection to Finite Sets (Operads)

The structure of this universal category is closely related to the combinatorics of finite sets.

**Definition 3.13.** Let $\mathrm{Fin}^{\mathrm{surj}}$ be the category of finite sets with surjective maps.

The tensor product of functors $F, G : \mathrm{Fin}^{\mathrm{surj}} \to \mathcal{A}$ is given by the convolution formula:

$$(F \otimes G)(I) = \bigoplus_{J \subseteq I} F(J) \otimes G(I \setminus J).$$

This formalism leads to the duality in operad theory (Koszul Duality), where we have an equivalence of derived categories:

$$D(\mathcal{E}) \cong D(\mathcal{E}').$$