

# A Study in Quasi-Coherent Sheaves and Tannaka Duality

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## 1 Preliminaries

### 1.1 $\infty$ -categories

**Definition 1.1** ( $\infty$ -Category). A simplicial set  $K$  is an  **$\infty$ -category** if for every  $n > 1$  and every **inner** index  $0 < i < n$ , every map of simplicial sets  $f_0 : \Lambda_i^n \rightarrow K$  admits an extension to an  $n$ -simplex  $f : \Delta^n \rightarrow K$ .

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{f_0} & K \\ \downarrow & \nearrow f & \\ \Delta^n & & \end{array}$$

**Definition 1.2** (Simplicial Category). A **simplicial category** (or  $\text{Set}_\Delta$ -enriched category)  $\mathcal{C}$  is a category where:

1. For any two objects  $X, Y \in \mathcal{C}$ , the collection of morphisms between them is not a set, but a **simplicial set**  $\text{Map}_{\mathcal{C}}(X, Y)$ .
2. For any three objects  $X, Y, Z \in \mathcal{C}$ , the composition map

$$\text{Map}_{\mathcal{C}}(Y, Z) \times \text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{\mathcal{C}}(X, Z)$$

is a morphism of simplicial sets and satisfies the usual associativity and identity axioms.

A simplicial category  $\mathcal{C}$  is **locally Kan** if for every pair of objects  $X, Y \in \text{Ob}(\mathcal{C})$ , the mapping simplicial set  $\text{Map}_{\mathcal{C}}(X, Y)$  is a Kan complex.

**Definition 1.3** (Simplicial Nerve  $N_\Delta$ ). The **simplicial nerve**  $N_\Delta(\mathcal{C})$  is the simplicial set defined by the assignment:

$$N_\Delta(\mathcal{C})_n = \text{Hom}_{\text{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$$

where  $\mathfrak{C}[\Delta^n]$  is the **rigidification** of the  $n$ -simplex  $\Delta^n$  into a simplicial category.

**Definition 1.4** ( $\infty$ -category via  $N_\Delta$ ). An  **$\infty$ -category** (or quasicategory) is a simplicial set  $K$  that is equivalent to the simplicial nerve of some locally Kan simplicial category  $\mathcal{C}$ .

$$K \simeq N_\Delta(\mathcal{C})$$

**Theorem 1.5** (Joyal-Lurie). There exists a Quillen equivalence between the Joyal model structure on  $\text{Set}_\Delta$  (modeling quasicategories) and the Bergner model structure on  $\text{Cat}_\Delta$  (modeling simplicial categories):

$$\mathfrak{C}[\cdot] : \text{Set}_\Delta \rightleftarrows \text{Cat}_\Delta : N_\Delta.$$

Specifically, for any simplicial category  $\mathcal{C}$  where mapping spaces are Kan complexes, its simplicial nerve  $N_\Delta(\mathcal{C})$  is a quasicategory.

**Definition 1.6** (Free Cocompletion). Let  $\mathcal{C}$  be a small  $\infty$ -category. An  $\infty$ -category  $\mathcal{P}(\mathcal{C})$  is called the **free cocompletion** of  $\mathcal{C}$  if it satisfies the following universal property:

1.  $\mathcal{P}(\mathcal{C})$  admits all small colimits.
2. There exists a functor  $j : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$  (called the Yoneda embedding) such that for any  $\infty$ -category  $\mathcal{D}$  which admits small colimits, composition with  $j$  induces an equivalence of  $\infty$ -categories:

$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Here,  $\mathrm{Fun}^L$  denotes the full subcategory of functors that preserve small colimits (left adjoints).

**Definition 1.7** (The  $\infty$ -category of Spaces). Let  $\mathcal{S}$  denote the  $\infty$ -category of spaces. It is defined in two equivalent ways:

**1. Via Dwyer-Kan Localization:**

Let  $W$  be the class of weak homotopy equivalences in  $\mathrm{Set}_\Delta$ . We define  $\mathcal{S}$  as the homotopy coherent nerve of the simplicial localization:

$$\mathcal{S} := N(\mathrm{Set}_\Delta[W^{-1}]).$$

Equivalently, via Kan complexes:  $\mathcal{S} \simeq N(\mathbf{Kan})$ .

**2. Via Free Cocompletion:**

The  $\infty$ -category  $\mathcal{S}$  is the free cocompletion of the point  $*$ . That is, it is the category of presheaves:

$$\mathcal{S} \simeq \mathcal{P}(*)$$

Universal Property: For any cocomplete  $\infty$ -category  $\mathcal{C}$ , there is an equivalence  $\mathrm{Fun}^L(\mathcal{S}, \mathcal{C}) \simeq \mathcal{C}$ .

**Theorem 1.8.** Let  $K$  be an  $\infty$ -category (quasi-category). Let  $\mathcal{C} = \mathfrak{C}[K]$  be its associated simplicial category (via rigidification). The construction of the presheaf  $\infty$ -category commutes with the nerve construction in the following sense:

1. **Simplicial Side:** Consider the category of simplicial presheaves  $\mathcal{P}_\Delta(\mathcal{C}) := \mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathcal{S}_{\mathrm{Kan}})$ . This category admits a simplicial model structure (projective structure).
2. **Infinity Side:** Consider the  $\infty$ -category of presheaves  $\mathcal{P}(K) := \mathrm{Fun}(K^{op}, \mathcal{S})$ .
3. **Equivalence:** There is an equivalence of  $\infty$ -categories:

$$\mathcal{P}(K) \simeq N(\mathcal{P}_\Delta(\mathcal{C})^{\mathrm{cf}})$$

where  $\mathcal{P}_\Delta(\mathcal{C})^{\mathrm{cf}}$  denotes the full simplicial subcategory of fibrant-cofibrant objects in the model category of simplicial presheaves.

In summary, the presheaf of an  $\infty$ -category is modeled by the nerve of the strictly cocomplete simplicial category of enriched presheaves.

**Remark 1.9** (Homotopy Category via Fibrant-Cofibrant Objects). To correctly construct the homotopy category  $\text{Ho}(\mathcal{M})$  from a simplicial model category  $\mathcal{M}$ , one cannot simply take the path components  $\pi_0$  of the mapping spaces between arbitrary objects.

Instead, one must restrict attention to the full subcategory of **fibrant-cofibrant objects**, denoted  $\mathcal{M}_{cf}$ . It is only within this subcategory that the simplicial mapping spaces  $\text{Map}_{\mathcal{M}}(X, Y)$  are guaranteed to be Kan complexes representing the correct derived mapping spaces. The morphisms in the homotopy category are thus given by:

$$[X, Y]_{\text{Ho}(\mathcal{M})} \cong \pi_0 \text{Map}_{\mathcal{M}}(X, Y) \quad \text{for } X, Y \in \mathcal{M}_{cf}.$$

For general objects  $X, Y$ , one must first replace them with weakly equivalent fibrant-cofibrant objects (via cofibrant replacement  $QX$  and fibrant replacement  $RY$ ) to compute this group.

## 1.2 Stable $\infty$ -Category

**Definition 1.10** (Loop Object). For any object  $X \in \mathcal{C}$ , the loop object  $\Omega X$  is the limit of the diagram  $0 \rightarrow X \leftarrow 0$ . It fits into the following pullback square:

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & X \end{array}$$

Intuitively,  $\Omega X \simeq 0 \times_X 0$ .

**Definition 1.11** (Suspension Object). For any object  $X \in \mathcal{C}$ , the suspension object  $\Sigma X$  is the colimit of the diagram  $0 \leftarrow X \rightarrow 0$ . It fits into the following pushout square:

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \S \\ 0 & \longrightarrow & \Sigma X \end{array}$$

Intuitively,  $\Sigma X \simeq 0 \amalg_X 0$ .

**Definition 1.12** (Stable  $\infty$ -Category). An  $\infty$ -category  $\mathcal{C}$  is called **stable** if it satisfies the following conditions:

1. There exists a zero object  $0 \in \mathcal{C}$  (i.e.,  $\mathcal{C}$  is pointed).
2. Every morphism in  $\mathcal{C}$  admits a kernel and a cokernel.
3. A triangle in  $\mathcal{C}$  is a pushout square if and only if it is a pullback square.

**Definition 1.13** (The Stabilization of an  $\infty$ -Category). Let  $\mathcal{C}$  be an  $\infty$ -category admitting finite limits. The process of constructing the stable  $\infty$ -category associated to  $\mathcal{C}$  proceeds in two stages:

### 1. Pointed View (Formation of $\mathcal{C}_*$ ):

First, we construct the *pointed*  $\infty$ -category  $\mathcal{C}_*$ . Assuming  $\mathcal{C}$  has a terminal object  $*$ ,  $\mathcal{C}_*$  is defined as the under-category of the terminal object:

$$\mathcal{C}_* := \mathcal{C}_{*/} \cong \text{Fun}(\Delta^1, \mathcal{C}) \times_{\text{Fun}(\{0\}, \mathcal{C})} \{*\}$$

Objects in  $\mathcal{C}_*$  are morphisms  $* \rightarrow X$  in  $\mathcal{C}$  (i.e., objects equipped with a base point). In  $\mathcal{C}_*$ , the object corresponding to the identity  $* \rightarrow *$  serves as a *zero object* (both initial and terminal). Consequently, the loop functor  $\Omega : \mathcal{C}_* \rightarrow \mathcal{C}_*$  is well-defined by  $\Omega X = * \times_X *$ .

**2. Stabilization (Formation of  $\text{Sp}(\mathcal{C})$ ):**

The *stabilization* of  $\mathcal{C}$ , denoted as  $\text{Sp}(\mathcal{C})$  (or  $\text{Stab}(\mathcal{C})$ ), is defined as the  $\infty$ -category of spectrum objects in  $\mathcal{C}_*$ . It is constructed as the homotopy limit of the tower of loop functors:

$$\text{Sp}(\mathcal{C}) := \varprojlim \left( \dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$

Explicitly, an object  $E \in \text{Sp}(\mathcal{C})$  consists of a sequence  $\{E_n\}_{n \geq 0}$  of objects in  $\mathcal{C}_*$  together with equivalences  $E_n \xrightarrow{\sim} \Omega E_{n+1}$  for each  $n$ . This construction forces the suspension functor  $\Sigma$  to be an equivalence, rendering  $\text{Sp}(\mathcal{C})$  a stable  $\infty$ -category.

**Theorem 1.14** (Universal Property of Stabilization). Let  $\mathcal{C}$  be an  $\infty$ -category with finite limits and a terminal object  $*$ . Let  $\mathcal{C}_* = \mathcal{C}_{*/}$  be its pointed version. The stabilization  $\text{Sp}(\mathcal{C})$  is characterized by the following equivalent descriptions:

1. **Internal Construction (Loop Towers):**  $\text{Sp}(\mathcal{C})$  is the homotopy limit of the sequence of loop functors:

$$\text{Sp}(\mathcal{C}) \simeq \varprojlim \left( \dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$

An object in  $\text{Sp}(\mathcal{C})$  is an  $\Omega$ -spectrum, i.e., a sequence  $\{E_n\}_{n \geq 0}$  in  $\mathcal{C}_*$  with equivalences  $E_n \simeq \Omega E_{n+1}$ .

2. **External Construction (Excision):**  $\text{Sp}(\mathcal{C})$  is equivalent to the  $\infty$ -category of pointed excisive functors from the category of finite pointed spaces  $\mathcal{S}_*^{\text{fin}}$  to  $\mathcal{C}$ :

$$\text{Sp}(\mathcal{C}) \simeq \text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$$

A functor  $F : \mathcal{S}_*^{\text{fin}} \rightarrow \mathcal{C}$  belongs to  $\text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$  if  $F(*) \simeq *$  and  $F$  maps every pushout square in  $\mathcal{S}_*^{\text{fin}}$  to a pullback square in  $\mathcal{C}$ .

Furthermore, the stabilization  $\text{Sp}(\mathcal{C})$  is the universal stable  $\infty$ -category under  $\mathcal{C}$ : for any stable  $\infty$ -category  $\mathcal{D}$ , the functor  $\text{Sp}(\mathcal{C}) \rightarrow \mathcal{D}$  induces an equivalence of  $\infty$ -categories  $\text{Fun}^{\text{lex}}(\text{Sp}(\mathcal{C}), \mathcal{D}) \simeq \text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{D})$ , where  $\text{Fun}^{\text{lex}}$  denotes the  $\infty$ -category of left exact functors.

**Remark 1.15** (Distinction between  $\text{Stab}(\mathcal{C})$  and  $\text{Sp}$ ). It is essential to distinguish between the abstract process of stabilization and the specific category of spectra:

1. **The Category of Spectra ( $\text{Sp}$ ):** Historically and by convention,  $\text{Sp}$  refers specifically to the stabilization of the  $\infty$ -category of pointed spaces  $\mathcal{S}_*$ . That is:

$$\text{Sp} \simeq \text{Stab}(\mathcal{S}) \simeq \varprojlim \left( \dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right)$$

This category serves as the unit object in the  $\infty$ -category of stable  $\infty$ -categories and provides the ground for stable homotopy theory.

2. **Stabilization of an Arbitrary  $\infty$ -Category ( $\text{Stab}(\mathcal{C})$ ):** For any  $\infty$ -category  $\mathcal{C}$  with finite limits,  $\text{Stab}(\mathcal{C})$  is the stable  $\infty$ -category constructed as the limit of the tower of loop functors:

$$\text{Stab}(\mathcal{C}) = \varprojlim \left( \dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$

While  $\text{Stab}(\mathcal{C})$  is always a stable  $\infty$ -category, it may not possess a symmetric monoidal structure (like the smash product) unless  $\mathcal{C}$  itself is equipped with a compatible monoidal structure.

3. **The Relation:** Any stable  $\infty$ -category  $\mathcal{D}$  is naturally tensored over  $\text{Sp}$ . In this sense,  $\text{Sp}$  plays a role analogous to the ring of integers  $\mathbb{Z}$  in abelian groups: for any  $D \in \mathcal{D}$  and  $E \in \text{Sp}$ , there is a well-defined object  $E \otimes D \in \mathcal{D}$ .

**Definition 1.16** (Internal Mapping Spectrum in  $\text{Stab}(\mathcal{C})$ ). Let  $\mathcal{C}$  be a **closed symmetric monoidal  $\infty$ -category**  $(\mathcal{C}, \otimes, \mathbf{1})$  that admits finite limits. Assume further that the tensor product  $\otimes$  is compatible with the stabilization (i.e., it preserves colimits in each variable).

Let  $\mathcal{D} = \text{Stab}(\mathcal{C})$  be the resulting stable  $\infty$ -category, equipped with the induced symmetric monoidal structure  $\otimes_{\mathcal{D}}$ . For any objects  $X, Y \in \mathcal{D}$ , the **internal mapping spectrum** is defined as the object  $\underline{\text{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$  that satisfies the following conditions:

1. **Adjunction Property:** It is the right adjoint to the tensor product functor. For any object  $Z \in \mathcal{D}$ , there is a natural equivalence of mapping spaces:

$$\underline{\text{Map}}_{\mathcal{D}}(Z \otimes_{\mathcal{D}} X, Y) \simeq \underline{\text{Map}}_{\mathcal{D}}(Z, \underline{\text{Map}}_{\mathcal{D}}(X, Y))$$

2. **Spectrum Level Structure:** In terms of the sequence of objects  $\{E_n\} \in \mathcal{C}_*$  representing the spectrum, the  $n$ -th level of the internal mapping spectrum is given by:

$$\underline{\text{Map}}_{\mathcal{D}}(X, Y)_n \simeq \underline{\text{Map}}_{\mathcal{C}}(X, \Sigma^n Y)$$

where  $\underline{\text{Map}}_{\mathcal{C}}$  denotes the internal Hom in the underlying category  $\mathcal{C}$  (if it exists) or the corresponding enrichment.

**Definition 1.17** (Mapping Spectrum). Let  $X$  and  $Y$  be spectra in  $\text{Sp}$ . The **mapping spectrum** from  $X$  to  $Y$ , denoted as  $\underline{\text{Map}}(X, Y) \in \text{Sp}$ , is the unique spectrum (up to equivalence) characterized by the following properties:

1. **Adjunction (Internal Hom):** For any spectrum  $Z$ , there is a natural equivalence of mapping spaces:

$$\underline{\text{Map}}_{\text{Sp}}(Z \wedge X, Y) \simeq \underline{\text{Map}}_{\text{Sp}}(Z, \underline{\text{Map}}(X, Y))$$

This identifies  $\underline{\text{Map}}(X, Y)$  as the right adjoint to the smash product functor  $(-\wedge X)$ .

2. **Omega-Spectrum Structure:** The  $n$ -th space of the mapping spectrum is equivalent to the space of maps from  $X$  to the  $n$ -th suspension of  $Y$ :

$$\underline{\text{Map}}(X, Y)_n \simeq \underline{\text{Map}}_{\text{Sp}}(X, \Sigma^n Y)$$

The structure maps  $\Sigma \underline{\text{Map}}(X, Y)_n \rightarrow \underline{\text{Map}}(X, Y)_{n+1}$  are induced by the stability of  $\text{Sp}$ .

**Definition 1.18** (Homotopy Groups in  $\text{Stab}(\mathcal{C})$ ). Let  $\mathcal{C}$  be a closed symmetric monoidal  $\infty$ -category with finite limits, and let  $\mathcal{D} = \text{Stab}(\mathcal{C})$  be its stabilization with unit object  $\mathbf{1}_{\mathcal{D}}$ .

1. **Homotopy Groups of an Object:** For any object  $E \in \mathcal{D}$  and  $n \in \mathbb{Z}$ , the  $n$ -th homotopy group of  $E$  is defined as the abelian group of homotopy classes of maps from the  $n$ -shifted unit object:

$$\pi_n(E) := [\Sigma^n \mathbf{1}_{\mathcal{D}}, E]_{\mathcal{D}}$$

2. **Homotopy Groups of the Mapping Spectrum:** Let  $\underline{\text{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$  be the internal mapping spectrum between  $X, Y \in \mathcal{D}$ . Its homotopy groups characterize the graded morphisms between the two objects:

$$\pi_n \underline{\text{Map}}_{\mathcal{D}}(X, Y) \cong [X, \Sigma^n Y]_{\mathcal{D}} \cong [\Sigma^{-n} X, Y]_{\mathcal{D}}$$

where  $[-, -]_{\mathcal{D}}$  denotes the set of homotopy classes i.e. the 0-th homotopy group of the kan complex  $\underline{\text{Map}}_{\mathcal{D}}(-, -)$ .

**Remark 1.19.** The distinction lies in the target category:

- $\text{Map}_{\mathcal{D}}(X, Y) \in \mathcal{S}$  is a **space** (Kan complex). It represents the mapping space in the  $\infty$ -categorical sense.
- $\underline{\text{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$  is a **spectrum** (Internal Hom). It is an object of the stable category  $\mathcal{D}$  that stabilizes the mapping space.

In terms of homotopy groups:  $\pi_n \text{Map}_{\mathcal{D}}(X, Y) \cong \pi_n \underline{\text{Map}}_{\mathcal{D}}(X, Y)$  for  $n \geq 0$ , because every spectrum is  $\Omega$ -spectrum in  $\text{Stab}(\mathcal{C})$ .

**Construction 1.20** (Stabilization of a Suspension Spectrum). Let  $X \in \mathcal{S}_*$  be a pointed space (or generally an object in a pointed  $\infty$ -category  $\mathcal{C}$  with finite colimits). The construction of its associated **suspension spectrum**  $\Sigma^\infty X \in \text{Sp}$  proceeds as follows:

1. **The Prespectrum Construction:** First, we form a *prespectrum*  $P_X$  by iterating the suspension functor  $\Sigma$  on  $X$ . This is given by the sequence of spaces:

$$(P_X)_n := \Sigma^n X, \quad \text{for } n \geq 0$$

together with the structural maps (identities):

$$\sigma_n : \Sigma((P_X)_n) = \Sigma(\Sigma^n X) \xrightarrow{id} \Sigma^{n+1} X = (P_X)_{n+1}$$

2. **Spectrification (The L-functor):** Since  $P_X$  is not necessarily an  $\Omega$ -spectrum (i.e., the adjoint maps  $(P_X)_n \rightarrow \Omega(P_X)_{n+1}$  are not equivalences), we apply the *spectrification functor*  $L$  (or stabilization) to convert it into a true spectrum. The resulting object is the suspension spectrum:

$$\Sigma^\infty X := L(P_X)$$

Conceptually, the  $k$ -th space of this stable object is the colimit:

$$(\Sigma^\infty X)_k \simeq \underset{m \rightarrow \infty}{\text{colim}} \Omega^m \Sigma^{m+k} X$$

3. **Universal Property (Adjunction):** The construction defines the left adjoint functor  $\Sigma^\infty$  in the stabilization adjunction:

$$\begin{array}{ccc} & \Sigma^\infty & \\ \mathcal{S}_* & \begin{array}{c} \nearrow \\ \searrow \end{array} & \text{Sp} \\ & \Omega^\infty & \end{array}$$

where for any spectrum  $E$ , the right adjoint is given by  $\Omega^\infty E := E_0$  (the 0-th space of the  $\Omega$ -spectrum  $E$ ).

**Remark 1.21** (Bousfield-Friedlander Structure and Stabilization). The Bousfield-Friedlander model structure  $\mathcal{M}_{\text{BF}}$  on the category of prespectra is the left Bousfield localization of the strict model structure  $\mathcal{M}_{\text{strict}}$ . The three classes of morphisms in  $\mathcal{M}_{\text{BF}}$  are characterized as follows:

- **Cofibrations:** These are exactly the same as the strict cofibrations (levelwise inclusions that satisfy the appropriate cell complex conditions).
- **Weak Equivalences:** These are the *stable weak equivalences*, i.e., maps  $f : X \rightarrow Y$  that induce isomorphisms on stable homotopy groups  $\pi_n^S(X) \cong \pi_n^S(Y)$  for all  $n \in \mathbb{Z}$ .

- **Fibrations:** These are the maps that satisfy the right lifting property with respect to acyclic cofibrations. Specifically, a map  $p : E \rightarrow B$  is a BF-fibration if it is a levelwise fibration and the square

$$\begin{array}{ccc} E_n & \longrightarrow & \Omega E_{n+1} \\ \downarrow p_n & & \downarrow \Omega p_{n+1} \\ B_n & \longrightarrow & \Omega B_{n+1} \end{array}$$

is a homotopy pullback for all  $n$ .

The transition from  $\mathcal{M}_{\text{strict}}$  to  $\mathcal{M}_{\text{BF}}$  captures the essence of stabilization. Since the fibrant objects in this structure are exactly the  $\Omega$ -spectra, the **fibrant replacement** of a prespectrum  $X$  in  $\mathcal{M}_{\text{BF}}$  is precisely its **stabilization** (spectrification).

If  $R_{\text{BF}}$  denotes the fibrant replacement functor, we have a natural stable equivalence  $j : X \xrightarrow{\sim} R_{\text{BF}}(X)$ , where  $R_{\text{BF}}(X)$  is an  $\Omega$ -spectrum. In the stable homotopy category, this is equivalent to the classical stabilization  $QX = \Omega^\infty \Sigma^\infty X$ :

$$\begin{array}{ccc} X & \xrightarrow{j} & R_{\text{BF}}(X) \\ \parallel & & \downarrow \simeq \\ X & \xrightarrow{\text{Stabilization}} & QX \end{array}$$

Thus, the Bousfield-Friedlander model structure provides the formal homotopy-theoretic machinery where "becoming an  $\Omega$ -spectrum" is equivalent to "becoming fibrant."

**Definition 1.22.** The **homotopy category functor**  $ho$  is the change-of-base functor induced by the path-components functor  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$ . For a category  $\mathcal{C}$  enriched over simplicial sets,  $ho(\mathcal{C})$  is the **Set**-enriched category with the same objects as  $\mathcal{C}$  and morphism sets defined by:

$$\text{Hom}_{ho(\mathcal{C})}(X, Y) = \pi_0(\text{Map}_{\mathcal{C}}(X, Y))$$

Composition in  $ho(\mathcal{C})$  is inherited from the enriched composition in  $\mathcal{C}$  via the product-preserving property of  $\pi_0$ .

**Proposition 1.23.** Let  $\mathcal{C}$  be a locally Kan simplicial category. Let  $N_\Delta$  denote the homotopy coherent nerve and  $N$  denote the classical nerve. There is a natural isomorphism of simplicial sets (or categories):

$$ho(N_\Delta(\mathcal{C})) \cong N(ho(\mathcal{C}))$$

**Example 1.24** (The Stable Homotopy Category). Let  $\mathcal{Sp}$  be the stable  $\infty$ -category of spectra. The classical **stable homotopy category**  $\text{SHC}$  is precisely its homotopy category:

$$\text{SHC} \cong ho(\mathcal{Sp})$$

Under the  $ho$  functor, the enriched mapping spaces  $\text{Map}_{\mathcal{Sp}}(X, Y)$  are replaced by their sets of path-components  $\pi_0$ . The stable property of  $\mathcal{Sp}$  (the equivalence of fiber and cofiber sequences) ensures that  $ho(\mathcal{Sp})$  inherits the structure of a **triangulated category**.

### 1.3 $\infty$ -Operad

**Definition 1.25** ( $\infty$ -Operator Category). An  **$\infty$ -operator category** is an  $\infty$ -category  $\mathcal{B}$  equipped with a specified subcategory of *inert morphisms*  $\mathcal{B}^{\text{inert}}$ , satisfying the following two structural axioms illustrated by commutative diagrams:

- 1. Active-Inert Factorization:** There exists a class of *active morphisms* such that every morphism  $f : X \rightarrow Z$  in  $\mathcal{B}$  factors essentially uniquely as an active morphism followed by an inert morphism.

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & \searrow^{f^{\text{act}} \text{ (Mix)}} & \swarrow^{f^{\text{inert}} \text{ (Select)}} \\ & Y & \end{array}$$

Here,  $f^{\text{act}}$  performs the operations (combining inputs), and  $f^{\text{inert}}$  performs the structural projection.

- 2. Elementary Decomposition:** There exists a set of *elementary objects* (or colors)  $\{U_\alpha\}$ . Every object  $X \in \mathcal{B}$  is determined by its inert projections to these elementary objects. Specifically, the collection of inert morphisms  $\{\rho^i\}$  exhibits  $X$  as a product:

$$\begin{array}{ccc} X & & \\ \rho^1 \swarrow & \downarrow \rho^2 & \searrow \rho^n \\ U_{i_1} & U_{i_2} & U_{i_n} \end{array}$$

This implies an equivalence  $X \simeq U_{i_1} \times U_{i_2} \times \dots \times U_{i_n}$ , ensuring that complex objects are merely aggregates of elementary slots.

**Definition 1.26** ( $\infty$ -Operator Category). An  **$\infty$ -operator category** is an  $\infty$ -category  $\mathcal{B}$  equipped with a specified factorization structure, consisting of two subcategories: *active morphisms* ( $\mathcal{B}^{\text{act}}$ ) and *inert morphisms* ( $\mathcal{B}^{\text{inert}}$ ). These satisfy the following two axioms:

- 1. Active-Inert Factorization System:** The pair  $(\mathcal{B}^{\text{act}}, \mathcal{B}^{\text{inert}})$  forms an *orthogonal factorization system* on  $\mathcal{B}$ .

This means that every morphism  $f : X \rightarrow Z$  in  $\mathcal{B}$  factors essentially uniquely as an active morphism followed by an inert morphism:

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & \searrow^{f^{\text{act}}} & \swarrow^{f^{\text{inert}}} \\ & Y & \end{array}$$

Here, the intermediate object  $Y$  and the active map  $f^{\text{act}}$  are not arbitrary; they are **determined** by the image of  $f$  under this factorization system.

- 2. Elementary Decomposition (Segal Core):** There exists a set of elementary objects  $\mathcal{E} \subset \text{Ob}(\mathcal{B})$ . For any object  $X \in \mathcal{B}$ , let  $\Lambda_X$  be the set of all inert morphisms targeting  $\mathcal{E}$ :

$$\Lambda_X := \{\rho : X \rightarrow U \mid \rho \in \mathcal{B}^{\text{inert}}, U \in \mathcal{E}\}$$

We require that the canonical map induced by these morphisms is an equivalence:

$$X \xrightarrow{\sim} \prod_{\rho \in \Lambda_X} \text{codom}(\rho)$$

**Definition 1.27** ( $\mathcal{B}$ -Operad). Let  $\mathcal{B}$  be an  $\infty$ -operator category. A  **$\mathcal{B}$ -operad** is a map of simplicial sets  $p : \mathcal{C}^\otimes \rightarrow \mathcal{B}$  satisfying the following three conditions:

1. **Inner Fibration:** The map  $p$  is an inner fibration of simplicial sets. That is, for every  $0 < k < n$ ,  $p$  has the right lifting property with respect to the inner horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$ .
2. **Inert Lifting Property:** For every inert morphism  $f : X \rightarrow Y$  in  $\mathcal{B}$  and every object  $C \in \mathcal{C}^\otimes$  such that  $p(C) = X$ , there exists a  $p$ -coCartesian edge  $\bar{f} : C \rightarrow C'$  in  $\mathcal{C}^\otimes$  such that  $p(\bar{f}) = f$ .
3. **Segal Condition:** For every object  $X \in \mathcal{B}$ , let  $\{X \xrightarrow{f_i} U_i\}_{i \in I}$  be the collection of inert morphisms decomposing  $X$  into elementary objects (as dictated by the structure of  $\mathcal{B}$ ). The functor induced by the  $p$ -coCartesian lifts of these inert morphisms,

$$\mathcal{C}_X^\otimes \xrightarrow{\sim} \prod_{i \in I} \mathcal{C}_{U_i}^\otimes,$$

is an equivalence of  $\infty$ -categories.

**Definition 1.28** ( $\mathcal{O}$ -Algebra Object). Let  $\mathcal{B}$  be an  $\infty$ -operator category. Let  $p : \mathcal{O}^\otimes \rightarrow \mathcal{B}$  and  $q : \mathcal{C}^\otimes \rightarrow \mathcal{B}$  be  $\mathcal{B}$ -operad. An  **$\mathcal{O}$ -algebra object in  $\mathcal{C}$**  is a map of  $\infty$ -operads over  $\mathcal{B}$ . Explicitly, it is a functor

$$A : \mathcal{O}^\otimes \longrightarrow \mathcal{C}^\otimes$$

satisfying two conditions:

1. **Commutativity over Base (Compatibility):** The functor  $A$  respects the projection to the base category  $\mathcal{B}$ . The following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}^\otimes & \xrightarrow{A} & \mathcal{C}^\otimes \\ p \searrow & & \swarrow q \\ & \mathcal{B} & \end{array}$$

(i.e.,  $q \circ A = p$ ).

2. **Inert Preservation (The Operad Map Condition):** The functor  $A$  carries inert morphisms to inert morphisms.

Specifically, if  $f$  is an *inert morphism* in  $\mathcal{O}^\otimes$  (meaning  $f$  is a  $p$ -coCartesian lift of an inert map in  $\mathcal{B}$ ), then its image  $A(f)$  must be an *inert morphism* in  $\mathcal{C}^\otimes$  (meaning  $A(f)$  is a  $q$ -coCartesian lift of that same map in  $\mathcal{B}$ ).

The  $\infty$ -category of all such algebras is denoted by  $\text{Alg}_{\mathcal{O}}(\mathcal{C})$ .

**Example 1.29 (The Zoo of Operads and Their Bases).** We classify common algebraic structures by specifying the underlying **Base Category**  $\mathcal{B}$  (which dictates the geometry of inputs) and the **Operad**  $\mathcal{O}^\otimes$  (which dictates the operations) as a fibration  $p : \mathcal{O}^\otimes \rightarrow \mathcal{B}$ .

### 1. The Commutative Case ( $E_\infty$ )

- **Base:**  $\mathcal{B} = N(\text{Fin}_*)$  (Symmetric/Unordered inputs).
- **Operad:**  $\mathcal{O}^\otimes = \text{Comm}^\otimes := N(\text{Fin}_*)$ .
- **Structure Map:** The identity map  $\text{id} : N(\text{Fin}_*) \rightarrow N(\text{Fin}_*)$ .
- **Resulting Algebra: Commutative  $\infty$ -Algebra.**

- **Note:** Since the map is the identity, the fiber over any operation is a point. There is essentially only one way to combine inputs (order doesn't matter).

## 2. The Associative Case ( $A_\infty$ )

- **Base:**  $\mathcal{B} = N(\text{Fin}_*)$ .
- **Operad:**  $\mathcal{O}^\otimes = \text{Ass}^\otimes$  (The Associative Operad).
- **Structure Map:** The "forgetful" functor that forgets the linear ordering of the fibers.
- **Resulting Algebra:** **Associative  $\infty$ -Algebra**.
- **Note:** The fiber over  $\langle n \rangle \rightarrow \langle 1 \rangle$  is equivalent to the symmetric group  $\Sigma_n$ . This allows inputs to be permuted (by the base), but the operation distinguishes the order of multiplication ( $x_1x_2 \neq x_2x_1$ ).

## 3. The Little $k$ -Disks Case ( $E_k$ )

- **Base:**  $\mathcal{B} = N(\text{Fin}_*)$ .
- **Operad:**  $\mathcal{O}^\otimes = \mathbb{E}_k^\otimes$ .
- **Structure Map:** The projection from the space of disk embeddings.
- **Resulting Algebra:**  $E_k$ -**Algebra**.
- **Note:** Interpolates between Associative ( $k = 1$ ) and Commutative ( $k = \infty$ ).

## 4. The Lie Case ( $L_\infty$ )

- **Base:**  $\mathcal{B} = N(\text{Fin}_*)$ .
- **Operad:**  $\mathcal{O}^\otimes = \text{Lie}^\otimes$ .
- **Resulting Algebra:**  $L_\infty$ -**Algebra** (Homotopy Lie Algebra).
- **Note:** Typically considered over a stable target category (like chain complexes).

## 5. The Non-Symmetric / Planar Case

- **Base:**  $\mathcal{B} = N(\Delta)^{op}$  (The Simplex Category; Linear/Ordered inputs).
- **Operad:**  $\mathcal{O}^\otimes = N(\Delta)^{op}$ .
- **Structure Map:** The identity map.
- **Resulting Algebra:** **Associative Monoid** (in the strict sense).
- **Note:** Here, the base category itself forbids permutation. There is no symmetric group action to even consider.

**Definition 1.30** (Endomorphism  $\infty$ -Category). Let  $\mathcal{C}$  be an  $\infty$ -category. The **Endomorphism  $\infty$ -Category**, denoted by  $\text{End}(\mathcal{C})$ , is defined as the functor  $\infty$ -category  $\text{Fun}(\mathcal{C}, \mathcal{C})$ .

It forms a **monoidal  $\infty$ -category** where the monoidal structure is determined by the composition of endofunctors:

1. The tensor product is given by composition:  $F \otimes G := F \circ G$ .
2. The unit object is given by the identity functor:  $\mathbb{I} := \text{Id}_{\mathcal{C}}$ .

**Definition 1.31** (Spaces of Monads and Comonads). Let  $\mathcal{C}$  be an  $\infty$ -category, and let  $\text{End}(\mathcal{C})$  denote the monoidal  $\infty$ -category of endofunctors on  $\mathcal{C}$  (equipped with the composition product). The  $\infty$ -categories (or spaces) of Monads and Comonads as the categories of associative algebra objects in  $\text{End}(\mathcal{C})$  and its opposite, respectively:

1. The  **$\infty$ -category of Monads** is defined as:

$$\text{Mnd}(\mathcal{C}) := \text{Alg}_{\mathcal{A}\text{ss}}(\text{End}(\mathcal{C}))$$

2. The  **$\infty$ -category of Comonads** is defined as:

$$\text{CoMnd}(\mathcal{C}) := \text{Alg}_{\mathcal{A}\text{ss}}(\text{End}(\mathcal{C})^{op})$$

The objects of  $\text{Mnd}(\mathcal{C})$  are referred to as **Monads** on  $\mathcal{C}$ , and the objects of  $\text{CoMnd}(\mathcal{C})$  are referred to as **Comonads** on  $\mathcal{C}$ .

**Definition 1.32** (Reedy Category). A small category  $\mathcal{R}$  is a **Reedy category** if it is equipped with a degree function  $d : \text{Ob}(\mathcal{R}) \rightarrow \lambda$  (where  $\lambda$  is an ordinal) and two subcategories  $\vec{\mathcal{R}}$  (the direct category) and  $\overleftarrow{\mathcal{R}}$  (the inverse category), such that:

1. Every non-identity morphism in  $\vec{\mathcal{R}}$  raises the degree.
2. Every non-identity morphism in  $\overleftarrow{\mathcal{R}}$  lowers the degree.
3. Every morphism  $f$  in  $\mathcal{R}$  factors uniquely as  $f = g \circ h$ , where  $h \in \overleftarrow{\mathcal{R}}$  and  $g \in \vec{\mathcal{R}}$ .

**Definition 1.33** (The Reedy Model Structure). Let  $\mathcal{M}$  be a model category and  $\mathcal{R}$  be a Reedy category. The category of diagrams  $\text{Fun}(\mathcal{R}, \mathcal{M})$  is equipped with the **Reedy model structure**, where a morphism  $f : X \rightarrow Y$  is defined to be:

1. A **Weak Equivalence** if it is a levelwise weak equivalence. That is, for every object  $\alpha \in \mathcal{R}$ , the map  $f_\alpha : X_\alpha \rightarrow Y_\alpha$  is a weak equivalence in  $\mathcal{M}$ .
2. A **Cofibration** if for every  $\alpha \in \mathcal{R}$ , the **relative latching map**  $\lambda_\alpha(f)$  is a cofibration in  $\mathcal{M}$ . Here,  $\lambda_\alpha(f)$  is the map induced by the pushout of the latching objects:

$$\lambda_\alpha(f) : X_\alpha \amalg_{L_\alpha X} L_\alpha Y \longrightarrow Y_\alpha$$

where the *latching object* is defined as  $L_\alpha X = \text{colim}_{\partial \vec{\mathcal{R}}/\alpha} X$ .

3. A **Fibration** if for every  $\alpha \in \mathcal{R}$ , the **relative matching map**  $\mu_\alpha(f)$  is a fibration in  $\mathcal{M}$ . Here,  $\mu_\alpha(f)$  is the map induced into the pullback of the matching objects:

$$\mu_\alpha(f) : X_\alpha \longrightarrow M_\alpha X \times_{M_\alpha Y} Y_\alpha$$

where the *matching object* is defined as  $M_\alpha X = \lim_{\alpha/\partial \vec{\mathcal{R}}} X$ .

**Proposition 1.34** (Latch and Matching as Monadic Structures). Let  $\mathcal{R}$  be a Reedy category. For any degree  $\alpha$ , consider the truncation inclusion of the category of degrees strictly lower than  $\alpha$ :

$$u : \mathcal{R}_{<\alpha} \hookrightarrow \mathcal{R}_{\leq \alpha}$$

Let  $u^*$  be the restriction functor. We identify the Latching and Matching objects via the adjunctions defining the skeleton and coskeleton:

1. **Latching as a Monad (Skeleton):** The pair  $u_! \dashv u^*$  generates a **Monad**  $T = u_! \circ u^*$  (Left Kan Extension followed by restriction). The Latching object is the value of this monad:

$$L_\alpha X \cong (T(X))_\alpha = (\text{Lan}_u(u^* X))_\alpha$$

The canonical map  $L_\alpha X \rightarrow X_\alpha$  corresponds to the monad algebra structure map (or the counit of the adjunction).

2. **Matching as a Comonad (Coskeleton):** The pair  $u^* \dashv u_*$  generates a **Comonad**  $G = u_* \circ u^*$  (Right Kan Extension followed by restriction). The Matching object is the value of this comonad:

$$M_\alpha X \cong (G(X))_\alpha = (\text{Ran}_u(u^* X))_\alpha$$

The canonical map  $X_\alpha \rightarrow M_\alpha X$  corresponds to the comonad coalgebra structure map (or the unit of the adjunction).

**Definition 1.35** (Monadic and Comonadic Resolution). Let  $\mathcal{C}$  be a category and  $X$  an object in  $\mathcal{C}$ . We define the canonical resolutions generated by monads and comonads as follows:

1. **Monadic Resolution:** Let  $(T, \mu, \eta)$  be a **Monad** on  $\mathcal{C}$ . The **Bar Construction** provides an augmented simplicial object  $B_\bullet(X)$  resolving  $X$ :

$$\dots \rightrightarrows T^3 X \rightleftarrows T^2 X \rightrightarrows TX \xrightarrow{\epsilon} X$$

The face maps  $d_i$  are given by the multiplication  $\mu$ , and degeneracy maps  $s_i$  by the unit  $\eta$ . This construction typically serves as a **cofibrant replacement** of  $X$ .

2. **Comonadic Resolution:** Let  $(G, \delta, \epsilon)$  be a **Comonad** on  $\mathcal{C}$ . The **Cobar Construction** provides an augmented cosimplicial object  $C^\bullet(X)$  resolving  $X$ :

$$X \xrightarrow{\eta} GX \rightrightarrows G^2 X \rightleftarrows G^3 X \rightrightarrows \dots$$

The coface maps  $d^i$  are given by the comultiplication  $\delta$ , and codegeneracy maps  $s^i$  by the counit  $\epsilon$ . This construction typically serves as a **fibrant replacement** of  $X$ .

**Example 1.36 (Reedy Latching and Matching).** In a Reedy category  $\mathcal{R}$ , resolutions arise from Kan extensions along the filtration  $u : \mathcal{R}_{<\alpha} \hookrightarrow \mathcal{R}_{\leq\alpha}$ .

- **Latching (Monad):** The Latching object  $L_\alpha X$  is generated by the **Skeleton Monad**  $T = u_! u^*$  (Left Kan extension followed by restriction).

$$L_\alpha X \cong (TX)_\alpha$$

- **Matching (Comonad):** The Matching object  $M_\alpha X$  is generated by the **Coskeleton Comonad**  $G = u_* u^*$  (Right Kan extension followed by restriction).

$$M_\alpha X \cong (GX)_\alpha$$

**Example 1.37 (The Cotangent Complex (André-Quillen)).** Used to define the derived cotangent complex  $\mathbb{L}_{A/k}$ .

- **Monad:** The **Free Algebra Monad**  $T$  on the category of  $k$ -modules (or sets).

$$T(V) = \text{Sym}_k(V) \quad (\text{Polynomial Algebra})$$

- **Resolution:** The simplicial resolution  $P_\bullet \rightarrow A$  is the Bar construction  $B_\bullet(T, T, A)$ . The cotangent complex is derived from applying differentials  $\Omega_{P_\bullet/k}^1 \otimes_{P_\bullet} A$ .

**Example 1.38 (The Postnikov Tower).** Decomposing a space  $X$  into its homotopy types.

- **Monad:** The  $n$ -Truncation Monad  $\tau_{\leq n}$  (or  $P_n$ ).

$$T_n(X) = \tau_{\leq n}(X)$$

This is an *idempotent* monad (localization). The tower is the limit sequence  $\cdots \rightarrow T_n X \rightarrow T_{n-1} X$ .

- *Note:* Dually, the **Whitehead Tower** uses the  $n$ -connected cover Comonad  $\tau_{>n}$ .

**Example 1.39 (Projective and Injective Resolution).** Let  $R$  be a ring and  $M$  an  $R$ -module.

1. **Projective Resolution (The Bar Construction):** Using the free-forgetful adjunction  $F \dashv U$ , we define the **Free Monad**  $T = F \circ U$ . Since  $T(M)$  is a free module, the associated Bar construction yields a canonical projective resolution:

$$\cdots \rightarrow T^3 M \rightarrow T^2 M \rightarrow TM \xrightarrow{\epsilon} M \rightarrow 0$$

The boundary maps are alternating sums of the monad multiplication  $\mu : T^2 \rightarrow T$ .

2. **Injective Resolution (The Cobar Construction):** Using the forgetful-cofree adjunction  $U \dashv C$  (where  $C(A) = \text{Hom}_{\mathbb{Z}}(R, A)$ ), we define the **Cofree Comonad**  $G = C \circ U$ . Since  $G(M)$  is an injective module, the associated Cobar construction yields a canonical injective resolution:

$$0 \rightarrow M \xrightarrow{\eta} GM \rightarrow G^2 M \rightarrow G^3 M \rightarrow \dots$$

The boundary maps are alternating sums of the comonad comultiplication  $\delta : G \rightarrow G^2$ .

**Example 1.40 (The Spectrification Monad on Prespectra).** Let  $\mathcal{P}$  be the category of Prespectra (sequences of spaces with maps  $\Sigma E_n \rightarrow E_{n+1}$ ). The process of converting a naive suspension spectrum into a genuine  $\Omega$ -spectrum is governed by the **Spectrification Monad**  $\mathbb{L}$ .

1. **The Level-wise Monad:** We define a Monad  $\mathbb{L} : \mathcal{P} \rightarrow \mathcal{P}$  by applying the spatial stabilization monad  $Q = \Omega^\infty \Sigma^\infty$  to *each level* of the spectrum independently:

$$(\mathbb{L}E)_n := Q(E_n) = \operatorname{colim}_k \Omega^k \Sigma^k E_n$$

2. **Application to Suspension Spectra:** If  $E = \Sigma^\infty X$  is the suspension spectrum of  $X$  (where  $E_n = \Sigma^n X$ ), applying this monad yields:

$$(\mathbb{L}(\Sigma^\infty X))_n = Q(\Sigma^n X)$$

The result  $\mathbb{L}(\Sigma^\infty X)$  is an  **$\Omega$ -spectrum**. This is the *fibrant replacement* of  $\Sigma^\infty X$  in the stable model structure.

3. **Distinction from Adams Resolution:**

- The **Adams/Bousfield-Kan resolution** builds a tower  $X \rightarrow QX \rightarrow Q^2 X \dots$  to resolve the *space*  $X$ .
- The **Spectrification Monad**  $\mathbb{L}$  acts once (essentially as a completion) to fix the *structure* of the spectrum, ensuring the adjoint structure maps  $E_n \rightarrow \Omega E_{n+1}$  become weak equivalences.

**Example 1.41 (The  $R$ -Completion of a Space).** Let  $R$  be a commutative ring (typically  $\mathbb{Z}_p$  or  $\mathbb{Q}$ ). The Bousfield-Kan resolution constructs the " $R$ -completion" of a space  $X$ , effectively translating algebraic information (homology with coefficients in  $R$ ) into homotopy information.

1. **The Monad ( $R$ -Linearization):** Let  $R : \mathcal{S} \rightarrow \mathcal{S}$  be the monad that assigns to a simplicial set  $K$  the free simplicial  $R$ -module generated by  $K$  (forgetting the module structure back to a simplicial set).

$$X \xrightarrow{\eta} R(X)$$

Intuitively, this replaces every simplex of  $X$  with the free  $R$ -module generated by its vertices.

2. **The Cosimplicial Space (Bar Construction):** Applying the Monad iteratively generates a **cosimplicial space**  $R^\bullet X$ :

$$X \xrightarrow{\eta} R(X) \rightrightarrows R(R(X)) \rightrightarrows R^3(X) \cdots$$

This tower resolves  $X$  by spaces that are algebraically simple (generalized Eilenberg-MacLane spaces).

3. **Totalization ( $R$ -Completion):** The **Totalization** (Homotopy Limit) of this cosimplicial space defines the  $R$ -completion of  $X$ :

$$X_R^\wedge := \text{Tot}(R^\bullet X) \simeq \underset{\Delta}{\text{holim}} R^\bullet X$$

4. **The Spectral Sequence:** This resolution yields the **Bousfield-Kan Spectral Sequence**, which computes the homotopy groups of the completion from the cohomology of  $X$ :

$$E_2^{s,t} \cong \text{Ext}_{\text{Comod}}^s(R, H_*(X; R))_t \implies \pi_{t-s}(X_R^\wedge)$$

For  $R = \mathbb{Z}_p$ , this computes the homotopy groups of the  $p$ -adic completion of  $X$ .

non-trival operator In hom/ delet push-out = pull-back = linear = stab lifting obstruction Tw(C) twisted fiber product spectral sequence Hochschild+Hodge Realization = /Cyclic Homology/Topological Cyclic Homology Ladder of Higher Coherence Bockstein, Transgression/k-invariant,Massey,Toda,Steenrod,Hurewicz,Samelson,Pontryagin,Whitehead,Cup Product,Dyer-Lashof,Adams,Novikov,Browder Bracket,Functional Cohomology Operations Transfer (Becker-Gottlieb Transfer) Geometric Filtration/Resolution Filtration/Coefficient Filtration corr sps Exact Couple/Filtered Complex The Rees Construction/Deformation non-trival operation = correlation of rees algebra Deformation Quantization The Dennis Trace:cat C to K(C) to Hochschild Lurie:K-theory is Universal invariant of additivity, tr is factor through bisimplicial

## 1.4 Obstructions and Traces

**Definition 1.42 ( $p$ -Cartesian Morphism).** Let  $p : \mathcal{E} \rightarrow \mathcal{B}$  be a functor. A morphism  $\phi : v \rightarrow u$  in  $\mathcal{E}$  is called  **$p$ -Cartesian** if it satisfies the following universal property:

For every object  $w \in \mathcal{E}$  and every morphism  $\psi : w \rightarrow u$ , given a morphism  $g : p(w) \rightarrow p(v)$  in  $\mathcal{B}$  such that  $p(\psi) = p(\phi) \circ g$ , there exists a **unique** morphism  $\tilde{g} : w \rightarrow v$  in  $\mathcal{E}$  such that  $p(\tilde{g}) = g$  and  $\phi \circ \tilde{g} = \psi$ .

This relationship is depicted in the following commutative diagram, where the vertical arrows represent the projection  $p$ , and the dashed arrow represents the unique lift:

$$\begin{array}{ccccc}
& w & \xrightarrow{\psi} & u & \\
\downarrow p & \nearrow \exists! \tilde{g} & & \downarrow p & \\
p(w) & \xrightarrow[p]{p(\psi)} & p(u) & & \\
& \searrow g & \nearrow p(\phi) & & \\
& p(v) & & &
\end{array}$$

**Definition 1.43** (Grothendieck Fibration). A functor  $p : \mathcal{E} \rightarrow \mathcal{B}$  is a **Grothendieck fibration** if, for every object  $u$  in  $\mathcal{E}$  and every morphism  $f : X \rightarrow p(u)$  in  $\mathcal{B}$ , there exists a morphism  $\phi : v \rightarrow u$  in  $\mathcal{E}$  satisfying the following conditions:

- **Lifting:**  $p(\phi) = f$ ;
- **Universality:**  $\phi$  is a  $p$ -Cartesian morphism.

This existence of a Cartesian lift is depicted by the diagram:

$$\begin{array}{ccc}
v & \xrightarrow[\exists \phi]{} & u \\
\downarrow p & & \downarrow p \\
X & \xrightarrow{f} & p(u)
\end{array}$$

**Definition 1.44** (Grothendieck Construction). Let  $\mathcal{C}$  be a category and  $F : \mathcal{C}^{op} \rightarrow \mathbf{Cat}$  be a contravariant functor (an indexed category). The **Grothendieck construction** of  $F$ , denoted by  $\int F$  (or  $\mathcal{C}_F$ ), is the category defined as follows:

- **Objects:** Pairs  $(c, x)$ , where  $c$  is an object of  $\mathcal{C}$  and  $x$  is an object of the category  $F(c)$ .
- **Morphisms:** A morphism from  $(c, x)$  to  $(d, y)$  is a pair  $(u, \alpha)$ , where:
  - $u : c \rightarrow d$  is a morphism in  $\mathcal{C}$ ;
  - $\alpha : x \rightarrow F(u)(y)$  is a morphism in the fiber category  $F(c)$ .

Note that since  $F$  is contravariant,  $F(u)$  is a functor  $F(d) \rightarrow F(c)$ , so  $F(u)(y)$  lies in  $F(c)$ .

- **Composition:** Given morphisms  $(u, \alpha) : (c, x) \rightarrow (d, y)$  and  $(v, \beta) : (d, y) \rightarrow (e, z)$ , the composite is defined by:

$$(v, \beta) \circ (u, \alpha) = (v \circ u, F(u)(\beta) \circ \alpha)$$

Here,  $F(u)(\beta)$  maps  $F(u)(y)$  to  $F(u)(F(v)(z)) = F(v \circ u)(z)$ .

There is a canonical projection functor  $p : \int F \rightarrow \mathcal{C}$  given by  $p(c, x) = c$  and  $p(u, \alpha) = u$ . This functor  $p$  makes  $\int F$  a **Grothendieck fibration** over  $\mathcal{C}$ , with the fiber over  $c$  isomorphic to  $F(c)$ .

**Definition 1.45** (Left and Right Fibrations). Let  $p : X \rightarrow S$  be a morphism of simplicial sets. The map  $p$  is classified based on the *right lifting property* against horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  as follows:

1.  $p$  is called a **left fibration** if it has the right lifting property for all  $0 \leq k < n$ .
2.  $p$  is called a **right fibration** if it has the right lifting property for all  $0 < k \leq n$ .

In either case, the condition asserts that for the specified range of  $k$  and any  $n \geq 1$ , given any commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \exists \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & S \end{array}$$

there exists a lift  $\Delta^n \rightarrow X$  making the diagram commute.

Intuitively, left fibrations model covariant functors (transporting fibers forward along paths starting at  $k = 0$ ), while right fibrations model contravariant functors (transporting fibers backward along paths ending at  $k = n$ ).

**Remark 1.46** (Directionality of Lifting). The distinction between left and right fibrations generalizes the classical categorical notions of **pushforward** and **pullback**. The choice of the horn inclusion  $\Lambda_k^1 \hookrightarrow \Delta^1$  determines the flow of information relative to a morphism  $u : x \rightarrow y$  in the base:

1. **Right Fibration ( $\Lambda_1^1$ ): Pullback / Cartesian.** The horn  $\Lambda_1^1$  fixes the *target* ( $y$ ) and the morphism  $u$ . Lifting implies existence of a domain over  $x$ .

$$\begin{array}{ccc} \bullet & \xrightarrow{\text{lift}} & \bullet \\ x \xrightarrow{u} y & \implies & \text{Information flows } y \rightarrow x \text{ (Contravariant } u^*) \end{array}$$

2. **Left Fibration ( $\Lambda_0^1$ ): Pushforward / Co-Cartesian.** The horn  $\Lambda_0^1$  fixes the *source* ( $x$ ) and the morphism  $u$ . Lifting implies existence of a codomain over  $y$ .

$$\begin{array}{ccc} \bullet & \xrightarrow{\text{lift}} & \bullet \\ x \xrightarrow{u} y & \implies & \text{Information flows } x \rightarrow y \text{ (Covariant } u_!) \end{array}$$

**Definition 1.47** (Twisted Arrow  $\infty$ -Category). Let  $\mathcal{C}$  be an  $\infty$ -category. The **twisted arrow category**  $\text{Tw}(\mathcal{C})$  is defined as the **Grothendieck construction** (or unstraightening) of the mapping space functor.

Explicitly, let  $\chi$  be the functor classifying the mapping spaces:

$$\begin{aligned} \chi : (\mathcal{C} \times \mathcal{C}^{op})^{op} &\longrightarrow \mathcal{S} \\ (X, Y) &\longmapsto \text{Map}_{\mathcal{C}}(X, Y). \end{aligned}$$

Then  $\text{Tw}(\mathcal{C}) \simeq \int \chi$ . Consequently,  $\text{Tw}(\mathcal{C})$  is characterized as the total space of the **right fibration**

$$p : \text{Tw}(\mathcal{C}) \longrightarrow \mathcal{C} \times \mathcal{C}^{op}$$

associated to  $\chi$ . Combinatorially, this is modeled by the simplicial set whose  $n$ -simplices are maps  $\Delta^n \star (\Delta^n)^{op} \rightarrow \mathcal{C}$ .

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{Tw}(\mathcal{C})) \cong \mathrm{Hom}_{\mathbf{sSet}}((\Delta^n)^{op} \star \Delta^n, \mathcal{C}),$$

For any pair of objects  $(x, y)$ , the fiber of  $p$  is the mapping space:

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(x, y) & \longrightarrow & \mathrm{Tw}(\mathcal{C}) \\ \downarrow & & \text{Right Fib} \downarrow p \\ \{(y, x)\} & \longrightarrow & \mathcal{C} \times \mathcal{C}^{op} \end{array}$$

**Remark 1.48** (Local Systems on  $\mathrm{Tw}(\mathcal{C})$  and Coends). A **local system** on the twisted arrow category is a functor  $\mathcal{F} : \mathrm{Tw}(\mathcal{C}) \rightarrow \mathcal{D}$  (where  $\mathcal{D}$  is a target  $\infty$ -category like  $\mathcal{S}$  or **Spectra**).

The significance of such a system lies in its role in computing **Coends** and **Traces**. Let  $M : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{D}$  be a bimodule (a functor of two variables). We can pull  $M$  back along the canonical projection  $p : \mathrm{Tw}(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{op}$  to obtain a local system  $p^*M$  on  $\mathrm{Tw}(\mathcal{C})$ .

The value of this local system on an object  $(f : A \rightarrow B) \in \mathrm{Tw}(\mathcal{C})$  is simply  $M(A, B)$ . The **Coend** of the bimodule  $M$  is then geometrically realized as the colimit of this local system over the twisted arrow category:

$$\int^{c \in \mathcal{C}} M(c, c) \simeq \underset{(f:A \rightarrow B) \in \mathrm{Tw}(\mathcal{C})}{\mathrm{colim}} M(A, B).$$

Intuitively,  $\mathrm{Tw}(\mathcal{C})$  provides the necessary "morphisms between morphisms" to glue the diagonal terms  $M(c, c)$  together, creating a global invariant (the Trace) of the category.

**Remark 1.49** (The Geometric Hierarchy of  $\mathrm{Tw}(\mathcal{C})$ ). In the  $\infty$ -categorical setting, the twisted arrow category  $\mathrm{Tw}(\mathcal{C})$  shifts the geometric dimension of objects in  $\mathcal{C}$  by one level. We can visualize the  $k$ -morphisms of  $\mathrm{Tw}(\mathcal{C})$  as  $(k + 1)$ -dimensional shapes in  $\mathcal{C}$ :

### 1. Objects (The Lines)

An object in  $\mathrm{Tw}(\mathcal{C})$  corresponds to a 1-simplex (an arrow) in  $\mathcal{C}$ .

$$A \xrightarrow{f} B$$

### 2. 1-Morphisms (The Twisted Squares)

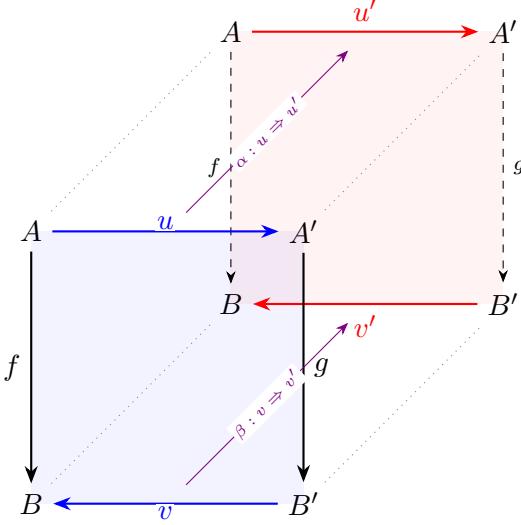
A morphism  $\phi : f \rightarrow g$  represents a factorization  $f \simeq v \circ g \circ u$ . Geometrically, this is a square filled by a homotopy  $\sigma$ .

$$\begin{array}{ccc} A & \xrightarrow{u} & A' \\ f \downarrow & \simeq & \downarrow g \\ B & \xleftarrow[v]{} & B' \end{array}$$

### 3. 2-Morphisms (The Homotopy Tunnels)

A 2-morphism represents a coherence between two factorizations. If we have a "blue" factorization  $\phi = (u, v)$  and a "red" factorization  $\phi' = (u', v')$ , a 2-morphism is the **deformation** (tunnel) connecting them.

In the diagram below, the vertical objects  $f$  and  $g$  form the fixed pillars, while the factorization maps flow from the front (blue) to the back (red) via the violet homotopies:



This entire solid represents a 3-simplex in  $\mathcal{C}$ , encoding the data that  $f \simeq v \circ g \circ u$  is homotopic to  $f \simeq v' \circ g \circ u'$ .

**Definition 1.50** (Local System on the Twisted Arrow  $\infty$ -Category). Let  $\mathcal{C}$  be an  $\infty$ -category and  $\mathcal{D}$  be a target  $\infty$ -category (e.g., the category of spectra  $\text{Sp}$ ).

A  $\mathcal{D}$ -valued local system on  $\text{Tw}(\mathcal{C})$  is a functor

$$\mathcal{L} : \text{Tw}(\mathcal{C}) \longrightarrow \mathcal{D}$$

satisfying the condition that  $\mathcal{L}$  sends every morphism in  $\text{Tw}(\mathcal{C})$  to an **equivalence** in  $\mathcal{D}$ .

Explicitly, this means:

1. **On Objects:** For every object in  $\text{Tw}(\mathcal{C})$ , which is a 1-morphism  $f : x \rightarrow y$  in  $\mathcal{C}$ , the local system assigns an object:

$$\mathcal{L}(f) \in \text{Obj}(\mathcal{D}).$$

2. **On Morphisms:** For every morphism  $\phi : f \rightarrow g$  in  $\text{Tw}(\mathcal{C})$ , which corresponds to a homotopy commutative square in  $\mathcal{C}$  (a factorization  $f \simeq v \circ g \circ u$ ):

$$\begin{array}{ccc} x & \xrightarrow{u} & x' \\ f \downarrow & & \downarrow g \\ y & \xleftarrow{v} & y' \end{array}$$

the induced map in  $\mathcal{D}$  is an equivalence:

$$\mathcal{L}(\phi) : \mathcal{L}(f) \xrightarrow{\sim} \mathcal{L}(g).$$

Consequently,  $\mathcal{L}$  factors through the fundamental  $\infty$ -groupoid of the twisted arrow category:

$$\mathcal{L} : \Pi_\infty(\text{Tw}(\mathcal{C})) \longrightarrow \mathcal{D}.$$

**Definition 1.51** (Topological Chiral Homology and Cohomology). Let  $\mathcal{C}$  be a spectral  $\infty$ -category and let  $\mathcal{F} : \text{Tw}(\mathcal{C}) \rightarrow \text{Sp}$  be a local system on its twisted arrow  $\infty$ -category (as defined previously).

We define the **Topological Chiral Homology** (also known as Factorization Homology over  $S^1$ ) of  $\mathcal{C}$  with coefficients in  $\mathcal{F}$  as the colimit of the local system:

$$\int_{S^1} \mathcal{F} := \text{colim}_{\text{Tw}(\mathcal{C})} \mathcal{F}.$$

Dually, we define the **Topological Chiral Cohomology** of  $\mathcal{C}$  with coefficients in  $\mathcal{F}$  as the limit of the local system:

$$\oint^{S^1} \mathcal{F} := \lim_{\text{Tw}(\mathcal{C})} \mathcal{F}.$$

**Example 1.52** (The Ladder of Theories on  $\text{Tw}(\mathcal{C})$ : Explicit Formulas). Here we detail the specific formulas for Homology (colim) and Cohomology (lim) at each level of complexity.

### 1. Level 1: Singular Theory (Topological)

*Context:*  $\mathcal{C}$  is a discrete category (or space  $X$ ).

- **Local System:** Constant functor  $\underline{\mathbb{Z}}$ .
- **Homology (Singular Chain Complex):**

$$H_*(BC; \mathbb{Z}) \cong \text{Tor}_*^{\mathbb{Z}[\mathcal{C}]}(\mathbb{Z}, \mathbb{Z}).$$

*Geometric Formula:*  $H_*(X) = \ker \partial / \text{im } \partial$  using the standard singular boundary operator  $\partial = \sum (-1)^i d_i$ .

- **Cohomology (Singular Cochain Complex):**

$$H^*(BC; \mathbb{Z}) \cong \text{Ext}_{\mathbb{Z}[\mathcal{C}]}^*(\mathbb{Z}, \mathbb{Z}).$$

*Tw(C) Formula:*  $\lim_{\text{Tw}(\mathcal{C})} \underline{\mathbb{Z}}$ .

### 2. Level 2: Classical Hochschild / Baues-Wirsching (Linear)

*Context:* Associative  $k$ -algebra  $A$  (or  $k$ -linear category).  $M$  is an  $A$ -bimodule.

- **Local System:** The bimodule  $M$  regarded as a functor on  $\text{Tw}(A)$ .
- **Homology (Bar Construction):**

$$HH_*(A; M) \cong \text{Tor}_*^{A \otimes A^{op}}(A, M).$$

*Explicit Complex:* The homology of the chain complex  $(C_*(A, M), b)$ :

$$C_n = M \otimes A^{\otimes n}, \quad b(m \otimes a_1 \dots) = ma_1 \otimes a_2 \dots + \sum \dots + (-1)^n a_n m \otimes \dots$$

- **Cohomology (Derivations/Extensions):**

$$HH^*(A; M) \cong \text{Ext}_{A \otimes A^{op}}^*(A, M).$$

*Explicit Complex:* The cohomology of  $(C^*(A, M), \delta)$ , where 1-cocycles are derivations:  $\delta f(a, b) = af(b) - f(ab) + f(a)b$ .

### 3. Level 3: Topological Hochschild (Spectral)

*Context:* Ring spectrum  $A$  (an  $S$ -algebra).

- **Local System:** The algebra  $A$  itself (as a spectral bimodule).

- **Homology (Derived Tensor Product):**

$$\mathrm{THH}(A) \simeq A \wedge_{A \wedge A^{op}} A \quad (\text{Symmetric Monoidal Product}).$$

*Geometric Formula:* The geometric realization of the cyclic nerve  $|N_\bullet^{cyc}(A)|$ .

- **Cohomology (Topological Center):**

$$\mathrm{THC}(A) \simeq F_{A \wedge A^{op}}(A, A) \simeq \mathrm{RHom}_{A \wedge A^{op}}(A, A).$$

This computes the **Derived Center** of the ring spectrum  $A$ .

#### 4. Level 4: Harrison / André-Quillen (Commutative)

*Context:* Commutative ring spectrum  $A$  ( $E_\infty$ -algebra).  $M$  is an  $A$ -module.

- **Local System:** Related to the symmetric algebra structure (killing shuffles).
- **Homology (Cotangent Complex):**

$$\mathrm{TAQ}_*(A; M) \cong \pi_*(M \wedge_A \mathbb{L}_A).$$

Here  $\mathbb{L}_A$  is the **Topological Cotangent Complex**. It classifies derivations:  $\mathrm{Map}_{Mod_A}(\mathbb{L}_A, M) \simeq \mathrm{Der}(A, M)$ .

- **Cohomology (Deformation Cohomology):**

$$D^*(A; M) \cong \pi_{-*} \mathrm{RHom}_A(\mathbb{L}_A, M).$$

Specifically classifies commutative deformations.

#### 5. Level 5: Factorization Homology (Manifolds)

*Context:*  $E_n$ -algebra  $A$ , Manifold  $M$  of dimension  $n$ .

- **Local System:** The algebra  $A$  assigned to disjoint disks.
- **Homology (The Integral):**

$$\int_M A \simeq \mathrm{colim}_{\{U_1 \sqcup \dots \sqcup U_k \hookrightarrow M\}} (A(U_1) \otimes \dots \otimes A(U_k)).$$

This is a colimit over the category of disk embeddings  $\mathrm{Disk}(M)$ .

- **Cohomology (Global Sections):**

$$\oint^M A \simeq \lim_{U \hookrightarrow M} A(U).$$

Often related to the mapping space  $\mathrm{Map}(M, B^n A^\times)$  (Non-abelian cohomology).

#### 6. Level 6: Topological Periodic / Cyclic (Tate)

*Context:*  $A$  is a ring spectrum. We use the  $S^1$ -action on  $\mathrm{THH}(A)$ .

- **Local System:**  $S^1$ -equivariant structure of  $A$ .
- **Homology (TP / Tate Construction):**

$$\mathrm{TP}(A) \simeq (\mathrm{THH}(A))^{tS^1} \simeq \mathrm{Cofiber} \left( \mathrm{Nm} : \mathrm{THH}(A)_{hS^1} \rightarrow \mathrm{THH}(A)^{hS^1} \right).$$

Defined as the Tate spectrum (mixing orbits and fixed points).

- **Cohomology (TC / Invariants):**

$$\mathrm{TC}(A) \simeq \mathrm{Map}_{Sp^{S^1}}(S^0, \mathrm{THH}(A)) \quad (\text{Roughly}).$$

Modern definition uses the fiber of the Frobenius map:  $\mathrm{TC}(A) \simeq \mathrm{fib}(\mathrm{TR}(A) \xrightarrow{1-F} \mathrm{TR}(A))$ .

**Remark 1.53** (The Operadic Unification and Geometric Duality). The various homology and cohomology theories discussed above (Singular, Hochschild, André-Quillen, etc.) are strictly special cases of the general **Operadic Homology and Cohomology**. This framework unifies the algebraic machinery with the geometric intuition of the "Tangent Bundle" as follows:

1. **The Tangent Structure (Beck Modules vs.  $\mathrm{Tw}(\mathcal{C})$ ):** For an algebra over an arbitrary operad  $\mathcal{O}$ , the intrinsic "tangent bundle" is the category of **Beck Modules** over the universal enveloping algebra  $U_{\mathcal{O}}(A)$ .
  - When  $\mathcal{O} = \mathrm{Ass}$  (Associative), the category of Beck Modules identifies with the category of **Bimodules** ( $A \otimes A^{\mathrm{op}}$ -modules).
  - Consequently,  $\mathrm{Tw}(\mathcal{C})$  is precisely the geometric realization (the loop-like skeleton) of the Beck Module structure for associative algebras.
2. **The Generating Object (Cotangent Complex):** The linearization of the operadic structure yields the **Cotangent Complex**  $\mathbb{L}_{A/\mathcal{O}}$ . This object plays the role of the generic "1-form" ( $\Omega^1$ ) on the algebraic variety defined by  $A$ .
3. **Geometric Duality (HKR Correspondence):** All theories can be classified by their interaction with the Cotangent Complex, reflecting the duality between **Vector Fields** and **Differential Forms**:

- **Cohomology (Ext  $\leftrightarrow$  Vector Fields):**

$$H_{\mathcal{O}}^*(A, M) \cong \mathrm{Ext}_{U_{\mathcal{O}}(A)}^*(\mathbb{L}_{A/\mathcal{O}}, M)$$

Geometrically, cohomology classifies **derivations** and **deformations**. Like a **Vector Field** (or Polyvector field), it describes the "directions" in which the algebraic structure can flow or be deformed (e.g.,  $H^2$  as deformation obstruction).

- **Homology (Tor  $\leftrightarrow$  Differential Forms):**

$$H_*^{\mathcal{O}}(A, M) \cong \mathrm{Tor}_*^{U_{\mathcal{O}}(A)}(\mathbb{L}_{A/\mathcal{O}}, M)$$

Geometrically, homology measures the **volume** or **trace** of the structure. Like a **Differential Form**, it integrates the linearized structure (Cotangent Complex) over the fundamental cycle of the category (e.g., the circle  $S^1$  in THH).

In summary,  $\mathrm{Tw}(\mathcal{C})$  provides the *integration domain* (geometry), the Local System provides the *coefficients* (physics), and the Operadic type determines whether we are integrating forms (Homology) or solving for fields (Cohomology).

**Remark 1.54 (Unified Sheaf-Theoretic Perspective).** The various homology and cohomology theories discussed can be unified as **Sheaf Cohomology** (for obstructions) and **Cosheaf Homology** (for traces) defined over distinct Grothendieck sites. In this view, algebra and geometry are distinguished only by the category on which the sheaf is defined.

## 1. The Site Definitions:

- **Singular Theory:** Defined on the site of open sets of a topological space  $X$ .
- **Hochschild Theory:** Defined on the **Cyclic Site** (or  $\text{Tw}(\mathcal{C})$ ), representing the geometry of the circle  $S^1$ .
- **Operadic Theory:** Defined on the **Dendroidal Site**  $\Omega$  (Category of Trees), representing the geometry of branching operations.

## 2. The Duality of Measurement:

- **Cohomology** ( $H^* \simeq R\Gamma$ ): Computes the derived global sections of a **Sheaf**. It classifies the **lifting obstructions** of local structures to global ones (e.g.,  $k$ -invariants).
- **Homology** ( $H_* \simeq \mathbb{L}\text{colim}$ ): Computes the derived colimit of a **Cosheaf**. It measures the **global trace** or "volume" of the algebraic structure (e.g., Factorization Homology).

Any "twisting" in the theory is simply the non-triviality of the coefficient sheaf (Local System) over the respective site.

**Definition 1.55 (Universal Monodromy Object via Geometric Tannaka Duality).** Let  $\mathcal{X}$  be an  $\infty$ -topos and let  $\text{Loc}(\mathcal{X})$  denote the  $\infty$ -category of local systems on  $\mathcal{X}$  (valued in the category of spaces  $\mathcal{S}$ ).

The **Universal Monodromy Object** of  $\mathcal{X}$ , formally known as the **Fundamental  $\infty$ -Groupoid** or **Shape**  $\Pi_\infty(\mathcal{X})$ , is defined as the unique object in  $\mathcal{S}$  determined by the following equivalence of  $\infty$ -categories:

$$\text{Loc}(\mathcal{X}) \xrightarrow{\sim} \text{Fun}(\Pi_\infty(\mathcal{X}), \mathcal{S}).$$

**Interpretation as Tannaka Duality:** This definition is an instance of **Geometric Tannaka Duality**. Here,  $\text{Loc}(\mathcal{X})$  serves as the "category of representations," and  $\Pi_\infty(\mathcal{X})$  is the "group" (or groupoid) reconstructed from these representations. Thus,  $\Pi_\infty(\mathcal{X})$  is the universal source of monodromy: to give a local system on  $\mathcal{X}$  is equivalent to giving a monodromy representation of  $\Pi_\infty(\mathcal{X})$ .

**Example 1.56 (Construction of the Twisted Postnikov Tower).** Let  $X$  be a path-connected CW-complex. We define the Postnikov tower  $\{P_n(X)\}$  inductively. The construction relies on the definition of local coefficient systems and the pullback of the universal path fibration.

### 1. The Coefficient Functor $\underline{\pi}_n$

The  $n$ -th homotopy group is defined as a functor from the fundamental groupoid of  $X$  to the category of abelian groups:

$$\underline{\pi}_n : \Pi_1(X) \longrightarrow \mathbf{Ab}.$$

This functor is specified by the following data:

- **Objects:** For each  $x \in X$ ,  $\underline{\pi}_n(x) := \pi_n(X, x)$ .
- **Morphisms:** For each path class  $[\gamma] : x \rightarrow y$ , the map  $\underline{\pi}_n([\gamma])$  is the change-of-basepoint isomorphism  $\gamma_\# : \pi_n(X, x) \rightarrow \pi_n(X, y)$  induced by path lifting.

This functor  $\underline{\pi}_n$  constitutes the local system of coefficients required for the subsequent cohomology groups.

## 2. The $k$ -invariant and Pullback Definition

The transition from the  $(n-1)$ -th stage to the  $n$ -th stage is determined by the  $k$ -invariant, a cohomology class  $k_{n+1} \in H^{n+1}(P_{n-1}(X); \underline{\pi}_n)$ . This class is represented by a continuous map:

$$k_{n+1} : P_{n-1}(X) \longrightarrow K(\underline{\pi}_n, n+1),$$

where  $K(\underline{\pi}_n, n+1)$  is the classifying space for the twisted cohomology (a fibration over  $K(\pi_1(X), 1)$ ).

We define  $P_n(X)$  as the pullback of the universal path space  $\mathcal{PK}(\underline{\pi}_n, n+1)$  along the map  $k_{n+1}$ . Formally, we write:

$$P_n(X) := k_{n+1}^*(\mathcal{PK}(\underline{\pi}_n, n+1)).$$

This definition identifies  $P_n(X)$  with the limit of the following cospan diagram:

$$\begin{array}{ccc} P_n(X) & \longrightarrow & \mathcal{PK}(\underline{\pi}_n, n+1) \\ p_n \downarrow & \lrcorner & \downarrow \\ P_{n-1}(X) & \xrightarrow{k_{n+1}} & K(\underline{\pi}_n, n+1) \end{array}$$

Here,  $\mathcal{PK}(\underline{\pi}_n, n+1)$  is the contractible path space of the twisted Eilenberg-MacLane space. The resulting space  $P_n(X)$  is the total space of a fibration over  $P_{n-1}(X)$  with fiber  $K(\underline{\pi}_n, n)$ , where the twisting is fully encoded by the map  $k_{n+1}$ .

**Theorem 1.57** (Homological Whitehead Theorem with Twisted Coefficients). Let  $f : X \rightarrow Y$  be a continuous map between connected CW complexes. Suppose that the following two conditions hold:

1. The map induces an isomorphism on the fundamental groups, i.e.,  $f_* : \pi_1(X) \xrightarrow{\cong} \pi_1(Y)$ . Let  $G = \pi_1(Y)$ .
2. For every local system of coefficients  $\mathcal{L}$  on  $Y$  (regarded as a  $\mathbb{Z}[G]$ -module), the induced homomorphism on twisted homology

$$f_* : H_*(X; f^*\mathcal{L}) \xrightarrow{\cong} H_*(Y; \mathcal{L})$$

is an isomorphism.

Then  $f$  is a weak homotopy equivalence; that is,  $f_* : \pi_n(X) \rightarrow \pi_n(Y)$  is an isomorphism for all  $n \geq 1$ .

**Remark 1.58 (Operations as Monodromy via Spectral Sequence Dynamics).** The diverse zoology of homotopy operations can be unified as manifestations of *Universal Monodromy* within the differential dynamics ( $d_r$ ) and multiplicative structures of specific spectral sequences. Each operation represents an obstruction or a higher-order coherence constraint.

- **Geometric Monodromy (Serre & EHP Spectral Sequences):** The **Serre SS** (for a fibration  $F \rightarrow E \rightarrow B$ ) encodes the twisting of the fiber over the base.
  - The **Transgression** is realized as the differential  $d_n : E_n^{0,n-1} \rightarrow E_n^{n,0}$ , representing the obstruction to extending a class from the fiber to the total space.
  - **$k$ -invariants** appear as the first non-trivial differentials in the spectral sequence of the Postnikov tower, explicitly measuring the twisting of the fibration.
  - The **Hurewicz map** sits on the edge homomorphisms, relating homotopy (fiber) to homology (total space).

- **Whitehead products** are detected by differentials in the **EHP spectral sequence** (or Serre SS of wedge sums), measuring the failure of the sphere spectrum to be commutative.
- **Algebraic Monodromy (Eilenberg-Moore & Bockstein Spectral Sequences):**
  - The **Cup Product** and **Pontryagin Product** constitute the multiplicative structure of the  $E_2$  page in the Cohomology and Homology **Serre SS**, respectively.
  - The **Bockstein Operation** ( $\beta$ ) is the differential  $d_1$  of the **Bockstein SS** associated with the extension  $0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \rightarrow 0$ .
  - **Samelson and Browder brackets** appear as commutator obstructions in the spectral sequences governing H-spaces and loop spaces.
- **Stable & Higher Monodromy (Adams & Adams-Novikov Spectral Sequences):** These spectral sequences converge to stable homotopy groups, where the filtration degree corresponds to the "resolution depth."
  - **Steenrod Operations** and **Dyer-Lashof Operations** act on the  $E_2$  page of the **Adams SS**, encoding the internal symmetry of the coefficients ( $\mathbb{F}_p$ ).
  - **Landweber-Novikov Operations** play the analogous role in the **Adams-Novikov SS** (based on  $MU$ ).
  - **Massey Products** on the  $E_r$  page converge to **Toda Brackets** in the stable homotopy groups. These represent higher-order obstructions ("ghosts" of vanished lower differentials).
  - **Functional Cohomology Operations** arise as secondary differentials when primary operations (primary monodromy) vanish.

In summary, all these operations are residues of the **Universal Monodromy**. They are detected either as the *source* of a differential (an obstruction) or as the *result* of a convergence (a higher composition), dictating how local algebraic data is glued into a global geometric structure.

**Remark 1.59 (Taxonomy of Spectral Sequences and the Unification via Monodromy).** It is illuminating to classify spectral sequences into three fundamental types, each corresponding to a distinct strategy of decomposition in homotopy theory. Despite their differences, they share a common cohomological essence.

### 1. Geometric Filtration (Geometric Decomposition):

- *Examples:* Serre, Leray, EHP spectral sequences.
- *Mechanism:* Corresponds to a decomposition of the **domain space** (e.g., via a CW-skeleton or a fiber bundle  $F \rightarrow E \rightarrow B$ ).
- *Philosophy:* We reconstruct the global topology by gluing local spatial slices (fibers).

### 2. Resolution Filtration (Algebraic Decomposition):

- *Examples:* Adams, Adams-Novikov, Eilenberg-Moore spectral sequences.
- *Mechanism:* Corresponds to a decomposition of the **category/structure** (e.g., resolving a spectrum by Eilenberg-MacLane spectra or free modules).
- *Philosophy:* We reconstruct the stable homotopy type by gluing algebraic layers (resolutions).

### 3. Coefficient Filtration (Numerical Decomposition):

- *Examples:* Bockstein spectral sequence.
- *Mechanism:* Corresponds to a decomposition of the **codomain/values** (e.g.,  $0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \dots$ ).
- *Philosophy:* We refine the precision of the invariants by gluing value ranges.

**Remark 1.60 (Spectral Sequences as Sheaf Cohomology and Operational Monodromy).** Fundamentally, the computation of a spectral sequence is the computation of **Sheaf Cohomology**. The  $E_2$ -page, typically of the form  $H^p(B; \mathcal{H}^q(F))$ , calculates the cohomology of the base with coefficients in the **local system** determined by the fiber.

In this framework, **Monodromy** transcends the classical role of  $\pi_1$ -action; it becomes the universal source of all **non-trivial twisted operations**.

- **The Setup ( $E_2$ ):** The local system itself encodes the "static" twist (the geometric monodromy).
- **The Dynamics ( $d_r$  & Operations):** The higher differentials and associated cohomology operations (such as **Steenrod squares**, **Massey products**, **Toda brackets**, and **k-invariants**) are the dynamic manifestations of this twist.

These operations quantify the obstruction to extending local sections to global ones. Specifically, phenomena like the **Bockstein** or **Transgression** arise precisely because the coefficient sheaf is not constant (or the filtration does not split). Thus, every non-trivial operation listed previously is essentially a measurement of the *failure* of the global structure to be a simple product, a failure dictated entirely by the underlying Monodromy.

**Remark 1.61 (Stability as Linearized Monodromy).** The essence of a **stable object** is that it embodies the **linearization** of Universal Monodromy.

Unlike unstable spaces where gluings are geometrically "curved," stable objects reside in an **additive category** where finite products and coproducts coincide ( $X \times Y \simeq X \vee Y$ ). Consequently:

- **Linearization:** The Monodromy action transforms from a non-linear geometric diffeomorphism into a linear algebraic operator.
- **Extension vs. Twisting:** The geometric "twisting" of a bundle simplifies to an algebraic **extension class** (in Ext groups).

Physically, if unstable homotopy theory studies the global manifold, stable homotopy theory studies its **tangent space**.

**Remark 1.62 (Differentials as Monodromy Residues in the Rees Algebra).** The Rees algebra perspective offers a rigorous algebraic geometry for spectral sequences, framing the differential  $d_r$  as the **residue** of the underlying Monodromy.

Let  $(C, \partial, F^\bullet)$  be a filtered complex. The Rees algebra  $\mathcal{R}(C) = \bigoplus_p F^p C \cdot t^{-p}$  encodes the filtration as a graded module over  $k[t]$ . We view  $t$  as a deformation parameter where  $t = 0$  corresponds to the associated graded object ( $E_1$  page) and  $t = 1$  to the total filtered complex.

1. **The Expansion of the Boundary Operator:** Within the Rees algebra, the boundary operator  $\partial$  does not simply commute with  $t$ . Instead, the interaction between the topology ( $\partial$ ) and the filtration ( $t$ ) can be expanded as a formal series. If  $x \in F^p C$ , the condition that  $x$  survives to the  $r$ -th page means that its boundary is "deep" in the filtration:

$$\partial x \in F^{p+r} C.$$

- 2. The Differential as a Residue:** In terms of the Rees algebra elements  $\tilde{x} = xt^{-p}$ , this condition translates to the equation:

$$\partial_{\mathcal{R}}(\tilde{x}) = t^r \cdot \tilde{y}$$

Here, the power  $t^r$  measures the "distance" or "twist" between the filtration levels. The differential  $d_r$  is precisely the operator that extracts the coefficient  $\tilde{y}$  modulo  $t$ :

$$d_r([x]) \equiv \text{Coeff}_{t^r}(\partial_{\mathcal{R}}(\tilde{x})) \pmod{t}.$$

- 3. Interpretation:** Thus,  $d_r$  acts as a **Residue of the Monodromy**.

- The **Monodromy** is the global incompatibility between the filtration and the boundary operator as we deform from  $t = 0$  to  $t = 1$ .
- The spectral sequence is the Taylor expansion of this Monodromy.
- The differential  $d_r$  is the **first non-vanishing term** (the leading residue) in this expansion. It detects exactly where the "local" algebraic structure ( $E_r$ ) fails to extend "globally" to the next order of precision ( $E_{r+1}$ ), blocked by the twist of the bundle.