A Study in Quasi-Coherent Sheaves and Tannaka Duality

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1 Preliminaries

1.1 ∞-categories

Definition 1.1 (∞-Category)**.** A simplicial set K is an ∞**-category** if for every $n > 1$ and every **inner** index $0 < i < n$, every map of simplicial sets $f_0 : \Lambda^n_i \to K$ admits an extension to an n-simplex $f : \Delta^n \to K$.

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f_0"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Definition 1.2 (Simplicial Category)**.** A **simplicial category** (or Set_Δ-enriched category) $\mathcal{C}$ is a category where:

1. For any two objects $X, Y \in \mathcal{C}$, the collection of morphisms between them is not a set, but a **simplicial set** $\mathrm{Map}_{\mathcal{C}}(X, Y)$.
2. For any three objects $X, Y, Z \in \mathcal{C}$, the composition map

$$\mathrm{Map}_{\mathcal{C}}(Y, Z) \times \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$$

   is a morphism of simplicial sets and satisfies the usual associativity and identity axioms.

A simplicial category $\mathcal{C}$ is **locally Kan** if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the mapping simplicial set $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex.

Definition 1.3 (Simplicial Nerve N_Δ)**.** The **simplicial nerve** $N_\Delta(\mathcal{C})$ is the simplicial set defined by the assignment:

$$N_\Delta(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$$

where $\mathfrak{C}[\Delta^n]$ is the **rigidification** of the n-simplex Δ^n into a simplicial category.

Definition 1.4 (∞-category via N_Δ)**.** An ∞**-category** (or quasicategory) is a simplicial set K that is equivalent to the simplicial nerve of some locally Kan simplicial category $\mathcal{C}$.

$$K \simeq N_\Delta(\mathcal{C})$$

Theorem 1.5 (Joyal-Lurie)**.** There exists a Quillen equivalence between the Joyal model structure on Set_Δ (modeling quasicategories) and the Bergner model structure on Cat_Δ (modeling simplicial categories):

$$\mathfrak{C}[\cdot] : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Cat}_\Delta : N_\Delta.$$

Specifically, for any simplicial category $\mathcal{C}$ where mapping spaces are Kan complexes, its simplicial nerve $N_\Delta(\mathcal{C})$ is a quasicategory.

Definition 1.6 (Free Cocompletion)**.** Let $\mathcal{C}$ be a small ∞-category. An ∞-category $\mathcal{P}(\mathcal{C})$ is called the **free cocompletion** of $\mathcal{C}$ if it satisfies the following universal property:

1. $\mathcal{P}(\mathcal{C})$ admits all small colimits.
2. There exists a functor $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ (called the Yoneda embedding) such that for any ∞-category $\mathcal{D}$ which admits small colimits, composition with j induces an equivalence of ∞-categories:
$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Here, Fun^L denotes the full subcategory of functors that preserve small colimits (left adjoints).

Definition 1.7 (The ∞-category of Spaces)**.** Let $\mathcal{S}$ denote the ∞-category of spaces. It is defined in two equivalent ways:

1. **Via Dwyer-Kan Localization:**
Let W be the class of weak homotopy equivalences in Set_Δ. We define $\mathcal{S}$ as the homotopy coherent nerve of the simplicial localization:
$$\mathcal{S} := N\big(\mathrm{Set}_\Delta[W^{-1}]\big).$$
Equivalently, via Kan complexes: $\mathcal{S} \simeq N(\mathbf{Kan})$.
2. **Via Free Cocompletion:**
The ∞-category $\mathcal{S}$ is the free cocompletion of the point $*$. That is, it is the category of presheaves:
$$\mathcal{S} \simeq \mathcal{P}(*).$$
Universal Property: For any cocomplete ∞-category $\mathcal{C}$, there is an equivalence $\mathrm{Fun}^L(\mathcal{S}, \mathcal{C}) \simeq \mathcal{C}$.

Theorem 1.8. Let K be an ∞-category (quasi-category). Let $\mathcal{C} = \mathfrak{C}[K]$ be its associated simplicial category (via rigidification). The construction of the presheaf ∞-category commutes with the nerve construction in the following sense:

1. **Simplicial Side:** Consider the category of simplicial presheaves $\mathcal{P}_\Delta(\mathcal{C}) := \mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathcal{S}_{\mathrm{Kan}})$. This category admits a simplicial model structure (projective structure).
2. **Infinity Side:** Consider the ∞-category of presheaves $\mathcal{P}(K) := \mathrm{Fun}(K^{op}, \mathcal{S})$.
3. **Equivalence:** There is an equivalence of ∞-categories:
$$\mathcal{P}(K) \simeq N\left(\mathcal{P}_\Delta(\mathcal{C})^{\mathrm{cf}}\right)$$
where $\mathcal{P}_\Delta(\mathcal{C})^{\mathrm{cf}}$ denotes the full simplicial subcategory of fibrant-cofibrant objects in the model category of simplicial presheaves.

In summary, the presheaf of an ∞-category is modeled by the nerve of the strictly cocomplete simplicial category of enriched presheaves.

Remark 1.9 (Homotopy Category via Fibrant-Cofibrant Objects)**.** To correctly construct the homotopy category $\mathrm{Ho}(\mathcal{M})$ from a simplicial model category $\mathcal{M}$, one cannot simply take the path components π_0 of the mapping spaces between arbitrary objects.

Instead, one must restrict attention to the full subcategory of **fibrant-cofibrant objects**, denoted $\mathcal{M}_{cf}$. It is only within this subcategory that the simplicial mapping spaces $\mathrm{Map}_{\mathcal{M}}(X, Y)$ are guaranteed to be Kan complexes representing the correct derived mapping spaces. The morphisms in the homotopy category are thus given by:

$$[X, Y]_{\mathrm{Ho}(\mathcal{M})} \cong \pi_0 \mathrm{Map}_{\mathcal{M}}(X, Y) \quad \text{for } X, Y \in \mathcal{M}_{cf}.$$

For general objects X, Y, one must first replace them with weakly equivalent fibrant-cofibrant objects (via cofibrant replacement QX and fibrant replacement RY) to compute this group.

1.2 Stable ∞-Category

Definition 1.10 (Loop Object)**.** For any object $X \in \mathcal{C}$, the loop object ΩX is the limit of the diagram $0 \to X \leftarrow 0$. It fits into the following pullback square:

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

Intuitively, $\Omega X \simeq 0 \times_X 0$.

Definition 1.11 (Suspension Object)**.** For any object $X \in \mathcal{C}$, the suspension object ΣX is the colimit of the diagram $0 \leftarrow X \to 0$. It fits into the following pushout square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[dr, phantom, "\S"] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \Sigma X \end{tikzcd}$$

Intuitively, $\Sigma X \simeq 0 \amalg_X 0$.

Definition 1.12 (Stable ∞-Category)**.** An ∞-category $\mathcal{C}$ is called **stable** if it satisfies the following conditions:

1. There exists a zero object $0 \in \mathcal{C}$ (i.e., $\mathcal{C}$ is pointed).
2. Every morphism in $\mathcal{C}$ admits a kernel and a cokernel.
3. A triangle in $\mathcal{C}$ is a pushout square if and only if it is a pullback square.

Definition 1.13 (The Stabilization of an ∞-Category)**.** Let $\mathcal{C}$ be an ∞-category admitting finite limits. The process of constructing the stable ∞-category associated to $\mathcal{C}$ proceeds in two stages:

1. **Pointed View (Formation of $\mathcal{C}_*$):**
   First, we construct the *pointed* ∞-category $\mathcal{C}_*$. Assuming $\mathcal{C}$ has a terminal object $*$, $\mathcal{C}_*$ is defined as the under-category of the terminal object:

   $$\mathcal{C}_* := \mathcal{C}_{*/} \cong \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathrm{Fun}(\{0\}, \mathcal{C})} \{*\}$$

   Objects in $\mathcal{C}_*$ are morphisms $* \to X$ in $\mathcal{C}$ (i.e., objects equipped with a base point). In $\mathcal{C}_*$, the object corresponding to the identity $* \to *$ serves as a *zero object* (both initial and terminal). Consequently, the loop functor $\Omega : \mathcal{C}_* \to \mathcal{C}_*$ is well-defined by $\Omega X = * \times_X *$.

2. **Stabilization (Formation of Sp($\mathcal{C}$)):**
   The *stabilization* of $\mathcal{C}$, denoted as $\mathrm{Sp}(\mathcal{C})$ (or $\mathrm{Stab}(\mathcal{C})$), is defined as the ∞-category of spectrum objects in $\mathcal{C}_*$. It is constructed as the homotopy limit of the tower of loop functors:
   $$\mathrm{Sp}(\mathcal{C}) := \varprojlim \left(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$
   Explicitly, an object $E \in \mathrm{Sp}(\mathcal{C})$ consists of a sequence $\{E_n\}_{n\geq 0}$ of objects in $\mathcal{C}_*$ together with equivalences $E_n \xrightarrow{\sim} \Omega E_{n+1}$ for each n. This construction forces the suspension functor Σ to be an equivalence, rendering $\mathrm{Sp}(\mathcal{C})$ a stable ∞-category.

Theorem 1.14 (Universal Property of Stabilization)**.** Let $\mathcal{C}$ be an ∞-category with finite limits and a terminal object $*$. Let $\mathcal{C}_* = \mathcal{C}_{*/}$ be its pointed version. The stabilization $\mathrm{Sp}(\mathcal{C})$ is characterized by the following equivalent descriptions:

1. **Internal Construction (Loop Towers):** $\mathrm{Sp}(\mathcal{C})$ is the homotopy limit of the sequence of loop functors:
   $$\mathrm{Sp}(\mathcal{C}) \simeq \varprojlim \left(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$
   An object in $\mathrm{Sp}(\mathcal{C})$ is an Ω-spectrum, i.e., a sequence $\{E_n\}_{n\geq 0}$ in $\mathcal{C}_*$ with equivalences $E_n \simeq \Omega E_{n+1}$.

2. **External Construction (Excision):** $\mathrm{Sp}(\mathcal{C})$ is equivalent to the ∞-category of pointed excisive functors from the category of finite pointed spaces $\mathcal{S}_*^{\mathrm{fin}}$ to $\mathcal{C}$:
   $$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$$
   A functor $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ belongs to $\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ if $F(*) \simeq *$ and F maps every pushout square in $\mathcal{S}_*^{\mathrm{fin}}$ to a pullback square in $\mathcal{C}$.

Furthermore, the stabilization $\mathrm{Sp}(\mathcal{C})$ is the universal stable ∞-category under $\mathcal{C}$: for any stable ∞-category $\mathcal{D}$, the functor $\mathrm{Sp}(\mathcal{C}) \to \mathcal{D}$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{lex}}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$, where $\mathrm{Fun}^{\mathrm{lex}}$ denotes the ∞-category of left exact functors.

Remark 1.15 (Distinction between $\mathrm{Stab}(\mathcal{C})$ and Sp)**.** It is essential to distinguish between the abstract process of stabilization and the specific category of spectra:

1. **The Category of Spectra (Sp):** Historically and by convention, Sp refers specifically to the stabilization of the ∞-category of pointed spaces $\mathcal{S}_*$. That is:
   $$\mathrm{Sp} \simeq \mathrm{Stab}(\mathcal{S}) \simeq \varprojlim(\ldots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)$$
   This category serves as the unit object in the ∞-category of stable ∞-categories and provides the ground for stable homotopy theory.

2. **Stabilization of an Arbitrary ∞-Category (Stab($\mathcal{C}$)):** For any ∞-category $\mathcal{C}$ with finite limits, $\mathrm{Stab}(\mathcal{C})$ is the stable ∞-category constructed as the limit of the tower of loop functors:
   $$\mathrm{Stab}(\mathcal{C}) = \varprojlim(\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*)$$
   While $\mathrm{Stab}(\mathcal{C})$ is always a stable ∞-category, it may not possess a symmetric monoidal structure (like the smash product) unless $\mathcal{C}$ itself is equipped with a compatible monoidal structure.

3. **The Relation:** Any stable ∞-category $\mathcal{D}$ is naturally tensored over Sp. In this sense, Sp plays a role analogous to the ring of integers $\mathbb{Z}$ in abelian groups: for any $D \in \mathcal{D}$ and $E \in \mathrm{Sp}$, there is a well-defined object $E \otimes D \in \mathcal{D}$.

Definition 1.16 (Internal Mapping Spectrum in $\mathrm{Stab}(\mathcal{C})$)**.** Let $\mathcal{C}$ be a **closed symmetric monoidal ∞-category** $(\mathcal{C}, \otimes, \mathbf{1})$ that admits finite limits. Assume further that the tensor product $\otimes$ is compatible with the stabilization (i.e., it preserves colimits in each variable).

Let $\mathcal{D} = \mathrm{Stab}(\mathcal{C})$ be the resulting stable ∞-category, equipped with the induced symmetric monoidal structure $\otimes_{\mathcal{D}}$. For any objects $X, Y \in \mathcal{D}$, the **internal mapping spectrum** is defined as the object $\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$ that satisfies the following conditions:

1. **Adjunction Property:** It is the right adjoint to the tensor product functor. For any object $Z \in \mathcal{D}$, there is a natural equivalence of mapping spaces:

$$\mathrm{Map}_{\mathcal{D}}(Z \otimes_{\mathcal{D}} X, Y) \simeq \mathrm{Map}_{\mathcal{D}}(Z, \underline{\mathrm{Map}}_{\mathcal{D}}(X, Y))$$

2. **Spectrum Level Structure:** In terms of the sequence of objects $\{E_n\} \in \mathcal{C}_*$ representing the spectrum, the n-th level of the internal mapping spectrum is given by:

$$\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y)_n \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(X, \Sigma^n Y)$$

where $\underline{\mathrm{Map}}_{\mathcal{C}}$ denotes the internal Hom in the underlying category $\mathcal{C}$ (if it exists) or the corresponding enrichment.

Definition 1.17 (Mapping Spectrum)**.** Let X and Y be spectra in Sp. The **mapping spectrum** from X to Y, denoted as $\underline{\mathrm{Map}}(X, Y) \in \mathrm{Sp}$, is the unique spectrum (up to equivalence) characterized by the following properties:

1. **Adjunction (Internal Hom):** For any spectrum Z, there is a natural equivalence of mapping spaces:

$$\mathrm{Map}_{\mathrm{Sp}}(Z \wedge X, Y) \simeq \mathrm{Map}_{\mathrm{Sp}}(Z, \underline{\mathrm{Map}}(X, Y))$$

This identifies $\underline{\mathrm{Map}}(X, Y)$ as the right adjoint to the smash product functor $(- \wedge X)$.

2. **Omega-Spectrum Structure:** The n-th space of the mapping spectrum is equivalent to the space of maps from X to the n-th suspension of Y:

$$\underline{\mathrm{Map}}(X, Y)_n \simeq \mathrm{Map}_{\mathrm{Sp}}(X, \Sigma^n Y)$$

The structure maps $\Sigma \underline{\mathrm{Map}}(X, Y)_n \to \underline{\mathrm{Map}}(X, Y)_{n+1}$ are induced by the stability of Sp.

Definition 1.18 (Homotopy Groups in $\mathrm{Stab}(\mathcal{C})$)**.** Let $\mathcal{C}$ be a closed symmetric monoidal ∞-category with finite limits, and let $\mathcal{D} = \mathrm{Stab}(\mathcal{C})$ be its stabilization with unit object $\mathbf{1}_{\mathcal{D}}$.

1. **Homotopy Groups of an Object:** For any object $E \in \mathcal{D}$ and $n \in \mathbb{Z}$, the n-th homotopy group of E is defined as the abelian group of homotopy classes of maps from the n-shifted unit object:

$$\pi_n(E) := [\Sigma^n \mathbf{1}_{\mathcal{D}}, E]_{\mathcal{D}}$$

2. **Homotopy Groups of the Mapping Spectrum:** Let $\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$ be the internal mapping spectrum between $X, Y \in \mathcal{D}$. Its homotopy groups characterize the graded morphisms between the two objects:

$$\pi_n \underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \cong [X, \Sigma^n Y]_{\mathcal{D}} \cong [\Sigma^{-n} X, Y]_{\mathcal{D}}$$

where $[-, -]_{\mathcal{D}}$ denotes the set of homotopy classes i.e. the 0-th homotopy group of the kan complex $\mathrm{Map}_{\mathcal{D}}(-, -)$.

Remark 1.19. The distinction lies in the target category:

- $\mathrm{Map}_{\mathcal{D}}(X, Y) \in \mathcal{S}$ is a **space** (Kan complex). It represents the mapping space in the ∞-categorical sense.
- $\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$ is a **spectrum** (Internal Hom). It is an object of the stable category $\mathcal{D}$ that stabilizes the mapping space.

In terms of homotopy groups: $\pi_n \mathrm{Map}_{\mathcal{D}}(X, Y) \cong \pi_n \underline{\mathrm{Map}}_{\mathcal{D}}(X, Y)$ for $n \geq 0$, because every spectrum is Ω-spectrum in $\mathrm{Stab}(\mathcal{C})$.

Construction 1.20 (Stabilization of a Suspension Spectrum)**.** Let $X \in \mathcal{S}_*$ be a pointed space (or generally an object in a pointed ∞-category $\mathcal{C}$ with finite colimits). The construction of its associated **suspension spectrum** $\Sigma^\infty X \in \mathrm{Sp}$ proceeds as follows:

1. **The Prespectrum Construction:** First, we form a *prespectrum* P_X by iterating the suspension functor Σ on X. This is given by the sequence of spaces:

$$(P_X)_n := \Sigma^n X, \quad \text{for } n \geq 0$$

   together with the structural maps (identities):

$$\sigma_n : \Sigma((P_X)_n) = \Sigma(\Sigma^n X) \xrightarrow{id} \Sigma^{n+1} X = (P_X)_{n+1}$$

2. **Spectrification (The L-functor):** Since P_X is not necessarily an Ω-spectrum (i.e., the adjoint maps $(P_X)_n \to \Omega(P_X)_{n+1}$ are not equivalences), we apply the *spectrification functor* L (or stabilization) to convert it into a true spectrum. The resulting object is the suspension spectrum:

$$\Sigma^\infty X := L(P_X)$$

   Conceptually, the k-th space of this stable object is the colimit:

$$(\Sigma^\infty X)_k \simeq \operatorname*{colim}_{m \to \infty} \Omega^m \Sigma^{m+k} X$$

3. **Universal Property (Adjunction):** The construction defines the left adjoint functor Σ^∞ in the stabilization adjunction:

$$\begin{tikzcd} \mathcal{S}_* \arrow[r, bend left, "\Sigma^\infty"] & \mathrm{Sp} \arrow[l, bend left, "\Omega^\infty"] \end{tikzcd}$$

   where for any spectrum E, the right adjoint is given by $\Omega^\infty E := E_0$ (the 0-th space of the Ω-spectrum E).

Remark 1.21 (Bousfield-Friedlander Structure and Stabilization)**.** The Bousfield-Friedlander model structure $\mathcal{M}_{\mathrm{BF}}$ on the category of prespectra is the left Bousfield localization of the strict model structure $\mathcal{M}_{\mathrm{strict}}$. The three classes of morphisms in $\mathcal{M}_{\mathrm{BF}}$ are characterized as follows:

- **Cofibrations:** These are exactly the same as the strict cofibrations (levelwise inclusions that satisfy the appropriate cell complex conditions).
- **Weak Equivalences:** These are the *stable weak equivalences*, i.e., maps $f : X \to Y$ that induce isomorphisms on stable homotopy groups $\pi_n^S(X) \cong \pi_n^S(Y)$ for all $n \in \mathbb{Z}$.

- **Fibrations:** These are the maps that satisfy the right lifting property with respect to acyclic cofibrations. Specifically, a map $p : E \to B$ is a BF-fibration if it is a levelwise fibration and the square

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "p_n"] & \Omega E_{n+1} \arrow[d, "\Omega p_{n+1}"] \\ B_n \arrow[r] & \Omega B_{n+1} \end{tikzcd}$$

  is a homotopy pullback for all n.

The transition from $\mathcal{M}_{\text{strict}}$ to $\mathcal{M}_{\text{BF}}$ captures the essence of stabilization. Since the fibrant objects in this structure are exactly the Ω-spectra, the **fibrant replacement** of a prespectrum X in $\mathcal{M}_{\text{BF}}$ is precisely its **stabilization** (spectrification).

If R_{BF} denotes the fibrant replacement functor, we have a natural stable equivalence $j : X \xrightarrow{\sim} R_{\text{BF}}(X)$, where $R_{\text{BF}}(X)$ is an Ω-spectrum. In the stable homotopy category, this is equivalent to the classical stabilization $QX = \Omega^\infty \Sigma^\infty X$:

$$\begin{tikzcd} X \arrow[r, "j", "\sim_{\text{stable}}"'] \arrow[d, equal] & R_{\text{BF}}(X) \arrow[d, "\simeq"] \\ X \arrow[r, "\text{Stabilization}"] & QX \end{tikzcd}$$

Thus, the Bousfield-Friedlander model structure provides the formal homotopy-theoretic machinery where "becoming an Ω-spectrum" is equivalent to "becoming fibrant."