

# A Study in Advanced Talks

Mathematical Notes Collection

Notes & Expositions

# Contents

<b>Perfectoid Spaces I: The Weight-Monodromy Conjecture</b>	<b>2</b>
1.1 Introduction . . . . .	3
1.2 The Setup and The Monodromy Operator . . . . .	3
1.2.1 Geometric Setup . . . . .	3
1.2.2 The Monodromy Theorem . . . . .	3
1.3 The Weight-Monodromy Conjecture . . . . .	3
1.4 Examples and Evidence . . . . .	4
1.4.1 Case 1: Good Reduction . . . . .	4
1.4.2 Case 2: The Tate Curve . . . . .	4
1.5 Strategy: Reduction to Equal Characteristic . . . . .	5
1.5.1 Known Results . . . . .	5
1.5.2 Rapoport's Suggestion . . . . .	5
1.5.3 The Obstruction and Perfectoid Spaces . . . . .	5
<b>Algebraic Geometry in Mixed Characteristic</b>	<b>6</b>
2.1 Motivation: The Defect of Classical Cohomology . . . . .	7
2.2 The Geometric Backdrop: Hensel's $p$ -adic World . . . . .	7
2.3 Prismatic Cohomology . . . . .	8
2.4 The Shape of Prismatic Cohomology . . . . .	8
2.5 Detailed Applications . . . . .	8
2.5.1 Algebraic K-Theory . . . . .	9
2.5.2 Kodaira Vanishing in Mixed Characteristic . . . . .	9
2.5.3 Other Recent Developments . . . . .	10

# Perfectoid Spaces I: The Weight-Monodromy Conjecture

*Speaker: Peter Scholze*

## 1.1 Introduction

These notes reconstruct the motivations leading to the theory of Perfectoid Spaces, rooted in Peter Scholze's undergraduate years in Bonn (circa 2007)[cite: 2]. The central motivation discussed here is the **Weight-Monodromy Conjecture** (WMC) by Deligne.

## 1.2 The Setup and The Monodromy Operator

### 1.2.1 Geometric Setup

Let  $K = \mathbb{Q}_p$  be the  $p$ -adic field. Let  $X$  be a smooth projective scheme over  $\mathbb{Q}_p$ [cite: 2]. Fix a prime  $\ell \neq p$ [cite: 3]. We are interested in the  $\ell$ -adic cohomology of  $X$ :

$$V := H_{\text{ét}}^i(X_{\overline{\mathbb{Q}_p}}, \overline{\mathbb{Q}_\ell}).$$

This vector space admits a continuous action of the absolute Galois group  $G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ [cite: 4].

### 1.2.2 The Monodromy Theorem

To state the conjecture, we must first define the structure of this Galois representation. The inertia group  $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$  acts on  $V$ . Even if  $X$  does not have good reduction, Grothendieck's  $\ell$ -adic Monodromy Theorem describes this action.

**Theorem 1.1** (Grothendieck's  $\ell$ -adic Monodromy). *There exists a nilpotent operator  $N : V \rightarrow V(-1)$ , called the **Monodromy Operator**, such that on an open subgroup of the inertia  $I_{\mathbb{Q}_p}$ , the action of  $\sigma \in I_{\mathbb{Q}_p}$  is given by*

$$\rho(\sigma) = \exp(t_\ell(\sigma)N),$$

where  $t_\ell : I_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_\ell$  is the  $\ell$ -adic tame character[cite: 10, 11].

**Definition 1.2** (Weight Decomposition). The representation  $V$  admits a decomposition based on the eigenvalues of the Geometric Frobenius  $\Phi$ . We say  $V$  has a **weight decomposition**[cite: 8]:

$$V = \bigoplus_{j=0}^{2i} V_j,$$

where the eigenvalues of  $\Phi$  acting on  $V_j$  are Weil numbers of weight  $j$ .

The monodromy operator  $N$  respects this structure in a specific way. Since  $N$  maps  $V$  to  $V(-1)$  (a Tate twist), it lowers the weight by 2[cite: 13]:

$$N : V_j \longrightarrow V_{j-2}(-1).$$

## 1.3 The Weight-Monodromy Conjecture

Deligne proposed the following conjecture regarding the interplay between the filtration by weights and the monodromy operator.

**Conjecture 1.3** (Deligne’s Weight-Monodromy Conjecture). *For any  $j \geq 0$ , the power of the monodromy operator induces an isomorphism[cite: 17, 25]:*

$$N^j : V_{i+j} \xrightarrow{\sim} V_{i-j}(-j).$$

Intuitively, this suggests that the monodromy operator acts like the hard Lefschetz operator, reflecting the weight filtration across the center  $i$ .

## 1.4 Examples and Evidence

### 1.4.1 Case 1: Good Reduction

Assume  $X$  has good reduction, meaning there exists a smooth projective model  $\mathcal{X}$  over  $\mathbb{Z}_p$  such that  $\mathcal{X}_{\mathbb{Q}_p} \cong X$ [cite: 19].

**Lemma 1.4.** *If  $X$  has good reduction, the action of the inertia group  $I_{\mathbb{Q}_p}$  is trivial[cite: 22].*

Consequently, the monodromy operator is trivial ( $N = 0$ )[cite: 23]. In this case, Conjecture 1.3 implies that  $V_j = 0$  for all  $j \neq i$ . Thus  $V = V_i$ . This reduces to the **Weil Conjectures** for the reduction  $X_{\mathbb{F}_p}$ , which are known[cite: 26].

### 1.4.2 Case 2: The Tate Curve

Consider an elliptic curve  $E$  over  $\mathbb{Q}_p$  with split multiplicative reduction. As a rigid analytic space,  $E$  can be uniformized as the **Tate Curve**[cite: 30]:

$$E(\overline{\mathbb{Q}_p}) \cong \mathbb{G}_m(\overline{\mathbb{Q}_p})/q^{\mathbb{Z}}, \quad \text{with } |q| < 1.$$

We compute  $H^1(E_{\overline{\mathbb{Q}_p}}, \overline{\mathbb{Q}_\ell})$ . Using the Hochschild-Serre spectral sequence (or the Kummer sequence for  $\mathbb{G}_m$ ), we have an exact sequence[cite: 39]:

$$0 \longrightarrow \overline{\mathbb{Q}_\ell}(0) \longrightarrow H^1(E, \overline{\mathbb{Q}_\ell}) \longrightarrow \overline{\mathbb{Q}_\ell}(-1) \longrightarrow 0.$$

Here:

- The subspace  $V_0 = \overline{\mathbb{Q}_\ell}(0)$  has weight 0.
- The quotient  $V_2 = \overline{\mathbb{Q}_\ell}(-1)$  has weight 2.
- The center weight is  $i = 1$ .

The Monodromy operator  $N$  maps the weight 2 part to the weight 0 part:

$$N : V_2 \xrightarrow{\sim} V_0(-1).$$

This verifies the conjecture for  $j = 1$ :  $N^1 : V_{1+1} \rightarrow V_{1-1}(-1)$  is an isomorphism[cite: 40].

## 1.5 Strategy: Reduction to Equal Characteristic

### 1.5.1 Known Results

The conjecture is known in the following cases:

1. Dimension of  $X$  is  $\leq 2$  [cite: 47].
2. **Equal Characteristic:** If  $X$  is defined over a function field  $F \cong \mathbb{F}_p((t))$  rather than  $\mathbb{Q}_p$ , the conjecture was proved by Deligne using  $L$ -functions [cite: 52, 53].

### 1.5.2 Rapoport's Suggestion

The strategy to prove WMC in mixed characteristic  $(\mathbb{Q}_p)$  is to reduce it to the equal characteristic case  $(\mathbb{F}_p((t)))$ .

- **Idea:** Base change to a highly ramified extension  $K/\mathbb{Q}_p$  [cite: 60].
- Let  $e$  be the ramification index. The ring of integers behaves like:

$$\mathcal{O}_K/p \cong \mathbb{F}_q[t]/t^e.$$

- As  $e \rightarrow \infty$ , this ring approximates  $\mathbb{F}_q[[t]]$  [cite: 63, 65].

### 1.5.3 The Obstruction and Perfectoid Spaces

However, for any finite  $e$ , the approximation is insufficient. There are algebraic obstructions to deforming  $X_{\mathcal{O}_K/p}$  to a scheme over  $\mathbb{F}_q[[t]]$  [cite: 72, 76].

This failure suggests that we must pass to the limit  $e = \infty$ . This leads to the definition of **Perfectoid Fields** (fields with infinite ramification). In the world of perfectoid spaces, we can construct a rigorous isomorphism (Tilting) between geometric objects in mixed characteristic and equal characteristic:

$$X_{\text{perfectoid}} \longleftrightarrow X_{\text{equal char}}^{\flat}.$$

This correspondence allows the transfer of cohomological results (like Deligne's theorem) back to  $\mathbb{Q}_p$ , as realized in Scholze's work [cite: 78-81].

# Algebraic Geometry in Mixed Characteristic

*Speaker: Bhargav Bhatt*

## 2.1 Motivation: The Defect of Classical Cohomology

The motivating goal for modern  $p$ -adic geometry is to establish a geometric understanding of integral cohomology classes, particularly torsion, in an algebraic setting.

**Theorem 2.1** (de Rham 1931, Hodge 1941). *Let  $X$  be a compact complex Kähler manifold (e.g.,  $X \subset \mathbb{CP}^m$ ). The integration of differential forms over cycles yields the following isomorphisms:*

$$H^n(X; \mathbb{C}) \simeq H_{\text{dR}}^n(X; \mathbb{C}) \simeq \bigoplus_{i+j=n} H^{i,j}(X).$$

This classical result provides a bidirectional bridge between topology and geometry:

- **Geometry to Topology:** The symmetry  $H^{i,j} \simeq \overline{H^{j,i}}$  implies constraints on the topological Betti numbers (e.g., if  $n$  is odd,  $\dim H^n(X; \mathbb{C})$  must be even).
- **Topology to Geometry:** Topological constraints, such as  $\pi_1(X) = 0$ , force the vanishing of holomorphic 1-forms ( $H^{1,0}(X) \subset H^1(X; \mathbb{C}) = 0$ ).

*Remark 2.2* (The Defect). The comparison above relies on coefficients in  $\mathbb{C}$ . Consequently, it completely misses the torsion information in  $H^*(X; \mathbb{Z})$ ! The goal of mixed characteristic geometry is to understand  $H^*(X; \mathbb{Z}/p^n)$  geometrically for algebraic varieties.

## 2.2 The Geometric Backdrop: Hensel's $p$ -adic World

To bridge characteristic 0 and characteristic  $p$ , we utilize the ring of  $p$ -adic integers  $\mathbb{Z}_p$ .

**Definition 2.3** (Mixed Characteristic Setting). We define the following base rings and fields:

- $\mathbb{Z}_p = \{\sum_{i \geq 0} a_i p^i \mid a_i \in \{0, \dots, p-1\}\}$ : The bridge between characteristics.
- $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$ : Characteristic 0.
- $\mathbb{C}_p = \widehat{\mathbb{Q}_p}$ : The  $p$ -adic analog of  $\mathbb{C}$ .
- $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$ : Characteristic  $p$ .

For a smooth projective variety  $X$  over  $\mathbb{Z}_p$ , we obtain a correspondence between the generic fiber (analytic/topological side) and the special fiber (algebraic side):

$$\begin{array}{ccccc} X_{\mathbb{C}_p} & & X & & X_{\mathbb{F}_p} \\ \text{generic fiber} \downarrow & & \downarrow & & \downarrow \text{special fiber} \\ \text{Spec}(\mathbb{C}_p) & \longrightarrow & \text{Spec}(\mathbb{Z}_p) & \longleftarrow & \text{Spec}(\mathbb{F}_p) \end{array}$$

Historically, the comparison principle (Tate 1966, Grothendieck 1970, Fontaine 1978) suggested that the mod  $p^n$  topology of  $X_{\mathbb{C}_p}$  should be recoverable from the algebraic geometry of  $X_{\mathbb{F}_p}$  augmented with specific linear algebra data (e.g., Frobenius actions).



## 2.3 Prismatic Cohomology

Prismatic cohomology serves as a 21st-century upgrade to this principle, unifying various cohomology theories.

**Theorem 2.4** (Bhatt-Morrow-Scholze, 2016, 2018). *Let  $X$  be a smooth formal scheme over  $\mathbb{Z}_p$ . There exists a cohomology theory  $H_\Delta^*(X; \mathbb{F}_p) \in \text{Mod}^{fg}(\mathbb{F}_p[[T]])$  satisfying the following comparisons:*

1. **Topological Comparison (Generic Fiber):**

$$H_\Delta^*(X; \mathbb{F}_p)[1/T] \simeq H^*(X_{\mathbb{C}_p}; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p((T))$$

*This recovers the cohomology of the generic fiber.*

2. **Differential Comparison (Special Fiber):**

$$H_\Delta^*(X; \mathbb{F}_p)/T \simeq H_{\text{dR}}^*(X_{\mathbb{F}_p})$$

*This recovers the de Rham cohomology of the reduction mod  $p$ .*

There is also an integral variant  $H_\Delta^*(X; \mathbb{Z}_p) \in \text{Mod}^{fg}(\mathbb{Z}_p[[T]])$ .

## 2.4 The Shape of Prismatic Cohomology

One of the most powerful aspects of prismatic cohomology is how it organizes different cohomology theories into a single geometric family over the parameter space  $\text{Spec}(\mathbb{Z}_p[[T]])$ .

The following table illustrates the correspondence between the specializations of the "prism" and classical cohomology theories:

Table 2.1: Specializations of  $H_\Delta^*(X; \mathbb{Z}_p)$  (Fibers over  $\text{Spec}(\mathbb{Z}_p[[T]])$ )

Locus	Condition	Recovered Theory
$p$ -axis	$T = 0$	<b>Crystalline / de Rham</b> (Geometry of $X_{\mathbb{F}_p}$ )
$T$ -axis	$p = 0$	<b>Mod <math>p</math> Étale</b> (Char $p$ topology)
Hodge-Tate Locus	$T = p$ (approx.)	<b>Hodge-Tate</b>
Generic Locus	$T$ inverted	<b>Étale Cohomology (<math>p</math>-adic)</b> (Topology of $X_{\mathbb{C}_p}$ )

This structure effectively upgrades the classical Hodge filtration to a deformation over the variable  $T$ .

## 2.5 Detailed Applications

The algebraic construction of prismatic cohomology has provided new tools to address long-standing questions where torsion classes were previously problematic.

### 2.5.1 Algebraic K-Theory

Classical results by Atiyah, Hirzebruch, and Bott established a spectral sequence relating the topological K-theory  $K(X)$  to singular cohomology for nice spaces. A major open question (Beilinson 1982) was to find an analog for the algebraic K-theory of rings.

**Theorem 2.5** (Clausen-Mathew-Morrow 2018, B-Morrow-Scholze 2018). *For a  $p$ -complete commutative ring  $R$ , there exists a natural "motivic" filtration on the  $p$ -adic K-theory  $K_{\text{et}}(R; \mathbb{Z}_p)$  where the graded pieces are given by **syntomic cohomology**:*

$$\text{gr}^i K_{\text{et}}(R; \mathbb{Z}_p) \simeq H_{\text{syn}}^*(R, \mathbb{Z}_p(i))[2i].$$

*Syntomic cohomology is a new object determined entirely by prismatic cohomology.*

This structural result has led to concrete calculations that were previously out of reach:

1. **Odd Vanishing (B-Scholze 2019):** For many "large" rings, such as  $\mathcal{O}_{\mathbb{C}_p}/p^n$ , the odd homotopy groups vanish:  $\pi_{\text{odd}} K(\mathcal{O}_{\mathbb{C}_p}/p^n) = 0$ . This relies on  $q$ -de Rham complexes and André's flatness lemma.
2. **Even Vanishing (Antieau-Krause-Nikolaus 2022):** Using absolute prismatic cohomology, it is shown that:

$$\pi_{2k} K(\mathbb{Z}/p^n) = 0 \quad \text{for all } k \gg 0.$$

### 2.5.2 Kodaira Vanishing in Mixed Characteristic

The classic Kodaira vanishing theorem states that for  $X \subset \mathbb{P}_{\mathbb{C}}^n$  smooth projective of dimension  $d$ ,  $H^{<d}(X, \mathcal{O}(-1)) = 0$ . This is known to fail in positive characteristic (Raynaud 1978, Totaro 2021).

**Theorem 2.6** (Global KV up to Finite Covers, Bhatt 2020). *For  $X \subset \mathbb{P}_{\mathbb{Z}_p}^n$  projective of relative dimension  $d$ , there exists a finite cover  $\pi : Y \rightarrow X$  such that the torsion part of the cohomology is annihilated by the pullback map:*

$$\text{Image} \left( H^{<d}(X, \mathcal{O}(-1))_{\text{tors}} \xrightarrow{\pi^*} H^{<d}(Y, \pi^* \mathcal{O}(-1))_{\text{tors}} \right) = 0.$$

This theorem is established using prismatic cohomology and Riemann-Hilbert constructions for perverse  $\mathbb{F}_p$ -sheaves. It also has a purely local commutative algebra formulation:

**Theorem 2.7** (Local KV / Splinter Property, Bhatt 2020). *Fix a finite extension  $\mathbb{Z}_p[x_1, \dots, x_n] \subset R$ . Then there exists a finite extension  $R \subset S$  such that any relation  $\sum a_i x_i = 0$  in  $R/p$  becomes trivial in  $S/p$ . Conceptually, if  $R^+$  is the integral closure of  $R$  in  $\text{Frac}(R)$ , then  $R^+/p$  is Cohen-Macaulay over  $R/p$ .*

*Remark 2.8* (Application to Minimal Model Program). These vanishing results were key ingredients in establishing the Minimal Model Program for arithmetic 3-folds with residue characteristic  $p > 5$  (B-Ma-Patakfalvi-Schwede-Tucker-Waldron-Witaszek 2020).

### 2.5.3 Other Recent Developments

Prismatic cohomology has also been instrumental in several other major results:

- **Tate Conjecture in Char 2:** Proved for K3 surfaces in characteristic 2 by Madapusi Pera (2016) and Ito-Ito-Koshikawa (2018).
- **$p$ -adic Upper Half Space:** Colmez-Dospinescu-Nizioł (2019) proved that  $H^*(\Omega; \mathbb{Z}_p)$  is  $p$ -torsionfree for Drinfeld's  $p$ -adic upper half space.
- **Essential Dimension:** Farb-Kisin-Wolfson (2021) proved that for an abelian variety  $A/\mathbb{C}$ , the multiplication map  $[p]$  has essential dimension  $\dim(A)$  for  $p \gg 0$ .
- **Poincaré Duality:** Established for  $\mathbb{Z}_p$ -étale cohomology in  $p$ -adic analytic geometry by Zavyalov (2021).