The review of Smooth Manifolds

Mathematical Notes Collection

Notes & Expositions

Contents

Lie Groups

Definition 1.1. A **Lie group** is a set G endowed with the structure of a smooth manifold and a group, such that the group operations are smooth. That is, the multiplication map

$$\mu : G \times G \to G, \quad (g, h) \mapsto gh$$

and the inversion map

$$\iota : G \to G, \quad g \mapsto g^{-1}$$

are both smooth maps.

Definition 1.2. Let G be a Lie group and let $g \in G$ be a fixed element.

- The **left translation** by g is the map $L_g : G \to G$ defined by

$$L_g(h) = gh \quad \text{for all } h \in G.$$

- The **right translation** by g is the map $R_g : G \to G$ defined by

$$R_g(h) = hg \quad \text{for all } h \in G.$$

Since the group multiplication is smooth, both L_g and R_g are diffeomorphisms of the manifold G.

Definition 1.3. Let G and H be Lie groups. A map $\phi : G \to H$ is called a **Lie group homomorphism** if:

1. ϕ is a group homomorphism, meaning $\phi(gh) = \phi(g)\phi(h)$ for all $g, h \in G$; and

2. ϕ is a smooth map between the manifolds G and H.

If ϕ is in addition a diffeomorphism, then ϕ is called a **Lie group isomorphism**.

Theorem 1.4. Let G and H be Lie groups, and let $\phi : G \to H$ be a Lie group homomorphism. Then ϕ has **constant rank**.

Specifically, for any $g \in G$, the rank of the differential $d\phi_g : T_gG \to T_{\phi(g)}H$ is equal to the rank of the differential at the identity, $d\phi_e$.

Definition 1.5. Let G be a Lie group. A subset $H \subseteq G$ is called a **Lie subgroup** if:

1. H is a subgroup of G in the algebraic sense;

2. H is endowed with a topology and a smooth structure that make it an **immersed submanifold** of G (meaning the inclusion map $\iota : H \hookrightarrow G$ is a smooth immersion); and

3. H is a Lie group with respect to this smooth structure.

It is important to note that the topology on an immersed submanifold H is not necessarily the subspace topology induced from G.

Theorem 1.6 (Closed Subgroup Theorem)**.** Let G be a Lie group and let H be a subgroup of G. If H is a closed subset of G, then H is an **embedded Lie subgroup** of G.

Definition 1.7. Let G be a Lie group and let M be a smooth manifold. A **left Lie group action** of G on M is a smooth map

$$\theta : G \times M \to M, \quad (g, p) \mapsto g \cdot p,$$

satisfying the following two axioms:

1. **Identity:** $e \cdot p = p$ for all $p \in M$, where e is the identity of G.

2. **Associativity:** $g \cdot (h \cdot p) = (gh) \cdot p$ for all $g, h \in G$ and $p \in M$.

Since the map θ is smooth, for every $g \in G$, the map $\theta_g : M \to M$ given by $p \mapsto g \cdot p$ is a diffeomorphism.

Definition 1.8. Let M and N be smooth manifolds endowed with smooth left actions of a Lie group G. Let $\theta_g^M : M \to M$ and $\theta_g^N : N \to N$ denote the diffeomorphisms associated with the action of an element $g \in G$.

A smooth map $F : M \to N$ is called G-**equivariant** if it commutes with the G-action. That is, for every $g \in G$, the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\ F\ } & N \\
{\scriptstyle \theta_g^M} \downarrow & & \downarrow {\scriptstyle \theta_g^N} \\
M & \xrightarrow{\ F\ } & N
\end{array}
$$

In algebraic terms, this condition is written as:

$$F(g \cdot p) = g \cdot F(p) \quad \text{for all } g \in G \text{ and } p \in M.$$

Theorem 1.9 (Equivariant Rank Theorem)**.** Let M and N be smooth manifolds endowed with smooth actions of a Lie group G. Let $F : M \to N$ be a smooth G-equivariant map. If the action of G on M is **transitive**, then F has **constant rank**.

Consequently, if the action is transitive:

- The image $F(M)$ is an **immersed submanifold** of N; and

- The fibers $F^{-1}(y)$ (for $y \in F(M)$) are closed **embedded submanifolds** of M.

Theorem 1.10 (Rank Theorem)**.** Let M and N be smooth manifolds of dimension m and n, respectively, and let $F : M \to N$ be a smooth map. Suppose that F has **constant rank** k in a neighborhood of a point $p \in M$.

Then there exist smooth charts (U, φ) centered at p (with coordinates $x^1, \ldots, x^m$) and (V, ψ) centered at $F(p)$ (with coordinates $y^1, \ldots, y^n$) such that the local coordinate representation of F,

$$\hat{F} = \psi \circ F \circ \varphi^{-1},$$

takes the following canonical form:

$$\hat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, \underbrace{0, \ldots, 0}_{n-k}).$$

Consequences:

- The level set $F^{-1}(F(p)) \cap U$ is a smooth submanifold of M of dimension $m - k$ (Kernel).

- The image $F(U)$ is an immersed submanifold of N of dimension k (Image).

Remark 1.11. **Computation via Jacobian Matrices.** Let X and Y be vector fields on $\mathbb{R}^n$ (or on a local chart $U \subset \mathbb{R}^n$), identified with column vectors of component functions $X = (X^1, \ldots, X^n)^T$ and $Y = (Y^1, \ldots, Y^n)^T$. The Lie bracket $[X, Y]$ can be explicitly computed using the Jacobian matrices $J_X = \left(\frac{\partial X^i}{\partial x^j} \right)$ and $J_Y = \left(\frac{\partial Y^i}{\partial x^j} \right)$ via the formula:

$$[X, Y] = J_Y X - J_X Y.$$

Remark 1.12. **Tangent Space as the Kernel of the Jacobian.** Let $M \subset \mathbb{R}^n$ be a submanifold defined implicitly by the independent equations $f_1(x) = \cdots = f_k(x) = 0$. At any point $p \in M$, the differentials $df_1|_p, \ldots, df_k|_p$ form a basis for the *conormal space* $N_p^* M$. These differentials appear explicitly as the rows of the Jacobian matrix $J_f(p)$:

$$J_f(p) = \begin{pmatrix} \nabla f_1(p)^T \\ \vdots \\ \nabla f_k(p)^T \end{pmatrix}.$$

Since a tangent vector $v \in T_p M$ must represent a direction in which the defining functions do not change (to first order), the tangent space is precisely the **kernel** (or null space) of this Jacobian matrix:

$$T_p M = \ker(dF_p) = \left\{ v \in \mathbb{R}^n \mid J_f(p) \cdot v = 0 \right\}.$$

Geometrically, this means the tangent space consists of all vectors orthogonal to the gradients $\{\nabla f_1, \ldots, \nabla f_k\}$.

Remark 1.13. **Decomposition of Volume Forms under Multiple Constraints.** Let N be an ambient manifold with a global volume form Ω, and let $M \subset N$ be a submanifold of codimension k defined by the vanishing of k independent functions, $f_1 = \cdots = f_k = 0$.

The relationship between the ambient geometry and the submanifold geometry is established by the unique decomposition:

$$\Omega = df_1 \wedge df_2 \wedge \cdots \wedge df_k \wedge \omega_M,$$

where:

- The k-form $df_1 \wedge \cdots \wedge df_k$ represents the **conormal volume**, generated by the constraint differentials (the rows of the Jacobian) which annihilate the tangent space.

- The $(n - k)$-form ω_M is the **induced volume form** (or boundary form) on M, measuring the volume of the tangent space $TM = \bigcap_{i=1}^{k} \ker(df_i)$.

In the context of complex geometry or residue theory, this relationship is often denoted using the *Leray residue* notation, intuitively expressing the induced form as a "division" of the ambient form by the defining equations:

$$\omega_M = \left. \frac{\Omega}{df_1 \wedge \cdots \wedge df_k} \right|_M.$$

Definition 1.14. Orientation Compatibility for Submanifolds.

Let X be an oriented smooth n-manifold with volume form Ω_X. Let $M \subset X$ be a submanifold of dimension m (codimension $k = n - m$).

Suppose the **normal bundle** $N(M)$ of M is oriented. Let ν be a non-vanishing k-form along M representing this normal orientation (often called the *transverse volume form*).

The **induced orientation** on M is defined by the unique m-form ω_M satisfying the compatibility condition:

$$\Omega_X\big|_M = \nu \wedge \omega_M.$$

In the specific case where M is defined regular level set of independent functions $f_1, \ldots, f_k$ (i.e., $M = \{f^{-1}(0)\}$), the normal orientation is naturally given by $\nu = df_1 \wedge \cdots \wedge df_k$. The condition becomes:

$$\Omega_X = (df_1 \wedge \cdots \wedge df_k) \wedge \omega_M.$$

Note: This generalizes the boundary case. For a boundary, $k = 1$, and ν corresponds to the single outward normal form. For a general submanifold, ν represents the "volume" of the directions perpendicular to M.

Definition 1.15. Riemannian Volume Form.

Let (M, g) be an oriented Riemannian manifold of dimension n. The **Riemannian volume form**, denoted by dV_g (or vol_g), is the unique differential n-form on M satisfying the following geometric condition:

$$dV_g(e_1, e_2, \ldots, e_n) = 1,$$

for any positively oriented orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM at any point $p \in M$.

In terms of any positively oriented local coordinate chart $(U, x^1, \ldots, x^n)$, the volume form is given explicitly by:

$$dV_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ are the components of the metric tensor, and $\sqrt{\det(g_{ij})}$ is the volume density factor.

Remark 1.16. **Converting Scalar Functions to Volume Forms via the Hodge Star.** Let (M, g) be an oriented Riemannian n-manifold. The mapping from a scalar function $f \in C^\infty(M)$ (a 0-form) to a top-degree form (an n-form) is canonically realized by the **Hodge star operator** $*$:

$$* : \Omega^0(M) \to \Omega^n(M).$$

This isomorphism is defined by multiplication with the Riemannian volume form dV_g:

$$*f = f \wedge (*1) = f \cdot dV_g.$$

In local coordinates, this transformation encapsulates the density factor:

$$*f = f(x)\sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

Consequently, the integral of a function f over M is rigorously defined as the integral of its Hodge dual:

$$\int_M f := \int_M *f.$$