

A Study in Quasi-Coherent Sheaves and Tannaka Duality

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1 Preliminaries

1.1 ∞ -categories

Definition 1.1 (∞ -Category). A simplicial set K is an **∞ -category** if for every $n > 1$ and every **inner** index $0 < i < n$, every map of simplicial sets $f_0 : \Lambda_i^n \rightarrow K$ admits an extension to an n -simplex $f : \Delta^n \rightarrow K$.

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{f_0} & K \\ \downarrow & \nearrow f & \\ \Delta^n & & \end{array}$$

Definition 1.2 (Simplicial Category). A **simplicial category** (or Set_Δ -enriched category) \mathcal{C} is a category where:

1. For any two objects $X, Y \in \mathcal{C}$, the collection of morphisms between them is not a set, but a **simplicial set** $\text{Map}_{\mathcal{C}}(X, Y)$.
2. For any three objects $X, Y, Z \in \mathcal{C}$, the composition map

$$\text{Map}_{\mathcal{C}}(Y, Z) \times \text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{\mathcal{C}}(X, Z)$$

is a morphism of simplicial sets and satisfies the usual associativity and identity axioms.

A simplicial category \mathcal{C} is **locally Kan** if for every pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, the mapping simplicial set $\text{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex.

Definition 1.3 (Simplicial Nerve N_Δ). The **simplicial nerve** $N_\Delta(\mathcal{C})$ is the simplicial set defined by the assignment:

$$N_\Delta(\mathcal{C})_n = \text{Hom}_{\text{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$$

where $\mathfrak{C}[\Delta^n]$ is the **rigidification** of the n -simplex Δ^n into a simplicial category.

Definition 1.4 (∞ -category via N_Δ). An **∞ -category** (or quasicategory) is a simplicial set K that is equivalent to the simplicial nerve of some locally Kan simplicial category \mathcal{C} .

$$K \simeq N_\Delta(\mathcal{C})$$

Theorem 1.5 (Joyal-Lurie). There exists a Quillen equivalence between the Joyal model structure on Set_Δ (modeling quasicategories) and the Bergner model structure on Cat_Δ (modeling simplicial categories):

$$\mathfrak{C}[\cdot] : \text{Set}_\Delta \rightleftarrows \text{Cat}_\Delta : N_\Delta.$$

Specifically, for any simplicial category \mathcal{C} where mapping spaces are Kan complexes, its simplicial nerve $N_\Delta(\mathcal{C})$ is a quasicategory.

Definition 1.6 (Free Cocompletion). Let \mathcal{C} be a small ∞ -category. An ∞ -category $\mathcal{P}(\mathcal{C})$ is called the **free cocompletion** of \mathcal{C} if it satisfies the following universal property:

1. $\mathcal{P}(\mathcal{C})$ admits all small colimits.
2. There exists a functor $j : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$ (called the Yoneda embedding) such that for any ∞ -category \mathcal{D} which admits small colimits, composition with j induces an equivalence of ∞ -categories:

$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Here, Fun^L denotes the full subcategory of functors that preserve small colimits (left adjoints).

Definition 1.7 (The ∞ -category of Spaces). Let \mathcal{S} denote the ∞ -category of spaces. It is defined in two equivalent ways:

1. Via Dwyer-Kan Localization:

Let W be the class of weak homotopy equivalences in Set_Δ . We define \mathcal{S} as the homotopy coherent nerve of the simplicial localization:

$$\mathcal{S} := N(\mathrm{Set}_\Delta[W^{-1}]).$$

Equivalently, via Kan complexes: $\mathcal{S} \simeq N(\mathbf{Kan})$.

2. Via Free Cocompletion:

The ∞ -category \mathcal{S} is the free cocompletion of the point $*$. That is, it is the category of presheaves:

$$\mathcal{S} \simeq \mathcal{P}(*)$$

Universal Property: For any cocomplete ∞ -category \mathcal{C} , there is an equivalence $\mathrm{Fun}^L(\mathcal{S}, \mathcal{C}) \simeq \mathcal{C}$.

Theorem 1.8. Let K be an ∞ -category (quasi-category). Let $\mathcal{C} = \mathfrak{C}[K]$ be its associated simplicial category (via rigidification). The construction of the presheaf ∞ -category commutes with the nerve construction in the following sense:

1. **Simplicial Side:** Consider the category of simplicial presheaves $\mathcal{P}_\Delta(\mathcal{C}) := \mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathcal{S}_{\mathrm{Kan}})$. This category admits a simplicial model structure (projective structure).
2. **Infinity Side:** Consider the ∞ -category of presheaves $\mathcal{P}(K) := \mathrm{Fun}(K^{op}, \mathcal{S})$.
3. **Equivalence:** There is an equivalence of ∞ -categories:

$$\mathcal{P}(K) \simeq N(\mathcal{P}_\Delta(\mathcal{C})^{\mathrm{cf}})$$

where $\mathcal{P}_\Delta(\mathcal{C})^{\mathrm{cf}}$ denotes the full simplicial subcategory of fibrant-cofibrant objects in the model category of simplicial presheaves.

In summary, the presheaf of an ∞ -category is modeled by the nerve of the strictly cocomplete simplicial category of enriched presheaves.

Remark 1.9 (Homotopy Category via Fibrant-Cofibrant Objects). To correctly construct the homotopy category $\text{Ho}(\mathcal{M})$ from a simplicial model category \mathcal{M} , one cannot simply take the path components π_0 of the mapping spaces between arbitrary objects.

Instead, one must restrict attention to the full subcategory of **fibrant-cofibrant objects**, denoted \mathcal{M}_{cf} . It is only within this subcategory that the simplicial mapping spaces $\text{Map}_{\mathcal{M}}(X, Y)$ are guaranteed to be Kan complexes representing the correct derived mapping spaces. The morphisms in the homotopy category are thus given by:

$$[X, Y]_{\text{Ho}(\mathcal{M})} \cong \pi_0 \text{Map}_{\mathcal{M}}(X, Y) \quad \text{for } X, Y \in \mathcal{M}_{cf}.$$

For general objects X, Y , one must first replace them with weakly equivalent fibrant-cofibrant objects (via cofibrant replacement QX and fibrant replacement RY) to compute this group.

1.2 Stable ∞ -Category

Definition 1.10 (Loop Object). For any object $X \in \mathcal{C}$, the loop object ΩX is the limit of the diagram $0 \rightarrow X \leftarrow 0$. It fits into the following pullback square:

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & X \end{array}$$

Intuitively, $\Omega X \simeq 0 \times_X 0$.

Definition 1.11 (Suspension Object). For any object $X \in \mathcal{C}$, the suspension object ΣX is the colimit of the diagram $0 \leftarrow X \rightarrow 0$. It fits into the following pushout square:

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \S \\ 0 & \longrightarrow & \Sigma X \end{array}$$

Intuitively, $\Sigma X \simeq 0 \amalg_X 0$.

Definition 1.12 (Stable ∞ -Category). An ∞ -category \mathcal{C} is called **stable** if it satisfies the following conditions:

1. There exists a zero object $0 \in \mathcal{C}$ (i.e., \mathcal{C} is pointed).
2. Every morphism in \mathcal{C} admits a kernel and a cokernel.
3. A triangle in \mathcal{C} is a pushout square if and only if it is a pullback square.

Definition 1.13 (The Stabilization of an ∞ -Category). Let \mathcal{C} be an ∞ -category admitting finite limits. The process of constructing the stable ∞ -category associated to \mathcal{C} proceeds in two stages:

1. Pointed View (Formation of \mathcal{C}_*):

First, we construct the *pointed* ∞ -category \mathcal{C}_* . Assuming \mathcal{C} has a terminal object $*$, \mathcal{C}_* is defined as the under-category of the terminal object:

$$\mathcal{C}_* := \mathcal{C}_{*/} \cong \text{Fun}(\Delta^1, \mathcal{C}) \times_{\text{Fun}(\{0\}, \mathcal{C})} \{*\}$$

Objects in \mathcal{C}_* are morphisms $* \rightarrow X$ in \mathcal{C} (i.e., objects equipped with a base point). In \mathcal{C}_* , the object corresponding to the identity $* \rightarrow *$ serves as a *zero object* (both initial and terminal). Consequently, the loop functor $\Omega : \mathcal{C}_* \rightarrow \mathcal{C}_*$ is well-defined by $\Omega X = * \times_X *$.

2. Stabilization (Formation of $\text{Sp}(\mathcal{C})$):

The *stabilization* of \mathcal{C} , denoted as $\text{Sp}(\mathcal{C})$ (or $\text{Stab}(\mathcal{C})$), is defined as the ∞ -category of spectrum objects in \mathcal{C}_* . It is constructed as the homotopy limit of the tower of loop functors:

$$\text{Sp}(\mathcal{C}) := \varprojlim \left(\dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$

Explicitly, an object $E \in \text{Sp}(\mathcal{C})$ consists of a sequence $\{E_n\}_{n \geq 0}$ of objects in \mathcal{C}_* together with equivalences $E_n \xrightarrow{\sim} \Omega E_{n+1}$ for each n . This construction forces the suspension functor Σ to be an equivalence, rendering $\text{Sp}(\mathcal{C})$ a stable ∞ -category.

Theorem 1.14 (Universal Property of Stabilization). Let \mathcal{C} be an ∞ -category with finite limits and a terminal object $*$. Let $\mathcal{C}_* = \mathcal{C}_{*/}$ be its pointed version. The stabilization $\text{Sp}(\mathcal{C})$ is characterized by the following equivalent descriptions:

1. **Internal Construction (Loop Towers):** $\text{Sp}(\mathcal{C})$ is the homotopy limit of the sequence of loop functors:

$$\text{Sp}(\mathcal{C}) \simeq \varprojlim \left(\dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$

An object in $\text{Sp}(\mathcal{C})$ is an Ω -spectrum, i.e., a sequence $\{E_n\}_{n \geq 0}$ in \mathcal{C}_* with equivalences $E_n \simeq \Omega E_{n+1}$.

2. **External Construction (Excision):** $\text{Sp}(\mathcal{C})$ is equivalent to the ∞ -category of pointed excisive functors from the category of finite pointed spaces $\mathcal{S}_*^{\text{fin}}$ to \mathcal{C} :

$$\text{Sp}(\mathcal{C}) \simeq \text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$$

A functor $F : \mathcal{S}_*^{\text{fin}} \rightarrow \mathcal{C}$ belongs to $\text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$ if $F(*) \simeq *$ and F maps every pushout square in $\mathcal{S}_*^{\text{fin}}$ to a pullback square in \mathcal{C} .

Furthermore, the stabilization $\text{Sp}(\mathcal{C})$ is the universal stable ∞ -category under \mathcal{C} : for any stable ∞ -category \mathcal{D} , the functor $\text{Sp}(\mathcal{C}) \rightarrow \mathcal{D}$ induces an equivalence of ∞ -categories $\text{Fun}^{\text{lex}}(\text{Sp}(\mathcal{C}), \mathcal{D}) \simeq \text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{D})$, where Fun^{lex} denotes the ∞ -category of left exact functors.

Remark 1.15 (Distinction between $\text{Stab}(\mathcal{C})$ and Sp). It is essential to distinguish between the abstract process of stabilization and the specific category of spectra:

1. **The Category of Spectra (Sp):** Historically and by convention, Sp refers specifically to the stabilization of the ∞ -category of pointed spaces \mathcal{S}_* . That is:

$$\text{Sp} \simeq \text{Stab}(\mathcal{S}) \simeq \varprojlim \left(\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right)$$

This category serves as the unit object in the ∞ -category of stable ∞ -categories and provides the ground for stable homotopy theory.

2. **Stabilization of an Arbitrary ∞ -Category ($\text{Stab}(\mathcal{C})$):** For any ∞ -category \mathcal{C} with finite limits, $\text{Stab}(\mathcal{C})$ is the stable ∞ -category constructed as the limit of the tower of loop functors:

$$\text{Stab}(\mathcal{C}) = \varprojlim \left(\dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$

While $\text{Stab}(\mathcal{C})$ is always a stable ∞ -category, it may not possess a symmetric monoidal structure (like the smash product) unless \mathcal{C} itself is equipped with a compatible monoidal structure.

3. **The Relation:** Any stable ∞ -category \mathcal{D} is naturally tensored over Sp . In this sense, Sp plays a role analogous to the ring of integers \mathbb{Z} in abelian groups: for any $D \in \mathcal{D}$ and $E \in \text{Sp}$, there is a well-defined object $E \otimes D \in \mathcal{D}$.

Definition 1.16 (Internal Mapping Spectrum in $\text{Stab}(\mathcal{C})$). Let \mathcal{C} be a **closed symmetric monoidal ∞ -category** $(\mathcal{C}, \otimes, \mathbf{1})$ that admits finite limits. Assume further that the tensor product \otimes is compatible with the stabilization (i.e., it preserves colimits in each variable).

Let $\mathcal{D} = \text{Stab}(\mathcal{C})$ be the resulting stable ∞ -category, equipped with the induced symmetric monoidal structure $\otimes_{\mathcal{D}}$. For any objects $X, Y \in \mathcal{D}$, the **internal mapping spectrum** is defined as the object $\underline{\text{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$ that satisfies the following conditions:

1. **Adjunction Property:** It is the right adjoint to the tensor product functor. For any object $Z \in \mathcal{D}$, there is a natural equivalence of mapping spaces:

$$\underline{\text{Map}}_{\mathcal{D}}(Z \otimes_{\mathcal{D}} X, Y) \simeq \underline{\text{Map}}_{\mathcal{D}}(Z, \underline{\text{Map}}_{\mathcal{D}}(X, Y))$$

2. **Spectrum Level Structure:** In terms of the sequence of objects $\{E_n\} \in \mathcal{C}_*$ representing the spectrum, the n -th level of the internal mapping spectrum is given by:

$$\underline{\text{Map}}_{\mathcal{D}}(X, Y)_n \simeq \underline{\text{Map}}_{\mathcal{C}}(X, \Sigma^n Y)$$

where $\underline{\text{Map}}_{\mathcal{C}}$ denotes the internal Hom in the underlying category \mathcal{C} (if it exists) or the corresponding enrichment.

Definition 1.17 (Mapping Spectrum). Let X and Y be spectra in Sp . The **mapping spectrum** from X to Y , denoted as $\underline{\text{Map}}(X, Y) \in \text{Sp}$, is the unique spectrum (up to equivalence) characterized by the following properties:

1. **Adjunction (Internal Hom):** For any spectrum Z , there is a natural equivalence of mapping spaces:

$$\underline{\text{Map}}_{\text{Sp}}(Z \wedge X, Y) \simeq \underline{\text{Map}}_{\text{Sp}}(Z, \underline{\text{Map}}(X, Y))$$

This identifies $\underline{\text{Map}}(X, Y)$ as the right adjoint to the smash product functor $(-\wedge X)$.

2. **Omega-Spectrum Structure:** The n -th space of the mapping spectrum is equivalent to the space of maps from X to the n -th suspension of Y :

$$\underline{\text{Map}}(X, Y)_n \simeq \underline{\text{Map}}_{\text{Sp}}(X, \Sigma^n Y)$$

The structure maps $\Sigma \underline{\text{Map}}(X, Y)_n \rightarrow \underline{\text{Map}}(X, Y)_{n+1}$ are induced by the stability of Sp .

Definition 1.18 (Homotopy Groups in $\text{Stab}(\mathcal{C})$). Let \mathcal{C} be a closed symmetric monoidal ∞ -category with finite limits, and let $\mathcal{D} = \text{Stab}(\mathcal{C})$ be its stabilization with unit object $\mathbf{1}_{\mathcal{D}}$.

1. **Homotopy Groups of an Object:** For any object $E \in \mathcal{D}$ and $n \in \mathbb{Z}$, the n -th homotopy group of E is defined as the abelian group of homotopy classes of maps from the n -shifted unit object:

$$\pi_n(E) := [\Sigma^n \mathbf{1}_{\mathcal{D}}, E]_{\mathcal{D}}$$

2. **Homotopy Groups of the Mapping Spectrum:** Let $\underline{\text{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$ be the internal mapping spectrum between $X, Y \in \mathcal{D}$. Its homotopy groups characterize the graded morphisms between the two objects:

$$\pi_n \underline{\text{Map}}_{\mathcal{D}}(X, Y) \cong [X, \Sigma^n Y]_{\mathcal{D}} \cong [\Sigma^{-n} X, Y]_{\mathcal{D}}$$

where $[-, -]_{\mathcal{D}}$ denotes the set of homotopy classes i.e. the 0-th homotopy group of the kan complex $\underline{\text{Map}}_{\mathcal{D}}(-, -)$.

Remark 1.19. The distinction lies in the target category:

- $\text{Map}_{\mathcal{D}}(X, Y) \in \mathcal{S}$ is a **space** (Kan complex). It represents the mapping space in the ∞ -categorical sense.
- $\underline{\text{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$ is a **spectrum** (Internal Hom). It is an object of the stable category \mathcal{D} that stabilizes the mapping space.

In terms of homotopy groups: $\pi_n \text{Map}_{\mathcal{D}}(X, Y) \cong \pi_n \underline{\text{Map}}_{\mathcal{D}}(X, Y)$ for $n \geq 0$, because every spectrum is Ω -spectrum in $\text{Stab}(\mathcal{C})$.

Construction 1.20 (Stabilization of a Suspension Spectrum). Let $X \in \mathcal{S}_*$ be a pointed space (or generally an object in a pointed ∞ -category \mathcal{C} with finite colimits). The construction of its associated **suspension spectrum** $\Sigma^\infty X \in \text{Sp}$ proceeds as follows:

1. **The Prespectrum Construction:** First, we form a *prespectrum* P_X by iterating the suspension functor Σ on X . This is given by the sequence of spaces:

$$(P_X)_n := \Sigma^n X, \quad \text{for } n \geq 0$$

together with the structural maps (identities):

$$\sigma_n : \Sigma((P_X)_n) = \Sigma(\Sigma^n X) \xrightarrow{id} \Sigma^{n+1} X = (P_X)_{n+1}$$

2. **Spectrification (The L-functor):** Since P_X is not necessarily an Ω -spectrum (i.e., the adjoint maps $(P_X)_n \rightarrow \Omega(P_X)_{n+1}$ are not equivalences), we apply the *spectrification functor* L (or stabilization) to convert it into a true spectrum. The resulting object is the suspension spectrum:

$$\Sigma^\infty X := L(P_X)$$

Conceptually, the k -th space of this stable object is the colimit:

$$(\Sigma^\infty X)_k \simeq \underset{m \rightarrow \infty}{\text{colim}} \Omega^m \Sigma^{m+k} X$$

3. **Universal Property (Adjunction):** The construction defines the left adjoint functor Σ^∞ in the stabilization adjunction:

$$\begin{array}{ccc} & \Sigma^\infty & \\ \mathcal{S}_* & \begin{array}{c} \nearrow \\ \searrow \end{array} & \text{Sp} \\ & \Omega^\infty & \end{array}$$

where for any spectrum E , the right adjoint is given by $\Omega^\infty E := E_0$ (the 0-th space of the Ω -spectrum E).

Remark 1.21 (Bousfield-Friedlander Structure and Stabilization). The Bousfield-Friedlander model structure \mathcal{M}_{BF} on the category of prespectra is the left Bousfield localization of the strict model structure $\mathcal{M}_{\text{strict}}$. The three classes of morphisms in \mathcal{M}_{BF} are characterized as follows:

- **Cofibrations:** These are exactly the same as the strict cofibrations (levelwise inclusions that satisfy the appropriate cell complex conditions).
- **Weak Equivalences:** These are the *stable weak equivalences*, i.e., maps $f : X \rightarrow Y$ that induce isomorphisms on stable homotopy groups $\pi_n^S(X) \cong \pi_n^S(Y)$ for all $n \in \mathbb{Z}$.

- **Fibrations:** These are the maps that satisfy the right lifting property with respect to acyclic cofibrations. Specifically, a map $p : E \rightarrow B$ is a BF-fibration if it is a levelwise fibration and the square

$$\begin{array}{ccc} E_n & \longrightarrow & \Omega E_{n+1} \\ \downarrow p_n & & \downarrow \Omega p_{n+1} \\ B_n & \longrightarrow & \Omega B_{n+1} \end{array}$$

is a homotopy pullback for all n .

The transition from $\mathcal{M}_{\text{strict}}$ to \mathcal{M}_{BF} captures the essence of stabilization. Since the fibrant objects in this structure are exactly the Ω -spectra, the **fibrant replacement** of a prespectrum X in \mathcal{M}_{BF} is precisely its **stabilization** (spectrification).

If R_{BF} denotes the fibrant replacement functor, we have a natural stable equivalence $j : X \xrightarrow{\sim} R_{\text{BF}}(X)$, where $R_{\text{BF}}(X)$ is an Ω -spectrum. In the stable homotopy category, this is equivalent to the classical stabilization $QX = \Omega^\infty \Sigma^\infty X$:

$$\begin{array}{ccc} X & \xrightarrow{j} & R_{\text{BF}}(X) \\ \parallel & & \downarrow \simeq \\ X & \xrightarrow{\text{Stabilization}} & QX \end{array}$$

Thus, the Bousfield-Friedlander model structure provides the formal homotopy-theoretic machinery where "becoming an Ω -spectrum" is equivalent to "becoming fibrant."

Definition 1.22. The **homotopy category functor** ho is the change-of-base functor induced by the path-components functor $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$. For a category \mathcal{C} enriched over simplicial sets, $ho(\mathcal{C})$ is the **Set**-enriched category with the same objects as \mathcal{C} and morphism sets defined by:

$$\text{Hom}_{ho(\mathcal{C})}(X, Y) = \pi_0(\text{Map}_{\mathcal{C}}(X, Y))$$

Composition in $ho(\mathcal{C})$ is inherited from the enriched composition in \mathcal{C} via the product-preserving property of π_0 .

Proposition 1.23. Let \mathcal{C} be a locally Kan simplicial category. Let N_Δ denote the homotopy coherent nerve and N denote the classical nerve. There is a natural isomorphism of simplicial sets (or categories):

$$ho(N_\Delta(\mathcal{C})) \cong N(ho(\mathcal{C}))$$

Example 1.24 (The Stable Homotopy Category). Let \mathcal{Sp} be the stable ∞ -category of spectra. The classical **stable homotopy category** SHC is precisely its homotopy category:

$$\text{SHC} \cong ho(\mathcal{Sp})$$

Under the ho functor, the enriched mapping spaces $\text{Map}_{\mathcal{Sp}}(X, Y)$ are replaced by their sets of path-components π_0 . The stable property of \mathcal{Sp} (the equivalence of fiber and cofiber sequences) ensures that $ho(\mathcal{Sp})$ inherits the structure of a **triangulated category**.

1.3 ∞ -Operad

Definition 1.25 (∞ -Operator Category). An **∞ -operator category** is an ∞ -category \mathcal{B} equipped with a specified subcategory of *inert morphisms* $\mathcal{B}^{\text{inert}}$, satisfying the following two structural axioms illustrated by commutative diagrams:

- 1. Active-Inert Factorization:** There exists a class of *active morphisms* such that every morphism $f : X \rightarrow Z$ in \mathcal{B} factors essentially uniquely as an active morphism followed by an inert morphism.

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & \searrow^{f^{\text{act}} \text{ (Mix)}} & \swarrow^{f^{\text{inert}} \text{ (Select)}} \\ & Y & \end{array}$$

Here, f^{act} performs the operations (combining inputs), and f^{inert} performs the structural projection.

- 2. Elementary Decomposition:** There exists a set of *elementary objects* (or colors) $\{U_\alpha\}$. Every object $X \in \mathcal{B}$ is determined by its inert projections to these elementary objects. Specifically, the collection of inert morphisms $\{\rho^i\}$ exhibits X as a product:

$$\begin{array}{ccc} X & & \\ \rho^1 \swarrow & \downarrow \rho^2 & \searrow \rho^n \\ U_{i_1} & U_{i_2} & U_{i_n} \end{array}$$

This implies an equivalence $X \simeq U_{i_1} \times U_{i_2} \times \dots \times U_{i_n}$, ensuring that complex objects are merely aggregates of elementary slots.

Definition 1.26 (∞ -Operator Category). An **∞ -operator category** is an ∞ -category \mathcal{B} equipped with a specified factorization structure, consisting of two subcategories: *active morphisms* (\mathcal{B}^{act}) and *inert morphisms* ($\mathcal{B}^{\text{inert}}$). These satisfy the following two axioms:

- 1. Active-Inert Factorization System:** The pair $(\mathcal{B}^{\text{act}}, \mathcal{B}^{\text{inert}})$ forms an *orthogonal factorization system* on \mathcal{B} .

This means that every morphism $f : X \rightarrow Z$ in \mathcal{B} factors essentially uniquely as an active morphism followed by an inert morphism:

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & \searrow^{f^{\text{act}}} & \swarrow^{f^{\text{inert}}} \\ & Y & \end{array}$$

Here, the intermediate object Y and the active map f^{act} are not arbitrary; they are **determined** by the image of f under this factorization system.

- 2. Elementary Decomposition (Segal Core):** There exists a set of elementary objects $\mathcal{E} \subset \text{Ob}(\mathcal{B})$. For any object $X \in \mathcal{B}$, let Λ_X be the set of all inert morphisms targeting \mathcal{E} :

$$\Lambda_X := \{\rho : X \rightarrow U \mid \rho \in \mathcal{B}^{\text{inert}}, U \in \mathcal{E}\}$$

We require that the canonical map induced by these morphisms is an equivalence:

$$X \xrightarrow{\sim} \prod_{\rho \in \Lambda_X} \text{codom}(\rho)$$

Definition 1.27 (\mathcal{B} -Operad). Let \mathcal{B} be an ∞ -operator category. A **\mathcal{B} -operad** is a map of simplicial sets $p : \mathcal{C}^\otimes \rightarrow \mathcal{B}$ satisfying the following three conditions:

1. **Inner Fibration:** The map p is an inner fibration of simplicial sets. That is, for every $0 < k < n$, p has the right lifting property with respect to the inner horn inclusion $\Lambda_k^n \hookrightarrow \Delta^n$.
2. **Inert Lifting Property:** For every inert morphism $f : X \rightarrow Y$ in \mathcal{B} and every object $C \in \mathcal{C}^\otimes$ such that $p(C) = X$, there exists a p -coCartesian edge $\bar{f} : C \rightarrow C'$ in \mathcal{C}^\otimes such that $p(\bar{f}) = f$.
3. **Segal Condition:** For every object $X \in \mathcal{B}$, let $\{X \xrightarrow{f_i} U_i\}_{i \in I}$ be the collection of inert morphisms decomposing X into elementary objects (as dictated by the structure of \mathcal{B}). The functor induced by the p -coCartesian lifts of these inert morphisms,

$$\mathcal{C}_X^\otimes \xrightarrow{\sim} \prod_{i \in I} \mathcal{C}_{U_i}^\otimes,$$

is an equivalence of ∞ -categories.

Definition 1.28 (\mathcal{O} -Algebra Object). Let \mathcal{B} be an ∞ -operator category. Let $p : \mathcal{O}^\otimes \rightarrow \mathcal{B}$ and $q : \mathcal{C}^\otimes \rightarrow \mathcal{B}$ be \mathcal{B} -operad. An **\mathcal{O} -algebra object in \mathcal{C}** is a map of ∞ -operads over \mathcal{B} . Explicitly, it is a functor

$$A : \mathcal{O}^\otimes \longrightarrow \mathcal{C}^\otimes$$

satisfying two conditions:

1. **Commutativity over Base (Compatibility):** The functor A respects the projection to the base category \mathcal{B} . The following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}^\otimes & \xrightarrow{A} & \mathcal{C}^\otimes \\ p \searrow & & \swarrow q \\ & \mathcal{B} & \end{array}$$

(i.e., $q \circ A = p$).

2. **Inert Preservation (The Operad Map Condition):** The functor A carries inert morphisms to inert morphisms.

Specifically, if f is an *inert morphism* in \mathcal{O}^\otimes (meaning f is a p -coCartesian lift of an inert map in \mathcal{B}), then its image $A(f)$ must be an *inert morphism* in \mathcal{C}^\otimes (meaning $A(f)$ is a q -coCartesian lift of that same map in \mathcal{B}).

The ∞ -category of all such algebras is denoted by $\text{Alg}_{\mathcal{O}}(\mathcal{C})$.

Example 1.29 (The Zoo of Operads and Their Bases). We classify common algebraic structures by specifying the underlying **Base Category** \mathcal{B} (which dictates the geometry of inputs) and the **Operad** \mathcal{O}^\otimes (which dictates the operations) as a fibration $p : \mathcal{O}^\otimes \rightarrow \mathcal{B}$.

1. The Commutative Case (E_∞)

- **Base:** $\mathcal{B} = N(\text{Fin}_*)$ (Symmetric/Unordered inputs).
- **Operad:** $\mathcal{O}^\otimes = \text{Comm}^\otimes := N(\text{Fin}_*)$.
- **Structure Map:** The identity map $\text{id} : N(\text{Fin}_*) \rightarrow N(\text{Fin}_*)$.
- **Resulting Algebra: Commutative ∞ -Algebra.**

- **Note:** Since the map is the identity, the fiber over any operation is a point. There is essentially only one way to combine inputs (order doesn't matter).

2. The Associative Case (A_∞)

- **Base:** $\mathcal{B} = N(\text{Fin}_*)$.
- **Operad:** $\mathcal{O}^\otimes = \text{Ass}^\otimes$ (The Associative Operad).
- **Structure Map:** The "forgetful" functor that forgets the linear ordering of the fibers.
- **Resulting Algebra:** **Associative ∞ -Algebra**.
- **Note:** The fiber over $\langle n \rangle \rightarrow \langle 1 \rangle$ is equivalent to the symmetric group Σ_n . This allows inputs to be permuted (by the base), but the operation distinguishes the order of multiplication ($x_1x_2 \neq x_2x_1$).

3. The Little k -Disks Case (E_k)

- **Base:** $\mathcal{B} = N(\text{Fin}_*)$.
- **Operad:** $\mathcal{O}^\otimes = \mathbb{E}_k^\otimes$.
- **Structure Map:** The projection from the space of disk embeddings.
- **Resulting Algebra:** E_k -**Algebra**.
- **Note:** Interpolates between Associative ($k = 1$) and Commutative ($k = \infty$).

4. The Lie Case (L_∞)

- **Base:** $\mathcal{B} = N(\text{Fin}_*)$.
- **Operad:** $\mathcal{O}^\otimes = \text{Lie}^\otimes$.
- **Resulting Algebra:** L_∞ -**Algebra** (Homotopy Lie Algebra).
- **Note:** Typically considered over a stable target category (like chain complexes).

5. The Non-Symmetric / Planar Case

- **Base:** $\mathcal{B} = N(\Delta)^{op}$ (The Simplex Category; Linear/Ordered inputs).
- **Operad:** $\mathcal{O}^\otimes = N(\Delta)^{op}$.
- **Structure Map:** The identity map.
- **Resulting Algebra:** **Associative Monoid** (in the strict sense).
- **Note:** Here, the base category itself forbids permutation. There is no symmetric group action to even consider.