The review of Smooth Manifolds

Mathematical Notes Collection

Notes & Expositions

Contents

Remark 0.1 (Dimension Identity). Let $X = \mathrm{Spec}(A)$ be an irreducible affine variety. For any point $x \in X$ (not necessarily closed) corresponding to a prime ideal $\mathfrak{p} \subset A$, let $Y = \overline{\{x\}} = V(\mathfrak{p})$ be the subvariety defined by $\mathfrak{p}$. The fundamental identity linking local geometry to algebra is:

$$\dim(\mathcal{O}_{X,x}) = \dim(A_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) = \mathrm{codim}_X(Y).$$

Consequently, the global dimension formula can be rewritten locally as:

$$\dim(X) = \dim(Y) + \dim(\mathcal{O}_{X,x}).$$

Remark 0.2 (Algorithm: Computing Blow-ups via Affine Charts). Let $X \subseteq \mathbb{A}^n$ be an affine variety and $Y = V(I) \subset X$ be the center of blowing up, where $I = \langle f_1, \dots, f_k \rangle$. The blow-up $\tilde{X} = \mathrm{Bl}_Y X$ is a closed subscheme of $X \times \mathbb{P}^{k-1}$. To determine the local equations of the *strict transform*:

1. **Select a Chart:** Consider the i-th affine chart $U_i \subset \mathbb{P}^{k-1}$ defined by $y_i \neq 0$. Introduce affine coordinates $u_j = y_j/y_i$ for $j \neq i$.

2. **Substitution (Total Transform):** The blow-up relations $y_j f_i = y_i f_j$ become $f_j = u_j f_i$. Geometrically, this treats f_i as the local generator of the exceptional divisor.

3. **Strict Transform:** For any defining equation $g \in I(X)$, substitute $f_j \mapsto u_j f_i$. Since $g \in I^d$ (where $d = \mathrm{ord}_Y(g)$), the term f_i^d factors out. The equation for the strict transform is:
$$g^{\mathrm{st}} = \frac{g(x_1, \dots, x_n)\big|_{f_j = u_j f_i}}{f_i^d} = 0.$$

Problem 0.3 (Global Polynomial Representation of Morphisms on $\mathbb{P}^n$)**.** Let k be an algebraically closed field and let n, m be positive integers with $n \leq m$. Consider an arbitrary morphism $\phi : \mathbb{P}^n \to \mathbb{P}^m$. Prove that ϕ is induced by a tuple of homogeneous polynomials $[F_0 : \cdots : F_m]$ of the same degree.
Proof Guidelines:

1. **Local Representation:** Consider the standard affine open covering $\{U_i\}_{i=0}^m$ of $\mathbb{P}^m$. Let $V_{ij} = \phi^{-1}(U_i) \cap \mathbb{A}^n_j$ be the preimage in the affine charts of $\mathbb{P}^n$. Show that locally, ϕ is determined by regular functions (polynomials) f_{ij}.

2. **Extension:** Prove that these local polynomial forms f_{ij} (viewed as maps from affine to projective space) can be homogenized and extended to morphisms defined on the entire projective space $\mathbb{P}^n$.

3. **Uniqueness (The Constant Factor):** Suppose there are two such global polynomial representations $F = [F_0 : \cdots : F_m]$ and $G = [G_0 : \cdots : G_m]$ that agree on a non-empty open subset. Consider the ratio function $h = F_k/G_k$ on the intersection of their non-vanishing loci.

   - Show that h extends to a global regular function on $\mathbb{P}^n$.
   - Conclude that h must be a constant, thereby proving $F_k = \lambda G_k$ for some $\lambda \in k^*$.

Remark 0.4 (Duality of f^* and f). Let $f : X \to Y$ be a morphism of affine varieties, and let $f^* : A(Y) \to A(X)$ be the induced homomorphism of coordinate rings. The algebraic properties of f^* correspond precisely to the geometric properties of f as follows:

1. **Injectivity of f^* (Dominance):**

$$f^* \text{ is injective } \iff f \text{ is dominant (i.e., } \overline{f(X)} = Y).$$

   Reasoning: If f^* has a non-trivial kernel, there exists a non-zero function $g \in A(Y)$ such that $g \circ f = 0$. This implies $f(X) \subseteq V(g) \subsetneq Y$, so the image is not dense. Conversely, if $f(X)$ is dense, no non-zero function can vanish on it.

2. **Surjectivity of f^* (Closed Immersion):**

$$f^* \text{ is surjective } \iff f \text{ is a closed immersion.}$$

   Reasoning: If f^* is surjective, then by the First Isomorphism Theorem, $A(X) \cong A(Y)/\ker(f^*)$. Geometrically, rings of the form $A(Y)/I$ correspond exactly to closed subvarieties $V(I) \subseteq Y$. Thus, X is isomorphic to a closed subset of Y.

Note: While algebraic surjectivity implies geometric injectivity (since embeddings are injective), algebraic injectivity does **not** imply geometric surjectivity (it only implies the image is dense).