A Study in Quasi-Coherent Sheaves and Tannaka Duality

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1 Preliminaries

1.1 ∞-categories

Definition 1.1 (∞-Category)**.** A simplicial set K is an ∞**-category** if for every $n > 1$ and every **inner** index $0 < i < n$, every map of simplicial sets $f_0 : \Lambda^n_i \to K$ admits an extension to an n-simplex $f : \Delta^n \to K$.

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f_0"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Definition 1.2 (Simplicial Category)**.** A **simplicial category** (or Set_Δ-enriched category) $\mathcal{C}$ is a category where:

1. For any two objects $X, Y \in \mathcal{C}$, the collection of morphisms between them is not a set, but a **simplicial set** $\mathrm{Map}_{\mathcal{C}}(X, Y)$.
2. For any three objects $X, Y, Z \in \mathcal{C}$, the composition map

   $$\mathrm{Map}_{\mathcal{C}}(Y, Z) \times \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$$

   is a morphism of simplicial sets and satisfies the usual associativity and identity axioms.

A simplicial category $\mathcal{C}$ is **locally Kan** if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the mapping simplicial set $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex.

Definition 1.3 (Simplicial Nerve N_Δ)**.** The **simplicial nerve** $N_\Delta(\mathcal{C})$ is the simplicial set defined by the assignment:

$$N_\Delta(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$$

where $\mathfrak{C}[\Delta^n]$ is the **rigidification** of the n-simplex Δ^n into a simplicial category.

Definition 1.4 (∞-category via N_Δ)**.** An ∞**-category** (or quasicategory) is a simplicial set K that is equivalent to the simplicial nerve of some locally Kan simplicial category $\mathcal{C}$.

$$K \simeq N_\Delta(\mathcal{C})$$

Theorem 1.5 (Joyal-Lurie)**.** There exists a Quillen equivalence between the Joyal model structure on Set_Δ (modeling quasicategories) and the Bergner model structure on Cat_Δ (modeling simplicial categories):

$$\mathfrak{C}[\cdot] : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Cat}_\Delta : N_\Delta.$$

Specifically, for any simplicial category $\mathcal{C}$ where mapping spaces are Kan complexes, its simplicial nerve $N_\Delta(\mathcal{C})$ is a quasicategory.

Definition 1.6 (Free Cocompletion)**.** Let $\mathcal{C}$ be a small ∞-category. An ∞-category $\mathcal{P}(\mathcal{C})$ is called the **free cocompletion** of $\mathcal{C}$ if it satisfies the following universal property:

1. $\mathcal{P}(\mathcal{C})$ admits all small colimits.
2. There exists a functor $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ (called the Yoneda embedding) such that for any ∞-category $\mathcal{D}$ which admits small colimits, composition with j induces an equivalence of ∞-categories:
$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Here, Fun^L denotes the full subcategory of functors that preserve small colimits (left adjoints).

Definition 1.7 (The ∞-category of Spaces)**.** Let $\mathcal{S}$ denote the ∞-category of spaces. It is defined in two equivalent ways:

1. **Via Dwyer-Kan Localization:**
Let W be the class of weak homotopy equivalences in Set_Δ. We define $\mathcal{S}$ as the homotopy coherent nerve of the simplicial localization:
$$\mathcal{S} := N(\mathrm{Set}_\Delta[W^{-1}]).$$
Equivalently, via Kan complexes: $\mathcal{S} \simeq N(\mathbf{Kan})$.
2. **Via Free Cocompletion:**
The ∞-category $\mathcal{S}$ is the free cocompletion of the point $*$. That is, it is the category of presheaves:
$$\mathcal{S} \simeq \mathcal{P}(*).$$
Universal Property: For any cocomplete ∞-category $\mathcal{C}$, there is an equivalence $\mathrm{Fun}^L(\mathcal{S}, \mathcal{C}) \simeq \mathcal{C}$.

Theorem 1.8. Let K be an ∞-category (quasi-category). Let $\mathcal{C} = \mathfrak{C}[K]$ be its associated simplicial category (via rigidification). The construction of the presheaf ∞-category commutes with the nerve construction in the following sense:

1. **Simplicial Side:** Consider the category of simplicial presheaves $\mathcal{P}_\Delta(\mathcal{C}) := \mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathcal{S}_{\mathrm{Kan}})$. This category admits a simplicial model structure (projective structure).
2. **Infinity Side:** Consider the ∞-category of presheaves $\mathcal{P}(K) := \mathrm{Fun}(K^{op}, \mathcal{S})$.
3. **Equivalence:** There is an equivalence of ∞-categories:
$$\mathcal{P}(K) \simeq N\left(\mathcal{P}_\Delta(\mathcal{C})^{\mathrm{cf}}\right)$$
where $\mathcal{P}_\Delta(\mathcal{C})^{\mathrm{cf}}$ denotes the full simplicial subcategory of fibrant-cofibrant objects in the model category of simplicial presheaves.

In summary, the presheaf of an ∞-category is modeled by the nerve of the strictly cocomplete simplicial category of enriched presheaves.

Remark 1.9 (Homotopy Category via Fibrant-Cofibrant Objects)**.** To correctly construct the homotopy category $\mathrm{Ho}(\mathcal{M})$ from a simplicial model category $\mathcal{M}$, one cannot simply take the path components π_0 of the mapping spaces between arbitrary objects.

Instead, one must restrict attention to the full subcategory of **fibrant-cofibrant objects**, denoted $\mathcal{M}_{cf}$. It is only within this subcategory that the simplicial mapping spaces $\mathrm{Map}_{\mathcal{M}}(X, Y)$ are guaranteed to be Kan complexes representing the correct derived mapping spaces. The morphisms in the homotopy category are thus given by:

$$[X, Y]_{\mathrm{Ho}(\mathcal{M})} \cong \pi_0 \mathrm{Map}_{\mathcal{M}}(X, Y) \quad \text{for } X, Y \in \mathcal{M}_{cf}.$$

For general objects X, Y, one must first replace them with weakly equivalent fibrant-cofibrant objects (via cofibrant replacement QX and fibrant replacement RY) to compute this group.

1.2 Stable ∞-Category

Definition 1.10 (Loop Object)**.** For any object $X \in \mathcal{C}$, the loop object ΩX is the limit of the diagram $0 \to X \leftarrow 0$. It fits into the following pullback square:

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

Intuitively, $\Omega X \simeq 0 \times_X 0$.

Definition 1.11 (Suspension Object)**.** For any object $X \in \mathcal{C}$, the suspension object ΣX is the colimit of the diagram $0 \leftarrow X \to 0$. It fits into the following pushout square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d, "\S"] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \Sigma X \end{tikzcd}$$

Intuitively, $\Sigma X \simeq 0 \amalg_X 0$.

Definition 1.12 (Stable ∞-Category)**.** An ∞-category $\mathcal{C}$ is called **stable** if it satisfies the following conditions:

1. There exists a zero object $0 \in \mathcal{C}$ (i.e., $\mathcal{C}$ is pointed).
2. Every morphism in $\mathcal{C}$ admits a kernel and a cokernel.
3. A triangle in $\mathcal{C}$ is a pushout square if and only if it is a pullback square.

Definition 1.13 (The Stabilization of an ∞-Category)**.** Let $\mathcal{C}$ be an ∞-category admitting finite limits. The process of constructing the stable ∞-category associated to $\mathcal{C}$ proceeds in two stages:

1. **Pointed View (Formation of $\mathcal{C}_*$):**
   First, we construct the *pointed* ∞-category $\mathcal{C}_*$. Assuming $\mathcal{C}$ has a terminal object $*$, $\mathcal{C}_*$ is defined as the under-category of the terminal object:

   $$\mathcal{C}_* := \mathcal{C}_{*/} \cong \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathrm{Fun}(\{0\}, \mathcal{C})} \{*\}$$

   Objects in $\mathcal{C}_*$ are morphisms $* \to X$ in $\mathcal{C}$ (i.e., objects equipped with a base point). In $\mathcal{C}_*$, the object corresponding to the identity $* \to *$ serves as a *zero object* (both initial and terminal). Consequently, the loop functor $\Omega : \mathcal{C}_* \to \mathcal{C}_*$ is well-defined by $\Omega X = * \times_X *$.

2. **Stabilization (Formation of Sp($\mathcal{C}$)):**
   The *stabilization* of $\mathcal{C}$, denoted as $\mathrm{Sp}(\mathcal{C})$ (or $\mathrm{Stab}(\mathcal{C})$), is defined as the ∞-category of spectrum objects in $\mathcal{C}_*$. It is constructed as the homotopy limit of the tower of loop functors:
   $$\mathrm{Sp}(\mathcal{C}) := \varprojlim \left(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$
   Explicitly, an object $E \in \mathrm{Sp}(\mathcal{C})$ consists of a sequence $\{E_n\}_{n\geq 0}$ of objects in $\mathcal{C}_*$ together with equivalences $E_n \xrightarrow{\sim} \Omega E_{n+1}$ for each n. This construction forces the suspension functor Σ to be an equivalence, rendering $\mathrm{Sp}(\mathcal{C})$ a stable ∞-category.

Theorem 1.14 (Universal Property of Stabilization)**.** Let $\mathcal{C}$ be an ∞-category with finite limits and a terminal object $*$. Let $\mathcal{C}_* = \mathcal{C}_{*/}$ be its pointed version. The stabilization $\mathrm{Sp}(\mathcal{C})$ is characterized by the following equivalent descriptions:

1. **Internal Construction (Loop Towers):** $\mathrm{Sp}(\mathcal{C})$ is the homotopy limit of the sequence of loop functors:
   $$\mathrm{Sp}(\mathcal{C}) \simeq \varprojlim \left(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$
   An object in $\mathrm{Sp}(\mathcal{C})$ is an Ω-spectrum, i.e., a sequence $\{E_n\}_{n\geq 0}$ in $\mathcal{C}_*$ with equivalences $E_n \simeq \Omega E_{n+1}$.

2. **External Construction (Excision):** $\mathrm{Sp}(\mathcal{C})$ is equivalent to the ∞-category of pointed excisive functors from the category of finite pointed spaces $\mathcal{S}_*^{\mathrm{fin}}$ to $\mathcal{C}$:
   $$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$$
   A functor $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ belongs to $\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ if $F(*) \simeq *$ and F maps every pushout square in $\mathcal{S}_*^{\mathrm{fin}}$ to a pullback square in $\mathcal{C}$.

Furthermore, the stabilization $\mathrm{Sp}(\mathcal{C})$ is the universal stable ∞-category under $\mathcal{C}$: for any stable ∞-category $\mathcal{D}$, the functor $\mathrm{Sp}(\mathcal{C}) \to \mathcal{D}$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{lex}}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$, where $\mathrm{Fun}^{\mathrm{lex}}$ denotes the ∞-category of left exact functors.

Remark 1.15 (Distinction between $\mathrm{Stab}(\mathcal{C})$ and Sp)**.** It is essential to distinguish between the abstract process of stabilization and the specific category of spectra:

1. **The Category of Spectra (Sp):** Historically and by convention, Sp refers specifically to the stabilization of the ∞-category of pointed spaces $\mathcal{S}_*$. That is:
   $$\mathrm{Sp} \simeq \mathrm{Stab}(\mathcal{S}) \simeq \varprojlim(\ldots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)$$
   This category serves as the unit object in the ∞-category of stable ∞-categories and provides the ground for stable homotopy theory.

2. **Stabilization of an Arbitrary ∞-Category** ($\mathrm{Stab}(\mathcal{C})$)**:** For any ∞-category $\mathcal{C}$ with finite limits, $\mathrm{Stab}(\mathcal{C})$ is the stable ∞-category constructed as the limit of the tower of loop functors:
   $$\mathrm{Stab}(\mathcal{C}) = \varprojlim(\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*)$$
   While $\mathrm{Stab}(\mathcal{C})$ is always a stable ∞-category, it may not possess a symmetric monoidal structure (like the smash product) unless $\mathcal{C}$ itself is equipped with a compatible monoidal structure.

3. **The Relation:** Any stable ∞-category $\mathcal{D}$ is naturally tensored over Sp. In this sense, Sp plays a role analogous to the ring of integers $\mathbb{Z}$ in abelian groups: for any $D \in \mathcal{D}$ and $E \in \mathrm{Sp}$, there is a well-defined object $E \otimes D \in \mathcal{D}$.

Definition 1.16 (Internal Mapping Spectrum in $\mathrm{Stab}(\mathcal{C})$)**.** Let $\mathcal{C}$ be a **closed symmetric monoidal ∞-category** $(\mathcal{C}, \otimes, \mathbf{1})$ that admits finite limits. Assume further that the tensor product $\otimes$ is compatible with the stabilization (i.e., it preserves colimits in each variable).

Let $\mathcal{D} = \mathrm{Stab}(\mathcal{C})$ be the resulting stable ∞-category, equipped with the induced symmetric monoidal structure $\otimes_{\mathcal{D}}$. For any objects $X, Y \in \mathcal{D}$, the **internal mapping spectrum** is defined as the object $\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$ that satisfies the following conditions:

1. **Adjunction Property:** It is the right adjoint to the tensor product functor. For any object $Z \in \mathcal{D}$, there is a natural equivalence of mapping spaces:

$$\mathrm{Map}_{\mathcal{D}}(Z \otimes_{\mathcal{D}} X, Y) \simeq \mathrm{Map}_{\mathcal{D}}(Z, \underline{\mathrm{Map}}_{\mathcal{D}}(X, Y))$$

2. **Spectrum Level Structure:** In terms of the sequence of objects $\{E_n\} \in \mathcal{C}_*$ representing the spectrum, the n-th level of the internal mapping spectrum is given by:

$$\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y)_n \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(X, \Sigma^n Y)$$

where $\underline{\mathrm{Map}}_{\mathcal{C}}$ denotes the internal Hom in the underlying category $\mathcal{C}$ (if it exists) or the corresponding enrichment.

Definition 1.17 (Mapping Spectrum)**.** Let X and Y be spectra in Sp. The **mapping spectrum** from X to Y, denoted as $\underline{\mathrm{Map}}(X, Y) \in \mathrm{Sp}$, is the unique spectrum (up to equivalence) characterized by the following properties:

1. **Adjunction (Internal Hom):** For any spectrum Z, there is a natural equivalence of mapping spaces:

$$\mathrm{Map}_{\mathrm{Sp}}(Z \wedge X, Y) \simeq \mathrm{Map}_{\mathrm{Sp}}(Z, \underline{\mathrm{Map}}(X, Y))$$

This identifies $\underline{\mathrm{Map}}(X, Y)$ as the right adjoint to the smash product functor $(- \wedge X)$.

2. **Omega-Spectrum Structure:** The n-th space of the mapping spectrum is equivalent to the space of maps from X to the n-th suspension of Y:

$$\underline{\mathrm{Map}}(X, Y)_n \simeq \mathrm{Map}_{\mathrm{Sp}}(X, \Sigma^n Y)$$

The structure maps $\Sigma \underline{\mathrm{Map}}(X, Y)_n \to \underline{\mathrm{Map}}(X, Y)_{n+1}$ are induced by the stability of Sp.

Definition 1.18 (Homotopy Groups in $\mathrm{Stab}(\mathcal{C})$)**.** Let $\mathcal{C}$ be a closed symmetric monoidal ∞-category with finite limits, and let $\mathcal{D} = \mathrm{Stab}(\mathcal{C})$ be its stabilization with unit object $\mathbf{1}_{\mathcal{D}}$.

1. **Homotopy Groups of an Object:** For any object $E \in \mathcal{D}$ and $n \in \mathbb{Z}$, the n-th homotopy group of E is defined as the abelian group of homotopy classes of maps from the n-shifted unit object:

$$\pi_n(E) := [\Sigma^n \mathbf{1}_{\mathcal{D}}, E]_{\mathcal{D}}$$

2. **Homotopy Groups of the Mapping Spectrum:** Let $\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$ be the internal mapping spectrum between $X, Y \in \mathcal{D}$. Its homotopy groups characterize the graded morphisms between the two objects:

$$\pi_n \underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \cong [X, \Sigma^n Y]_{\mathcal{D}} \cong [\Sigma^{-n} X, Y]_{\mathcal{D}}$$

where $[-, -]_{\mathcal{D}}$ denotes the set of homotopy classes i.e. the 0-th homotopy group of the kan complex $\mathrm{Map}_{\mathcal{D}}(-, -)$.

Remark 1.19. The distinction lies in the target category:

- $\mathrm{Map}_{\mathcal{D}}(X, Y) \in \mathcal{S}$ is a **space** (Kan complex). It represents the mapping space in the ∞-categorical sense.

- $\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \in \mathcal{D}$ is a **spectrum** (Internal Hom). It is an object of the stable category $\mathcal{D}$ that stabilizes the mapping space.

In terms of homotopy groups: $\pi_n \mathrm{Map}_{\mathcal{D}}(X, Y) \cong \pi_n \underline{\mathrm{Map}}_{\mathcal{D}}(X, Y)$ for $n \geq 0$, because every spectrum is Ω-spectrum in $\mathrm{Stab}(\mathcal{C})$.

Construction 1.20 (Stabilization of a Suspension Spectrum)**.** Let $X \in \mathcal{S}_*$ be a pointed space (or generally an object in a pointed ∞-category $\mathcal{C}$ with finite colimits). The construction of its associated **suspension spectrum** $\Sigma^\infty X \in \mathrm{Sp}$ proceeds as follows:

1. **The Prespectrum Construction:** First, we form a *prespectrum* P_X by iterating the suspension functor Σ on X. This is given by the sequence of spaces:

$$(P_X)_n := \Sigma^n X, \quad \text{for } n \geq 0$$

together with the structural maps (identities):

$$\sigma_n : \Sigma((P_X)_n) = \Sigma(\Sigma^n X) \xrightarrow{id} \Sigma^{n+1} X = (P_X)_{n+1}$$

2. **Spectrification (The L-functor):** Since P_X is not necessarily an Ω-spectrum (i.e., the adjoint maps $(P_X)_n \to \Omega(P_X)_{n+1}$ are not equivalences), we apply the *spectrification functor* L (or stabilization) to convert it into a true spectrum. The resulting object is the suspension spectrum:

$$\Sigma^\infty X := L(P_X)$$

Conceptually, the k-th space of this stable object is the colimit:

$$(\Sigma^\infty X)_k \simeq \operatorname*{colim}_{m \to \infty} \Omega^m \Sigma^{m+k} X$$

3. **Universal Property (Adjunction):** The construction defines the left adjoint functor Σ^∞ in the stabilization adjunction:

$$\begin{tikzcd} \mathcal{S}_* \arrow[r, bend left, "\Sigma^\infty"] & \mathrm{Sp} \arrow[l, bend left, "\Omega^\infty"] \end{tikzcd}$$

where for any spectrum E, the right adjoint is given by $\Omega^\infty E := E_0$ (the 0-th space of the Ω-spectrum E).

Remark 1.21 (Bousfield-Friedlander Structure and Stabilization)**.** The Bousfield-Friedlander model structure $\mathcal{M}_{\mathrm{BF}}$ on the category of prespectra is the left Bousfield localization of the strict model structure $\mathcal{M}_{\mathrm{strict}}$. The three classes of morphisms in $\mathcal{M}_{\mathrm{BF}}$ are characterized as follows:

- **Cofibrations:** These are exactly the same as the strict cofibrations (levelwise inclusions that satisfy the appropriate cell complex conditions).

- **Weak Equivalences:** These are the *stable weak equivalences*, i.e., maps $f : X \to Y$ that induce isomorphisms on stable homotopy groups $\pi_n^S(X) \cong \pi_n^S(Y)$ for all $n \in \mathbb{Z}$.

- **Fibrations:** These are the maps that satisfy the right lifting property with respect to acyclic cofibrations. Specifically, a map $p : E \to B$ is a BF-fibration if it is a levelwise fibration and the square
$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "p_n"] & \Omega E_{n+1} \arrow[d, "\Omega p_{n+1}"] \\ B_n \arrow[r] & \Omega B_{n+1} \end{tikzcd}$$
is a homotopy pullback for all n.

The transition from $\mathcal{M}_{\text{strict}}$ to $\mathcal{M}_{\text{BF}}$ captures the essence of stabilization. Since the fibrant objects in this structure are exactly the Ω-spectra, the **fibrant replacement** of a prespectrum X in $\mathcal{M}_{\text{BF}}$ is precisely its **stabilization** (spectrification).

If R_{BF} denotes the fibrant replacement functor, we have a natural stable equivalence $j : X \xrightarrow{\sim} R_{\text{BF}}(X)$, where $R_{\text{BF}}(X)$ is an Ω-spectrum. In the stable homotopy category, this is equivalent to the classical stabilization $QX = \Omega^\infty \Sigma^\infty X$:

$$\begin{tikzcd} X \arrow[r, "j", "\sim_{\text{stable}}"'] \arrow[d, equal] & R_{\text{BF}}(X) \arrow[d, "\simeq"] \\ X \arrow[r, "\text{Stabilization}"] & QX \end{tikzcd}$$

Thus, the Bousfield-Friedlander model structure provides the formal homotopy-theoretic machinery where "becoming an Ω-spectrum" is equivalent to "becoming fibrant."

Definition 1.22. The **homotopy category functor** *ho* is the change-of-base functor induced by the path-components functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$. For a category $\mathcal{C}$ enriched over simplicial sets, $ho(\mathcal{C})$ is the **Set**-enriched category with the same objects as $\mathcal{C}$ and morphism sets defined by:
$$\mathrm{Hom}_{ho(\mathcal{C})}(X, Y) = \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y))$$
Composition in $ho(\mathcal{C})$ is inherited from the enriched composition in $\mathcal{C}$ via the product-preserving property of π_0.

Proposition 1.23. Let $\mathcal{C}$ be a locally Kan simplicial category. Let N_Δ denote the homotopy coherent nerve and N denote the classical nerve. There is a natural isomorphism of simplicial sets (or categories):
$$ho(N_\Delta(\mathcal{C})) \cong N(ho(\mathcal{C}))$$

Example 1.24 (The Stable Homotopy Category)**.** Let $\mathcal{S}p$ be the stable ∞-category of spectra. The classical **stable homotopy category** SHC is precisely its homotopy category:
$$\mathsf{SHC} \cong ho(\mathcal{S}p)$$
Under the *ho* functor, the enriched mapping spaces $\mathrm{Map}_{\mathcal{S}p}(X, Y)$ are replaced by their sets of path-components π_0. The stable property of $\mathcal{S}p$ (the equivalence of fiber and cofiber sequences) ensures that $ho(\mathcal{S}p)$ inherits the structure of a **triangulated category**.

1.3 ∞-Operad

Definition 1.25 (∞-Operator Category)**.** An **∞-operator category** is an ∞-category $\mathcal{B}$ equipped with a specified subcategory of *inert morphisms* $\mathcal{B}^{\text{inert}}$, satisfying the following two structural axioms illustrated by commutative diagrams:

1. **Active-Inert Factorization:** There exists a class of *active morphisms* such that every morphism $f : X \to Z$ in $\mathcal{B}$ factors essentially uniquely as an active morphism followed by an inert morphism.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f^{\mathrm{act}}\ \text{(Mix)}"'] & & Z \\ & Y \arrow[ur, "f^{\mathrm{inert}}\ \text{(Select)}"'] & \end{tikzcd}$$

Here, f^{act} performs the operations (combining inputs), and f^{inert} performs the structural projection.

2. **Elementary Decomposition:** There exists a set of *elementary objects* (or colors) $\{U_\alpha\}$. Every object $X \in \mathcal{B}$ is determined by its inert projections to these elementary objects. Specifically, the collection of inert morphisms $\{\rho^i\}$ exhibits X as a product:

$$\begin{tikzcd} & X \arrow[dl, "\rho^1"'] \arrow[d, "\rho^2"] \arrow[dr, "\rho^n"] & \\ U_{i_1} & U_{i_2} & U_{i_n} \end{tikzcd}$$

This implies an equivalence $X \simeq U_{i_1} \times U_{i_2} \times \cdots \times U_{i_n}$, ensuring that complex objects are merely aggregates of elementary slots.

Definition 1.26 (∞-Operator Category)**.** An ∞**-operator category** is an ∞-category $\mathcal{B}$ equipped with a specified factorization structure, consisting of two subcategories: *active morphisms* ($\mathcal{B}^{\mathrm{act}}$) and *inert morphisms* ($\mathcal{B}^{\mathrm{inert}}$). These satisfy the following two axioms:

1. **Active-Inert Factorization System:** The pair $(\mathcal{B}^{\mathrm{act}}, \mathcal{B}^{\mathrm{inert}})$ forms an *orthogonal factorization system* on $\mathcal{B}$.

   This means that every morphism $f : X \to Z$ in $\mathcal{B}$ factors essentially uniquely as an active morphism followed by an inert morphism:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f^{\mathrm{act}}"'] & & Z \\ & Y \arrow[ur, "f^{\mathrm{inert}}"'] & \end{tikzcd}$$

   Here, the intermediate object Y and the active map f^{act} are not arbitrary; they are **determined** by the image of f under this factorization system.

2. **Elementary Decomposition (Segal Core):** There exists a set of elementary objects $\mathcal{E} \subset \mathrm{Ob}(\mathcal{B})$. For any object $X \in \mathcal{B}$, let Λ_X be the set of all inert morphisms targeting $\mathcal{E}$:

$$\Lambda_X := \{\rho : X \to U \mid \rho \in \mathcal{B}^{\mathrm{inert}}, U \in \mathcal{E}\}$$

   We require that the canonical map induced by these morphisms is an equivalence:

$$X \xrightarrow{\simeq} \prod_{\rho \in \Lambda_X} \mathrm{codom}(\rho)$$

Definition 1.27 ($\mathcal{B}$-Operad)**.** Let $\mathcal{B}$ be an ∞-operator category. A $\mathcal{B}$**-operad** is a map of simplicial sets $p : \mathcal{C}^{\otimes} \to \mathcal{B}$ satisfying the following three conditions:

1. **Inner Fibration:** The map p is an inner fibration of simplicial sets. That is, for every $0 < k < n$, p has the right lifting property with respect to the inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$.

2. **Inert Lifting Property:** For every inert morphism $f : X \to Y$ in $\mathcal{B}$ and every object $C \in \mathcal{C}^{\otimes}$ such that $p(C) = X$, there exists a p-coCartesian edge $\bar{f} : C \to C'$ in $\mathcal{C}^{\otimes}$ such that $p(\bar{f}) = f$.

3. **Segal Condition:** For every object $X \in \mathcal{B}$, let $\{X \xrightarrow{f_i} U_i\}_{i\in I}$ be the collection of inert morphisms decomposing X into elementary objects (as dictated by the structure of $\mathcal{B}$). The functor induced by the p-coCartesian lifts of these inert morphisms,

$$\mathcal{C}^{\otimes}_X \xrightarrow{\;\simeq\;} \prod_{i\in I} \mathcal{C}^{\otimes}_{U_i},$$

is an equivalence of ∞-categories.

Definition 1.28 ($\mathcal{O}$-Algebra Object)**.** Let $\mathcal{B}$ be an ∞-operator category. Let $p : \mathcal{O}^{\otimes} \to \mathcal{B}$ and $q : \mathcal{C}^{\otimes} \to \mathcal{B}$ be $\mathcal{B}$**-operad**. An $\mathcal{O}$**-algebra object in** $\mathcal{C}$ is a map of ∞-operads over $\mathcal{B}$. Explicitly, it is a functor

$$A : \mathcal{O}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$$

satisfying two conditions:

1. **Commutativity over Base (Compatibility):** The functor A respects the projection to the base category $\mathcal{B}$. The following diagram commutes:

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "A"] \arrow[dr, "p"'] & & \mathcal{C}^{\otimes} \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd}$$

(i.e., $q \circ A = p$).

2. **Inert Preservation (The Operad Map Condition):** The functor A carries inert morphisms to inert morphisms.

   Specifically, if f is an *inert morphism* in $\mathcal{O}^{\otimes}$ (meaning f is a p-coCartesian lift of an inert map in $\mathcal{B}$), then its image $A(f)$ must be an *inert morphism* in $\mathcal{C}^{\otimes}$ (meaning $A(f)$ is a q-coCartesian lift of that same map in $\mathcal{B}$).

The ∞-category of all such algebras is denoted by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.

Example 1.29 (**The Zoo of Operads and Their Bases**)**.** We classify common algebraic structures by specifying the underlying **Base Category** $\mathcal{B}$ (which dictates the geometry of inputs) and the **Operad** $\mathcal{O}^{\otimes}$ (which dictates the operations) as a fibration $p : \mathcal{O}^{\otimes} \to \mathcal{B}$.

1. **The Commutative Case** (E_∞)

   - **Base:** $\mathcal{B} = N(\mathrm{Fin}_*)$ (Symmetric/Unordered inputs).
   - **Operad:** $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes} := N(\mathrm{Fin}_*)$.
   - **Structure Map:** The identity map $\mathrm{id} : N(\mathrm{Fin}_*) \to N(\mathrm{Fin}_*)$.
   - **Resulting Algebra: Commutative ∞-Algebra**.

- **Note:** Since the map is the identity, the fiber over any operation is a point. There is essentially only one way to combine inputs (order doesn't matter).

2. **The Associative Case (A_∞)**
   - **Base:** $\mathcal{B} = N(\mathrm{Fin}_*)$.
   - **Operad:** $\mathcal{O}^\otimes = \mathrm{Ass}^\otimes$ (The Associative Operad).
   - **Structure Map:** The "forgetful" functor that forgets the linear ordering of the fibers.
   - **Resulting Algebra: Associative ∞-Algebra**.
   - **Note:** The fiber over $\langle n \rangle \to \langle 1 \rangle$ is equivalent to the symmetric group Σ_n. This allows inputs to be permuted (by the base), but the operation distinguishes the order of multiplication ($x_1 x_2 \neq x_2 x_1$).

3. **The Little k-Disks Case (E_k)**
   - **Base:** $\mathcal{B} = N(\mathrm{Fin}_*)$.
   - **Operad:** $\mathcal{O}^\otimes = \mathbb{E}_k^\otimes$.
   - **Structure Map:** The projection from the space of disk embeddings.
   - **Resulting Algebra: E_k-Algebra**.
   - **Note:** Interpolates between Associative ($k = 1$) and Commutative ($k = \infty$).

4. **The Lie Case (L_∞)**
   - **Base:** $\mathcal{B} = N(\mathrm{Fin}_*)$.
   - **Operad:** $\mathcal{O}^\otimes = \mathrm{Lie}^\otimes$.
   - **Resulting Algebra: L_∞-Algebra** (Homotopy Lie Algebra).
   - **Note:** Typically considered over a stable target category (like chain complexes).

5. **The Non-Symmetric / Planar Case**
   - **Base:** $\mathcal{B} = N(\Delta)^{op}$ (The Simplex Category; Linear/Ordered inputs).
   - **Operad:** $\mathcal{O}^\otimes = N(\Delta)^{op}$.
   - **Structure Map:** The identity map.
   - **Resulting Algebra: Associative Monoid** (in the strict sense).
   - **Note:** Here, the base category itself forbids permutation. There is no symmetric group action to even consider.

Definition 1.30 (Endomorphism ∞-Category)**.** Let $\mathcal{C}$ be an ∞-category. The **Endomorphism ∞-Category**, denoted by $\mathrm{End}(\mathcal{C})$, is defined as the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

It forms a **monoidal ∞-category** where the monoidal structure is determined by the composition of endofunctors:

1. The tensor product is given by composition: $F \otimes G := F \circ G$.

2. The unit object is given by the identity functor: $\mathbb{I} := \mathrm{Id}_{\mathcal{C}}$.

Definition 1.31 (Spaces of Monads and Comonads)**.** Let $\mathcal{C}$ be an ∞-category, and let $\mathrm{End}(\mathcal{C})$ denote the monoidal ∞-category of endofunctors on $\mathcal{C}$ (equipped with the composition product). The ∞-categories (or spaces) of Monads and Comonads as the categories of associative algebra objects in $\mathrm{End}(\mathcal{C})$ and its opposite, respectively:

1. The ∞**-category of Monads** is defined as:

$$\mathrm{Mnd}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{A}ss}(\mathrm{End}(\mathcal{C}))$$

2. The ∞**-category of Comonads** is defined as:

$$\mathrm{CoMnd}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{A}ss}(\mathrm{End}(\mathcal{C})^{op})$$

The objects of $\mathrm{Mnd}(\mathcal{C})$ are referred to as **Monads** on $\mathcal{C}$, and the objects of $\mathrm{CoMnd}(\mathcal{C})$ are referred to as **Comonads** on $\mathcal{C}$.

Definition 1.32 (Reedy Category)**.** A small category $\mathcal{R}$ is a **Reedy category** if it is equipped with a degree function $d : \mathrm{Ob}(\mathcal{R}) \to \lambda$ (where λ is an ordinal) and two subcategories $\overrightarrow{\mathcal{R}}$ (the direct category) and $\overleftarrow{\mathcal{R}}$ (the inverse category), such that:

1. Every non-identity morphism in $\overrightarrow{\mathcal{R}}$ raises the degree.
2. Every non-identity morphism in $\overleftarrow{\mathcal{R}}$ lowers the degree.
3. Every morphism f in $\mathcal{R}$ factors uniquely as $f = g \circ h$, where $h \in \overleftarrow{\mathcal{R}}$ and $g \in \overrightarrow{\mathcal{R}}$.

Definition 1.33 (The Reedy Model Structure)**.** Let $\mathcal{M}$ be a model category and $\mathcal{R}$ be a Reedy category. The category of diagrams $\mathrm{Fun}(\mathcal{R}, \mathcal{M})$ is equipped with the **Reedy model structure**, where a morphism $f : X \to Y$ is defined to be:

1. A **Weak Equivalence** if it is a levelwise weak equivalence. That is, for every object $\alpha \in \mathcal{R}$, the map $f_\alpha : X_\alpha \to Y_\alpha$ is a weak equivalence in $\mathcal{M}$.
2. A **Cofibration** if for every $\alpha \in \mathcal{R}$, the **relative latching map** $\lambda_\alpha(f)$ is a cofibration in $\mathcal{M}$. Here, $\lambda_\alpha(f)$ is the map induced by the pushout of the latching objects:

   $$\lambda_\alpha(f) : X_\alpha \amalg_{L_\alpha X} L_\alpha Y \longrightarrow Y_\alpha$$

   where the *latching object* is defined as $L_\alpha X = \mathrm{colim}_{\partial\vec{\mathcal{R}}/\alpha} X$.
3. A **Fibration** if for every $\alpha \in \mathcal{R}$, the **relative matching map** $\mu_\alpha(f)$ is a fibration in $\mathcal{M}$. Here, $\mu_\alpha(f)$ is the map induced into the pullback of the matching objects:

   $$\mu_\alpha(f) : X_\alpha \longrightarrow M_\alpha X \times_{M_\alpha Y} Y_\alpha$$

   where the *matching object* is defined as $M_\alpha X = \lim_{\alpha/\partial\overleftarrow{\mathcal{R}}} X$.

Proposition 1.34 (Latching and Matching as Monadic Structures)**.** Let $\mathcal{R}$ be a Reedy category. For any degree α, consider the truncation inclusion of the category of degrees strictly lower than α:

$$u : \mathcal{R}_{<\alpha} \hookrightarrow \mathcal{R}_{\leq\alpha}$$

Let u^* be the restriction functor. We identify the Latching and Matching objects via the adjunctions defining the skeleton and coskeleton:

1. **Latching as a Monad (Skeleton):** The pair $u_! \dashv u^*$ generates a **Monad** $T = u_! \circ u^*$ (Left Kan Extension followed by restriction). The Latching object is the value of this monad:

   $$L_\alpha X \cong (T(X))_\alpha = (\mathrm{Lan}_u(u^* X))_\alpha$$

   The canonical map $L_\alpha X \to X_\alpha$ corresponds to the monad algebra structure map (or the counit of the adjunction).

2. **Matching as a Comonad (Coskeleton):** The pair $u^* \dashv u_*$ generates a **Comonad** $G = u_* \circ u^*$ (Right Kan Extension followed by restriction). The Matching object is the value of this comonad:
$$M_\alpha X \cong (G(X))_\alpha = (\mathrm{Ran}_u(u^* X))_\alpha$$
The canonical map $X_\alpha \to M_\alpha X$ corresponds to the comonad coalgebra structure map (or the unit of the adjunction).

Definition 1.35 (Monadic and Comonadic Resolution)**.** Let $\mathcal{C}$ be a category and X an object in $\mathcal{C}$. We define the canonical resolutions generated by monads and comonads as follows:

1. **Monadic Resolution:** Let (T, μ, η) be a **Monad** on $\mathcal{C}$. The **Bar Construction** provides an augmented simplicial object $B_\bullet(X)$ resolving X:
$$\dots \rightrightarrows T^3 X \rightrightarrows T^2 X \rightrightarrows TX \xrightarrow{\epsilon} X$$
The face maps d_i are given by the multiplication μ, and degeneracy maps s_i by the unit η. This construction typically serves as a **cofibrant replacement** of X.

2. **Comonadic Resolution:** Let (G, δ, ϵ) be a **Comonad** on $\mathcal{C}$. The **Cobar Construction** provides an augmented cosimplicial object $C^\bullet(X)$ resolving X:
$$X \xrightarrow{\eta} GX \rightrightarrows G^2 X \rightrightarrows G^3 X \rightrightarrows \dots$$
The coface maps d^i are given by the comultiplication δ, and codegeneracy maps s^i by the counit ϵ. This construction typically serves as a **fibrant replacement** of X.

Example 1.36 (**Reedy Latching and Matching**)**.** In a Reedy category $\mathcal{R}$, resolutions arise from Kan extensions along the filtration $u : \mathcal{R}_{<\alpha} \hookrightarrow \mathcal{R}_{\leq\alpha}$.

- **Latching (Monad):** The Latching object $L_\alpha X$ is generated by the **Skeleton Monad** $T = u_! u^*$ (Left Kan extension followed by restriction).
$$L_\alpha X \cong (TX)_\alpha$$
- **Matching (Comonad):** The Matching object $M_\alpha X$ is generated by the **Coskeleton Comonad** $G = u_* u^*$ (Right Kan extension followed by restriction).
$$M_\alpha X \cong (GX)_\alpha$$

Example 1.37 (**The Cotangent Complex (André-Quillen)**)**.** Used to define the derived cotangent complex $\mathbb{L}_{A/k}$.

- **Monad:** The **Free Algebra Monad** T on the category of k-modules (or sets).
$$T(V) = \mathrm{Sym}_k(V) \quad \text{(Polynomial Algebra)}$$
- **Resolution:** The simplicial resolution $P_\bullet \to A$ is the Bar construction $B_\bullet(T, T, A)$. The cotangent complex is derived from applying differentials $\Omega^1_{P_\bullet/k} \otimes_{P_\bullet} A$.

Example 1.38 (**The Postnikov Tower**)**.** Decomposing a space X into its homotopy types.

- **Monad:** The n**-Truncation Monad** $\tau_{\leq n}$ (or P_n).

$$T_n(X) = \tau_{\leq n}(X)$$

This is an *idempotent* monad (localization). The tower is the limit sequence $\cdots \to T_n X \to T_{n-1} X$.

- *Note:* Dually, the **Whitehead Tower** uses the n-connected cover Comonad $\tau_{>n}$.

Example 1.39 (**Projective and Injective Resolution**). Let R be a ring and M an R-module.

1. **Projective Resolution (The Bar Construction):** Using the free-forgetful adjunction $F \dashv U$, we define the **Free Monad** $T = F \circ U$. Since $T(M)$ is a free module, the associated Bar construction yields a canonical projective resolution:

$$\cdots \longrightarrow T^3 M \longrightarrow T^2 M \longrightarrow TM \xrightarrow{\epsilon} M \longrightarrow 0$$

The boundary maps are alternating sums of the monad multiplication $\mu : T^2 \to T$.

2. **Injective Resolution (The Cobar Construction):** Using the forgetful-cofree adjunction $U \dashv C$ (where $C(A) = \mathrm{Hom}_{\mathbb{Z}}(R, A)$), we define the **Cofree Comonad** $G = C \circ U$. Since $G(M)$ is an injective module, the associated Cobar construction yields a canonical injective resolution:

$$0 \longrightarrow M \xrightarrow{\eta} GM \longrightarrow G^2 M \longrightarrow G^3 M \longrightarrow \ldots$$

The boundary maps are alternating sums of the comonad comultiplication $\delta : G \to G^2$.

Example 1.40 (**The Spectrification Monad on Prespectra**). Let $\mathcal{P}$ be the category of Prespectra (sequences of spaces with maps $\Sigma E_n \to E_{n+1}$). The process of converting a naive suspension spectrum into a genuine Ω-spectrum is governed by the **Spectrification Monad** $\mathbb{L}$.

1. **The Level-wise Monad:** We define a Monad $\mathbb{L} : \mathcal{P} \to \mathcal{P}$ by applying the spatial stabilization monad $Q = \Omega^\infty \Sigma^\infty$ to *each level* of the spectrum independently:

$$(\mathbb{L}E)_n := Q(E_n) = \operatorname*{colim}_k \Omega^k \Sigma^k E_n$$

2. **Application to Suspension Spectra:** If $E = \Sigma^\infty X$ is the suspension spectrum of X (where $E_n = \Sigma^n X$), applying this monad yields:

$$(\mathbb{L}(\Sigma^\infty X))_n = Q(\Sigma^n X)$$

The result $\mathbb{L}(\Sigma^\infty X)$ is an **Ω-spectrum**. This is the *fibrant replacement* of $\Sigma^\infty X$ in the stable model structure.

3. **Distinction from Adams Resolution:**
   - The **Adams/Bousfield-Kan resolution** builds a tower $X \to QX \to Q^2 X \ldots$ to resolve the *space* X.
   - The **Spectrification Monad** $\mathbb{L}$ acts once (essentially as a completion) to fix the *structure* of the spectrum, ensuring the adjoint structure maps $E_n \to \Omega E_{n+1}$ become weak equivalences.

Example 1.41 (**The R-Completion of a Space**). Let R be a commutative ring (typically $\mathbb{Z}_p$ or $\mathbb{Q}$). The Bousfield-Kan resolution constructs the "R-completion" of a space X, effectively translating algebraic information (homology with coefficients in R) into homotopy information.

1. **The Monad (R-Linearization):** Let $R : \mathcal{S} \to \mathcal{S}$ be the monad that assigns to a simplicial set K the free simplicial R-module generated by K (forgetting the module structure back to a simplicial set).
$$X \xrightarrow{\eta} R(X)$$
Intuitively, this replaces every simplex of X with the free R-module generated by its vertices.

2. **The Cosimplicial Space (Bar Construction):** Applying the Monad iteratively generates a **cosimplicial space** $R^\bullet X$:
$$X \xrightarrow{\eta} R(X) \rightrightarrows R(R(X)) \Rrightarrow R^3(X) \cdots$$
This tower resolves X by spaces that are algebraically simple (generalized Eilenberg-MacLane spaces).

3. **Totalization (R-Completion):** The **Totalization** (Homotopy Limit) of this cosimplicial space defines the R-completion of X:
$$X_R^\wedge := \mathrm{Tot}(R^\bullet X) \simeq \operatorname*{holim}_{\Delta} R^\bullet X$$

4. **The Spectral Sequence:** This resolution yields the **Bousfield-Kan Spectral Sequence**, which computes the homotopy groups of the completion from the cohomology of X:
$$E_2^{s,t} \cong \mathrm{Ext}^s_{\mathrm{Comod}}(R, H_*(X; R))_t \implies \pi_{t-s}(X_R^\wedge)$$
For $R = \mathbb{Z}_p$, this computes the homotopy groups of the p-adic completion of X.