

§ 1. Basic Definitions

1. Simplicial Sets

Let Δ be the category of finite ordinal numbers.

- **Objects:** Ordered sets $\mathbf{n} = \{0 \rightarrow 1 \rightarrow \dots \rightarrow n\}$ for $n \geq 0$.
- **Morphisms:** Order-preserving maps $\theta : \mathbf{m} \rightarrow \mathbf{n}$.

A **simplicial set** is a contravariant functor:

$$X : \Delta^{op} \rightarrow \mathbf{Sets}$$

It consists of sets X_n (n -simplices) and structure maps induced by morphisms in Δ .

2. Coface and Codegeneracies

Special morphisms in Δ generate all morphisms:

- **Coface maps** $d^i : \mathbf{n} - 1 \rightarrow \mathbf{n}$ ($0 \leq i \leq n$): The injective map missing i .
- **Codegeneracy maps** $s^j : \mathbf{n} + 1 \rightarrow \mathbf{n}$ ($0 \leq j \leq n$): The surjective map covering j twice.

Cosimplicial Identities (in Δ):

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && \text{if } i < j \\ s^j d^i &= d^i s^{j-1} && \text{if } i < j \\ s^j d^j &= \text{id} = s^j d^{j+1} \\ s^j d^i &= d^{i-1} s^j && \text{if } i > j + 1 \\ s^j s^i &= s^i s^{j+1} && \text{if } i \leq j \end{aligned}$$

Simplicial Identities (Structure maps of X): For a simplicial set X , let $d_i = X(d^i)$ (faces) and $s_j = X(s^j)$ (degeneracies):

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j \\ d_i s_j &= s_{j-1} d_i && \text{if } i < j \\ d_j s_j &= \text{id} = d_{j+1} s_j \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j + 1 \\ s_i s_j &= s_{j+1} s_i && \text{if } i \leq j \end{aligned}$$

3. Simplicial Abelian Groups

A **simplicial abelian group** is a functor:

$$A : \Delta^{op} \rightarrow \mathbf{Ab}$$

where \mathbf{Ab} is the category of abelian groups.

- Let $\mathbb{Z}Y$ be the free abelian group on a simplicial set Y .
- **Moore Complex:** A chain complex defined by:

$$(\mathbb{Z}Y)_n \xrightarrow{\partial} (\mathbb{Z}Y)_{n-1}$$

with boundary $\partial = \sum_{i=0}^n (-1)^i d_i$.

4. Classifying Space (Nerve)

Let \mathcal{C} be a small category. The **classifying space** (or nerve) BC is a simplicial set defined by:

$$BC_n = \text{hom}_{\mathbf{cat}}(\mathbf{n}, \mathcal{C})$$

An n -simplex is a string of composable arrows of length n :

$$a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_n$$

5. Standard n -Simplex & Yoneda Lemma

The **standard n -simplex** Δ^n is the representable functor:

$$\Delta^n = \text{hom}_{\Delta}(-, \mathbf{n})$$

Yoneda Lemma: There is a natural bijection between n -simplices of X and simplicial maps from Δ^n to X :

$$\text{hom}_S(\Delta^n, X) \cong X_n$$

Let $\iota_n = \text{id}_{\mathbf{n}} \in (\Delta^n)_n$. A map $\phi : \Delta^n \rightarrow X$ corresponds to the simplex $x = \phi(\iota_n)$. Conversely, $x \in X_n$ defines $\iota_x : \Delta^n \rightarrow X$.

6. Boundary ($\partial\Delta^n$) and Horns (Λ_k^n)

Boundary: $\partial\Delta^n \subset \Delta^n$ is the smallest subcomplex containing all faces $d_j(\iota_n)$.

Construction of $\partial\Delta^n$: The set of j -simplices $(\partial\Delta^n)_j$ is defined as:

$$(\partial\Delta^n)_j = \begin{cases} (\Delta^n)_j & \text{if } 0 \leq j \leq n-1 \\ \text{deg}_j & \text{if } j \geq n \end{cases}$$

where deg_j denotes the set of iterated degeneracies of elements in $(\Delta^n)_{n-1}$. Essentially, in dimensions $\geq n$, it contains only degenerate elements originating from lower dimensions.

Example: $\partial\Delta^3$ (Boundary of the 3-simplex).

- It contains all proper faces of Δ^3 but not the non-degenerate 3-simplex ι_3 (or its degeneracies).
- **Non-degenerate elements:**
 - 4 vertices (0-simplices).
 - 6 edges (1-simplices).
 - 4 faces (2-simplices): $d_0(\iota_3), d_1(\iota_3), d_2(\iota_3), d_3(\iota_3)$.
- Geometrically, it forms the surface of a tetrahedron.

k -th Horn: $\Lambda_k^n \subset \Delta^n$ ($n \geq 1, 0 \leq k \leq n$) is the subcomplex generated by all faces $d_j(\iota_n)$ **except** the k -th face $d_k(\iota_n)$.

Example: Λ_2^3 (The 2nd horn of the 3-simplex) is the subcomplex of Δ^3 generated by the faces d_0, d_1 , and d_3 (omitting d_2).