A Study in Quasi-Coherent Sheaves and Tannaka Duality

Notes on Lurie's DAG VIII

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1 Preliminaries

1.1 ∞-categories

Definition 1.1 (∞-Category)**.** A simplicial set K is an ∞**-category** if for every $n > 1$ and every **inner** index $0 < i < n$, every map of simplicial sets $f_0 : \Lambda_i^n \to K$ admits an extension to an n-simplex $f : \Delta^n \to K$.

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "f_0"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Definition 1.2 (Simplicial Category)**.** A **simplicial category** (or Set_Δ-enriched category) $\mathcal{C}$ is a category where:

1. For any two objects $X, Y \in \mathcal{C}$, the collection of morphisms between them is not a set, but a **simplicial set** $\mathrm{Map}_{\mathcal{C}}(X, Y)$.
2. For any three objects $X, Y, Z \in \mathcal{C}$, the composition map
$$\mathrm{Map}_{\mathcal{C}}(Y, Z) \times \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$$
is a morphism of simplicial sets and satisfies the usual associativity and identity axioms.

A simplicial category $\mathcal{C}$ is **locally Kan** if for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the mapping simplicial set $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex.

Definition 1.3 (Simplicial Nerve N_Δ)**.** The **simplicial nerve** $N_\Delta(\mathcal{C})$ is the simplicial set defined by the assignment:
$$N_\Delta(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$$
where $\mathfrak{C}[\Delta^n]$ is the **rigidification** of the n-simplex Δ^n into a simplicial category.

Definition 1.4 (∞-category via N_Δ)**.** An ∞**-category** (or quasicategory) is a simplicial set K that is equivalent to the simplicial nerve of some locally Kan simplicial category $\mathcal{C}$.

$$K \simeq N_\Delta(\mathcal{C})$$

Theorem 1.5 (Joyal-Lurie)**.** *There exists a Quillen equivalence between the Joyal model structure on* Set_Δ *(modeling quasicategories) and the Bergner model structure on* Cat_Δ *(modeling simplicial categories):*

$$\mathfrak{C}[\cdot] : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Cat}_\Delta : N_\Delta.$$

Specifically, for any simplicial category $\mathcal{C}$ *where mapping spaces are Kan complexes, its simplicial nerve* $N_\Delta(\mathcal{C})$ *is a quasicategory.*

Definition 1.6 (Free Cocompletion)**.** Let $\mathcal{C}$ be a small ∞-category. An ∞-category $\mathcal{P}(\mathcal{C})$ is called the **free cocompletion** of $\mathcal{C}$ if it satisfies the following universal property:

1. $\mathcal{P}(\mathcal{C})$ admits all small colimits.
2. There exists a functor $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ (called the Yoneda embedding) such that for any ∞-category $\mathcal{D}$ which admits small colimits, composition with j induces an equivalence of ∞-categories:
$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Here, Fun^L denotes the full subcategory of functors that preserve small colimits (left adjoints).

Definition 1.7 (The ∞-category of Spaces)**.** Let $\mathcal{S}$ denote the ∞-category of spaces. It is defined in two equivalent ways:

1. **Via Dwyer-Kan Localization:**
   Let W be the class of weak homotopy equivalences in Set_Δ. We define $\mathcal{S}$ as the homotopy coherent nerve of the simplicial localization:
$$\mathcal{S} := N\big(\mathrm{Set}_\Delta[W^{-1}]\big).$$
   Equivalently, via Kan complexes: $\mathcal{S} \simeq N(\mathbf{Kan})$.
2. **Via Free Cocompletion:**
   The ∞-category $\mathcal{S}$ is the free cocompletion of the point $*$. That is, it is the category of presheaves:
$$\mathcal{S} \simeq \mathcal{P}(*).$$
   Universal Property: For any cocomplete ∞-category $\mathcal{C}$, there is an equivalence $\mathrm{Fun}^L(\mathcal{S}, \mathcal{C}) \simeq \mathcal{C}$.