A Study in Advanced Talks

Mathematical Notes Collection

Notes & Expositions

Contents

Perfectoid Spaces I: The Weight-Monodromy Conjecture

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1.1 Introduction

These notes reconstruct the motivations leading to the theory of Perfectoid Spaces, rooted in Peter Scholze's undergraduate years in Bonn (circa 2007)[cite: 2]. The central motivation discussed here is the **Weight-Monodromy Conjecture** (WMC) by Deligne.

1.2 The Setup and The Monodromy Operator

1.2.1 Geometric Setup

Let $K = \mathbb{Q}_p$ be the p-adic field. Let X be a smooth projective scheme over $\mathbb{Q}_p$[cite: 2]. Fix a prime $\ell \neq p$[cite: 3]. We are interested in the ℓ-adic cohomology of X:

$$V := H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell).$$

This vector space admits a continuous action of the absolute Galois group $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$[cite: 4].

1.2.2 The Monodromy Theorem

To state the conjecture, we must first define the structure of this Galois representation. The inertia group $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ acts on V. Even if X does not have good reduction, Grothendieck's ℓ-adic Monodromy Theorem describes this action.

Theorem 1.1 (Grothendieck's ℓ-adic Monodromy)**.** *There exists a nilpotent operator $N : V \to V(-1)$, called the **Monodromy Operator**, such that on an open subgroup of the inertia $I_{\mathbb{Q}_p}$, the action of $\sigma \in I_{\mathbb{Q}_p}$ is given by*

$$\rho(\sigma) = \exp(t_\ell(\sigma)N),$$

where $t_\ell : I_{\mathbb{Q}_p} \to \mathbb{Z}_\ell$ is the ℓ-adic tame character[cite: 10, 11].

Definition 1.2 (Weight Decomposition)**.** The representation V admits a decomposition based on the eigenvalues of the Geometric Frobenius Φ. We say V has a **weight decomposition**[cite: 8]:

$$V = \bigoplus_{j=0}^{2i} V_j,$$

where the eigenvalues of Φ acting on V_j are Weil numbers of weight j.

The monodromy operator N respects this structure in a specific way. Since N maps V to $V(-1)$ (a Tate twist), it lowers the weight by 2[cite: 13]:

$$N : V_j \longrightarrow V_{j-2}(-1).$$

1.3 The Weight-Monodromy Conjecture

Deligne proposed the following conjecture regarding the interplay between the filtration by weights and the monodromy operator.

Conjecture 1.3 (Deligne's Weight-Monodromy Conjecture)**.** *For any $j \geq 0$, the power of the monodromy operator induces an isomorphism[cite: 17, 25]:*

$$N^j : V_{i+j} \xrightarrow{\sim} V_{i-j}(-j).$$

Intuitively, this suggests that the monodromy operator acts like the hard Lefschetz operator, reflecting the weight filtration across the center i.

1.4 Examples and Evidence

1.4.1 Case 1: Good Reduction

Assume X has good reduction, meaning there exists a smooth projective model $\mathcal{X}$ over $\mathbb{Z}_p$ such that $\mathcal{X}_{\mathbb{Q}_p} \cong X$[cite: 19].

Lemma 1.4. *If X has good reduction, the action of the inertia group $I_{\mathbb{Q}_p}$ is trivial[cite: 22].*

Consequently, the monodromy operator is trivial ($N = 0$)[cite: 23]. In this case, Conjecture 1.3 implies that $V_j = 0$ for all $j \neq i$. Thus $V = V_i$. This reduces to the **Weil Conjectures** for the reduction $X_{\mathbb{F}_p}$, which are known[cite: 26].

1.4.2 Case 2: The Tate Curve

Consider an elliptic curve E over $\mathbb{Q}_p$ with split multiplicative reduction. As a rigid analytic space, E can be uniformized as the **Tate Curve**[cite: 30]:

$$E(\overline{\mathbb{Q}}_p) \cong \mathbb{G}_m(\overline{\mathbb{Q}}_p)/q^{\mathbb{Z}}, \quad \text{with } |q| < 1.$$

We compute $H^1(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$. Using the Hochschild-Serre spectral sequence (or the Kummer sequence for $\mathbb{G}_m$), we have an exact sequence[cite: 39]:

$$0 \longrightarrow \overline{\mathbb{Q}}_\ell(0) \longrightarrow H^1(E, \overline{\mathbb{Q}}_\ell) \longrightarrow \overline{\mathbb{Q}}_\ell(-1) \longrightarrow 0.$$

Here:

- The subspace $V_0 = \overline{\mathbb{Q}}_\ell(0)$ has weight 0.
- The quotient $V_2 = \overline{\mathbb{Q}}_\ell(-1)$ has weight 2.
- The center weight is $i = 1$.

The Monodromy operator N maps the weight 2 part to the weight 0 part:

$$N : V_2 \xrightarrow{\sim} V_0(-1).$$

This verifies the conjecture for $j = 1$: $N^1 : V_{1+1} \to V_{1-1}(-1)$ is an isomorphism[cite: 40].

1.5 Strategy: Reduction to Equal Characteristic

1.5.1 Known Results

The conjecture is known in the following cases:

1. Dimension of X is ≤ 2[cite: 47].
2. **Equal Characteristic:** If X is defined over a function field $F \cong \mathbb{F}_p((t))$ rather than $\mathbb{Q}_p$, the conjecture was proved by Deligne using L-functions[cite: 52, 53].

1.5.2 Rapoport's Suggestion

The strategy to prove WMC in mixed characteristic ($\mathbb{Q}_p$) is to reduce it to the equal characteristic case ($\mathbb{F}_p((t))$).

- **Idea:** Base change to a highly ramified extension $K/\mathbb{Q}_p$[cite: 60].
- Let e be the ramification index. The ring of integers behaves like:

$$\mathcal{O}_K/p \cong \mathbb{F}_q[t]/t^e.$$

- As $e \to \infty$, this ring approximates $\mathbb{F}_q[[t]]$[cite: 63, 65].

1.5.3 The Obstruction and Perfectoid Spaces

However, for any finite e, the approximation is insufficient. There are algebraic obstructions to deforming $X_{\mathcal{O}_K/p}$ to a scheme over $\mathbb{F}_q[[t]]$[cite: 72, 76].

This failure suggests that we must pass to the limit $e = \infty$. This leads to the definition of **Perfectoid Fields** (fields with infinite ramification). In the world of perfectoid spaces, we can construct a rigorous isomorphism (Tilting) between geometric objects in mixed characteristic and equal characteristic:

$$X_{\text{perfectoid}} \longleftrightarrow X^{\flat}_{\text{equal char}}.$$

This correspondence allows the transfer of cohomological results (like Deligne's theorem) back to $\mathbb{Q}_p$, as realized in Scholze's work [cite: 78-81].