On the Approximability of Multistage Min-Sum Set Cover

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16/07/2021

Outline

Introduction - Static Version(MSSC)

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- e.g a customer may be interested in sports and gardening, another in science, chess and videogames.
- We want to maintain an ordering of products, so that customers find something they like quickly.
- In this example $S_1, S_2, \ldots, S_m$ model the users' interests and π is the ordering we're looking for.

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Greedy Algorithm for Min Sum Set Cover

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- This is optimal unless P=NP (NP-Hard to approximate with factor $4 - \epsilon$).
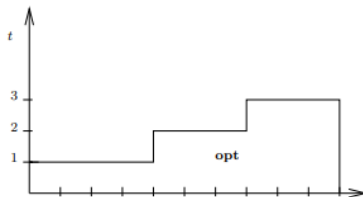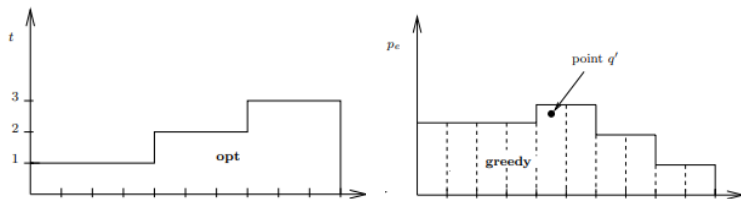
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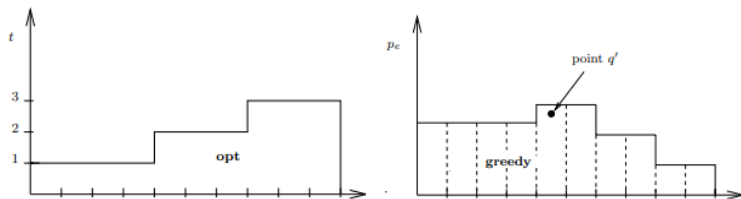- Map Greedy and Optimal Solution to Histograms.

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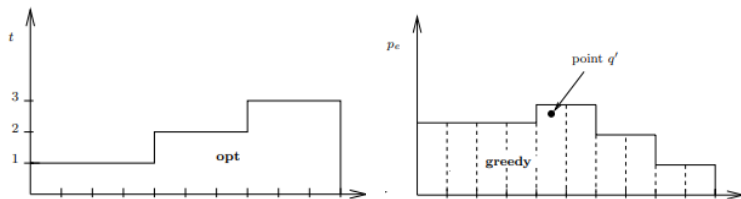
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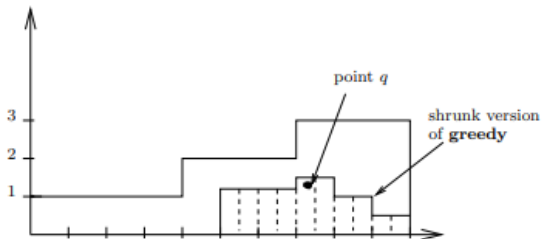- Map Greedy and Optimal Solution to Histograms.

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- LP-Rounding Approach, Skutella and Williamson [6] improve it to 28-apx.

GMSSC - Fractional LP

$$\min \ \sum_{t=1}^{n} \sum_{i=1}^{m} (1 - y_{i,t})$$

$$\text{s.t} \ \sum_{e \in U} x_{e,t} = 1 \quad \text{for all } t \leq n$$

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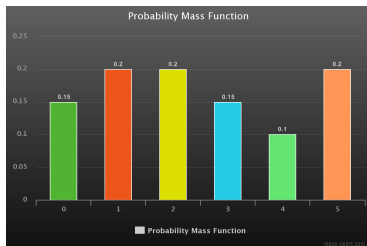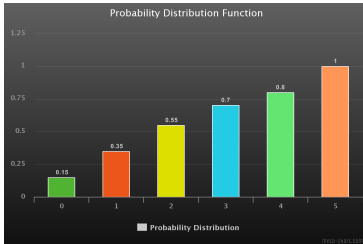Use $x_{e,t}$ to order the elements and construct π.

GMSSC - Randomized Rounding
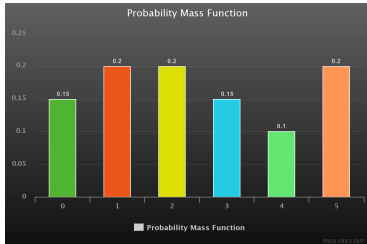
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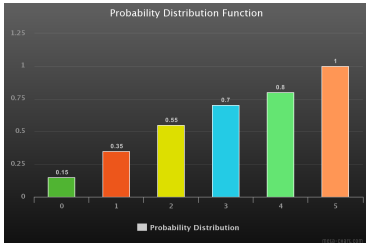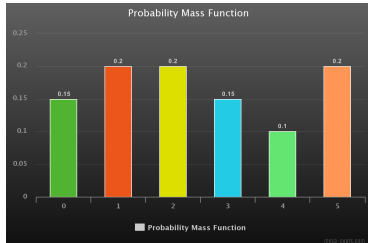
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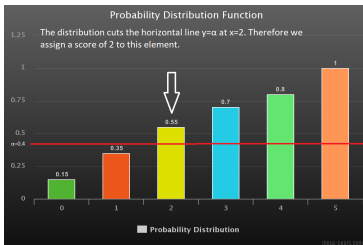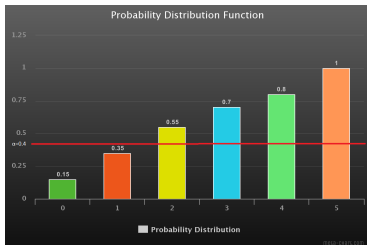
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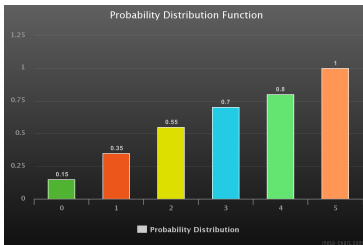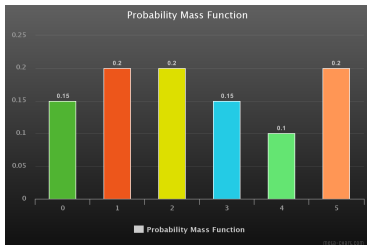
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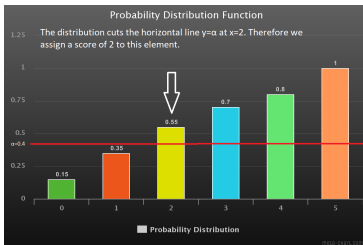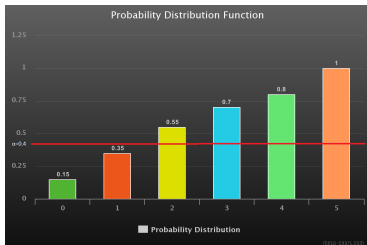
Sort in order of non-decreasing score.

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Mult-MSSC - **The Offline Version**

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- Here d_{KT} denotes the Kendall-Tau distance - number of inv. to transform π^{t-1} to π^t.

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Mult-MSSC and Move-To-Front are approximately the same. That is:

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$$\sum_{e \in S_t} A_{e1}^t = 1 \quad t = 1, \ldots, T$$

$$A^0 = \pi^0$$

$$A_{ei}^t \geq 0 \quad e \in U, \ i = 1, \ldots, n \text{ and } t = 1, \ldots, T$$

Move-To-Front - Linear Relaxation

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Footrule Distance d_{FR} - Optimal Value of the Transportation Problem:

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Move-to-Front - Linear Relaxation

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Let the stochastic matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/4 & 0 & 3/4 \end{pmatrix}$.

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Move-To-Front - Randomized Rounding

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Fact

From the Linear Relaxation of MTF we know:
$\sum_{t=1}^{T} d_{FR}(A_t, A_{t-1}) \leq 4 \cdot OPT_{\text{Mult-MSSC}}$.

Move-To-Front - Randomized Rounding

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A Randomized Algorithm for Mult-MSSC

1: Find the optimal solution $A^0 = \pi^0, A^1, \ldots, A^T$ for $\mathrm{Fractional - MTF}$.
2: **for** each element $e \in U$ **do**
3:     Select α_e uniformly at random in $[0, 1]$.
4: **end for**
5: **for** $t = 1 \ldots T$ **do**
6:     **for** all elements $e \in U$ **do**
7:         $I_e^t := \mathrm{argmin}_{1 \leq i \leq n} \{\log n \cdot \sum_{s=1}^{i} A_{es}^t \geq \alpha_e\}$.
8:     **end for**
9:     $\pi^t :=$ sort elements according to I_e^t with ties being broken lexicographically.
10: **end for**

Analysis of Randomized Rounding

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Theorem - Approximation Ratio

The Randomized Rounding Algorithm 18 is $O(log^2 n)$ approximation for Mult-MSSC.

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- Basic idea: Decompose matrix to sequence of neighboring matrices(matrices that differ only in 2 entries).

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Deterministic Rounding Algorithm for r-bounded sequences

Input: A request sequence $R_1, \ldots, R_T$ with $|R_t| \leq r$ and an initial permutation π^0.
Output: A sequence of permutations $\pi^1, \ldots, \pi^T$.
1: Find the optimal solution $A^0 = \pi^0, A^1, \ldots, A^T$ for $\mathrm{Fractional - MTF}$.
2: **for** $t = 1 \ldots T$ **do**
3:    $\pi^t :=$ in π^{t-1}, move to the first position an element $e \in R_t$ such that $A_{e1}^t \geq 1/r$
4: **end for**

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Algorithm 20 is a $O(r^2)$-approximation algorithm for $\mathrm{Mult\text{-}MSSC}$.

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- Prove $O(r^2)$-approximation if matrices A^t are semi-integral.
- Prove that we can transform any sequence of matrices to semi-integral with the same moving cost.

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- You can find the implementation of our algorithms and relevant experiments in this github repo.

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