On the Approximability of Multistage Min-Sum Set Cover

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Outline

Introduction - Static Version(MSSC)

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- We want to maintain an ordering of products, so that customers find something they like quickly.
- In this example $S_1, S_2, \ldots, S_m$ model the users' interests and π is the ordering we're looking for.

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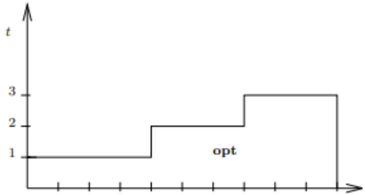

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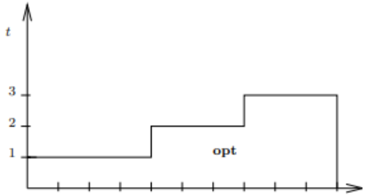

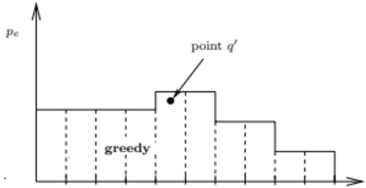

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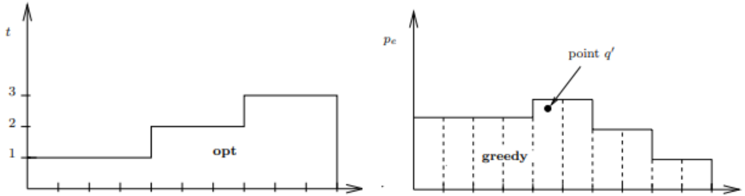

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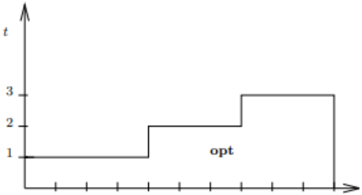

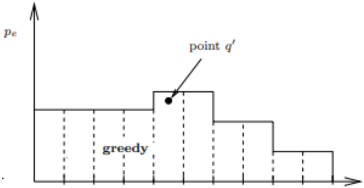

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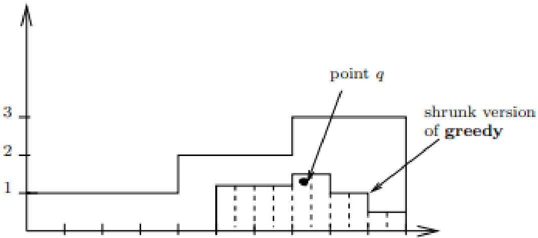

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- LP-Rounding Approach, Skutella and Williamson [6] improve it to 28-apx.

GMSSC - Fractional LP

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\begin{aligned}
\text{m}in \quad & \sum_{t=1}^{n}\sum_{i=1}^{m}(1-y_{i,t}) \\
\text{s.t} \quad & \sum_{e\in U} x_{e,t} = 1 \qquad \text{for all } t \leq n \\
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Use $x_{e,t}$ to order the elements and construct π.

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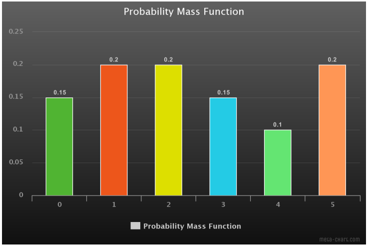

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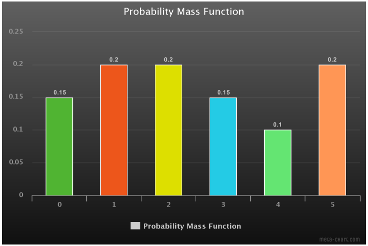

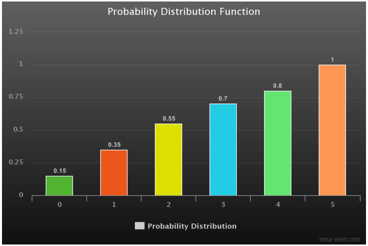

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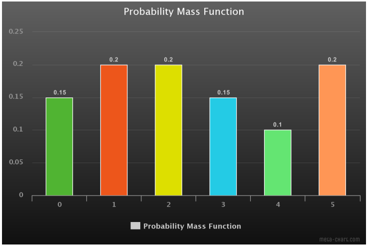

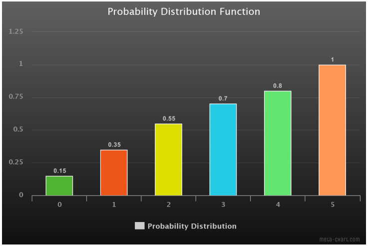

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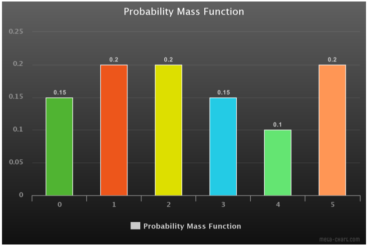

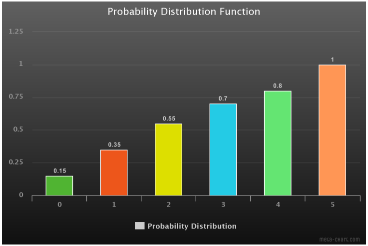

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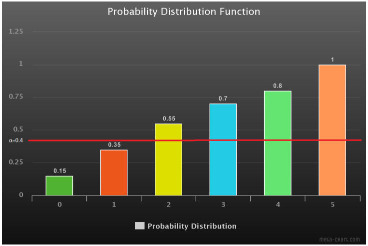

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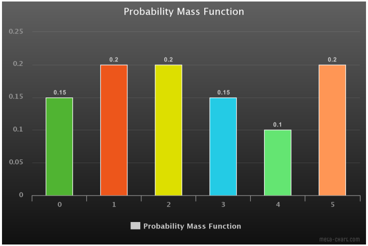

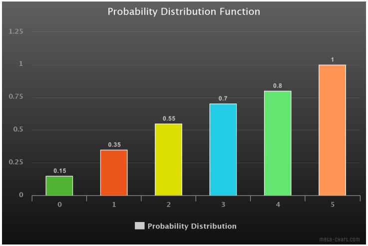

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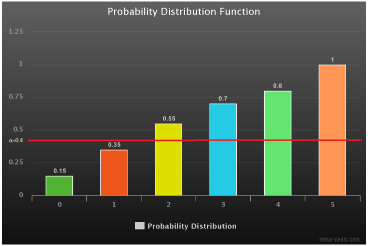

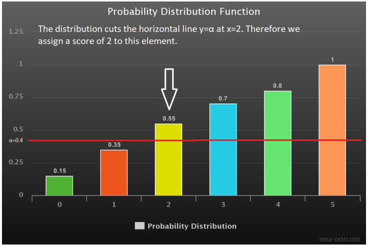

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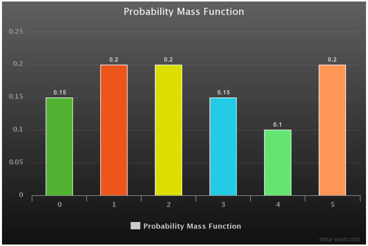

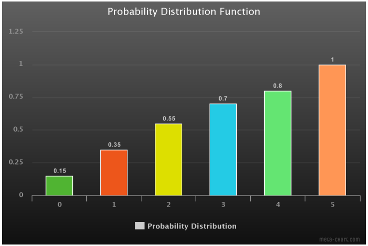

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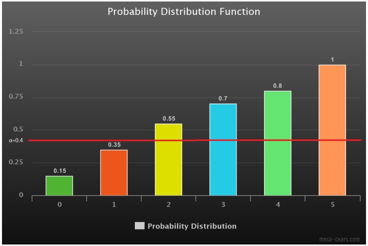

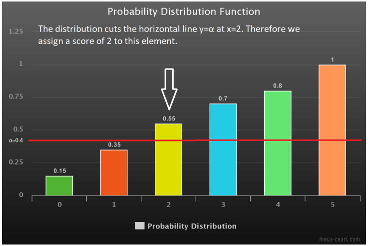

Sort in order of non-decreasing score.

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- Bridge the Gap? $\rightarrow$ Construct approximation algorithms for the offline version.

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- Here d_{KT} denotes the Kendall-Tau distance - number of inv. to transform π^{t-1} to π^t.

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2. The moving cost $\sum_{t=1}^{T} d_{KT}(\pi^t, \pi^{t-1})$ is minimized.

Lemma

Mult-MSSC and Move-To-Front are approximately the same. That is:

Mult-MSSC - **The Move-To-Front Version**

INPUT: A Request Sequence $S_1, S_2, \ldots, S_T \subseteq U$.
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- $OPT_{MTF} \leq 4 \cdot OPT_{\text{Mult-MSSC}}$

Move-To-Front - Linear Relaxation

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Any permutation can be represented with a 0-1 doubly stochastic matrix.

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LP Relaxation of Move-to-Front

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\begin{aligned}
\min \quad & \sum_{t=1}^{T} \mathrm{d}_{\mathrm{FR}}(A^t, A^{t-1}) \\
\text{s.t} \quad & \sum_{i=1}^{n} A^t_{ei} = 1 \qquad e \in U \text{ and } t = 1, \dots, T \\
& \sum_{e \in U} A^t_{ei} = 1 \qquad i = 1, \dots, n \text{ and } t = 1, \dots, T \\
& \sum_{e \in S_t} A^t_{e1} = 1 \quad t = 1, \dots, T \\
& A^0 = \pi^0 \\
& A^t_{ei} \geq 0 \qquad\quad e \in U,\ i = 1, \dots, n \text{ and } t = 1, \dots, T
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Let the stochastic matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/4 & 0 & 3/4 \end{pmatrix}$.

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Move-To-Front - Randomized Rounding

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Fact

From the Linear Relaxation of MTF we know:
$\sum_{t=1}^{T} d_{FR}(A_t, A_{t-1}) \leq 4 \cdot OPT_{\text{Mult-MSSC}}$.

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A Randomized Algorithm for $\mathrm{Mult\text{-}MSSC}$

Find the optimal solution $A^0 = \pi^0, A^1, \ldots, A^T$ for $\mathrm{Fractional - MTF}$.
for each element $e \in U$ **do**
  Select α_e uniformly at random in $[0, 1]$.
end for
for $t = 1 \ldots T$ **do**
  for all elements $e \in U$ **do**
    $I_e^t := \mathrm{argmin}_{1 \leq i \leq n}\{\log n \cdot \sum_{s=1}^{i} A_{es}^t \geq \alpha_e\}$.
  end for
  $\pi^t :=$ sort elements according to I_e^t with ties being broken lexicographically.
end for

Analysis of Randomized Rounding

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Theorem - Approximation Ratio

The Randomized Rounding Algorithm 18 is $O(log^2 n)$ approximation for Mult-MSSC.

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  2. $\underbrace{\mathbb{E}[\pi^t(S_t)]}_{\text{Access Cost of set } S_t} \leq 2 \cdot \underbrace{\pi_*^t(S_t)}_{\text{Access Cost of the Optimal Permutation}}$.

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- Basic idea: Decompose matrix to sequence of neighboring matrices(matrices that differ only in 2 entries).

The Deterministic Rounding Algorithm for MTF

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- Sequence of requests $S_1, S_2, \ldots, S_T$ is r-bounded iff $|S_t| \leq r,$ for all t.

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Deterministic Rounding Algorithm for r-bounded sequences

Input: A request sequence $R_1, \ldots, R_T$ with $|R_t| \leq r$ and an initial permutation π^0.
Output: A sequence of permutations $\pi^1, \ldots, \pi^T$.

1: Find the optimal solution $A^0 = \pi^0, A^1, \ldots, A^T$ for $\mathrm{Fractional-MTF}$.
2: **for** $t = 1 \ldots T$ **do**
3: $\quad \pi^t :=$ in π^{t-1}, move to the first position an element $e \in R_t$ such that $A^t_{e1} \geq 1/r$
4: **end for**

Analysis of Deterministic Rounding Algorithm 20

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Algorithm 20 is a $O(r^2)$-approximation algorithm for $\mathrm{Mult\text{-}MSSC}$.

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Analysis of Deterministic Rounding Algorithm 20

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- Prove $O(r^2)$-approximation if matrices A^t are semi-integral.

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- Prove $O(r^2)$-approximation if matrices A^t are semi-integral.
- Prove that we can transform any sequence of matrices to semi-integral with the same moving cost.

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- You can find the implementation of our algorithms and relevant experiments in this github repo.

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