21 April 2021

MH4900 Final Year Project

Diffusion Processes for the Density of the Integral of the Quadratic Brownian Bridge & Asian Options

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Agenda

Section I: Introduction

Diffusion Processes, Brownian Motion and Asian Options, Aim and Structure of Project

Stochastic Processes

Diffusion Process

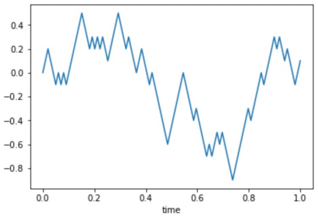
A Diffusion Process is a continuous time Markov process which satisfies a stochastic differential equation. [2]

Brownian Motion

Brownian Motion is a continuous time stochastic process such that:

- $B_0 = 0$
- B_t is continuous; independent increments
- $B_t - B_s \sim N(0, t-s)$ for $0 \le s \le t$
- B_t solves the SDE $dX_t = dB_t$

Brownian Bridge: $B_s \mid B_t = a$, for $0 \le s \le t$

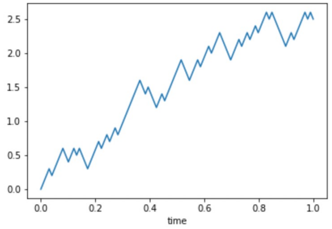


Geometric Brownian Motion (Asset Price Process)

Geometric Brownian Motion is the exponential of B_t

- Models the price of a risky asset with return μ and volatility σ.
- S_t is lognormally distributed
- It satisfies the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$



Time Integral of Quadratic Brownian Motion, Quadratic Brownian Bridge and Asset Price Process

We are interested in studying the following processes in regards to probability density comparisons. The asset price process will also be applied to pricing Asian Options in the second part of the project.

$$\int_0^t |B_s|^2 ds$$

$$\int_0^t |B_s|^2 ds \mid B_t = a$$

$$\int_0^t S_u du$$

Asian Options

An option is a financial derivative which gives the holder a right to buy (call option) or sell (put option) the underlying asset at a given price K over a given period of time. [1]

European Options

- The most basic vanilla option.
- Exercised at maturity T.
- Payoff at maturity f_E: $(S_T - K)^+$ (call) and $(K - S_T)^+$ (put)
- Can be solved by the famous Black Scholes PDE or by evaluating the expectation, since the asset price process is lognormally distributed.

$$^*Price\ at\ time\ t:\ \boldsymbol{e^{-r(T-t)}E^*}\left[(S_T - K)^+ | \boldsymbol{\mathcal{F}_t}\right]$$

Asian Options

- An Exotic option
- Uses the average of the underlying asset price: $\frac{1}{T}\int_0^T S_u du$ to compute the payoff instead of terminal asset price S_T
- Asian options protect the holder from market manipulation of prices at expiry
- Payoff at maturity f_A: $(\frac{1}{T}\int_0^T S_u du - K)^+$ (call) and $(K - \frac{1}{T}\int_0^T S_u du)^+$ (put)

$$Price\ at\ time\ t: \boldsymbol{e^{-r(T-t)}E^*}\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ | \mathcal{F}_{\boldsymbol{t}}\right]$$

The distribution of a sum of lognormal random variables has no closed form solution!
Thus, the Expectation for the Asian Option Price cannot be solved easily, so we use various approximation techniques to obtain Asian Option Price

* Price of call option; [11]
** $\mathcal{F}_t$ is the filtration at time t, models the information known at time t.

Aim of this Project

This project has two separate goals, with some overlapping methods under the common theme of diffusion processes.

Density of Time Integrals

We are interested in obtaining the true and approximated probability density of two processes:

- Time integral of quadratic Brownian motion $\int_0^t B_s ds$
- Time integral of quadratic Brownian bridge $\int_0^t B_s ds \mid B_t = a$

Asian Option Approximation

Our primary goal in this section is to estimate the Asian option price and compare different approximations. We study two approaches:

- Probabilistic methods
- PDE methods

Moment Matching

- Conditional and unconditional moment matching
- Estimate a gamma and lognormal approximation to the distribution of the time integrals: $\int_0^t B_s ds$ and $\int_0^t B_s ds \mid B_t = a$.

- Unconditional moment matching
- Estimate the gamma and lognormal approximation to the distribution of the asset price process.
- This is discounted to give an approximation for the Asian option price.

Monte Carlo Methods and PQL Diffusion Density

- Monte Carlo simulation gives the true density of both time integrals.
- A conditional density as well as a joint density is found for the Brownian bridge terminal values and the time integral.

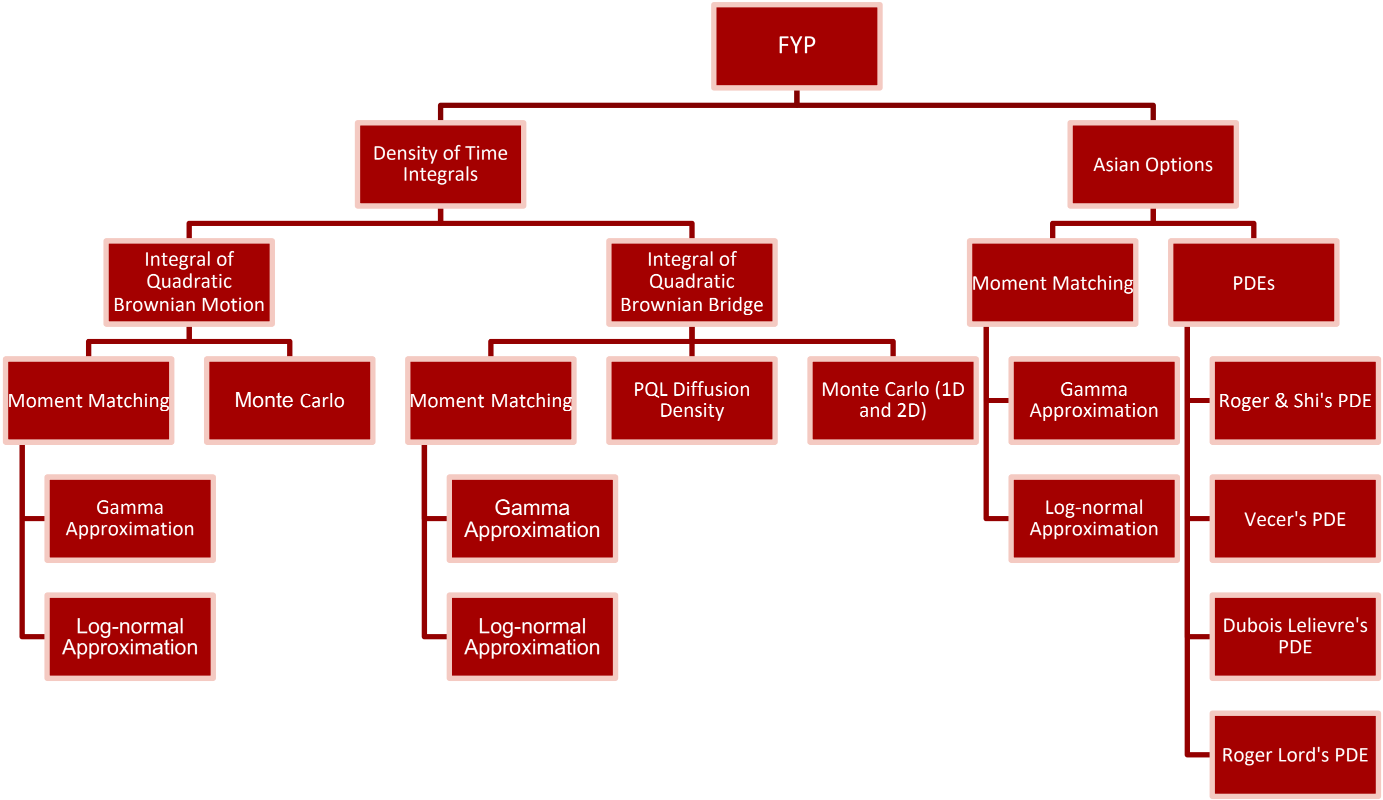
- The Planar quadratic Langevin diffusion density (PQL) from Franchi's paper will also be compared to both approximations and the true density.

Diffusion Process for PDEs

- Vast literature available on PDEs for Asian options
- Brown's Generalization idea [8]: use a diffusion process
- Extend Brown's work to Roger Lord and Vecer's PDE.

- Compare numerical results across PDEs and both approaches

Structure of Contents



Section II:
Relevant Literature & Preliminaries

Literature Review, Preliminaries

Relevant Literature

Time Integrals of Quadratic Brownian Motion/Bridge

Conditional Moment Matching

- Conditional Moment Matching for CIR process (Prayoga & Privault) [5]
- Gamma Approximation for Stochastic Integrals (Nicolas Susanto Tjandra) [16]
- CIR Process for stochastic interest rates (Privault) [17]

Kernel Density

- Planar Quadratic Langevin Diffusion Density (Franchi) [3]
- Heat kernels (Coste) [18]

Asian Option Pricing

Probabilistic Methods

- Curran Approximation [4]
- Moment Matching (Prayoga & Privault), (Roger Lord) [5]

PDEs

- Ingersoll's 2 dimensional PDE [12]

1-dimensional PDEs [1][8]:

- Roger & Shi's PDE
- Vecer's PDE
- Vecer's New PDE
- Dubois & Lelievre's PDE
- Brown's PDE
- Roger Lord's PDE

Other

Bounds

- Curran's lower bound [8]
- Roger & Shi's lower bound [8]
- Thompson's Upper bound [8]

- Laplace inversion (Geman & Yor) [15]
- Monte Carlo Methods [8]

Preliminaries

Itô's Lemma	For any stochastic process X_t, $df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2$ For any stochastic processes X_t and Y_t , $d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$ [11]
Laplace Transform	For a random variable X, the Laplace transform of X is :$\mathcal{L}_X(c) = E[e^{-cX}]$ First moment of X: $-\frac{d}{dc}\mathcal{L}_X(c)\,\vert_{c=0} = E[X]$ Second moment of X: $\frac{d^2}{dc^2}\mathcal{L}_X(c)\,\vert_{c=0} = E[X^2]$
Conditional Density Function	The conditional density function for a random variable X, given $Y = y$: $f_{X\vert Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$, where $f(x, y)$ is the joint density function.
Distribution of S_t	The asset price process S_t with volatility σ and return r is lognormally distributed with parameters: [11] $E[\ln S_t] = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)t$ and $Var(\ln S_t) = \sigma^2 t$
Vanilla Option Pricing Formula	For an option with payoff C and time to maturity T, the option price at time 0 can be given as: [11] $e^{-rT}E[C]$

Section III:
Densities for Time Integrals of Quadratic Brownian Motion/Bridge

Moment Matching, PQL Diffusion Density, Monte Carlo Simulation

Moment Matching

Moment Matching

- Let $y_t = f(B_t)$ and let $Y_t = \int_0^t y_s ds$
- Use the moments to compute approximate parameters for gamma and log-normal distribution for Y_t.

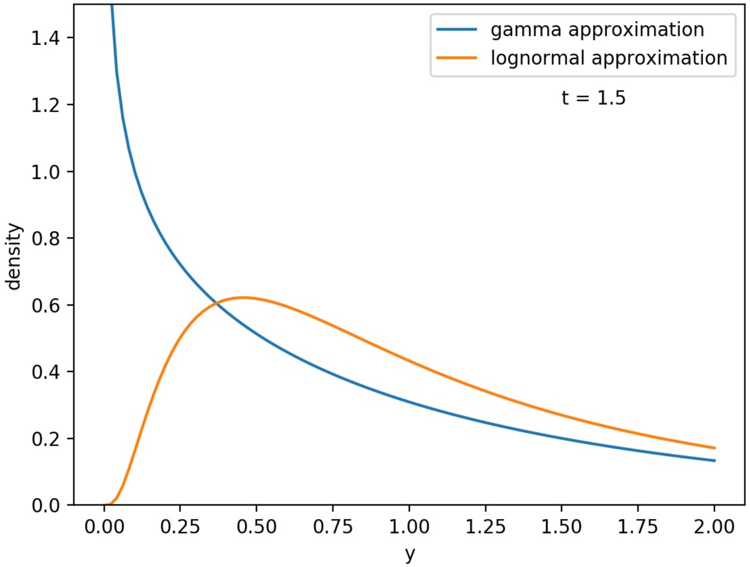
Gamma Approximation	$f_{Y_t \mid B_t=a} = \frac{e^{-y/\alpha}}{\alpha} \frac{(\frac{y}{\alpha})^{\beta-1}}{\Gamma(\beta)}$ $\alpha = \frac{Var[Y_t]}{E[Y_t]}, \beta = \frac{E[Y_t]}{\alpha}$ in the unconditional case, & $\alpha = \frac{Var[Y_t \mid B_t=a]}{E[Y_t \mid B_t=a]}, \beta = \frac{E[Y_t \mid B_t=a]}{\alpha}$ in the conditional case[5]
Log-normal Approximation	$f_{Y_t \mid B_t=a} = \frac{1}{y\eta\sqrt{2\pi t}} e^{-(-\mu+\ln y)^2/(2t\eta^2)}$ $\eta = \sqrt{\ln\left(1 + \frac{Var[Y_t]}{E[Y_t]^2}\right)}, \mu = -\frac{\eta^2}{2} + \ln Var\,[Y_t]$ in the unconditional case, & $\eta = \sqrt{\ln\left(1 + \frac{Var[Y_t \mid B_t=a]}{E[Y_t \mid B_t=a]^2}\right)}, \mu = -\frac{\eta^2}{2} + \ln Var\,[Y_t \mid B_t = a]$ in the conditional case [5]
Joint Distribution Approximation	B_t is normally distributed with mean 0 and variance t. $f_{B_t,Y_t}(a,y) = f_{Y_t \mid B_t=a}(y)\, f_{B_t}(a)$

From here on, set $y_t = |B_t|^2$ to obtain the quadratic Brownian motion.

Moment Matching

Density Approximation: Time integral of Quadratic Brownian Motion (Unconditional)

First two Moments for $\int_0^t \lvert B_s\rvert^2 ds$	$E\left[\int_0^t \lvert B_s\rvert^2 ds\right] = \frac{t^2}{2}$ $Var\left[\int_0^t \lvert B_s\rvert^2 ds\right] = \frac{t^4}{3}$
Gamma Approximation	$f_{\int_0^t \lvert B_s\rvert^2 ds} = \frac{e^{-y/\alpha}}{\alpha} \frac{(\frac{y}{\alpha})^{\beta-1}}{\Gamma(\beta)}$ $\alpha = \frac{2t^2}{3}, \beta = \frac{3}{4}$
Log-normal Approximation	$f_{\int_0^t \lvert B_s\rvert^2 ds} = \frac{1}{y\eta\sqrt{2\pi t}} e^{-(-\mu+\ln y)^2/(2t\eta^2)}$ $\eta = \sqrt{\ln\left(\frac{7}{3}\right)}, \quad \mu = -\frac{1}{2}\ln\left(\frac{7}{3}\right) + \ln\frac{t^4}{3}$



Moment Matching

Density Approximation: Time integral of Quadratic Brownian Bridge

First two Moments for $\int_0^t |B_s|^2 ds \mid B_t = a$

$$E\left[\int_0^t |B_s|^2 ds \mid B_t = a\right] = \frac{a^2 t}{3} + \frac{t^2}{6}$$

$$Var\left[\int_0^t |B_s|^2 ds \mid B_t = a\right] = \frac{1}{3}(4t^3a^2 + t^4)$$ (by Laplace transform)

Gamma Approximation (conditional)

$$f_{\int_0^t |B_s|^2 ds \mid B_t = a} = \frac{e^{-y/\alpha}}{\alpha} \frac{(\frac{y}{\alpha})^{\beta-1}}{\Gamma(\beta)}$$

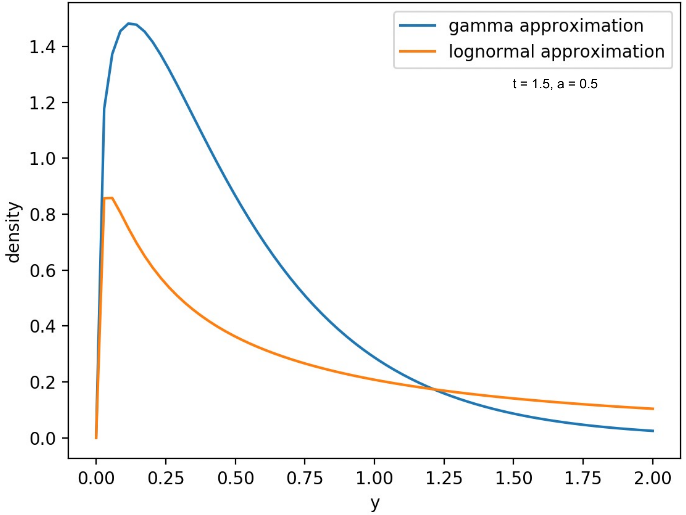
$$\alpha = \frac{2(4t^3a^2+t^4)}{15\,(2a^2t+t^2)},\quad \beta = \frac{5}{4} + \frac{5}{4}\left(\frac{4a^4t^2}{4t^3a^2+t^4}\right)$$

Log-normal Approximation (conditional)

$$f_{\int_0^t |B_s|^2 ds \mid B_t = a} = \frac{1}{y\eta\sqrt{2\pi t}} e^{-(-\mu+\ln y)^2/(2t\eta^2)}$$

$$\eta = \sqrt{\ln\left(1 + \frac{4}{5}\left(\frac{4ta^2+t^2}{4a^4+t^2+4a^2t}\right)\right)},$$

$$\mu = -\frac{1}{2}\ln\left(1 + \frac{4}{5}\left(\frac{4ta^2+t^2}{4a^4+t^2+4a^2t}\right)\right) + \ln\left(\frac{1}{45}(4t^3a^2+t^4)\right)$$



Moment Matching

Density Approximation: Time integral of Quadratic Brownian Bridge

First two Moments for $\int_0^t |B_s|^2 ds \mid B_t = a$

$$E\left[\int_0^t |B_s|^2 ds \mid B_t = a\right] = \frac{a^2 t}{3} + \frac{t^2}{6}$$

$$Var\left[\int_0^t |B_s|^2 ds \mid B_t = a\right] = \frac{1}{3}(4t^3a^2 + t^4)$$ (by Laplace transform)

Gamma Approximation (joint density)

$$f_{B_t, \int_0^t |B_s|^2 ds}(a, y) = \frac{e^{-a^2/2t}}{\sqrt{2\pi t}} \frac{e^{-y/\alpha}}{\alpha} \frac{(\frac{y}{\alpha})^{\beta-1}}{\Gamma(\beta)}$$

$$\alpha = \frac{2(4t^3a^2+t^4)}{15\,(2a^2t+t^2)}, \quad \beta = \frac{5}{4} + \frac{5}{4}\left(\frac{4a^4t^2}{4t^3a^2+t^4}\right)$$

Log-normal Approximation (joint density)

$$f_{B_t, \int_0^t |B_s|^2 ds}(a, y) = \frac{e^{-a^2/2t}}{2\pi t\, y\eta} e^{-(-\mu + \ln y)^2/(2t\eta^2)}$$

$$\eta = \sqrt{\ln\left(1 + \frac{4}{5}\left(\frac{4ta^2 + t^2}{4a^4 + t^2 + 4a^2 t}\right)\right)},$$

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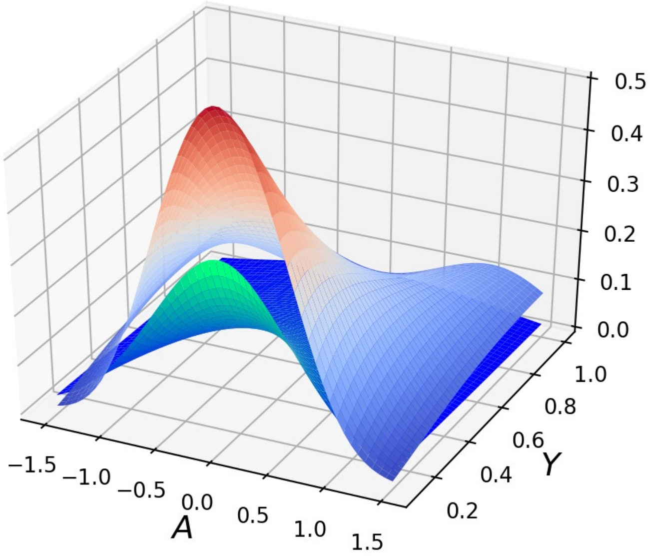


Blue green: Log-normal approximation
Cool warm: Gamma approximation
t = 1.5

PQL Diffusion Density

Franchi [3] uses the planar quadratic Langevin (PQL) diffusion $X_t = \left(B_t, \int_0^t B_s ds\right)$.

Franchi's Results: Theorem 2.1

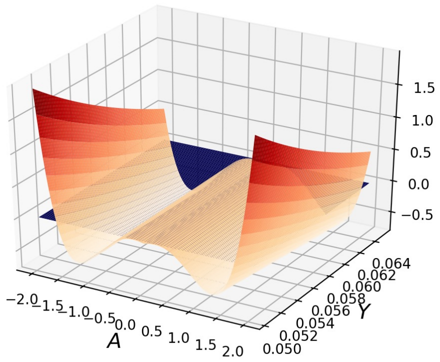
For small time $\epsilon \to 0$, the joint density p_ϵ of $\left(B_\epsilon, \int_0^\epsilon B_s^2\right)$ is found.

- Hormander's Criterion [3] : If an SDE satisfies the criterion, its solution admits a smooth density.
- From Franchi, we take it that the PQL satisfies Hormander's criterion and thus admits a smooth density.

The density [3] is given by

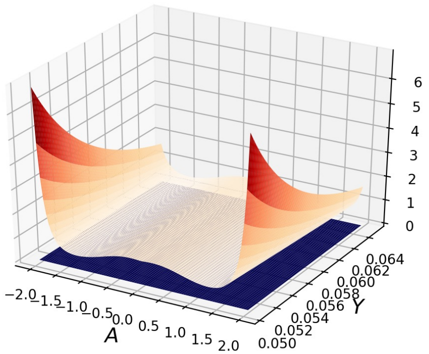
$$p_\epsilon\big(0;(a,y)\big) = \frac{1+O(\sqrt{\epsilon})}{\sqrt{8y\epsilon^3}} \exp \underbrace{\left[-\frac{\pi^2 y}{2\epsilon^2} + \frac{\pi\sqrt{2a^2y}}{\epsilon^{\frac{3}{2}}} - \frac{3a^2}{4\epsilon} - \frac{(4\pi^2+3)|a^2|}{24\pi\sqrt{2y\epsilon}} + \frac{(2\pi^2+3)a^4}{48\pi^2 y}\right]}_{z(a,y)}$$

Structure of PQL Diffusion Density



Orange Red: Polynomial $z(a,y)$
ϵ = 1.5

- $z(a,y)$ is a quartic polynomial
- Intersects plane $Z = 0$ 4 times: distinct roots
- Approximate roots at planes: $\mathrm{A} = -1.83,\ A = 1.83$



Orange Red: $\exp\left(z(a,y)\right)$
ϵ = 1.5

- $\exp(z(a,y))$ is non-negative, lies above $Z = 0$
- The rate at which $a \to -\infty, a \to \infty$ increases exponentially.
- Singularity like structures begin at Roots $A = -1.83, 1.83$

PQL Diffusion Density

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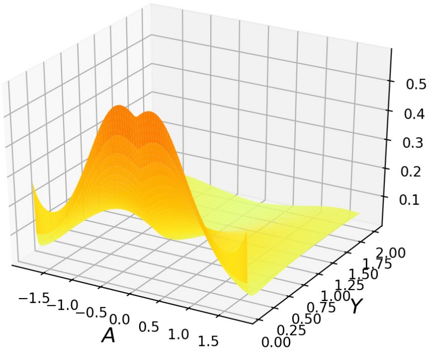
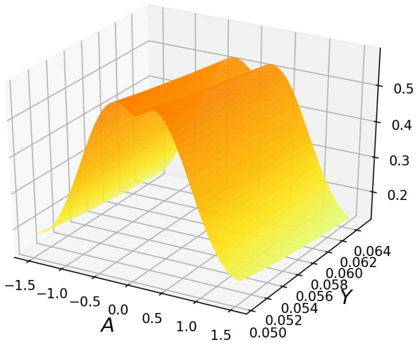
$$p_\epsilon(0;(a,y)) = \frac{1+O(\sqrt{\epsilon})}{\sqrt{8y\epsilon^3}} \exp \underbrace{\left[-\frac{\pi^2 y}{2\epsilon^2} + \frac{\pi\sqrt{2a^2y}}{\epsilon^{\frac{3}{2}}} - \frac{3a^2}{4\epsilon} - \frac{(4\pi^2+3)|a^2|}{24\pi\sqrt{2y\epsilon}} + \frac{(2\pi^2+3)a^4}{48\pi^2 y}\right]}_{z(a,y)}$$

Structure of PQL Diffusion Density



- Multiply $\exp(z(a,y))$ by first term.
- As $y \to \infty, \frac{1+O(\sqrt{\epsilon})}{\sqrt{8y\epsilon^3}} \to 0$.
- Singularity peaks begin from roots.

Yellow: PQL diffusion density p_ϵ
ϵ = 1.5



- Scaled down graph
- Focus on small range of y, non singularity range of a.
- Cross section is similar to a normal bell curve

Orange Red: $\exp(z(a,y))$
ϵ = 1.5

Monte Carlo Simulation

General Idea

Algorithms that use repeated random sampling to generate numerical results for random processes.

- Values of a quadratic Brownian motion are calculated for a given time.
- Run over multiple trials
- Results are aggregated into histograms.
- **This density is the closest to the true density of the time integral.**
- Very time expensive

Brownian Motion Algorithm

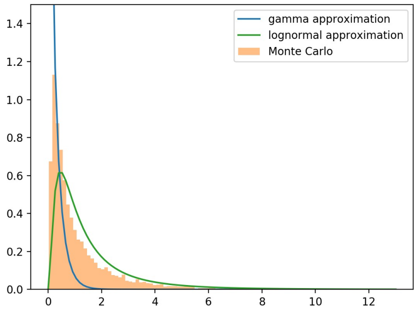
```

t = 1.5
n = 100
trials = 60000
points = 100
p1 = np.zeros([trials])
A = np.zeros([trials])

alpha = 2*t**2/3
beta = 3/4
sigma = math.sqrt(math.log(7/3))
mu = -0.5*(math.log(7/3))**0.5 + math.log(t**4/3)

def brownsum(t,n):
    Bs = np.zeros(n)
    ds = t/n
    for i in range(1,n):
        sign = np.random.choice([-1,1])
        Bs[i] = Bs[i-1] + sign*np.sqrt(ds)
    p1 = np.dot(Bs,Bs.T)*ds
    return Bs, p1

start = time.time()

for i in range(trials):
    p1[i] = brownsum(t,n)[1]

end = time.time()
```

For our Monte Carlo Simulation:

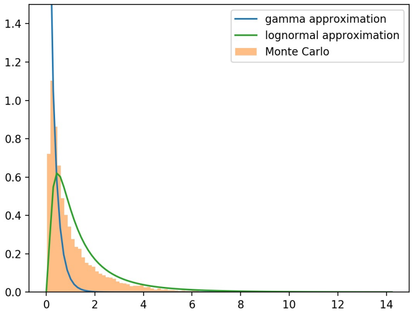
- Run on a 2020 MacBook Pro with M1 Chip
- Bins = 100
- Granularity = 100
- Number of trials: between 10000 – 60000
- Higher number of trials, more accurate results, more time expensive

1D Monte Carlo Simulation

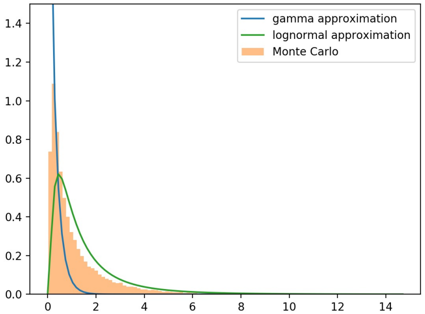
Time integral of Quadratic Brownian Motion: Monte Carlo for t = 1.5



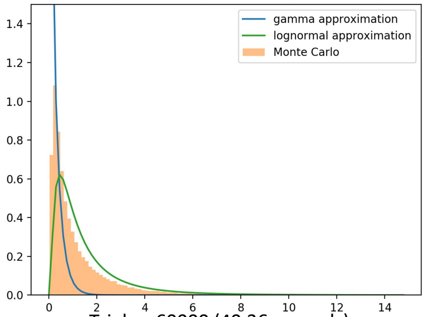
Trials = 10000 (6.9 seconds)



Trials = 20000 (13.68 seconds)



Trials = 40000 (27.29 seconds)



Trials = 60000 (40.36 seconds)

Kolmogorov Smirnov Test
Nonparametric test to compare samples and reference distribution

trials	**10000**	**20000**	**40000**	**60000**
Gamma KS Test Statistic (p-value)	0.5042 (0.0)	0.5022 (0.0)	0.5047 (0.0)	0.5044 (0.0)
Log-normal KS Test Statistic (p-value)	0.2061 (0.0)	0.1986 (0.0)	0.2056 (0.0)	0.2078 (0.0)

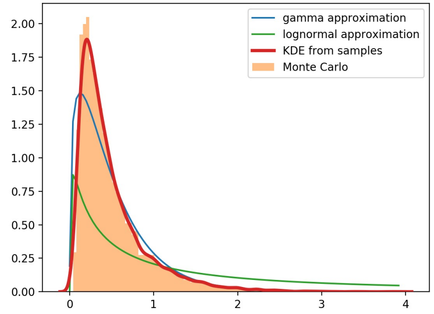
- As number of trials increase, the fit for both gets slightly better
- $H0$: The distribution of the samples is the same as the reference distribution
- P values are zero, thus the true density from Monte Carlo is not the same as the gamma or log-normal distribution.

Gamma and Log-normal Approximation are **not a good enough fit** for the true density of $\int_0^{1.5} |B_s|^2 ds$

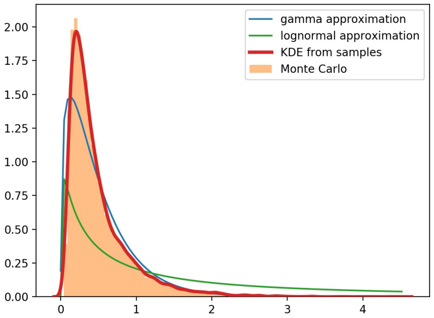
1D Monte Carlo Simulation

Time integral of Quadratic Brownian Bridge: 1D Monte Carlo for t = 1.5, a = 0.5

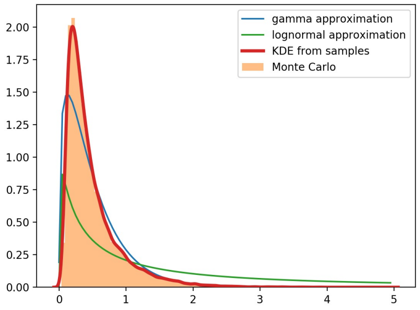
Kernel Density Estimator (KDE): non-parametric estimate of probability density by shape smoothing



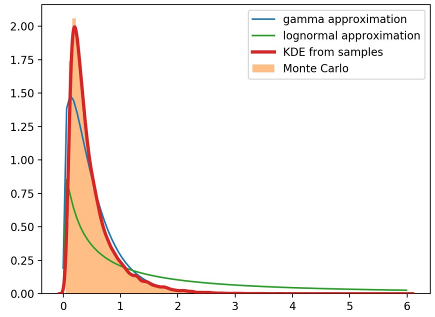
Trials = 10000 (8.97 seconds)



Trials = 20000 (17.22 seconds)



Trials = 40000 (34.79 seconds)



Trials = 60000 (54.12 seconds)

Kolmogorov Smirnov Test

Nonparametric test to compare samples and reference distribution

trials	10000	20000	40000	60000
Gamma KS Test Statistic (p-value)	0.2090 (0.0)	0.2090 (0.0)	0.2090 (0.0)	0.2090 (0.0)
Log-normal KS Test Statistic (p-value)	0.2715 (0.0)	0.2715 (0.0)	0.2715 (0.0)	0.2715 (0.0)

Since P values are zero, true density from Monte Carlo is not the same as the gamma or log-normal distribution.

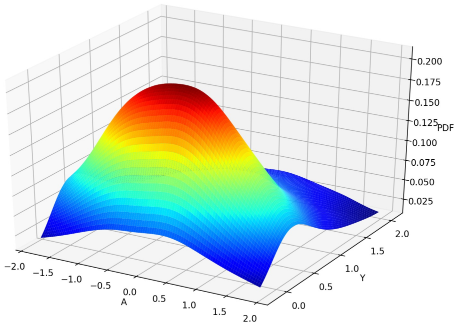
Gamma and Log-normal Approximation are **not a good enough fit** for the true density of $\int_0^{1.5} |B_s|^2 ds \mid B_t = 0.5$

2D Monte Carlo Simulation

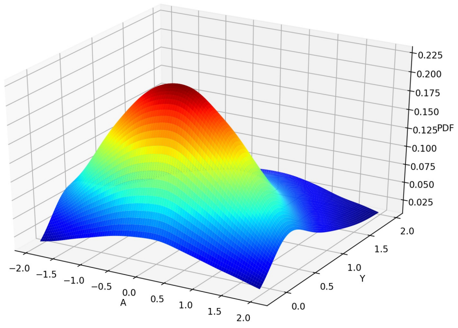
Time integral of Quadratic Brownian Bridge: 2D Monte Carlo for t = 1.5

General Idea

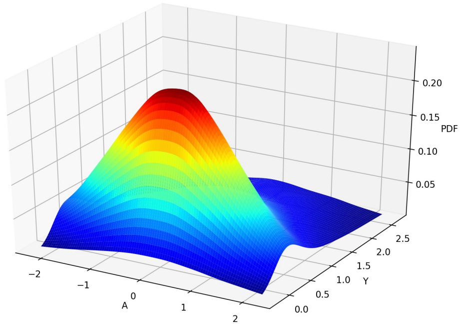
- Run a Monte Carlo Simulation over a 2D sample: $\left(a = B_{1.5}\,, y = \int_0^{1.5} |B_s|^2 ds\right)$
- Samples are aggregated into 2 dimensional 'square' bins
- KDE is applied to smooth out the histogram
- **This gives us the true joint density of $\boldsymbol{B_t}$, $\boldsymbol{\int_0^t |B_s|^2 ds}$**



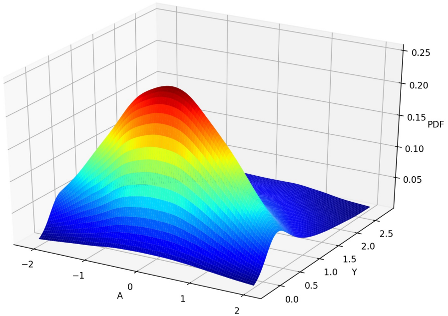
Rainbow: 2D Monte Carlo KDE
Trials = 10000 (14.53 seconds)



Rainbow: 2D Monte Carlo KDE
Trials = 20000 (28.98 seconds)



Rainbow: 2D Monte Carlo KDE
Trials = 40000 (57.68 seconds)



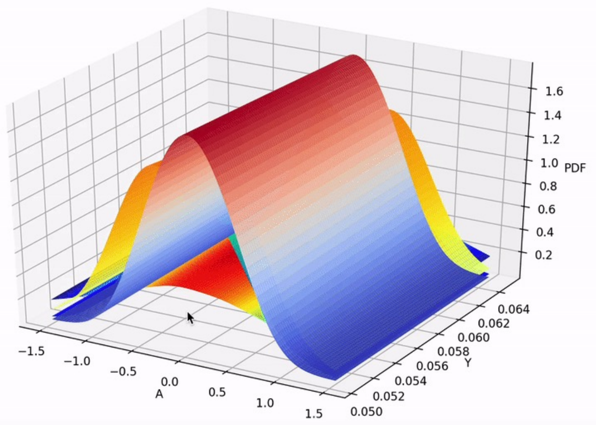
Rainbow: 2D Monte Carlo KDE
Trials = 60000 (86.16 seconds)

No Clear method to test whether this true joint density is drawn from the gamma or lognormal joint density approximation.

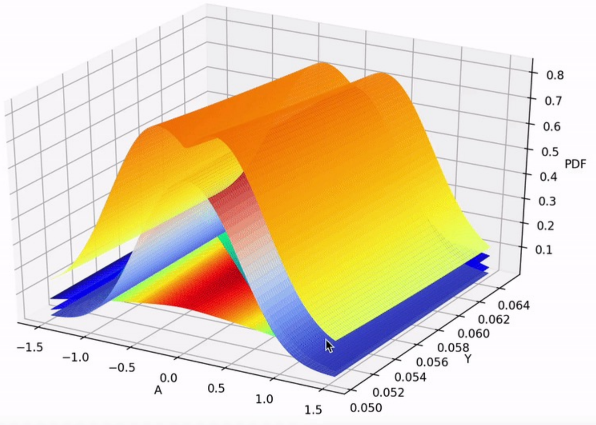
We graphically compare all joint density plots

Graphical Comparisons (Large values of t)

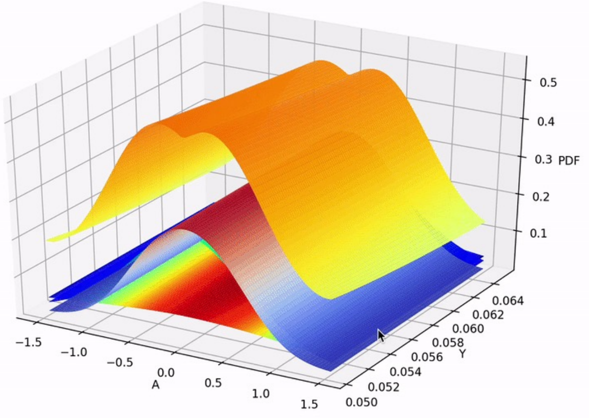
Rainbow: 2D Monte Carlo KDE
Yellow: PQL Diffusion Density
Cool warm: Gamma Joint Density Approximation
Green: Lognormal Joint Density Approximation



t = 1.0



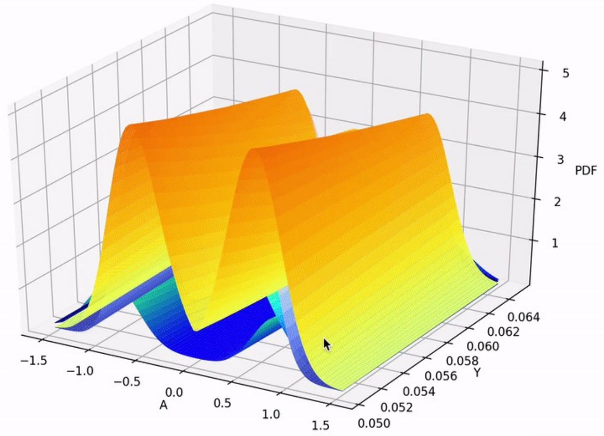
t = 1.5



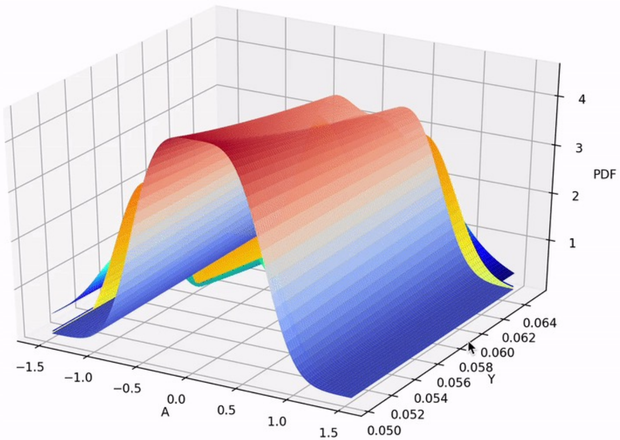
t = 2.0

Graphical Comparisons (Small values of t)

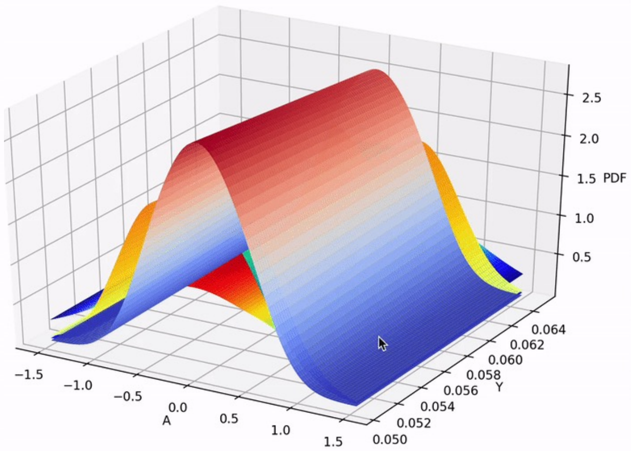
Rainbow: 2D Monte Carlo KDE
Yellow: PQL Diffusion Density
Cool warm: Gamma Joint Density Approximation
Green: Lognormal Joint Density Approximation



t = 0.3



t = 0.5



t = 0.8

Section IV:
Asian Options

Moment Matching, Diffusion Process for PDEs, Numerical Results

Moment matching for Asian call options

CIR Model

The Cox-Ingersoll-Ross (CIR) Model with interest rate r and volatility of the underlying asset σ: $dS_t = rS_t dt + \sigma\sqrt{S_t}dB_t$

Let S_t be the model for the underlying asset and let $Y_t = \int_0^t S_u du$. From [5] we have the following moments:

Moments of S_t

$E[S_T] = S_0 e^{rT}, Var[S_T] = -S_0 \frac{\sigma^2}{r}(e^{rT} - e^{2rT})$

Moments of Y_t

$E[Y_T] = \frac{S_0(e^{rT}-1)}{r}, Var[Y_T] = -\sigma^2 S_0 \frac{\sigma^2}{r}\frac{1+2rTe^{rT}-e^{2rT}}{r^3}$

Asian Option price at time $t = 0$: $\boldsymbol{e^{-rT}E^*}\left[\left(\frac{Y_T}{T} - K\right)^+\right]$

In both the gamma and Log-normal case, the density approximations by moment matching can be used to compute the Expectation in the Asian option price.

Gamma Approximation

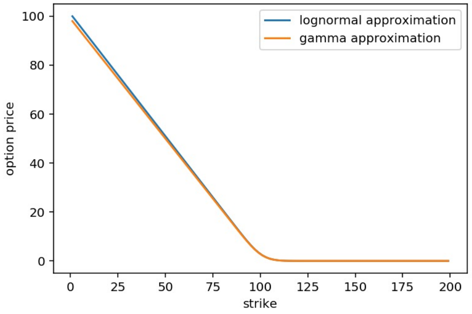
$$c = \frac{e^{-rT}}{T}\frac{1}{\Gamma(\beta)}\left(\int_{KT/\alpha}^{\infty} e^{-x}x^{\beta}dx - KT\alpha\int_{KT/\alpha}^{\infty} e^{-x}x^{\beta-1}dx\right)$$

Log-normal Approximation

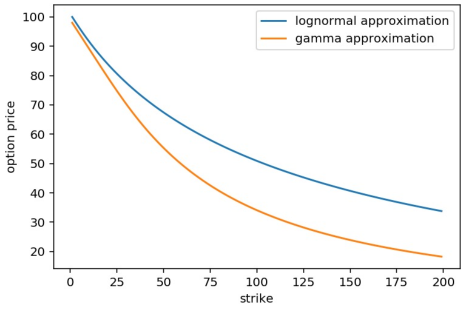
$$c = \frac{e^{-rT}}{T}\left(e^{-\mu\eta^2 T/2+\eta^2 T/2}\,\phi(d_1) - KT\phi(d_2)\right),$$

$d_1 = \frac{\ln(\frac{E[Y_t]}{KT})}{\eta\sqrt{T}}$ and $d_2 = d_1 - \eta\sqrt{T}$

Constant Parameters for Numerical Analysis: $S_0 = 100, K = 95, r = 0.02$



Long Term ($T = 1$), Lower Volatility ($\sigma = 0.1$)



Long Term ($T = 1$), Higher Volatility ($\sigma = 2$)

Partial Differential Equations for Asian options

Asian option price at time t: $S_t g(t, Z_t)$

Roger & Shi [6] (1995)

$$\frac{\partial g}{\partial t} + \left(\frac{1}{T} - rZ_t\right)\frac{\partial g}{\partial z} + \frac{1}{2}Z_t^2\sigma^2\frac{\partial^2 g}{\partial z^2} = 0$$

Vecer [7] (2001)

$$\frac{\partial g}{\partial t} + r(q_t - Z_t)\frac{\partial g}{\partial z} + \frac{1}{2}(q_t - z)^2\sigma^2\frac{\partial^2 g}{\partial z^2} = 0$$

Where stock position $q_t = 1 - \frac{t}{T}$

Dubois Lelievre [13] (2004)

$$\frac{\partial g}{\partial t} + r(q_t - Z_t)\frac{\partial g}{\partial z} + \frac{1}{2}(q_t - z)^2\sigma^2\frac{\partial^2 g}{\partial z^2} = 0$$

Where stock position $q_t = \frac{t}{T}$

Roger Lord [1] (2006)

$$\frac{\partial g}{\partial t} + r(1 - Z_t)\frac{\partial g}{\partial z} + \frac{1}{2}(1 - z)^2\sigma^2\frac{\partial^2 g}{\partial z^2} = 0$$

Using a Diffusion Process to generalise all PDEs

Key Points in Brown's Paper [8]

Let $\theta_t = a(t) + b(t)\frac{\alpha_t - Ke^{-r(T-t)}}{\tilde{S}_t}$

- $a(t), b(t)$ are arbitrary functions
- α_t is the average asset process as defined by Brown
- $\tilde{S}_t$ is the discounted asset price process
- $p(t) = \frac{1 - e^{-r(T-t)}}{rT}$

Main Results

- θ_t is a diffusion process
- θ_t satisfies the SDE:

$$\underbrace{\left(a'(t) + \frac{b'(t)\,(\theta_t - a(t))}{b(t)}\right)}_{\mu(t,\theta_t)} dt + \underbrace{\sigma(a(t) + p(t)b(t) - \theta_t)}_{\sigma(t,\theta_t)} dB_t$$

- Then the value of an Asian option is given by $S_t\phi(t, \theta_t)$ where [14]:

$$\frac{\partial \phi}{\partial t} + \mu(t, \theta_t)\frac{\partial \phi}{\partial \theta} + \frac{1}{2}\sigma(t, \theta_t)^2\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Partial Differential Equations for Asian options

Asian option price at time t: $S_t g(t, Z_t)$

Using a Diffusion Process to generalise all PDEs

Roger & Shi [6] (1995)

$$\frac{\partial g}{\partial t} + \left(\frac{1}{T} - rZ_t\right)\frac{\partial g}{\partial z} + \frac{1}{2}Z_t^2\sigma^2\frac{\partial^2 g}{\partial z^2} = 0$$

$$\frac{\partial \phi}{\partial t} + \left(r\theta_t - \frac{1}{T}\right)\frac{\partial \phi}{\partial \theta} + \frac{1}{2}\theta_t^2\sigma^2\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$a(t) = p(t)e^{r(T-t)}$ and $b(t) = -e^{r(T-t)}$

Vecer [7] (2001)

$$\frac{\partial g}{\partial t} + r(q_t - Z_t)\frac{\partial g}{\partial z} + \frac{1}{2}(q_t - z)^2\sigma^2\frac{\partial^2 g}{\partial z^2} = 0$$

Where stock position $q_t = 1 - \frac{t}{T}$

$$\frac{\partial \phi}{\partial t} + \left(1 - \frac{t}{T} - \theta_t\right)\frac{\partial \phi}{\partial \theta} + \frac{1}{2}\left(1 - \frac{t}{T} - \theta_t\right)^2\sigma^2\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$a(t) = 1 - \frac{t}{T} - p(t)b(t)$ and $b(t) = e^{r(T-t)}$

Dubois Lelievre [13] (2004)

$$\frac{\partial g}{\partial t} + r(q_t - Z_t)\frac{\partial g}{\partial z} + \frac{1}{2}(q_t - z)^2\sigma^2\frac{\partial^2 g}{\partial z^2} = 0$$

Where stock position $q_t = \frac{t}{T}$

$$\frac{\partial \phi}{\partial t} + \left(\frac{t}{T} - \theta_t\right)\frac{\partial \phi}{\partial \theta} + \frac{1}{2}\left(\frac{t}{T} - \theta_t\right)^2\sigma^2\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$a(t) = \frac{t}{T} - p(t)b(t)$ and $b(t) = e^{r(T-t)}$

Roger Lord [1] (2006)

$$\frac{\partial g}{\partial t} + r(1 - Z_t)\frac{\partial g}{\partial z} + \frac{1}{2}(1 - z)^2\sigma^2\frac{\partial^2 g}{\partial z^2} = 0$$

$$\frac{\partial \phi}{\partial t} + (1 - \theta_t)\frac{\partial \phi}{\partial \theta} + \frac{1}{2}(1 - \theta_t)^2\sigma^2\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

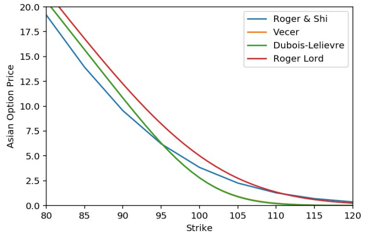
$a(t) = 1 - p(t)b(t)$ and $b(t) = e^{r(T-t)}$

Finite Difference Methods (FDM)

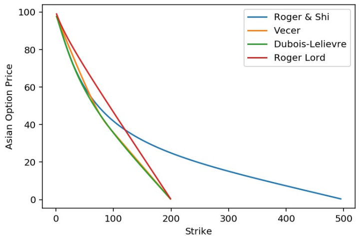
- Discretize the differential
- Solve PDE over a mesh grid
- Use Sparse Matrix methods

Constant Parameters for Numerical Analysis: $S_0 = 100, K = 95, r = 0.02$

Implicit Scheme



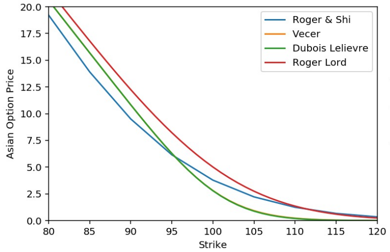
Long Term $(T = 1)$, Lower Volatility $(\sigma = 0.1)$



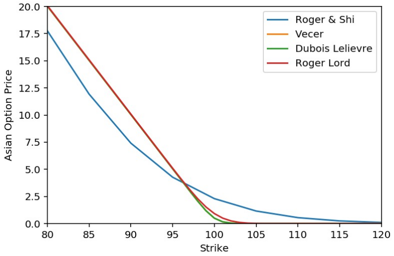
Long Term $(T = 1)$, Higher Volatility $(\sigma = 1)$

- Gives stable results across Maturities and volatilities
- Vecer & Dubois-Lelievre perform similarly for lower volatility
- They differ slightly for higher volatility and lower strike
- Roger Lord gives highest price

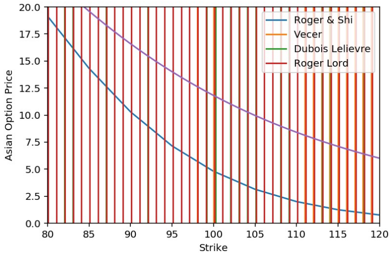
Explicit Scheme



Long Term $(T = 1)$,
Lower Volatility $(\sigma = 0.1)$



Short Term $(T = 0.05)$,
Lower Volatility $(\sigma = 0.1)$



Short Term $(T = 0.05)$,
Higher Volatility $(\sigma = 1)$

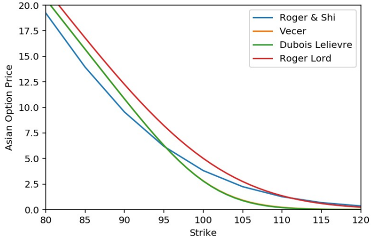
- Explosive Behavior
- For Higher Maturities and Volatilities
- Roger & Shi is the only PDE that performs well for all cases
- Roger Lord Oscillates for higher volatility even in short term maturity

Finite Difference Methods (FDM)

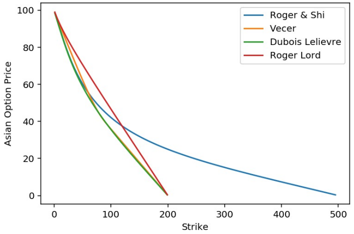
- Discretize the differential
- Solve PDE over a mesh grid
- Use Sparse Matrix methods

Constant Parameters for Numerical Analysis: $S_0 = 100, K = 95, r = 0.02$

Crank Nicolson Scheme



Long Term $(T = 1)$, Lower Volatility $(\sigma = 0.1)$



Long Term $(T = 1)$, Higher Volatility $(\sigma = 1)$

- Average of Implicit & Explicit Scheme
- Gives stable results across Maturities and volatilities
- Similar (Better) Results as Implicit Scheme.
- Circumvents Explosive Behavior of Explicit Scheme
- Roger Lord gives highest price

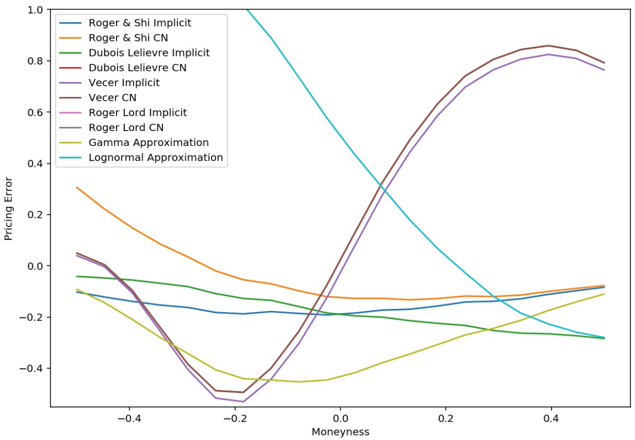
Summary of Numerical Results

Moneyness

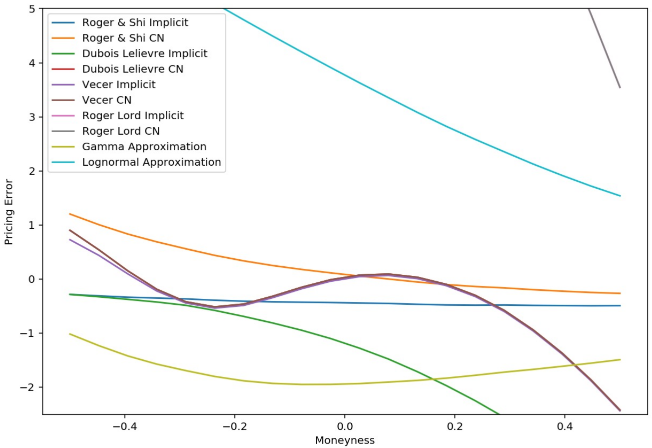
Intrinsic Value of the option; value of option if exercised immediately [1]

For Asian options, Moneyness at time t: $\left(\frac{1}{t}\int_0^t S_u du - K\right)^+$

Pricing Error (Relative to Monte Carlo Asian Option Price)



Pricing Error for $(T = 1)$



Pricing Error $(T = 5)$

- Vecer's PDE shows most volatile error
- Roger Lord's PDE shows the highest error (out of graph bounds)
- Roger & Shi Performs well
- Density Approximation error approaches ≈ 0
- Dubois Lelievre is stable for short term maturities.
- Overall, crank Nicolson performs better for PDEs

Section V: Conclusion

Conclusion

Density of Time Integrals

3 Methods to estimate the density of $\int_0^t |B_s|^2 ds,\ \int_0^t |B_s|^2 ds \mid B_t = a$

- Moment Matching: Gamma Approximation
- Moment Matching: Log-normal Approximation
- Monte Carlo Simulation

Joint Density : $\left(B_t, \int_0^t B_s ds\right)$

- Moment Matching
- PQL Diffusion Density
- Monte Carlo Kernel Density

Our Main Results:

- Density approximations perform poorly for 1D Monte Carlo simulations
- 2D Monte Carlo simulations over 2D samples gave true joint density
- PQL Diffusion Density performs well for not too small and not too high times

Asian option pricing

2 Approaches to approximate Asian option prices

- Moment Matching
- Partial Differential Equations

Our Main Results:

- Density Approximations perform well. Log-normal error increases for higher volatility, higher strike.
- A Diffusion Process was used to implement a generalized PDE framework.
- Explicit Scheme displays explosive behavior for long term maturity.
- Crank Nicolson Scheme performs the best
- Roger & Shi’s PDE is the most stable
- Roger Lord’s PDE has the highest pricing error

Section VI:
Areas for Further Research

Further Research

Density of Time Integrals

Moment Matching:

- Find appropriate statistical tests for 2D joint densities

Monte Carlo:

- Control variables and variance reduction to improve results

PQL :

- PQL Diffusion Density may be evaluated for larger times

Asian option pricing

Moment Matching:

Conditional Moment Matching might perform better due to added information.

Partial Differential Equations:

- Unusually high pricing error for Roger Lord's PDE in Brown's framework
- Laplace transforms can be used to solve PDEs
- Faster algorithms to solve Finite Difference Schemes
- Comparisons with bounds
- Monte Carlo price for error benchmark can be improved by variance reduction techniques

Thank you!

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