Week 1

Lecture 2
Introduction to Numbers
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Week 1

Overview Lecture

Subject Overview

Lecture 1

Introduction to cryptography.

Lecture 2

Introduction to Numbers

Quizz 1

Workshops start from Week 2

Lecture 2

2.1 Fundamentals

- Sets, Source of finite sets and functions.
- Basic facts and properties of numbers: Divisibiliy.

A set is a collection of objects. The objects are referred to as elements of the set.
Example:
$X = \{a, b, c\}$ is a set with three elements a, b and c.

Name	Set	Symbol Used
Natural Numbers	$\{0, 1, 2, 3, \cdots\}$	N
Integers	$\{\cdots, -2, -1, 0, +1, +2, \cdots\}$	Z
Positive Integers	$\{1, 2, 3, \cdots\}$	$Z+$
Negative Integers	$\{\cdots, -2, -1\}$	$Z-$

Table: Examples of Sets

Main Source of Finite Sets

The set of integers is a major source of finite sets.
For example, for a positive integer n, the set of numbers from 0 to $n-1$ form a finite set of n entities denoted by Z_n.

$$Z_n := \{0, 1, 2, \cdots n-1\}$$

The properties of such finite sets play a vital role in coding theory.

Functions

A function is defined by a triplet $< X, Y, f >$, where

- X: a set called domain;
- Y: a set called range or codomain and
- f: a rule which assigns to each element in X precisely one element in Y. It is denoted by $f : X \to Y$

Example: Encoding: E.

$$[0, 1]^K \to [0, 1]^N,$$

Where the message domain is all binary vectors of length K and the codomain is a space of N bit numbers.

Example from Cryptographic Functions

- Alphabet, $\mathcal{A}$: A finite set. For example, $\mathcal{A} = \{0, 1\}$, the binary alphabet.
- Message Space, $\mathcal{M}$: Consists of strings of symbols from an alphabet.
- Cipher Text Space, $\mathcal{C}$: Consists of strings of symbols from an alphabet which may differ form the alphabet of $\mathcal{M}$.
- Key space $\mathcal{K}$: A set of key space and an element of $\mathcal{K}$ is key.
- Encryption function, E_e:
$$C = E_e(M)$$
- Decryption function, D_d:
$$M = D_d(C)$$

2.2 Division and Remainders

- Divisibility.
- Division with Remainder.
- Finding Remainder and Modulo Operation
- Division Theorem

Divisibility

An integer "a" is said to be **divisible** by a positive integer "b", and this is written as $b|a$, if $a = b\ c$ for a third integer "c" and $c \neq 0$. (The above statement is also same as "b" divides "a".)
In the following statements, a, b, c are integers.

1. $a|a$,
2. $a|b$ and $b|c$ implies $a|c$,
3. $a|b$ and $b|a$ implies $a = \pm b$,
4. $a|b$ and $a|c$ implies $a|(b\ x + c\ y)$ for all integers x and y,
5. $a|b$ implies $ca\ |cb$, for any c.

Divisibility, cont

Proof of (4).

Since $a|b$, we have $b = ma$ for some integer m. Similarly since $a|c$, we can write $c = na$ for some integer n. Now consider $b\ x + c\ y = m\ a\ x + n\ a\ y = a(m\ x + n\ y)$. Therefore $a|b\ x + c\ y$. □

Division with Remainder

Let a, b be two integers, $a > b$

$$b \ \text{ does not divide } a \ ;$$

Then let c be the largest integer smaller than a and is multiple of b;

$$b|c,$$

where $c = q \ b < a$;
then

$$a = c + r = q \ b + r.$$

q is the quotient and r is the reminder called as **remainder modulo** b.

Finding Remainder and Modulo Operation

Let a be any integer b a positive integer which is not zero, then are unique integers q (quotient) and r (remainder) such that

$$a = qb + r, 0 \leq r < b.$$

The quotient q can be obtained by $q = \lfloor a/b \rfloor$, where $\lfloor x \rfloor$, represents the floor function which returns the largest integer less than or equal to x. The remainder r is written as

$$r = a \mod b.$$

Example: $12 \bmod 5 = 2$.
$-12 \bmod 5 = 3$.

Division Theorem

Theorem

Let a and b are integers and assume that b is positive. Then there exist integers q and r such that

$$a = qb + r, 0 \leq r < b.$$

Proof.

For fixed a and b, let X be the collection of integers of the form $a - xb$. Let r be the least non-negative integer in X, and let q be the corresponding integer, so that $a - q\ b\ =\ r$.
Claim: $0 \leq r < b$.
Note that this follows from the well-ordering principle.
Now we need to examine the uniqueness of q and r:

$\square$

Proof Cont.

Suppose they are not unique, then we have $q\ b + r\ =\ q'\ b + r'$.
WLG (Without loss of generality) : $r \leq r'$.
Then, $(q - q')\ b = (r' - r)$ and $r' - r \geq 0$.
If $(r' - r) \neq 0$, then necessarily $(q - q') > 0$
If so then

$$r' - r = (q - q')\ b \geq 1\ b$$

.
But $r' - r \leq r' < b$
So we have

$$b\ \leq\ r' - r\ < b$$

This is a contradiction to $r \neq r'$.
Therefore $r = r'$ and
subsequently, $q = q'$.

$\square$

2.3 Prime Numbers

- Prime and Composite Numbers
- Greatest Common Divisior(gcd)
- A useful theorem

Prime Numbers

Definition

A number is said to be **a prime number** *if $p > 1$ and p has no positive divisors except 1 and p.*

Definition

The numbers which are not prime numbers are referred as **composite numbers***.*

Fact

There are infinitely many prime numbers.

Can you prove this? There is a simple proof originally attributed to Euclid.

Prime Numbers

Fact

There are infinitely many prime numbers.

We know there are primes, eg $2, 3$, etc. Consider a set of first n primes: $\{p_1, p_2, \cdots, p_n\}$. We show how to construct a next bigger prime. Let $Q = 1 + p_1 \times p_2 \times \cdots p_n$. Clearly $Q > p_n$, the biggest prime in the set and none of them divides Q. If Q is a prime number, we are done with the proof. If not, there exist another prime q which divides Q. q cannot be one of the primes in the set and has to be a new prime greater than p_n. Now we are done with the proof.

Definition

If d divides two integers m and n, then d is called a common divisor. The greatest of common divisors of the integers is the GCD of m and n.

Definition

Numbers m and n are said to be relatively prime if the GCD of m and n is 1.

Example: $gcd(3, 5) = 1$
$gcd(2, 14) = 2$;

A useful theorem

Theorem

Let a, b, q, r be integers with such that $a = qb + r$. Then $gcd(a, b) = gcd(b, r)$.

Proof.

If a and b are identically zero, then $r = 0$ and the result is trivially true. Otherwise let $d = gcd(a, b)$. Since $d|a$ and $d|b$, we have $d|a - qb$ (the divisibility property (4)). So, $d|r$ and d is a common divisor of both b and r. Now let c be a divisor of b and r. i.e $c|b$ and $c|r$. Then again from the divisibility property (4), $c|qb + r$, so $c|a$. This means that c is a common divisor of a and b. So, $c \leq d$. This implies that $d = gcd(b, r)$.
Thus, we have proved $gcd(a, b) = gcd(b, r)$. $\square$

2.4 GCD Computation

- Key Fact for GCD computation
- Euclid's algorithm
- GCD Illustration through Manual Computations
- Modular Arithmetic
- Modular Multiplicative Inverse
- Fundamental Theorem of Arithmetic

Key Fact for GCD computation

There is an algorithm to compute gcd which is considered as one of the earliest known algorithms, familar in many cultures. It is known as Euclidean algorithm in modern textbooks.

Fact

Let $a > b > 0$. Then

$$gcd(a, b) = gcd(b, (a \ mod \ b)).$$

From the basic fact remaindering, we have $a = qb + r$, where $r = a \ mod \ b$ is the remainder. It is clear that a common divisor of a and b is divisor of r too and the result is obvious.

Euclid's algorithm

```
Euclid(a,b);
X:=a; y:=b;
while y > 0 do {
r = x mod y;
x:=y;
y:=r; }
return(x);
```

Euclid's algorithm

$$
\begin{array}{rcll}
 & & & gcd(33,21) \\
33 & = & 1 \times 21 + 12 & gcd(21,12) \\
21 & = & 1 \times 12 + 9 & gcd(12,9) \\
12 & = & 1 \times 9 + 3 & gcd(9,3) \\
9 & = & 3 \times 3 + 0 & gcd(3,0)
\end{array}
$$

Table: Determination of $gcd(33,21)$

GCD Illustration through Manual Computations

Consider $gcd(33, 21)$:

$$
\begin{array}{rcccc}
33 & = & 1 \times 21 + 12 & gcd(21, 12) & (A) \\
21 & = & 1 \times 12 + 9 & gcd(12, 9) & (B) \\
12 & = & 1 \times 9 + 3 & gcd(9, 3) & (C) \\
9 & = & 3 \times 3 + 0 & gcd(3, 0) &
\end{array}
$$

Table: Determine $gcd(33, 21)$

$$
\begin{array}{cc}
3 = 12 - 1 \times 9 & From(C) \\
3 = 12 - 1 \times (21 - 1 \times 12) & From(B) \\
3 = 2 \times 12 - 1 \times 21 & \\
3 = 2 \times (33 - 1 \times 21) - 1 \times 21 & From(A) \\
3 = 2 \times 33 + (-3) \times 21 & Simplification
\end{array}
$$

Note that the gcd (in this case 3) can be written as a function of its inputs (33 and 21). This is an extended Euclidean algorithm helps in computing inverses! We will study this fact next week.

Let a and b be integers and let n be a positive integer.
We say "a" is congruent to "b", modulo n and write

$$a \equiv b \ (mod \quad n),$$

if a and b differ by a multiple of n; i.e ; if n is a factor of $|b - a|$.
Every integer is congruent mod n to exactly one of the integers in the set

$$Z_n = \{0, 1, 2, \cdots, n - 1\}.$$

We can define the following operations:

$$X \oplus_n y = (x + y) \ mod \ n.$$

$$X \otimes_n y = (xy) \ mod \ n$$

When the context is clear we use the above special addition and multiplication symbols interchangeably with their counterpart regular symbols.

Modular Multiplicative Inverse

Definition

Let $x \in Z_n$, if there is an integer y such that

$$X \otimes_n y = 1,$$

then we say y is the multiplicative inverse of x. It is denoted by $y = x^{-1}$ usually.

Example: let $n = 5$, 2 is inverse of 3 in Z_5. Or in other words 2 is inverse of 3 modulo 5.

Determining multiplicative inverse

Fact

For any integers a and b, there exist integers x and y such that

$$gcd[a, b] := ax + by.$$

You can determine x and y by modifying Euclid's algorithm for gcd(a,b). Thus we can say that we can find inverse of a modulo n provided $gcd(a, n) = 1$. gcd can also determined from the next result. Can you think how?

Fundamental Theorem of Arithmetic

Fact

Every natural number $n > 1$ has a unique prime factorization or prime power factorization.

$$n = \Pi_{i=1}^{\tau} p_i^{a_i},$$

where τ is a positive number.

Example:
$15 =?$
$32 =?$
$2^{607} - 1 =?$
$3937 =?$

Fundamental Theorem of Arithmetic

Fact

Every natural number $n > 1$ has a unique prime factorization or prime power factorization.

$$n = \Pi_{i=1}^{\tau} p_i^{a_i},$$

where τ is a positive number.

Example:

$15 = 5 * 3$

$32 = 2^5$

$2^{607} - 1 = 1\ (2^{607} - 1)$

$3937 = 127 * 31$

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Introduction to Numbers

2.1 **Fundamentals**

2.2 **Division and Remainders**

2.3 **Prime Numbers**

2.4 **GCD computation**

Quizz 1

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