DR. BABASAHEB AMBEDKAR TECHNOLOGICAL UNIVERSITY, LONERE

Winter Examination – 2022

Course: B. Tech.	**Branch :**	**Semester :III**
Subject Code & Name: Engineering Mathematics(BTBS301)		
Max Marks: 60	**Date: / /20**	**Duration: 3 Hr.**

Instructions to the Students:

1. *All the questions are compulsory.*
2. *The level of question/expected answer as per OBE or the Course Outcome (CO) on which the question is based is mentioned in ( ) in front of the question.*
3. *Use of non-programmable scientific calculators is allowed.*
4. *Assume suitable data wherever necessary and mention it clearly.*

		(Level/CO)	Marks
Q. 1	**Solve Any Two of the following.**		**12**
A)	Find the Laplace transform of $e^{-2t}\int_0^t \frac{cos2t}{t}\,dt.$	Understand	6
B)	Find the Laplace transform of the periodic function, $f(t) = \frac{t}{T}$ for $0 < t < T$, & $f(t+T) = f(t)$.	Understand	6
C)	By using Laplace transform, evaluate $\int_0^\infty e^{-2t}t^2 sin3t\, dt$	Evalution	6
Q.2	**Solve Any Two of the following.**		**12**
A)	By using convolution theorem, find inverse Laplace transform of $\frac{s}{(s^2+1)(s^2+4)}$	Application	6
B)	Find inverse Laplace transform of $\cot^{-1}(\frac{s+3}{2})$	Application	6
C)	Using Laplace Transform , solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = e^{-t}$, given $y(0) = 1$ & $y'(0) = 0$	Application	6
Q. 3	**Solve Any Two of the following.**		**12**
A)	Using the Fourier integral representation ,show that i) $\int_0^\infty \frac{\omega \sin x\omega}{1+\omega^2}d\omega = \frac{\pi}{2}e^{-x}\ (x > 0)$ ii) $\int_0^\infty \frac{\cos x\omega}{1+\omega^2}d\omega = \frac{\pi}{2}e^{-x}\ (x \geq 0)$	Understand	6

B)	Find the Fourier transform of the function $f(x) = \begin{cases} 1 - x^2 & \lvert x\rvert \le 1 \\ 0 & \lvert x\rvert > 1 \end{cases}$ Hence evaluate $\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$	Evalution	6
C)	Using Parseval's identity, show that $\int_0^{\infty} \frac{t^2}{(4+t^2)(9+t^2)} dt = \frac{\pi}{10}$	Application	6
Q.4	**Solve Any Two of the following.**		12
A)	Form the partial differential equation by eliminating arbitrary function from $f(x + y + z, x^2 + y^2 + z^2) = 0$	Understand	6
B)	Solve the partial differential equation $$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$$	Application	6
C)	Find the temperature in a bar of length 2 units whose ends are kept at zero temperature & lateral surface insulated if the initial temperature is $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$	Application	6
Q. 5	**Solve Any Two of the following.** https://www.batuonline.com		12
A)	Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is a harmonic function & hence determine the corresponding analytic function, $f(z) = u + iv$.	Understand	6
B)	Evaluate , by using Cauchy's integral formula: i) $\oint_C \frac{e^{-z}}{z+1} dz$, where C is the circle $\lvert z\rvert = 2$. ii) $\oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$, where C is the circle $\lvert z\rvert = 1$.	Evalution	6
C)	Evaluate $\int_C \frac{2z-1}{z(z+1)(z-3)} dz$,where C is the circle $\lvert z\rvert = 2$	Evalution	6
	*** End ***		

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Model Answer

Q.1 A]

$$L\{\cos 2t\} = \frac{s}{s^2+2^2}$$

$$L\left\{\frac{\cos 2t}{t}\right\} = \int_s^\infty \frac{s}{s^2+2^2}\,ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+2^2}\,ds$$

$$= \frac{1}{2}\left[\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\infty - \log(s^2+4)\right]$$

$$L\left\{\frac{\cos 2t}{t}\right\} = -\frac{1}{2}\log(s^2+4)$$

$$L\left\{\int_0^t \frac{\cos 2t}{t}\,dt\right\} = \frac{1}{s}\left[-\frac{1}{2}\log(s^2+4)\right]$$

$$L\left\{e^{-2t}\int_0^t \frac{\cos 2t}{t}\,dt\right\} = \frac{1}{s+2}\left[-\frac{1}{2}\log\left((s+2)^2+4\right)\right]$$

$$= \frac{-1}{2(s+2)}\log(s^2+2s+4+4)$$

$$= -\frac{1}{2(s+2)}\log(s^2+2s+8)$$

B) $f(t) = \frac{t}{T} \quad 0<t<T$

Here $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \left(\frac{t}{T}\right) dt$$

$$= \frac{1}{T(1-e^{-sT})} \int_0^T e^{-st}\, t\, dt$$

$$= \frac{1}{T} \frac{1}{1-e^{-sT}} \left[-\frac{T}{s} e^{-sT} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{1}{T} \frac{1}{1-e^{-sT}} \left[-\frac{T}{s} e^{-sT} + \frac{1}{s^2} (1 - e^{-sT}) \right]$$

$$= -\frac{1}{s} \left(\frac{e^{-sT}}{1-e^{-sT}} \right) + \frac{1}{s^2 T}$$

$$L\{f(t)\} = \frac{1}{s^2 T} - \frac{1}{s} \left(\frac{e^{-sT}}{1-e^{-sT}} \right)$$

c] $\int_0^\infty e^{-2t} t^2 \sin 3t \, dt$.

We have

$$\int_0^\infty e^{-st} f(t) \, dt = L\{f(t)\}$$

$$\int_0^\infty e^{-2t} (t^2 \sin 3t) \, dt = L\{t^2 \sin 3t\}$$

& $s = 2$.

Now,

$$L\{\sin 3t\} = \frac{3}{s^2+3^2}.$$

Next,

$$L\{t^2 \sin 3t\} = (-1)^2 \frac{d^2}{ds^2}\left\{\frac{3}{s^2+9}\right\}, \quad s=2$$

$$= (1) \frac{d}{ds}\left\{\frac{d}{ds} \frac{3}{s^2+9}\right\}, \quad s=2$$

$$= \frac{d}{ds}\left\{\frac{-3}{s^2+9} \times 2s\right\}, \quad s=2$$

$$= -6 \frac{d}{ds}\left\{\frac{s}{(s^2+9)^2}\right\}, \quad s=2.$$

$$-6\left[\frac{(s^2+9)^2(1)-s(2(s^2+9))\times 2s}{(s^2+9)^4}\right], s=2$$

$$=-6\left[\frac{s^2+9-4s^2}{(s^2+9)^4}\right], s=2$$

$$=-6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right], s=2$$

$$=-6\left[\frac{-3s^2+9}{(s^2+9)^3}\right], s=2$$

$$=-6\left[\frac{-3(2)^2+9}{(2^2+9)^3}\right]$$

$$=-6\left[\frac{-12+9}{(4+9)^3}\right]$$

$$=\frac{-6(-3)}{(13)^3}$$

$$=\frac{18}{(13)^3}$$

$$=\frac{18}{2197}$$

$$\int_0^\infty e^{-2t}[t^2\sin 3t\,dt]=\frac{18}{2197}$$

Q.2.

A. Here $\bar{f}(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{1}{s^2+1} \cdot \frac{s}{s^2+4} = \bar{F}_1(s) \cdot \bar{F}_2(s)$

When $\bar{F}_1(s) = \frac{1}{s^2+1} \Rightarrow L^{-1}\{\bar{F}_1(s)\} = F_1(t) = L^{-1}\{\frac{1}{s^2+1}\} = \sin t$.

$\bar{F}_2(s) = \frac{s}{s^2+2^2} \Rightarrow L^{-1}\{\bar{F}_2(s)\} = F_2(t) = L^{-1}\{\frac{s}{s^2+4}\} = \cos 2t$.

Now.

$F_1(t) = \sin t \Rightarrow F_1(u) = \sin u$.

$F_2(t) = \cos 2t \Rightarrow F_2(t-u) = \cos 2(t-u) = \cos(2t-2u)$.

Hence, by the convolution thm.

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\{\bar{F}_1(s)\bar{F}_2(s)\} = \int_0^t F_1(u) F_2(t-u)\,du = \int_0^t \sin u \cos(2t-2u)\,du$$

$$= \frac{1}{2}\int_0^t 2 \sin u \cos(2t-2u)\,du.$$

$$= \frac{1}{2}\int_0^t \{\sin(u+2t-2u) + \sin(3u-2t)\,dt$$

$$= \frac{1}{2}\int_0^t \{\sin(2t-u) + \sin(3u-2t)\}\,du.$$

$$= \frac{1}{2}\left[\{\cos t - \cos 2t\} + \{-\frac{\cos t}{3} + \frac{\cos 2t}{3}\}\right]$$

$$= \frac{1}{2}\left[\frac{2}{3}\cos t - \frac{2}{3}\cos 2t\right]$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

$$F(t) = \frac{1}{3}(\cos t - \cos 2t)$$

(B) $f(s) = \cot^{-1}\left(\frac{s+3}{2}\right)$

gives, $\frac{d\bar{f}(s)}{ds} = -\frac{1}{1+\left(\frac{s+3}{2}\right)^2}\left(\frac{1}{2}\right)$

$$= -\left\{\frac{2}{(s+3)^2+(2)^2}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{d\bar{f}(s)}{ds}\right\} = -L^{-1}\left\{\frac{2}{(s+3)^2+(2)^2}\right\}$$

$$\Rightarrow -t f(t) = -e^{-3t} L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$-t f(t) = -e^{-3t} \sin 2t$$

$$f(t) = -e^{-3t}\frac{\sin 2t}{-t}$$

$$f(t) = \frac{e^{-3t}\sin 2t}{t}$$

(C)

given

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = e^{-t}, \; \& \; y(0) = 1 \; \& \; y'(0) = 0.$$

The given diff^n eq^n can be re-written as.

$$y''(t) - 4y'(t) + 3y(t) = e^{-t}; \; y(0) = 1, \; y'(0) = 0$$

$$\Rightarrow L\{y''(t)\} - 4L\{y'(t)\} + 3L\{y(t)\} = L\{e^{-t}\}.$$

$$\Rightarrow s^2 y(s) - s y(0) - y'(0) - 4\{s y(s) - y(0)\} + 3y(s) = \frac{1}{s+1}$$

$$s^2 y(s) - s(1) - 0 - 4\{s y(s) - 1\} + 3y(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2 y(s) - s - 4s y(s) + 4 + 3y(s) = \frac{1}{s+1}$$

$$\Rightarrow (s^2 - 4s + 3) y(s) = s - 4 + \frac{1}{s+1}$$

$$\Rightarrow (s-1)(s-3) y(s) = s - 4 + \frac{1}{s+1}$$

$$\Rightarrow y(s) = \frac{s-4}{(s-1)(s-3)} + \frac{1}{(s-1)(s-3)(s+1)}$$

$$= \frac{(s-4)(s+1)+1}{(s-1)(s-3)(s+1)}.$$

$$= \frac{s^2 - 3s - 3}{(s-1)(s-3)(s+1)}$$

let $$\frac{s^2 - 3s - 3}{(s-1)(s-3)(s+1)} = \frac{A}{s-1} + \frac{B}{s-3} + \frac{C}{s+1} \quad \text{--- (2)}$$

$$\Rightarrow s^2 - 3s - 3 = A(s-3)(s+1) + B(s-1)(s+1) + C(s-1)(s-3)$$

putting $s=1$ ~~s=1~~ we have

$$(1)^2-3(1)-3=A(-2)(2)$$

$$-5=-4A$$

$$A=\frac{5}{4} \quad \boxed{A=\frac{5}{4}}$$

put $s=3$

$$(3)^2-3(3)-3=B(2)(4) \Rightarrow B=-\frac{3}{8}$$

putting $s=-1$,

$$(-1)^2-3(-1)-3=C(-2)(-4) \Rightarrow C=1/8$$

From (2)

$$y(s)=\frac{5}{4}\left\{\frac{1}{s-1}\right\}-\frac{3}{8}\left\{\frac{1}{s-3}\right\}+\frac{1}{8}\left\{\frac{1}{s+1}\right\}$$

$$\Rightarrow L^{-1}\{y(s)\}=\frac{5}{4}L^{-1}\left\{\frac{1}{s-1}\right\}-\frac{3}{8}L^{-1}\left\{\frac{1}{s-3}\right\}+\frac{1}{8}\left\{\frac{1}{s+1}\right\}$$

$$y(t)=\frac{5}{4}e^{t}-\frac{3}{8}e^{3t}+\frac{1}{8}e^{-t}.$$

$$\boxed{y(t)=\frac{5}{4}e^{t}-\frac{3}{8}e^{-3t}+\frac{1}{8}e^{-t}}$$

Q 3
A. The Fourier integral representation is given by

$$f(x) = \int_0^{\infty} B(\lambda) \sin\lambda x \, d\lambda \quad -(1)$$

When $B(\lambda) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin\lambda x \, dx$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-x} \sin\lambda x \, dx \quad \text{if } f(x) = e^{-x}$$

$$= \frac{2}{\pi} \left[\frac{e^{-x}}{1+\lambda^2} (-\sin\lambda x - \lambda \cos\lambda x) \right]_0^{\infty}$$

$$= \frac{2\lambda}{\pi(1+\lambda^2)}$$

From (1)

$$e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{\lambda \sin\lambda x}{1+\lambda^2} \, d\lambda$$

$$\int_0^{\infty} \frac{\lambda \sin\lambda x}{1+\lambda^2} \, d\lambda = \frac{\pi}{2} e^{-x}$$

$$\Rightarrow \int_0^{\infty} \frac{w \sin wx}{1+w^2} \, dw = \frac{\pi}{2} e^{-x} \quad (x>0)$$

(II) The Fourier cosine representation of $f(x)$ is.

$$f(x) = \int_0^{\infty} A(\lambda) \cos\lambda x \, d\lambda$$

when $A(\lambda) = \frac{2}{\pi} \int_0^{\infty} f(x) \cos\lambda x \, dx$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-x} \cos\lambda x \, dx \quad \text{if } f(x) = e^{-x}.$$

$$= \frac{2}{\pi}\left[\frac{e^{-x}}{1+\lambda^2}\left(-\lambda\cos\lambda x + \lambda\sin\lambda x\right)\right]_0^{\infty}$$

$$= \frac{2}{\pi(1+\lambda^2)}$$

from ② we obtain

$$e^{-x} = \frac{2}{\pi}\int_0^{\infty}\frac{\cos\lambda x}{1+\lambda^2}\,d\lambda$$

$$\Rightarrow \int_0^{\infty}\frac{\cos\lambda x}{1+\lambda^2}\,d\lambda = \frac{\pi}{2}e^{-x}$$

$$\Rightarrow \int_0^{\infty}\frac{\cos xw}{1+w^2}\,dw = \frac{\pi}{2}e^{-x}\ (x\geq 0)$$

B] The Fourier transform of $f(x)$ is given by

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$= \int_{-1}^{1}(1-x^2)\, e^{isx}\,dx$$

$$= \int_{-1}^{1}(1-x^2)\left(\cos sx + i\sin sx\right)dx$$

$$= \int_{-1}^{1}(1-x^2)\cos sx\,dx + i\int_{-1}^{1}(1-x^2)\sin sx\,dx$$

$$= 2\int_0^1 (1-x^2)\cos sx\,dx \quad [1-x^2 \text{ is even fun}^n]$$

$$= -\frac{4}{s^3}\left(s\cos s - \sin s\right)$$

Now, by inversion formula, we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{4}{s^3}(s\cos s - \sin s)(\cos sx - i\sin sx)\, ds.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{4}{s^3}(s\cos s - \sin s)\cos sx\, ds.$$

put $x = \frac{1}{2} \Rightarrow f(x) = 3/4$

$$-\frac{4}{\pi}\int_0^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos\left(\frac{s}{2}\right) ds = \frac{3}{4}$$

$$\Rightarrow \int_0^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right) dx = \frac{-3}{16}.$$

c] Let

$$f(x) = \frac{x}{x^4+4} \quad \& \quad G(x) = \frac{x}{x^2+9} \quad \text{then}$$

$$F_s(s) = F_s\left(\frac{x}{x^4+4}\right) = \frac{\pi}{2} e^{-2s}.$$

$$G_s(s) = G_s\left(\frac{x}{x^2+9}\right) = \frac{\pi}{2} e^{-3s}.$$

by using Parsevals identity for Fourier sine transforms.

$$\frac{2}{\pi}\int_0^\infty F_c(s)\,G_c(s)\,ds = \int_0^\infty f(x)\,g(x)\,dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \left(\frac{\pi}{2}e^{-2s}\right)\left(\frac{\pi}{2}e^{-3s}\right)ds = \int_0^\infty \left(\frac{x}{x^2+4}\right)\left(\frac{x}{x^2+9}\right)dx$$

$$\Rightarrow \frac{\pi}{2}\int_0^\infty e^{-5s}\,ds = \int_0^\infty \frac{x^2}{(x^2+4)(x^2+9)}\,dx$$

$$\Rightarrow \frac{\pi}{2}\left[\frac{e^{-5s}}{-5}\right]_0^\infty = \int_0^\infty \frac{x^2}{(x^2+4)(x^2+9)}\,dx$$

$$\Rightarrow \frac{\pi}{2}\left[\left(0-\frac{1}{5}\right]\right. = \int_0^\infty \frac{x^2}{(x^2+4)(x^2+9)}\,dx$$

$$\frac{\pi}{10} = \int_0^\infty \frac{x^2}{(x^2+4)(x^2+9)}\,dx$$

$$\frac{\pi}{10} = \int_0^\infty \frac{t^2}{(t^2+4)(t^2+9)}\,dt$$

$$\Rightarrow \int_0^\infty \frac{t^2}{(t^2+4)(t^2+9)}\,dt = \frac{\pi}{10}$$

4. A] Let $u = x+y+z$ & $v = x^2+y^2+z^2$, then

$$F(u,v) = 0$$

diff^n partially w.r.to x & y

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

$$\Rightarrow \frac{\partial F}{\partial u}(1+p)+\frac{\partial F}{\partial v}(2x+2zp)=0 \quad \text{—— (1)}$$

$$\& \ \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0.$$

$$\Rightarrow \frac{\partial F}{\partial u}(1+q)+\frac{\partial F}{\partial v}(2y+2zq)=0 \quad \text{—— (2)}$$

Eliminating $\frac{\partial F}{\partial u}$ & $\frac{\partial F}{\partial v}$ from (1) & (2),

$$(1+p)(2y+2zq) = (1+q)(2x+2zp)$$

$$\Rightarrow (y-z)p+(z-x)q = x-y.$$

B] The partial differential equation

$$(x^2-yz)p+(y^2-zx)q = z^2-xy \quad \text{—— (1)}$$

is Lagrange's linear of the form

$$Pp+Qq = R$$

where $P = x^2-yz$, $Q = y^2-zx$ & $R = z^2-xy$.

The Lagrange's auxiliary equations are.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \quad \text{--- (2)}$$

From (2) We have

$$\frac{dx-dy}{(x^2-yz)-(y^2-zx)} = \frac{dy-dz}{(y^2-zx)-(z^2-xy)} = \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} \quad \text{--- (3)}$$

From first two ratios of (3)

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)}$$

$$\Rightarrow \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0$$

$$\Rightarrow \log(x-y) - \log(y-z) = \log a$$

$$\Rightarrow \frac{x-y}{y-z} = a \quad \text{--- (4)}$$

From the last two ratios of (3)

$$\frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\Rightarrow \frac{xdx+ydy+zdz}{x+y+z} = d(x+y+z).$$

$$x\,dx + y\,dy + z\,dz - (x+y+z)\,d(x+y+z) = 0$$

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$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} - \frac{(x+y+z)^2}{2} = c.$$

$$\Rightarrow xy + yz + zx = -c, = b \quad \text{--- (5)}$$

From (4) & (5) the general sol^n is.

$$\phi\left(\frac{x-y}{y-z}, xy + yz + zx\right) = 0.$$

c] one dimensional heat eq^n is.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

the sol^n of (1) consistent with the physical nature of the problems is given by

$$u = c_1 e^{-m^2c^2t}(c_2 \cos mx + c_3 \sin mx) \quad \text{--- (2)}$$

When $u(x,t) = 0$ at $x = 0$ --- (3)

$u(x,t) = 0$ at $x = 2$ --- (4)

$$u(x,t) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} \quad \text{at } t = 0 \quad \text{--- (5)}$$

Using the condition (3) in (2), we obtain.

$$0 = c_1 e^{-m^2c^2t}(c_2)$$

$$\Rightarrow c_2 = 0.$$

from ③ we have.

$$u = c_1 c_3 e^{-m^2 c^2 t} \sin mx \quad \text{—— ⑥}$$

using condition ④ in ③

$$0 = c_1 c_3 e^{-m^2 c^2 t} \sin 2m$$

$$\Rightarrow \sin 2m = 0 = \sin n\pi$$

$$\Rightarrow 2m = n\pi \quad \Rightarrow m = \frac{n\pi}{2}$$

Now, $u = c_1 c_3 e^{-\left(\frac{n\pi}{2}\right)^2 c^2 t} \sin \frac{n\pi x}{2}$ —— ⑦

The genral solⁿ is.

$$u = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{2}\right)^2 c^2 t} \sin \frac{n\pi x}{2} \quad \text{—— ⑧}$$

Using ⑤ & ⑧ $\sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$

$$= b_1 \sin \frac{\pi x}{2} + b_2 \sin \frac{2\pi x}{2} + b_3 \sin \frac{3\pi x}{2} + b_4 \sin \frac{4\pi x}{2} + \ldots$$

$$\Rightarrow b_1 = 1, \; b_5 = 3, \; b_2 = b_3 = b_4 = b_6 = b_7 = 0 = b_8 \ldots$$

From ⑧, we have,

$$u = b_1 e^{-\frac{\pi^2}{4} c^2 t} \sin \frac{\pi x}{2} + b_5 e^{-\left(\frac{5\pi}{2}\right)^2 c^2 t} \sin \frac{5\pi x}{2}.$$

i.e. $u = e^{-\frac{\pi^2}{4} c^2 t} \sin \frac{\pi x}{2} + 3 e^{-\left(\frac{5\pi}{2}\right)^2 c^2 t} \sin \frac{5\pi x}{2}$

Q.5

A) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

diffⁿ w.r.to x & y

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

$$\frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

i.e the function u is harmonic

Now, $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \frac{\partial v}{\partial y}$ (by using C-R eqⁿ)

$$\therefore V = 3x^2y - y^3 + 6xy + \phi(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = 6xy + 6y + \phi'(x) = 6xy + 6y \quad \left(\because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \phi(x) = C$$

$$\therefore V = 3x^2y - y^3 + 6xy + C.$$

Hence $f(z) = u + iv = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 + i(3x^2y - y^3 + 6xy + C)$

B. i) Here the fuc^n $F(z) = \bar{e}^{z}$ is an analytic fuc^n

Also Singular point $a = -1$ lies inside the circle $|z| = 2$.

by using cauchy's integral formula.

$$\oint_c \frac{\bar{e}^{z}}{z+1}\, dz = 2\pi i\, f(a) \Rightarrow 2\pi i\, \bar{e}^{(-1)}$$

$$= 2\pi i e$$

ii) The fuc^n $f(z) = \sin^2 z$ analytic inside & on the circle $|z| = 1$ & the Singular point $a = \pi/6$ lies inside the circle

by using cauchy's integral formula

$$F''(a) = \frac{2!}{2\pi i} \int_c \frac{f(z)}{(z-a)^3}\, dz$$

$$\oint_c \frac{\sin^2 z}{(z-\pi/6)^3}\, dz = \pi i \left[\frac{d^2}{dz^2}(\sin^2 z)\right]_{z=\pi/6}$$

$$= \pi i \left[2\cos 2z\right]_{z=\pi/6}$$

$$= \pi i \left[2\cos \pi/3\right]$$

$$= \pi i\,(1)$$

$$= \pi i$$

Here the function $F(z) = \frac{2z-1}{z(z+1)(z-3)}$ has three poles

$z = 0, -1, 3$. of which only $z = 0, -1$ lies. inside the circle $|z| = 2$.

$$\therefore \operatorname*{Res}_{z=0} F(z) = \lim_{z \to 0} z F(z) = \lim_{z \to 0} \frac{2z-1}{(z+1)(z-3)} = \frac{1}{3}$$

$$\operatorname*{Res}_{z=-1} F(z) = \lim_{z \to -1} (z+1) F(z) = \lim_{z \to -1} \frac{2z-1}{z(z-3)} = -\frac{3}{4}$$

Hence, by the residue the^m,

$$\oint_C \frac{2z-1}{z(z+1)(z-3)} dz = 2\pi i \{\text{sum of Residue}\}$$

$$= 2\pi i \left(\frac{1}{3} - \frac{3}{4}\right)$$

$$= -\frac{5\pi i}{6}.$$