Machine Learning: Vector Space (back-ground info for PCA)

by Elena Battini Sönmez
İstanbul Bilgi University

Vector:

A finite dimensional vector X may be written: $X = [x_1, x_2, \ldots X_n]'$ where the virgule is the transpose operation; i.e. X is a column vector

All x_i may be real or integer numbers

Vector Space S:

A vector space is a set of vectors that can be combined together using linear combination

Linear Vector Space S has the additive (+) and the multiplication (*) operations that sattisfy the following properties:

$$1.\ \forall x, y \in S,\ x + y \in S$$

$$2.\ \exists\, 0 : \forall\ x \in S,\ x + 0 = 0 + x = x$$

$$3.\ \forall\ x, \exists\ y : x + y = 0$$

$$4.\ \forall\ x, y, z : (x + y) + z = x + (y + z)$$

$$5.\ \forall\ a, b \in \Re, \forall\ x, y \in S : a \cdot x \in S,\ a \cdot (b \cdot x) = (a \cdot b) \cdot x,$$
$$(a + b) \cdot x = a \cdot x + b \cdot x,\ a \cdot (x + y) = a \cdot x + a \cdot y$$

$$6.\ \exists 1 : 1 \cdot x = x,\ \exists\, 0 : 0 \cdot x = 0$$

Example:

$\mathfrak{R}^n$ is one of the most familiar linear vector space. Considering n=2: x_1=[1, 3]', x_2=[0, 3]', ...

More generally, X = [x_1,x_2] and Y = [y_1,y_2]

Exercise: Prove that $\mathfrak{R}^2$ is a vector space.

Subspace:

Let V be a non empty subspace of a linear vector space S, if V satisfies the following conditions:

$$1.\ \forall\ x \in V, \alpha \in \Re : \alpha \cdot x \in V$$

$$2.\ \forall\ x, y \in V : x + y \in V$$

Then V is a subset of S

Note: Every vector space S must have the 0 vector

Exercises:

1. Prove that $\Re^3$ is a vector space
2. Let $S=\{(x_1,x_2)' \mid x_2=2x_1\}$. Prove that S is a subspace of $\Re^2$
3. Prove that $S=\{(x_1,x_2,x_3)' \mid x_1=x_2\}$ is a subspace of $\Re^3$
4. Prove that $S=\{(x_1,1)' \mid x$ real number$\}$ is NOT a subspace of $\Re^2$
5. Prove that $S=\{ A \in \Re^{2\times 2} \mid a_{1,2}=-a_{2,1}\}$ is a subspace of $\Re^{2\times 2}$

Span and Spanning Sets:

1. The linear combination of $\{v_1,v_2,\dots,v_n\}$ is: $c_1 \cdot v_1+c_2 \cdot v_2+\dots+c_n \cdot v_n$ where c_2 are scalars, little numbers

2. The vectors $\{v_1,v_2,\dots,v_n\}$ in a vector space V are said to be linearly independent if $c_1 \cdot v_1+c_2 \cdot v_2+\dots+c_n \cdot v_n=0$ iff $c_i=0$ for all $i=1,\dots,n$

3. The set of all linear combination of $\{v_1,v_2,\dots,v_n\}$ is its span

4. If $\{v_1,v_2,\dots,v_n\}$ are independent and $V=\text{span}(v_1,v_2,\dots,v_n)$ then:

- They are the minimum spanning set for V
- They are the basis for V

Basis:

The vectors $\{v_1,v_2,\ldots,v_n\}$ form a basis for the vector space V if and only if:

- $v_1,v_2,\ldots,v_n$ are linearly independent, and
- $V=span(v_1,v_2,\ldots,v_n)$

Exercises:

1. Prove that the following vectors are NOT linearly independent: $v_1=(1,-1,2)'$, $v_2=(-2,3,1)'$, $v_3=(-1,3,8)'$
2. Prove that $e_1=(1,0)'$, $e_2=(0,1)'$ is a basis for $\Re^2$
3. Prove that {(1,1,1)', (0,1,1)',(2,0,1)} and {(1,1,1)', (1,1,0)',(1,0,1)'} are both basis for $\Re^3$

Note: all basis of vector space S have the same cardinality (number of elements), which is also the dimension of S

Norm:

- It is the mathematical concept associated with the *length of a vector*
- Let S be a vector space with element x. The norm of x, $||x||$, is a real valued function which satisfies the following properties:

$$\left||x|\right| \geq 0, \forall\, x \in S$$

$$\left||x|\right| = 0 \; iff \; x = 0$$

$$\left||\propto \cdot\, x|\right| = |\propto| \, ||x||, \forall \propto \in \Re$$

$$\left||x + y|\right| \leq \left||x|\right| + ||y||$$

Useful norms:

$$1.\ l_1\ norm: \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$2.\ l_p\ norm: \|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$$

$$3.\ l_\infty\ norm: \|x\|_\infty = \max_{i=1,\dots,n} |x_i|$$

Exercise: Use the l_p formula to calculate the l_2 norm

Unit norm vector:

- A vector x is said to be normalized if $\|x\| = 1$
- A normalized vector is also referred to as unit norm vector
- It is always possible to normalize every vector except ….(?)
- That is: $\forall\ x, \exists\ y = \dfrac{x}{\|x\|}$

Exercises: Normalize x=[1,2]'

Inner Product:

Let S be a vector space defined over a scalar field R. An inner product is a function $\langle .,. \rangle : S \times S \to R$ having the following properties:

$$1. \langle x, y \rangle = \langle y, x \rangle$$

$$2. \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \; \forall \alpha \in \Re$$

$$3. \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

$$4. \langle x, x \rangle > 0 \; if \;\; x \neq 0 \; and \; \langle x, x \rangle = 0 \; iff \;\; x = 0$$

For finite dimensional vectors:

$$\langle x, y \rangle = x^t \cdot y = (x^{'} \cdot y) = y^t \cdot x = \sum_{i=1}^{n} x_i y_i = x \bullet y$$

Inner product, dot product, orthogonal vectors:

- Inner product: $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$
- Dot product: $x \bullet y = \sum_{i=1}^{n} x_i y_i$
- Vectors x and y are orthogonal vectors iff

$$\langle x, y \rangle = 0. We\ write: x \perp y$$

Ortho-normal vectors:

- A set of vectors $\{p_1, p_2, \ldots, p_n\}$ are ortho-normal if:
  - They are mutually orthogonal
  - They have unit norm

- Exercise: Prove that $e_1=(1,0)'$ and $e_2=(0,1)'$ are ortho-normal