Trigonometry

$\int \sin x\,dx = -\cos x + C$	$\int \sec x \tan x\,dx = \sec x + C$
$\int \cos x\,dx = \sin x + C$	$\int \cot x \csc x\,dx = -\csc x + C$
$\int \sec^2 x\,dx = \tan x$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$	$\int \frac{-1}{1+x^2}\,dx = \cot^{-1}x + C$
$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}x + C$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$	$\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \csc^{-1}x + C$

tricky:

$\int \frac{2}{(1+x)(1+x^2)}\,dx$

$= \int \frac{1}{(1+x)} + \frac{-x+1}{1+x^2}\,dx$

$= \ln|1+x| + \int \frac{1}{1+x^2} - \frac{x}{1+x^2}\,dx$

$= \ln|1+x| + \tan^{-1}x - \frac{1}{2}\ln|1+x^2|$

$\int \frac{\sin x + \cos x}{(\cos x - \sin x)^2}\,dx$

$= (\cos x - \sin x)^{-1} + C,\ C \in \mathbb{R}$

$\int \frac{1}{1-\cos x}\,dx$

$= \int \frac{1+\cos x}{\sin^2 x}\,dx$

$= \int \frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\,dx$

$= \int \csc^2 x + \cot x \csc x\,dx$

$= -\frac{1}{2}\cot x - \csc x + C$

$\int \tan^2 \frac{x}{2}\,dx$

$= \int 1 - \sec^2 \frac{x}{2}\,dx$

$= x - 2\tan\frac{x}{2} + C$

1. Substitution

↳ not recognizable

Steps

1) differentiate wrt new variable
2) replace 'x' with 'u'
   $\int(\dots)\,dx \to \int(\dots)\frac{dx}{du}\,dx$
3) integrate wrt 'u'
4) replace all 'u' with f(x)

$\theta = \sin^{-1}(x)$ etc.

$\frac{dx}{d\theta} = -a\sin\theta$

$\int \sqrt{a^2-x^2}\,dx$

$= \int \sqrt{a^2-x^2}\ -a\sin\theta\,d\theta$

$= \int \sqrt{a^2-(1-\cos^2\theta)}\ -a\sin\theta\,d\theta$

$= \int a\sqrt{\sin^2\theta}\ \ -a\sin\theta\,d\theta$

$= \int -a\sin^2\theta\,d\theta$

$= \int -a^2\left(\frac{1-\cos 2\theta}{2}\right)d\theta$

$= -\frac{a^2}{2}\left(\theta - \frac{1}{2}\cos 2\theta\right) + C$

$= -\frac{1}{2}a^2\theta + \frac{1}{4}a^2\sin 2\theta + C$

$= -\frac{1}{2}a^2\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{4}(2)\left(\sqrt{a^2-x^2}\right) + C$

$= -\frac{1}{2}a^2\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\sqrt{a^2-x^2} + C$

2. By parts

↳ product of 2 mutually exclusive parts

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}\,dx$$

(ilatec) ILAEC

Priority of 'u':
- inverse trig
- log funcs
- algebraic
- trigonometric funcs
- exponential funcs
- constants

1) assign 'u' & 'v'
2) repeat process until there are no integrals left

1. $\int \cos(\ln x)\,dx$

   u

   $v = x$

   $= x\cos(\ln x) - \int -\sin(\ln x)\,dx$

   $= x\cos(\ln x) + x\cos(\ln x) + C$

2. $\int e^{2x}\sin x\,dx$

   $\frac{dv}{dx}$ u

   $= \frac{1}{2}e^{2x}\sin x - \int \frac{1}{2}e^{2x}(-\cos x)\,dx$

   $= \frac{1}{2}e^{2x}\sin x + \frac{1}{2}\int e^{2x}\cos x\,dx$

   $= \frac{1}{2}e^{2x}\sin x + \frac{1}{2}\left[\frac{1}{2}e^{2x}\cos x - \int \frac{1}{2}e^{2x}\sin x\right]$

   $\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$

   $= \frac{e^{2x}}{5}(2\sin x - \cos x) + C,\ C \in \mathbb{R}$

3. By separation

$\frac{dy}{dx} = g(y)$ or $\frac{dy}{dx} = f(x)g(y)$

↳ divide both sides by g(y)

$\frac{dy}{dx} = 2 - y$

$\frac{1}{y-2}\frac{dy}{dx} = -1$

$\int \frac{1}{y-2}\,dy = -\int 1\,dx$

$\ln|y-2| = -x + C$

$|y-2| = e^{-x+C}$

$y - 2 = \pm e^{x-C}$

$y = Ae^{x} + 2$

↳ Absorbs − the ±

$\frac{dy}{dx} = e^x(\cos^2 y)$

$\frac{1}{\cos^2 y}\frac{dy}{dx} = e^x$

$\int \frac{1}{\cos^2 y}\frac{dy}{dx}\,dx = \int e^x\,dx$

$\int \sec^2 y\,dy = e^x + C$

$\tan y = e^x + C$

4. Integrating Factor

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$I(x) = e^{\int P(x)\,dx}$

1) write in standard form
2) Find I(x)
3) Multiply both sides by the I(x)
4) "Reverse" the differentiation

$x^3\frac{dy}{dx} + 3x^2y = 5x^4$

$\frac{dy}{dx}(x^3y) = 5x^4$

5) solve

$(x^2+1)\frac{dy}{dx} + xy = x^3$

$\frac{dy}{dx} + \frac{xy}{(x^2+1)} = \frac{x^3}{(x^2+1)}$

$I(x) = e^{\int \frac{x}{x^2+1}dx}$

$= e^{\frac{1}{2}\ln|x^2+1|}$

$= \sqrt{x^2+1}$

$\frac{dy}{dx}\sqrt{x^2+1} + x(x^2+1)^{-\frac{1}{2}}y = x^3(x^2+1)^{-\frac{1}{2}}$

$y\sqrt{x^2+1} = \int x^3(x^2+1)^{-\frac{1}{2}}\,dx$

$= \frac{1}{3}\int 2x\sqrt{x^2+1} - \frac{2x}{\sqrt{x^2+1}}\,dx$

$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} - \sqrt{x^2+1} + C$

$y = \frac{1}{3}x^2 - \frac{2}{3} + \frac{C}{\sqrt{x^2+1}}$

Real life applications

Newton's law of cooling

$\frac{dT}{dt} = -m(T-S)$

Decay

$\frac{dP}{dt} = kP$

$\frac{dP}{dt} = (\beta-\alpha)P$

net growth rate

Logistic growth

$\frac{dP}{dt} = \beta P - (\alpha + \gamma P)P$

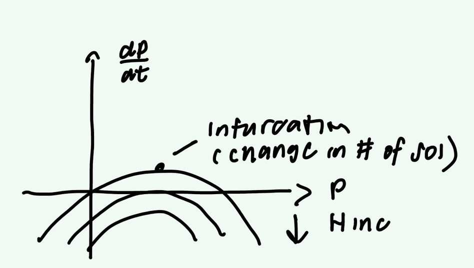
$= kP - \gamma P$

$= kP\left(1 - \frac{P}{N}\right)$

↳ carrying capacity

Harvesting

$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - H$ harvesting rate



Equil pts: $kP\left(1-\frac{P}{N}\right) - H = 0$

$-\frac{k}{N}P^2 + kP - H = 0$

$P = \frac{N}{2} \pm \sqrt{\frac{N^2}{4} - \frac{HN}{k}}$

when H↑ Equil ← (gets closer pk)

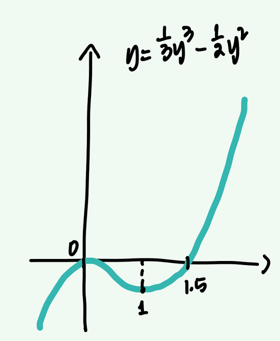
equilibrium points → autonomous

$\frac{dy}{dx} = f(y)$ when $f(c) = 0$

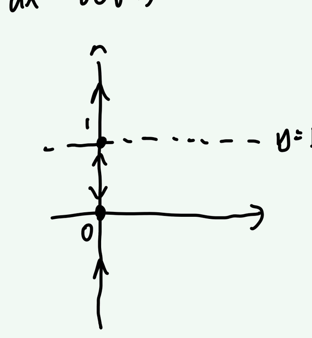
1) solve $\frac{dy}{dx} = 0$
2) Draw line, divide regions
3) If $f(y) > 0$ ↑, sol curve ↑
4) If $f(y) < 0$ ↓, sol curve ↓

solution curves can't cross

$y = \frac{1}{3}y^3 - \frac{1}{2}y^2$



$\frac{dy}{dx} = y(y-1)$



phase lines
- how a solution behaves as x ↑/↓
- doesn't show rate of change of sol curves
- $\frac{dy}{dx} = 0$ → counted as 1 asymptote