KALKULUA

Adierazpen geometrikoa
Katearen erregela

2015-2016 IKASTURTEA

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2.3 Adierazpen geometrikoa

Deribatu Partzialak

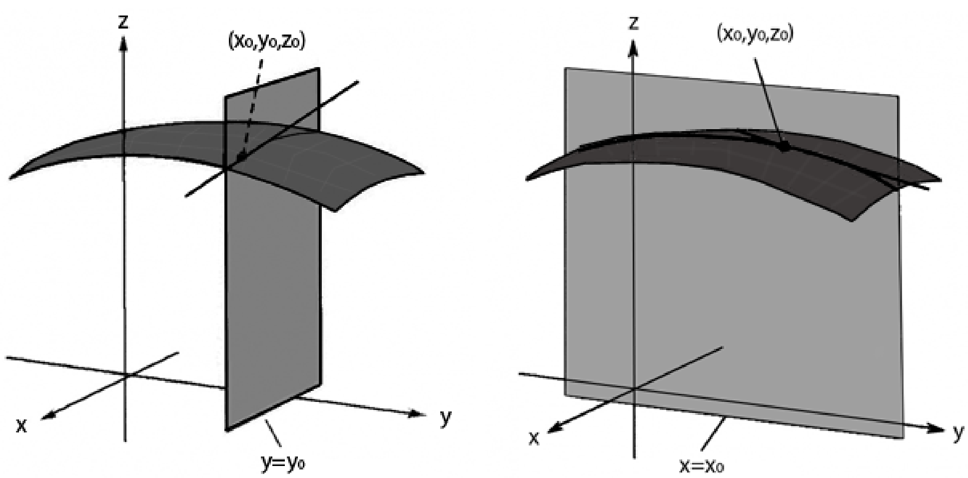
Izan bedi $f : \mathcal{A} \subseteq \mathbb{R}^2 \to \mathbb{R}$ funtzioa, deribatu partzialak kalkulatzerakoan, aldagai bat konstante gordetzen dugu.

x-rekiko deribatzean, $z = f(a, b)$ gainazala $y = b$ planoz ebakitzen dugu, $y = f(x, b)$ kurba lortuz.

x-rekiko deribatuak, $z = f(x, b)$ kurbaren ukitzailearen malda emango digu.

y-rekiko deribatzean, $z = f(x, y)$ gainazala $x = a$ planoz ebakitzen dugu, $y = f(a, y)$ kurba lortuz.

y-rekiko deribatuak, $z = f(a, y)$ kurbaren ukitzailearen malda emango digu.



Diferentzial totala

Deribatu partzialei dagokien zuzen ukitzaileak plano batean daude, $(a, b, f(a, b))$ puntutik igarotzen direlako. Galdera hau da: plano hori $z = f(x, y)$ gainazalaren plano ukitzailea al da $(a, b, f(a, b))$ puntuan?

$(a, b, f(a, b))$ puntutik igarotzen diren planoen sortak ekuazio hau du:

$$z - f(a, b) = \mathcal{A}(x - a) + \mathcal{B}(y - a).$$

Jakin nahi dugu horietatik zein den ukitzailea. Plano ukitzailea izateko baldintza hau izanik:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-z}{\|(x,y)-(a,b)\|} = 0 \quad \text{edo} \quad \lim_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)-\mathcal{A}(x-a)-\mathcal{B}(y-b)}{\|(x,y)-(a,b)\|} = 0$$

$f(x,y)$ diferentziagarria bada (a,b) puntuan, aurreko baldintza betetzen duten $\mathcal{A}$ eta $\mathcal{B}$ bakarrak dira eta $\mathcal{A} = D_1 f(a,b)$ eta $\mathcal{B} = D_2 f(a,b)$ dira.

Ondorioz, plano ukitzailearen ekuazioa hau da:

$$z - f(a,b) = D_1 f(a,b)(x-a) + D_2 f(a,b)(y-b)$$

Beraz, **diferentzial totala plano ukitzailearekin lotu dezakegu.**

<u>13 Adibidea</u>

$z = \frac{x^2}{y^3}$ gainazalaren plano ukitzailea $\left(a, b, \frac{a^2}{b^3}\right)$ puntuan, $b \neq 0$.

z-ren deribatu partzialak $D_1\ z = \frac{2x}{y^3}$; $D_2\ z = \frac{-3x^2}{y^4}$

Deribatu partzialak (a,b) puntuan: $D_1\ z(a,b) = \frac{2a}{b^3}$; $D_2\ z(a,b) = \frac{-3a^2}{b^4}$

Planoaren ekuazioa: $z - \frac{a^2}{b^3} = \frac{2a}{b^3}(x-a) + \frac{-3a^2}{b^3}(y-b)$

Adibidez, $(4,2,2)$ puntuan: $z - 2 = (x-4) - 3(y-2)$

2.4 Funtzio konposatuaren diferentziagarritasuna

<u>14 Teorema</u>

Izan bitez $f : \mathcal{A} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ eta $g : \mathcal{B} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ funtzioak, $g(\mathcal{A}) \subseteq \mathcal{B}$ izanik.

$f(x)$ diferentziagarria bada $a \in \mathcal{A}$ puntuan eta $g(y)$ diferentziagarria bada $b = f(a) \in \mathcal{B}$ puntuan, $h = gof : \mathcal{A} \subseteq \mathbb{R}^n \to \mathbb{R}^p$ funtzioa ere diferentziagarria izango da $a \in \mathcal{A}$ puntuan eta

$$Dh(a) = D(gof)(a) = Dg(f(a))Df(a) \text{ beteko da.}$$

15 Adibidea

2.1-1 ariketa

$f(u,v) = u^3v^3 + u + 1$, $u = x^2 + y^2$, $v = e^{x+y} - 1$, $\frac{\partial f}{\partial x}$ eta $\frac{\partial f}{\partial y}$

$$f : \mathbb{R}^2 \to \mathbb{R} \quad g : \mathbb{R}^2 \to \mathbb{R}^2$$

$D_h(x,y) = Dg(u,v) \cdot Df(x,y)$

$$f(x,y) = (x^2 + y^2, e^{x+y} - 1) \qquad g(u,v) = u^3v^3 + u + 1$$

$$D_1f(x,y) = (2x, e^{x+y}) \qquad D_2f(x,y) = (2y, e^{x+y})$$

$$D_1g(u,v) = 3u^2v^3 + 1 \qquad D_2g(u,v) = 3u^3v^2$$

$$Df(x,y) = \begin{pmatrix} 2x & 2y \\ e^{x+y} & e^{x+y} \end{pmatrix}$$

$$Dg(u,v) = \begin{pmatrix} 3u^2v^3 + 1, & 3u^3v^2 \end{pmatrix}$$

$$Dh(x,y) = (3u^2v^3 + 1, 3u^3v^2) \begin{pmatrix} 2x & 2y \\ e^{x+y} & e^{x+y} \end{pmatrix} =$$

$$= (6xu^2v^3 + 2x + 3u^2v^3e^{x+y} + e^{x+y} \qquad 6yu^2v^3 + 3u^2v^3e^{x+y})$$

$$= (6x(x^2+y^2)^2(e^{x+y}-1)^3 + 2x + 3(x^2+y^2)^2(e^{x+y}-1)^3e^{x+y} + e^{x+y}$$

$$6y(x^2+y^2)^2(e^{x+y}-1)^3 + 3(x^2+y^2)^2(e^{x+y}-1)^3e^{x+y})$$

Ariketak

1.3-2

Kalkulatu ezazu $\bigtriangledown f(x,y)$ existitzen den puntuetan.

$$f(x,y) = \begin{cases} xy \sin \frac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$(x,y) \neq (0,0)$ denean:

$\bigtriangledown f(x,y) = (D_1f(x,y), D_2(f,x))$

$$D_1f(x,y) = y \sin \tfrac{1}{x^2+y^2} + xy(\cos \tfrac{1}{x^2+y^2})(-1(x^2+y^2)^{-2}2x) = y(\sin \tfrac{1}{x^2+y^2}) - \tfrac{2x^2y}{(x^2+y^2)^2}(\cos \tfrac{1}{x^2+y^2})$$

$$D_2f(x,y) = x \sin \tfrac{1}{x^2+y^2} + xy \cos(\tfrac{1}{x^2+y^2}(-1(x^2+y^2)^{-2}2y)) = x(\sin \tfrac{1}{x^2+y^2}) - \tfrac{2xy^2}{(x^2+y^2)^2}\cos(\tfrac{1}{x^2+y^2})$$

$(x, y) = (0, 0)$ denean:

$$\frac{\partial f}{\partial x}(0,0) = D_{(1,0)}f(0,0) = \lim_{t\to 0}\frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0}\frac{f(t,0)-0}{t} = \lim_{t\to 0}\frac{0}{t} = 0$$
.

$$\frac{\delta f}{\delta y}(0,0) = D_{(0,1)}f(0,0) = \lim_{t\to 0}\frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0}\frac{f(0,t)-0}{t} = \lim_{t\to 0}\frac{0}{t} = 0$$
.

Beraz,

$$f(x,y) = \begin{cases} (0,0) & (x,y) = (0,0) \\ (y(\sin\frac{1}{x^2+y^2}) - \frac{2x^2y}{(x^2+y^2)^2}(\cos\frac{1}{x^2+y^2}),\, x(\sin\frac{1}{x^2+y^2}) - \frac{2xy^2}{(x^2+y^2)^2}\cos(\frac{1}{x^2+y^2})) & (x,y) \neq (0,0) \end{cases}$$

<u>2.1-2</u>

Kalkulatu itzazu eskatzen diren deribatu partzialak.

$h(u,v) = \ln(u^2+v)$, $u = e^{x+y^2}$, $v = x^2+y$, $\frac{\partial f}{\partial x}$ eta $\frac{\partial f}{\partial y}$

$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial u}\ \frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\ \frac{\partial v}{\partial x} = \frac{2u}{u^2+v}e^{x+y^2} + \frac{1}{u^2+v}2x = \frac{2e^{x+y^2}}{(e^{x+y^2})^2+x^2+y}e^{x+y^2} + \frac{2x}{(e^{x+y^2})^2+x^2+y} =$

$= \frac{2((e^{x+y^2})^2+x)}{(e^{x+y^2})^2+x^2+y}$.

$\frac{\partial h}{\partial y} = \frac{\partial h}{\partial u}\ \frac{\partial h}{\partial v}\ \frac{\partial v}{\partial y} = \frac{2u}{u^2+v}2ye^{x+y^2} + \frac{1}{u^2+v}1 =$

$= \frac{4y(e^{x+y^2})^2+1}{(e^{x+y^2})^2+x^2+y}$.

1.7-2

Aurki itzazu gainazal hauen plano ukitzaileen ekuazioak, emandako puntuetan:

$x^2 + xy^2 + y^3 + z + 1 = 0$ $(2, -3, 4)$ puntuan

$z = -x^2 - xy^2 - y^3 - 1$

z-ren deribatu partzialak:

$$D_1 z = -2x - y^2$$
$$D_2 z = -2xy - 3y^2$$

Deribatu partzialak $(2, -3)$ puntuan:

$$D_1 z = -2 \cdot 2 - (-3)^2 = -13$$
$$D_2 z = -2 \cdot 2(-3) - (-3)^2 = -15$$

Planoaren ekuazioa:

$$z - 4 = -13(x-2) - 15(y+3)$$

1.5-4

Egiaztatu itzazu berdintza hauek:

$f(x, y, z) = x + \frac{x-y}{y-z}, \quad y \neq z, \quad \frac{\delta f}{\delta x} + \frac{\delta f}{\delta y} + \frac{\delta f}{\delta z} = 1$

$\frac{\partial f}{\partial x} = 1 + \frac{1(y-z)-(x-y)0}{(y-z)^2} = 1 + \frac{y-z}{(y-z)^2} = 1 + \frac{1}{y-z} = \frac{y-z+1}{y-z}.$

$\frac{\partial f}{\partial y} = \frac{-1(y-z)-(x-y)1}{(y-z)^2} = \frac{-y+z-x+y}{(y-z)^2} = \frac{z-x}{(y-z)^2}.$

$\frac{\partial f}{\partial z} = \frac{0(y-z)-(x-y)(-1)}{(y-z)^2} = \frac{x-y}{(y-z)^2}.$

$$\frac{\delta f}{\delta x} + \frac{\delta f}{\delta y} + \frac{\delta f}{\delta z} = \frac{(y-z)(y-z+1)+z-x+x-y}{(y-z)^2} = \frac{y^2-yz+y-yz+z^2-z+z-y}{(y-z)^2} = \frac{y^2-2yz+z^2}{(y-z)^2} = \frac{(y-z)^2}{(y-z)^2} = 1$$

2.6-1

Idatz itzazu ekuazio hauek ematen diren aldagai berrietan:

$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z=0,\qquad u=x,\quad v=\frac{y}{x},\qquad z=z(x,y)$ izanik;

$$f:\mathbb{R}^2\to\mathbb{R}^2 \qquad Df(x,y)=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ \frac{-y}{x^2} & \frac{1}{x} \end{pmatrix}$$

$$g:\mathbb{R}^2\to\mathbb{R} \qquad Dg(u,v)=\left(\frac{\partial g}{\partial u}\ \frac{\partial g}{\partial v}\right)$$

$$z(x,y)=g(f(x,y))=g(u,v)$$

$$Dz(x,y)=Dg(f(x,y))\cdot Df(x,y)$$

$$\left(\frac{\partial z}{\partial x}\ \frac{\partial z}{\partial y}\right)=\left(\frac{\partial g}{\partial u}\ \frac{\partial g}{\partial v}\right)\begin{pmatrix} 1 & 0 \\ \frac{-y}{x^2} & \frac{1}{x} \end{pmatrix}=\left(\frac{\partial g}{\partial u}-\frac{y}{x^2}\frac{\partial g}{\partial u}\qquad \frac{1}{x}\frac{\partial g}{\partial v}\right)$$

$$\begin{cases} \frac{\partial z}{\partial x}=\frac{\partial g}{\partial u}-\frac{y}{x^2}\frac{\partial g}{\partial v}=\frac{\partial g}{\partial u}-\frac{v}{u}\frac{\partial g}{\partial v} \\ \frac{\partial z}{\partial y}=\frac{1}{u}\frac{\partial g}{\partial v} \end{cases}$$

$$u\left(\frac{\partial g}{\partial u}-\frac{v}{u}\frac{\partial g}{\partial u}\right)+v\cdot u\left(\frac{1}{u}\frac{\partial g}{\partial v}\right)-z=0\Rightarrow u\frac{\partial g}{\partial u}-v\frac{\partial g}{\partial v}+v\frac{\partial g}{\partial v}-z=0\Rightarrow z=u\frac{\partial g}{\partial u}$$