## 3.3 Segiden arteko eragiketak eta limiteak. Indeterminazioak

9 Def $\{a_n\}$ eta $\{b_n\}$ segidak emanik, beren arteko eragiketak honela definituko ditugu.

$$\{a_n\} \pm \{b_n\} = \{a_n \pm b_n\}$$

$$\{a_n\} \cdot \{b_n\} = \{a_n \cdot b_n\}$$

$$\{a_n\} : \{b_n\} = \{a_n / b_n\}, \quad b_n \neq 0$$

$$\log_k \{a_n\} = \{\log_k a_n\} \quad a_n > 0,\ k > 0$$

$$k^{\{a_n\}} = \{k^{a_n}\}, \quad k > 0$$

$$\{a_n\}^{\{b_n\}} = \{a_n^{\,b_n}\}, \quad a_n > 0$$

$$\{a_n\} = \{\underbrace{a_1, a_2, a_3,}_{\text{kopuru finitua}} \underbrace{\ldots}_{\text{batetik aurrera infinitu daude}}\}$$

Zenbaki batetik aurrera, beti ~~?~~ kopuru infinitu daude, horren aurretik ordea kopuru finitu daude, beraz ~~?~~ ~~?~~ zenbaki batek ez badu betetzen propietatea baina zenbaki horretatik aurrera beti betetzen bada, ~~?~~ zenbaki horretaraino dauden zenbaki guztiak baztertu dezakegu,

$$\{a_n\} \to a \quad \text{eta} \quad \{b_n\} \to b$$

Orokorrean "Segiden arteko eragiketen limiteak limiteen arteko eragiketak dira"

$k/0,\ \infty,\ -\infty,\ ?$

$\boxed{\infty - \infty}$ $\boxed{0 \cdot \infty}$ $\boxed{\frac{0}{0}}$ $\boxed{\frac{\infty}{\infty}}$ $\boxed{0^0}$ $\boxed{1^\infty}$ $\boxed{\infty^0}$

## 3.4 Indeterminazioak ebazteko metodoak

### 3.4.1 Baliokidetasuna

10Def $\{a_n\}$ eta $\{b_n\}$ segidak baliokideak dira $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ denean. Honela idatziko dugu. $\{a_n\} \sim \{b_n\}$

11Prop $\{a_n\} \sim \{b_n\}$ bada, $\lim_{n\to\infty} a_n = \lim b_n$ beteko da

12Adib Aztertu ditzagun $\{\frac{1}{n}\}$ eta $\{\frac{1}{n^2}\}$ segidak

$\lim_{n\to\infty} \frac{1}{n} = 0$ eta $\lim_{n\to\infty} \frac{1}{n^2} = 0$

$\lim_{n\to\infty} \frac{1/n}{1/n^2} = \lim_{n\to\infty} n = \infty \neq 1$, beraz, ez dira baliokideak

13Def

gerturatu

zerbait txikia → a hau da zerora hurbiltzea

a) $\{a_n\}$ segida infinitesimala da $\lim_{n\to\infty} a_n = 0$ bada

b) $\{a_n\}$ segida infinitua da $\lim_{n\to\infty} a_n = \infty$ bada

zerbait handia, infinitura hurbiltzea.

Baliokidetsak

Infinitesimalak $\lim_{n\to\infty} a_n = 0$ betetzen da.

- $\{a_n\} \sim \{\sin a_n\} \sim \{\tan a_n\} \sim \{\arcsin a_n\} \sim \{\arctan a_n\}$

⋮

Infinituak:

14Prop Ordezkapen - printzipioa

Segida baten limitea kalkulatzean adierazpenean agertzen den biderkagai edo zatitzaile bat bere baliokide bataz ordezka daiteke segidaren limitea aldatu gabe

④

c) $\{a_n\}$ bornatua eta $\lim_{n\to\infty} b_n = 0 \Rightarrow \lim_{n\to\infty}(a_n b_n) = 0$

Frogatu behar dugu,

$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} : \forall n \geq n_0 \ |a_n b_n| < \varepsilon$

Izan bedi, $\forall \varepsilon > 0$. $\{a_n\}$ bornatua denez, $\exists M > 0$ non $|a_n| < M$ den $\forall n \in \mathbb{N}$-rako.

$\lim_{n\to\infty} b_n = 0$ denez, $\exists n_1 \in \mathbb{N}$ non, $\forall n \geq n_1$, $|b_n| < \frac{\varepsilon}{M}$

Beraz, $n_0 = n_1$ izanik, $\forall n \geq n_0$; $|a_n b_n| = |a_n||b_n| < M \frac{\varepsilon}{M} = \varepsilon$

②

a) Segida konbergente oro konbergentea da.

Baieztapen hau okerra da, izan ere segida monotono konbergentea izan dadin bornatua izan behar da. (zet) Eta segida bornatu oro ez da bornatua.

Adib: $\{2^n\}$ segida

b) Segida bornatu oro konbergentea da. Ez da egia.

$\{\sin n\}$ bornatua eta oszilatzailea da, beraz, ezin da izan konbergentea.

c) Segida konbergente guztiak ez dira monotonoak. Egia.

$\{\frac{1}{n}\}$ ~~konbergentea eta monotonoa da,~~

$\left\{\frac{(-1)^n}{n}\right\} \to 0$

(16)

$$\lim_{n\to\infty} \frac{1^2+2^2+\dots+n^2}{n^3+\ln n}=\frac{1}{3}$$

(4)

$$\frac{1}{\ln n}\sum \sin\frac{\pi}{k} = \boxed{\pi}$$

(13)

$$\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{\sqrt{n}(2n)!}=\lim_{n\to\infty}\frac{2^{2n}\cdot(n^2\cdot e^n\sqrt{2\pi n})^2}{\sqrt{n}\cdot(2n)^{2n}e^{2n}\sqrt{2\pi 2n}}=\lim_{n\to\infty}\frac{2^{2n}(n^4\cdot e^{2n}\cdot}{}$$

$$\boxed{n!\sim(n^2\cdot e^n\sqrt{2\pi n})^2}$$

(Stirling)

$$=\lim_{n\to\infty}\frac{\cancel{2^{2n}}\cdot\cancel{n^{2n}}\cdot\cancel{e^{2n}}\cdot\sqrt{2\pi n}}{\sqrt{n}\cdot\cancel{2^{2n}n^{2n}}\cdot e^{2n}\sqrt{2\pi 2n}}=\lim_{n\to\infty}\frac{\cancel{2n}\;2\pi n}{\sqrt{n}\;2\sqrt{2\pi 2n}}=$$

$$=\lim_{n\to\infty}\frac{2\pi n}{2\sqrt{n^2}\cdot\sqrt{\pi}}=\lim_{n\to\infty}\frac{\pi}{\sqrt{\pi}}=\frac{\pi}{\sqrt{\pi}}=\sqrt{\pi}$$

(10)

$$\lim_{n\to\infty} n(1-\sqrt[n]{a})=\infty\cdot 0\ \to \text{Indeterminación}$$

$$\lim_{n\to\infty}-n(\sqrt[n]{a}-1)=\lim_{n\to\infty}-n(\ln\sqrt[n]{a})=\lim_{n\to\infty}-n\ln a^{\frac{1}{n}}=\lim_{n\to\infty}-\frac{\cancel{n}}{\cancel{n}}=$$

$$\{\sqrt[n]{a}-1\}\sim\{\ln\sqrt[n]{a}\}$$

$$=\boxed{-\ln a}$$

(5)

$$\lim_{n\to\infty} n^2e^{-\sqrt{n}}=\lim_{n\to\infty}\frac{n^2}{e^{\sqrt{n}}}=0$$

(8)

$$\lim_{m\to\infty}\left(\frac{m-1}{m+3}\right)^{m+2}=\lim_{m\to\infty}\left(\frac{m+3-4}{m+3}\right)^{m+2}=\lim_{n\to\infty}\left(\frac{m+3}{}1-\frac{4}{m+3}\right)^{m+2}=$$

$$\frac{m-1}{m+3}=\frac{m+3-4}{m+3}$$

$$=\lim_{n\to\infty}e^{-4}=\frac{1}{e^4}$$

(58)

$$\lim_{n\to\infty} \frac{1^2+2^2+3^2+\dots+n^2}{1+2+3+\dots+n} \tan\frac{1}{n} = \lim_{n\to\infty}\frac{1^2+2^2+3^2+\dots+n^2}{1+2+3+\dots+n}\cdot\frac{1}{n} =$$

$\tan\frac{1}{n} \sim \frac{1}{n}$

$$= \frac{1^2+2^2+3^2+\dots+n^2}{\cancel{n+2n}+n(1+2+3\dots+n)} = \frac{(n+1)^2}{n(\cancel{n+1})} = \lim_{n\to\infty} \frac{n+1}{n} = \boxed{1}$$

(19)

$$\lim_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[4]{n+1} = \sqrt[4]{n+1}\left(\frac{\sqrt[8]{n^2+1}}{\sqrt[4]{n+1}}-1\right) = \lim_{n\to\infty}\sqrt[4]{n+1}\left(\sqrt[8]{\frac{n^2+1}{(n+1)^2}}-1\right)=$$

$$\lim_{n\to\infty}\sqrt[4]{n+1}\left(\ln\sqrt[8]{\frac{n^2+1}{(n+1)^2}}\right) = \lim_{n\to\infty}\frac{1}{8}\sqrt[4]{n+1}\left(\ln\frac{n^2+1}{(n+1)^2}\right) = \lim_{n\to\infty}\frac{1}{8}\sqrt[4]{n+1}\left(\frac{n^2+1}{(n+1)^2}-1\right)=$$

$\sqrt[8]{\frac{n^2+1}{(n+1)^2}}-1 \sim \ln\sqrt[8]{\frac{n^2+1}{(n+1)^2}}$

$\ln\frac{n^2+1}{(n+1)^2}\sim\left(\frac{n^2+1}{(n+1)^2}-1\right)$

$$=\lim_{n\to\infty}\frac{1}{8}(n+1)^{\frac{1}{4}}\left(\frac{n^2+1-n^2-2n-1}{(n+1)^2}\right) = \lim_{n\to\infty}\frac{1}{8}(n+1)^{\frac{1}{4}}\left(\frac{-2n}{(n+1)^2}\right)=$$

$$=\frac{1}{8}\lim_{n\to\infty}-\frac{2n\sqrt{n+1}}{(n+1)^2} = \frac{1}{8}\lim -\frac{2n\sqrt{n+1}}{n^2+2n+1} = 0$$

32

$$\lim_{n\to\infty} \frac{1^2+2^2\sin\frac{1}{2}+\dots+n^2\sin\frac{1}{n}}{n^2}$$

$3.2\ \lim_{n\to\infty} = n^2 = \infty$

$\boxed{\text{Stolz}}$

2 $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ existitzen da

Emaitza: $\frac{1}{2}$

1 $\{b_n\} = \{n^2\} \to$ monotono gorakorra

$$\lim_{n\to\infty} \frac{\left(\cancel{1^2+2^2\sin\frac{1}{2}+\dots+n^2\sin\frac{1}{n}}+(n+1)^2\sin\frac{1}{n+1}\right)-\left(\cancel{1^2+2^2\sin\frac{1}{2}+\dots+n^2\sin\frac{1}{n}}\right)}{(n+1)^2-n^2} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^2\sin\frac{1}{n+1}}{(n+1)^2-n^2} = \lim_{n\to\infty}\frac{(n+1)^{\cancel{2}}\cdot\frac{1}{\cancel{n+1}}}{\cancel{n^2}+2n+1-\cancel{n^2}} = \lim_{n\to\infty}\frac{n+1}{2n+1} =$$

Baliok

$\sin\frac{1}{n+1} \sim \frac{1}{n+1}$

$= \boxed{\frac{1}{2}}$

37

$$\lim_{n\to\infty} \frac{1+2^2+3^2+\dots+n^2}{1+2+3+\dots+n}\tan\frac{1}{n} = \lim_{n\to\infty}\frac{(n+1)^2}{(n+1)}\cdot\lim_{n\to\infty}\tan\frac{1}{n} =$$

$\boxed{\text{Stolz}}$

$$= \lim_{n\to\infty} n+1\cdot\lim\tan\frac{1}{n} = \lim_{n\to\infty}(n+1)\tan\frac{1}{n} =$$

$\{a_n\} = 1+2^2+3^2+\dots+n^2$

$$= \lim_{n\to\infty}\frac{n+1}{n} =$$

$\{b_n\} = 1+2+3+\dots+n$

$\tan\frac{1}{n} \sim \frac{1}{n}$

1 $\{b_n\}$ monotono gorakorra da

$3.2\ \lim_{n\to\infty} b_n = \infty$

(1)

a)

$a_1 = \frac{5}{2}, \quad 5a_{n+1} = a_n^2 + 6$

a) $\forall n, \ 2 < a_n < 3$ bornatua

b) beherakorra

$\Big\} \{a_n\} \Rightarrow$ konbergentea $\Rightarrow \lim_{n\to\infty} a_n = l \in R$

$\lim_{n\to\infty} 5a_{n+1} = \lim_{n\to\infty} (a_n^2+6)$

$5l = l^2+6$

$l_1 = 3 \to$ baztertu egingo dugu beherakorra baita

$l_2 = 2$

(33)

$$\lim_{n\to\infty} \frac{\ln\left(\sqrt{a+\sqrt{b}}\cdot\sqrt[3]{a+\sqrt[3]{b}}\cdots\sqrt[n]{a+\sqrt[n]{b}}\right)}{\sin 1+\frac{\sin 1}{2}+\dots+\sin\frac{1}{n}} = \boxed{\text{Stolz}}$$

1. $\{b_n\}$ hertsiki monotonoa da;
2. $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ existitu
3. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$

$$= \lim_{n\to\infty} \frac{\ln\left(\sqrt{a+\sqrt{b}}\cdot\sqrt[3]{a+\sqrt[3]{b}}\cdots\sqrt[n]{a+\sqrt[n]{b}}\cdot\sqrt[n+1]{a+\sqrt[n+1]{b}}\right)-(a_n)}{\left(\sin 1+\sin\frac{1}{2}+\dots+\sin\frac{1}{n}+\sin\frac{1}{n+1}\right)-\left(\sin 1+\dots+\sin\frac{1}{n}\right)} =$$

$$= \lim_{n\to\infty} \frac{\ln\left(\sqrt[n+1]{a+\sqrt[n+1]{b}}\right)}{\sin\frac{1}{n+1}} = \lim_{n\to\infty} \frac{\ln\left(a+\sqrt[n+1]{b}\right)^{n+1}}{\sin\frac{1}{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{n+1}\ln\left(a+\sqrt[n+1]{b}\right)}{\sin\frac{1}{n+1}} = \left\{\sin\frac{1}{n+1}\right\}\sim\left\{\frac{1}{n+1}\right\}$$

$$= \lim_{n\to\infty} \frac{\cancel{\frac{1}{n+1}}\ln\left(a+\sqrt[n+1]{b}\right)}{\cancel{\frac{1}{n+1}}} = \lim_{n\to\infty} \ln\left(a+\sqrt[n+1]{b}\right) = \cancel{\lim_{n\to\infty} \ln a\cdot\ln\left(\sqrt[n+1]{b}\right)} =$$

$$\cancel{\neq \lim_{n\to\infty} \ln a\cdot\frac{1}{n+1}\cdot\ln b} =$$

$$\underset{\{a=1\}}{=} \lim_{n\to\infty} \ln\left(1+\sqrt[n+1]{b}\right) \underset{\left\{\ln\left(1+\sqrt[n+1]{b}\right)\right\}\sim\left\{\sqrt[n+1]{b}\right\}}{=} \lim_{n\to\infty} \sqrt[n+1]{b} = \boxed{1}$$

6)

c) $a_1 = a > 0\,,\quad a_{n+1} = \frac{n}{4n+1}\,a_n.$

b)

$$\lim_{n\to\infty} \frac{1}{e^n} = 0$$

Frogatu behar dugu $\forall \varepsilon > 0 \; \exists n_0(\varepsilon) \in \mathbb{N} / \forall n \geq n_0(\varepsilon)$, hau da,
$\left|\frac{1}{e^n} - 0\right| < \varepsilon$

$\frac{1}{e^n} > 0$ izanez, $\left|\frac{1}{e^n}\right| = \frac{1}{e^n}$

$\left|\frac{1}{e^n}\right| < \varepsilon$ izanik eta $\boxed{\left|\frac{1}{e^n}\right| > 0 \text{ izanez,}}$ $\frac{1}{e^n}$ jarrillo dugu, $\frac{1}{e^n} < \varepsilon$
$\frac{1}{\varepsilon} < e^n$ ~~beta~~ izanik, ondorio hau lortullo dugu,

$n_0(\varepsilon) = \min \{ n \in \mathbb{N} / \frac{1}{\varepsilon} < e^n \}$ hori izanik, $\forall n \geq n_0(\varepsilon)$, beraz
$\frac{1}{e^n} < \varepsilon$ betetzen da, hau da, $\left|\frac{1}{e^n}\right| < \varepsilon$ betetzen da, $\forall n \geq n_0(\varepsilon)$ izango da.

Arikketak. Segidak Aitzol Elu

Lehenengo arikketa zuzendu.

(1)

a) $\lim_{n\to\infty} \frac{n+1}{3n} = \frac{1}{3}$

Frogatu behar dugu, $\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N} / \forall n \geq n_0(\varepsilon)$, hau da,

$\left|\frac{n+1}{3n} - \frac{1}{3}\right| < \varepsilon$.

$\left|\frac{n+1}{3n} - \frac{1}{3}\right| < \varepsilon$ bete dadin, eskerreko atala sinplifikatu behar

ezkerrekoa sinplifikatzeko ez duzu eskuinekoa idatzi behar

dugu, $\left|\frac{\cancel{n}+1}{3n} - \frac{\cancel{n}}{3n}\right| < \varepsilon \Rightarrow \left|\frac{1}{3n}\right| < \varepsilon$, $\left|\frac{1}{3n}\right| > 0$ denez, $\frac{1}{3n} < \varepsilon$ idatzi

$=$ $=$ $=$

genezake.

$\frac{1}{\varepsilon} < 3n$ ~~betetz~~ izanezkero, $n_0(\varepsilon) = \min\{n \in \mathbb{N} / \frac{1}{\varepsilon} < 3n\}$ badago, eta

$\forall n \geq n_0(\varepsilon)$ $\frac{1}{3n} < \varepsilon$ betetzen da, hau da, $\left|\frac{n+1}{3n} - \frac{1}{3}\right| < \frac{1}{3}$ betetzen da, beraz

$\forall n \geq n_0(\frac{1}{3})$ izango da.

c) $\lim_{n\to\infty} \frac{2n^2+1}{n} = \infty$ $\quad \forall K > 0 \; \exists n_0 \in \mathbb{N} / \forall n \geq n_0 \; \left|\frac{2n^2+1}{n}\right| > K$

Frogatu behar dugu, $\forall \varepsilon > 0$ $\exists n_0(\varepsilon) \in \mathbb{N} /$ ~~non~~ $\forall n \geq n_0(\varepsilon)$,

$\left|\frac{2n^2+1}{n} - \infty\right| < \varepsilon \Rightarrow \left|\frac{2n^2+1}{n} - 2n\right| < \varepsilon \Rightarrow \left|\frac{2n^2+1}{n} - 2n\right| < \varepsilon \Rightarrow$

$n \to \infty$ denez $\infty = n$ jarri dugu

$\Rightarrow \left|\frac{2n^2+1}{n} - \frac{2n^2}{n}\right| < \varepsilon \Rightarrow \left|\frac{1}{n}\right| < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon$

$\left\{\left|\frac{1}{n}\right| > 0\right\}$

$\frac{1}{\varepsilon} < n$ izanik, $n_0(\varepsilon) = \min\{n \in \mathbb{N} / \frac{1}{\varepsilon} < n\}$ da $\forall n \geq n_0(\varepsilon)$ izanik $\frac{1}{n} < \varepsilon$

betetzen da beraz $\left|\frac{2n^2+1}{n} - \infty\right| < \varepsilon$ betetzen da. $\forall n \geq n_0(\infty)$ izango da.

④

a)

$$\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$$

$$\lim_{n\to\infty} |a_n| = 0 \underset{\substack{\lim\\ def}}{\Rightarrow} \forall \varepsilon > 0 \ \boxed{\exists n_0(\varepsilon) \in \mathbb{N}} \ \forall n \geq n_0(\varepsilon) \ \big||a_n| - 0\big| < \varepsilon \Rightarrow$$

$$\Rightarrow \forall \varepsilon > 0 \ \exists n_0(\varepsilon) \in \mathbb{N} \ \forall n \geq n_0(\varepsilon) \ \big||a_n|\big| < \varepsilon \Rightarrow$$

$$\Rightarrow \forall \varepsilon > 0 \ \exists n_0(\varepsilon) \in \mathbb{N} \ \forall n \geq n_0(\varepsilon) \ |a_n| < \varepsilon \Rightarrow$$

$$\Rightarrow \forall \varepsilon > 0 \ \boxed{\exists n_0(\varepsilon) \in \mathbb{N}} \ \forall n \geq n_0(\varepsilon) \ |a_n - 0| < \varepsilon \Rightarrow$$

$$\underset{\substack{\lim\\ def}}{\Rightarrow} \lim_{n\to\infty} a_n = 0$$

$\boxed{\phantom{xx}} = \boxed{\phantom{xx}}$ limitea betetzen denez, frogatuta geratzen da berdintza hori.

⑤

23)

$$\left(\frac{2}{n+1}\right)^{\frac{2}{2+\ln n}}$$

$$\{\ln \tfrac{a}{b}\} \sim \{\ln a - \ln b\}$$

$$\ln l = \lim_{n\to\infty} \frac{2}{2+\ln n} \ln\left(\frac{2}{n+1}\right) \Rightarrow \lim_{n\to\infty} \frac{2[\ln 2 - \ln(n+1)]}{2+\ln n} \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} \frac{2\ln 2 - \ln(n+1)}{2+\ln n} = \lim_{n\to\infty} \frac{2\ln 2}{2+\ln n} - \lim_{n\to\infty} \frac{-\ln(n+1)}{2+\ln(n)} =$$

$$= 0 - 2 = -2$$

$$\ln l = -2$$

$$e^{\ln l} = e^{-2}$$

$$l = e^{-2} = \frac{1}{e^2}$$

(4)

d)

$\lim_{n\to\infty} |a_n| = 1$ esateak ez du esan nahi $\lim_{n\to\infty} a_n = 1$ denik

Hau frogatzeko, nahikoa da $\lim_{n\to\infty} |a_n| = 1$ eta $\lim_{n\to\infty} a_n \neq 1$ duen.

Segida bat aurkitzea.

Adb, $\{a_n\} = \{-1\}$ (Segida konstantea)

$\lim_{n\to\infty} |-1| = 1$

$\lim_{n\to\infty} -1 = -1$

(5)

21)

$$\lim_{n\to\infty} \frac{\ln n}{\cot g\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\ln n}{tg\, n} = \lim_{n\to\infty} \frac{n-1}{n} = 1$$

$\{\cot g \frac{1}{n}\} \sim \{\tan n\}$ $\{\ln n\} \sim \{n-1\}$ $\{tg\, n\} \sim \{n\}$

$$\cot \frac{1}{n} = \frac{1}{\tan \frac{1}{n}}$$

$$= \lim_{n\to\infty} \ln n \cdot \tan \frac{1}{n} = \lim_{n\to\infty} \ln n \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$\boxed{\{\tan \frac{1}{n}\} \sim \{\frac{1}{n}\}}$

Polinomioek ordena altuagokoak ~~dute~~ dira

(4)

$$\frac{1}{\ln n}\sum \sin\frac{\pi}{k}$$

$$\lim_{n\to\infty}\frac{1}{\ln n}\sum \sin\frac{\pi}{k} = \lim_{n\to\infty}\frac{\sin\frac{\pi}{1}+\sin\frac{\pi}{2}+\sin\frac{\pi}{3}+\dots+\sin\frac{\pi}{n}}{\underbrace{\ln n}_{\{b_n\}}} =$$

$\boxed{\text{Stolz}}$

1) $\{b_n\}$ hertsilki monotonoa da.

3.2 $\lim_{n\to\infty}\{b_n\}=\infty$

$$= \lim_{n\to\infty}\frac{\left(\cancel{\sin\frac{\pi}{1}}+\cancel{\sin\frac{\pi}{2}}+\dots+\cancel{\sin\frac{\pi}{n}}+\sin\frac{\pi}{n+1}\right)-\left(\cancel{\sin\frac{\pi}{1}}+\cancel{\sin\frac{\pi}{2}}+\dots+\cancel{\sin\frac{\pi}{n}}\right)}{\ln(n+1)-\ln n} =$$

$$= \lim_{n\to\infty}\frac{\sin\frac{\pi}{n+1}}{\ln(n+1)-\ln n} = \lim_{n\to\infty}\frac{\sin\frac{\pi}{n+1}}{\ln\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty}\frac{\sin\frac{\pi}{n+1}}{\ln\left(1+\frac{1}{n}\right)} =$$

$\left\{\ln\left(1+\frac{1}{n}\right)\right\}\sim\left\{\frac{1}{n}\right\}$

$$= \lim_{n\to\infty}\frac{\sin\frac{\pi}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} n\sin\frac{\pi}{n+1} = \lim_{n\to\infty}\frac{n\pi}{n+1} = \boxed{\pi}$$

$\left\{\sin\frac{\pi}{n+1}\right\}\sim\left\{\frac{\pi}{n+1}\right\}$

(5)

$$n^2e^{-\sqrt{n}}$$

$$\lim_{n\to\infty} n^2e^{-\sqrt{n}} = \lim_{n\to\infty}\frac{n^2}{e^{\sqrt{n}}} = 0$$

Arillotalli . Segidall

Analisi Matematica

(2)

$$\lim_{n\to\infty} \underbrace{\left(\cos\sqrt{\frac{2\ln 5}{n}}\right)}_{a_n}^{n} = 1^{\infty} = \ell \to \text{Indeterminazioa}$$

$$\ln a_n = \ln \ell = \lim_{n\to\infty} \ln\left(\cos\sqrt{\frac{2\ln 5}{n}}\right)^n = \lim_{n\to\infty} n \ln \cos\sqrt{\frac{2\ln 5}{n}} =$$

$$= \lim_{n\to\infty} n \ln = \infty \cdot 0 = \text{Indeterminazioa}$$

$$= \lim_{n\to\infty} n\left[\cos\sqrt{\frac{2\ln 5}{n}} - 1\right] = \lim_{n\to\infty} n\left[\frac{-\left(\sqrt{\frac{2\ln 5}{n}}\right)^2}{2}\right] = \lim_{n\to\infty} \frac{\cancel{n}\,\cancel{2}\ln 5}{\cancel{n}\,\cancel{2}} =$$

$$\{a_n - 1\} \sim \{\ln a_n\} \qquad \{1 - \cos a_n\} \sim \left\{\frac{a_n^2}{2}\right\}$$

$$= \lim_{n\to\infty} -\ln 5 = -\ln 5 \Rightarrow \ln \ell = -\ln 5$$

$$\ell = 5^{-1} \Rightarrow \boxed{\ell = \frac{1}{5}}$$

(3)

$$\left(\frac{n}{3n^2+2}\right)^{\ln n}$$

$$\lim_{n\to\infty}\left(\frac{n}{3n^2+2}\right)^{\ln n} = \lim_{n\to\infty}\left(\frac{n}{3n^2+2}\right)^{(n-1)} = \lim_{n\to\infty}\left(\frac{n}{3n^2+2}\right)^{n-1}$$

$$\{n-1\} \sim \{\ln n\}$$

$$= \lim_{n\to\infty} ($$

$A^2 = A \times A \quad \{(a,b),(b,a),(a,a),(b,b)\}$

5.1

$$\frac{1/n}{\log_a(1+7/n)}$$

$$\lim_{n\to\infty} \frac{1/n}{\log_a\left(1+\frac{7}{n}\right)} = \lim_{n\to\infty} \frac{\ln \frac{n+1}{n}}{\log_a (}$$

⑦

$n^{-1/n} \ln\sqrt{n}$

$$\lim_{n\to\infty} \frac{\ln\sqrt{n}}{\sqrt[n]{n}} = \lim_{n\to\infty} \frac{\frac{1}{2}\ln n}{\sqrt[n]{n}} = \lim_{n\to\infty} \frac{n-1}{2n^{1/n}} = \boxed{\infty}$$

$$\boxed{\{\ln n\} \sim \{n-1\}}$$

⑫

$\left(\frac{1}{n}\right)^{n^2+1}$

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{n^2+1} = \cancel{\lim_{n\to\infty} \left(\ln\left(1+\frac{1}{n}\right)\right)^{n^2+1}} = \cancel{\lim_{n\to\infty} \left(\frac{1}{n}-1\right)^{n^2+1}}$$

$$\left\{\ln\left(1+\frac{1}{n}\right)\right\} \sim \left\{\frac{1}{n}\right\}$$

⑭

$\frac{n+1}{\ln n}$

$$\lim_{n\to\infty} \frac{n+1}{\ln n} = \lim_{n\to\infty} \frac{n+1}{n-1} = \boxed{1}$$

$$\{\ln n\} \sim \{n-1\}$$

⑮

$n - \ln n$

$$\lim_{n\to\infty} n - \ln n = \lim_{n\to\infty} n - (n-1) = \boxed{1}$$

$$\{\ln n\} \sim \{n-1\}$$

(20)

$$n\left(\sqrt{n^2+1}-n\right) \Rightarrow \lim_{n\to\infty} n^2\left(\sqrt{\frac{n^2+1}{n}}-1\right) = \lim_{n\to\infty} n^2\left(\sqrt{\frac{n^2+1}{n^2}}-1\right) = \lim_{n\to\infty} n^2 \ln\sqrt{\frac{n^2+1}{n^2}} =$$

$$\left\{\sqrt{\frac{n^2+1}{n^2}}-1\right\} \sim \left\{\ln\sqrt{\frac{n^2+1}{n^2}}\right\}$$

$$\cancel{\lim_{n\to\infty} n\left(\sqrt{n^2+1}-n\right) = n\left(\left(\ln\sqrt{n^2+1}-1\right)-n\right) = \frac{n}{2}\left(\ln n^2 \ldots -n\right) =}$$

$$\cancel{= \frac{n}{2}\, n\left(\left(\ln n\right)-n\right) = n\left(n-1-n\right) = n-1}$$

$$\cancel{n\left(\left(n^2+1\right)^{\frac{1}{2}}-n\right)}$$

(21)

$$\cancel{\frac{\ln n}{\cos\left(\frac{1}{n}\right)}}$$

$$= \lim_{n\to\infty} \frac{n^2}{2}\ln\frac{n^2+1}{n^2} = \lim_{n\to\infty}\frac{n^2}{2}\ln\left(1+\frac{1}{n^2}\right) =$$

$$\left\{\ln\left(1+\frac{1}{n^2}\right)\right\} \sim \left\{\frac{1}{n^2}\right\}$$

$$= \lim_{n\to\infty}\frac{n^2}{2}\cdot\frac{1}{n^2} = \boxed{\frac{1}{2}}$$

$$\cancel{\lim_{n\to\infty}\frac{\ln n}{\cos\left(\frac{1}{n}\right)} = \lim_{n\to\infty}}$$

(22)

$$n\left(\sqrt{\frac{n+1}{n}}-1\right)$$

$$\lim_{n\to\infty} n\left(\sqrt{\frac{n+1}{n}}-1\right) = \lim_{n\to\infty} n \ln\sqrt{\frac{n+1}{n}} = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n}\right)^{\frac{1}{2}} =$$

$$\left\{\sqrt{\frac{n+1}{n}}-1\right\} \sim \left\{\ln\sqrt{\frac{n+1}{n}}\right\}$$

$$= \lim_{n\to\infty}\frac{n}{2}\ln\frac{n+1}{n} = \frac{n}{2n} = \boxed{\frac{1}{2}}$$

$$\left\{\ln\frac{n+1}{n}\right\} \sim \left\{\frac{1}{n}\right\}$$

## 3.4.4 e Zenbakia

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty}\left(1+\frac{k}{n+e}\right)^{n+p} = e^k$$

$\{a_n\}\to 0$ bada, $\lim_{n\to\infty}(1+a_n)^{\frac{1}{a_n}} = e$

$\{a_n\}\to\pm\infty$ bada, $\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e$

19/1ab

$\lim\left(\frac{n+1}{n+3}\right)^{n+5}$ Kalkulatzeko dago

$$\lim\left(\frac{n+1}{n+3}\right)^{n+5} = \lim_{n\to\infty}\left(1-\frac{2}{n+3}\right)^{n+5} = e^{-2} = \frac{1}{e^2}$$

$\lim_{n\to\infty} \sqrt[n]{n} = l$ badugu, $\ln l = \ln \lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} \ln \sqrt[n]{n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$

izendatzaileko ~~ordena~~ infinitua orden handiagoa delako.

Hortaz, $\ln l = 0$ da eta, hortik, $l = e^0 = 1$ da, hots, $\lim_{n\to\infty} \sqrt[n]{n} = 1$ da.

### 3.4.3 Stolz-en irizpidea

12.T $\{a_n\}$ eta $\{b_n\}$ segidak emanik, baldintza hauek betetzen badira:

1) $\{b_n\}$ hertsiki monotonoa bada,

2) $\lim \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$ existitzen bada eta

3) hauetako bat betetzen bada

3.1 $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ 3.2 $\lim_{n\to\infty} b_n = \infty$

3.3 $\lim_{n\to\infty} b_n = -\infty$

berdintza hau betetzen da: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$

18 Adb

Kalkula dezagun $\left\{ \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\ln n} \right\}$ segidaren limitea

$\{b_n\} = \{\ln n\}$ hertsiki monotono gorakorra da eta $\lim_{n\to\infty} \ln n = +\infty$ da

$\{a_n\} = \left\{1 + \frac{1}{2} + \dots + \frac{1}{n}\right\}$ bada, $a_{n+1} - a_n = \left(1 + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+1}\right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) = \frac{1}{n+1}$

$b_{n+1} - b_n = \ln(n+1) - \ln(n) = \ln \frac{n+1}{n}$

beraz, $\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{1/n+1}{\ln \frac{n+1}{n}} = \lim_{n\to\infty} \frac{1/n+1}{1/n} = 1$

$\left\{\ln \frac{n+1}{n}\right\} \sim \left\{\frac{1}{n}\right\}$

ondorioz, $\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\ln n} = 1$ da

Froga segidaren adierazpena $\left\{\frac{a_n \cdot b_n}{c_n}\right\}$ bada eta $\{a_n\} \sim \{d_n\}$ eta $\{c_n\} \sim \{e_n\}$ badira, segidaren limitea honela geratuko da.

$$\lim_{n\to\infty} \frac{a_n b_n}{c_n} = \lim_{n\to\infty} \frac{a_n b_n d_n e_n}{c_n d_n e_n} = \lim_{n\to\infty} \left(\frac{a_n}{d_n} \cdot \frac{b_n d_n}{e_n} \cdot \frac{e_n}{c_n}\right) =$$

$$= \lim_{n\to\infty} \frac{a_n}{d_n} \lim_{n\to\infty} \frac{b_n d_n}{c_n} \lim_{n\to\infty} \frac{e_n}{c_n} = 1 \cdot \lim \frac{b_n a_n}{e_n} \cdot 1 = \lim_{n\to\infty} \frac{b_n d_n}{e_n}$$

15 Adb

$\left\{\frac{\sqrt[n]{10} - 1}{\sqrt[n]{100} - 1}\right\}$ segidaren limitea kalkulatuko dugu.

$$\lim_{n\to\infty} \frac{\sqrt[n]{10} - 1}{\sqrt[n]{100} - 1} = \lim_{n\to\infty} \frac{\ln \sqrt[n]{10}}{\ln \sqrt[n]{100}} = \lim_{n\to\infty} \frac{\frac{1}{n} \ln 10}{\frac{1}{n} \ln 100} = \lim_{n\to\infty} \frac{\ln 10}{\ln 100} = \lim_{n\to\infty} \frac{\ln 10}{\ln 10^2} =$$

$$\{\sqrt[n]{10} - 1\} \sim \{\ln \sqrt[n]{10}\}$$
$$\{\sqrt[n]{100} - 1\} \sim \{\ln \sqrt[n]{100}\}$$

$$= \lim_{n\to\infty} \frac{\ln 10}{2 \ln 10} = \frac{1}{2}$$

## 3.4.2 Infinituen-ordenak

Infinitura hurbiltzeko lau abiadura bereiziko ditugu.

| $\{(\ln a_n)^q\}$ | $\{(a_n)^p\}$ | $\{k^{a_n}\}$ | $\{a_n^{r a_n}\}$ | $q, p, r > 0$; $k > 1$ |
|---|---|---|---|---|
| logaritmikoa | polinomiala | esponentziala | berreketa | |

Infinituen ordenak erabiltzeko, segidaren adierazpena zatiketa moduan idatziko dugu.

16 Adb

$\lim_{n\to\infty} \sqrt[n]{n}$ kalkulatuko dugu.

betetzeko $\frac{1}{n}<3$ nahikoa da, $\frac{1}{\varepsilon}<n$ izatea.

Hortaz, $n_0(\varepsilon)=\min\{n\in\mathbb{N} / \frac{1}{\varepsilon}<n\}$ badago, eta $\forall n\geq n_0(\varepsilon)$ $\frac{1}{n}<\varepsilon$ betetzen da, hau da, $\left|\frac{2n+1}{n}-2\right|<2$ betetzen da, $\forall n\geq n_0(2)$

7 Def $\{a_n\}$ segida dibergentea dela esango diogu, baldin, $\infty$-ren edozein ingurune irekitarako segidaren gai batetik aurrera segidaren gai guztiak kanpoaldean badaude.

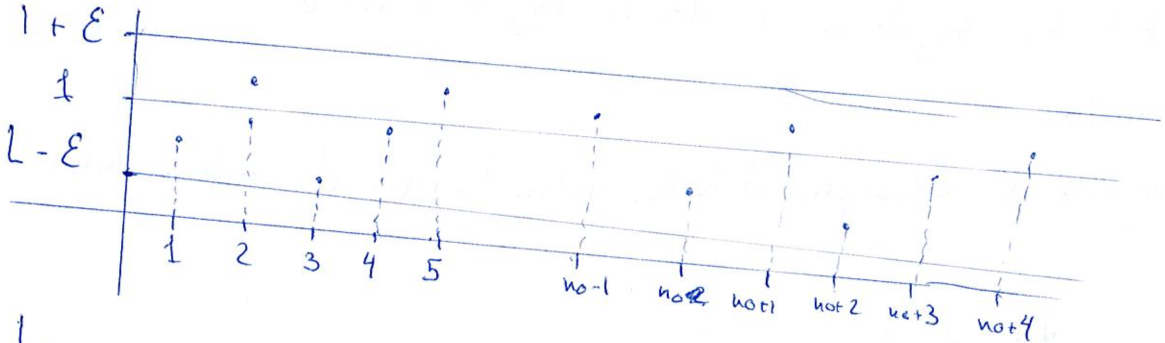
Horrela adierazten dugu $\lim_{n\to\infty} a_n=\infty$

$A = \{1\}$

$C_n = \{1\}$

5 Def $\{A_n\}$ Segida konbergentea da $\mathbb{R}$ multzoan, $l \in \mathbb{R}$ badago non $l$-ren edozein ingurune irekitan segidaren gai batetik aurrera segidaren gai guztiak baitaude.

Kasu horretan, $\{A_n\}$ segidaren limitea $l$ dela esango dugu eta $\lim_{n \to \infty} = a_n$ idatziko dugu, batzuetan $\{a_n\} \to l$ idatziko dugu.




$\lim_{n \to \infty} a_n = 1 \Leftrightarrow \forall \varepsilon = 0 \quad n_0(\varepsilon) \in \mathbb{N} \quad \forall n > n_0(\varepsilon)$

$a_n \in \mathcal{E}(1, \varepsilon)$

$d(1, a_n) < \varepsilon$

$a_n - 1 < \varepsilon$

6 Adb

$$\lim_{n \to \infty} \frac{2n+1}{n} = 2$$

Frogatu behar dugu, $\forall \varepsilon > 0 \ \exists n_0(\varepsilon) \in \mathbb{N} / \forall n \geq n_0(\varepsilon) \ \left| \frac{2n+1}{n} - 2 \right| < \varepsilon$

$\left| \frac{2n+1}{n} - 2 \right| < \varepsilon$ bete dadin, ezkerreko atala simplifikatuko dugu.

$\left| \frac{2n+1}{n} - \frac{2n}{n} \right| < \varepsilon \Rightarrow \left| \frac{1}{n} \right| < \varepsilon$ $\frac{1}{n}$ idatzi dezakegu 0 baino handiagoa

# 3. Gaia. Segidak $\mathbb{R}$ multzoan

## 3.1 Segida. Segiden limiteak

1 Def $\mathbb{R}$ multzoko segida bat $\mathbb{N}$ multzotik $\mathbb{R}$ multzora doan aplikazio bat da, non $n \in \mathbb{N}$ elementuari $f_n = a_n \in \mathbb{R}$ elementuen multzoak esleitzen zaion.

$$f: \mathbb{N} \to \mathbb{R} \quad f(n) = a_n \in \mathbb{R}$$

$f(n) = \{a_n\} \subset \mathbb{R}$ multzoa denez, $\{A\}$ segida $\mathbb{R}$-ren azpimultzo bat da segidaren multzoari segida deritze eta $a_n$ gaiari orokor derito.

Segidek bi ezaugarri dituzte, infinitu gai eta ordena batean daude.

$$\{1a, 2a, 3a, \dots\}$$

2 Adb

a)

$$\{a_n\} = \{\sin n\} = \sin 1, \sin 2, \sin 3, \dots, \sin n$$

$$\{b_n\} = -1, 1, -1, 1, -1, 1 \dots$$

$\sin(n) = \{1\}$ Segida konstantea da.

3 Def $\{b_n\}$ segida $\{a_n\}$ segidaren azpisegida da $\{b_n\}$ Segidaren gai guztiak, bestela ere segida batean jar genezake $\{b_n\} \leq \{a_n\}$.

4 Adb Bi azpisegida

$$\{d_n\} = \{\sin n / \sin n > \tfrac{1}{2}\} = \{\sin 1, S\}, \sin 8 \dots$$

$$\{e_n\} = \{\sin n / \sin n < \tfrac{1}{2} = S_4, S_5, S_6 \quad \sin_{10}, \sin_{12}, \sin_{13}, \sin_{15}\}$$

d) $\{a_n\}$ hertsiki monotono ~~g~~ beherakorra da $\forall n \geq n_0$ $a_{n+1} < a_n$ betetzen bada.

27 Adb

$\{1\}$ segida Monotono gorakorra eta beherakorra

$\left\{\frac{n^2-1}{n^2}\right\}$ segida hertsiki monotono ~~gorak beherakorra~~ gorakorra

$\left\{\frac{n^2+1}{n^2}\right\}$ segida. hertsiki monotono ~~gorakorra~~ beherakorra

28 teorema

Segida monotono bornatuak konbergenteak dira.

29 Adb

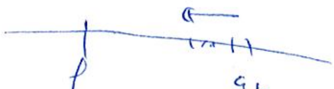
$\left\{\left(1+\frac{1}{n}\right)^n\right\} \to e$

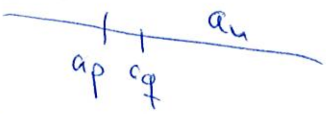
Froga daiteke ~~eta~~ $\forall n$ $2 \leq \left(1+\frac{1}{n}\right)^n \leq 3$ hau da, bornatua da. $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ segida hertsiki monotono gorakorra dela.

3.6 Cauchy-ren segidak

30 Def $\{a_n\}$ segida Cauchy-ren segida da baldin.

$\forall \varepsilon > 0 \; \exists n_0(\varepsilon) \in \mathbb{N} / \forall p,q \geq n_0(\varepsilon)$ $d(a_p, a_q) < \varepsilon$ betetzen da




31 T $\{a_n\}$ segida Cauchy-ren segida dugu baldin eta soilik baldin segida konbergentea da.

25 Adb

$\{\sin n\}$ aztertuko dugu. ~~Ez denez helduko ez infinitora ez minus infinitora, ezta izango konbergentea,~~

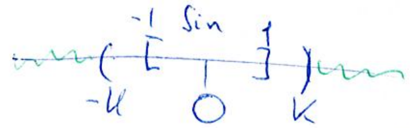
$$\{a_n\} = \{\sin n \mid \sin n > \tfrac{1}{2}\} = \{\sin 1, \sin 2, \sin 7 \dots\}$$

$$\{e_n\} = \{\sin n / \sin n < \tfrac{1}{2}\} = \{\sin 4, \sin 5, \sin 10 \dots\}$$

$\{a_n\}$, konbergentea balitz, $\frac{1}{2} \leq l_1 \leq 1$ ~~dela daukagu~~ litzateke
$\{e_n\}$, konbergentea balitz, $-1 \leq l_2 \leq -\frac{1}{2}$ litzateke

hortaz, $l_1 \neq l_2$ litzateke, ondorioz $\sin n$ ezin da konbergentea izan

Dibergentea izateko $\forall K > 0 \; \exists n_0(K) \in \mathbb{N} / \forall n \geq n_0(K) \; a_n \in ext(\mathcal{E}(0,K))$
$|a_n| > K$




(dibergentea izateko ~~-K eta K~~ baino +-K-tik ~~kanpo~~ inguruan ezin da egon)

Baina $\forall n \; \sin n \in (-2,2)$, beraz, ezin da dibergentea izan.

Ondorioz, $\{\sin n\}$ oszilatzailea da edo $\nexists \lim\limits_{n \to \infty} \sin n$

3.5.1 Segida Monotonoak

26 Def $\{a_n\} \subset \mathbb{R}$ segida emanik,

a) $\{a_n\}$ monotono gorakorra da gai bakoitzerako $a_n \leq a_{n+1}$ betetzen bada.

b) $\{a_n\}$ hertsiki monotono gorakorra da $\forall n \geq n_0 \; a_n < a_{n+1}$ betetzen bada.

c) $\{a_n\}$ monotono beherakorra da $\forall n \geq n_0 \; a_n > a_{n+1}$ betetzen bada.

§21 Prop $\{a_n\}$ segida konbergentea bada, bere azpisegida guztiak konbergenteak dira eta limite bera dute.

22 Prop $\{a_n\}$ segida konbergentea bada, segida bornatua da.

Froga Demagun $l$ dela $\{a_n\}$ segidaren limitea,

$\forall \varepsilon > 0 \ \exists n_0(\varepsilon) \in \mathbb{N} / \forall n \geq n_0(\varepsilon)$ $\underbrace{a_n \in E(l, \varepsilon)}$ edo $a_n \in E(l-\varepsilon, l+\varepsilon)$ edo

$\underline{l-\varepsilon < a_n < l+\varepsilon}$

↳ bi muturren artean egon behar da, horrela bi borne dituguu, goi borne bat eta behe borne bat

Horrela esan nahi du segidaren $\{a_{n_0}, a_{n_0+1}, \dots\}$ azpimultzoa bornaturik dagoela.

Bestalde, $\{a_1, a_2, \dots, a_{n_0-1}\}$ azpimultzo finitua da, beraz, bornatua da.

Ondorioz, $a_n$ segida bi azpimultzo bornatutan banatu dugu,

23 Prop $\{a_n\}$ segida konbergente baten limitea ez bada zero, segidaren gai batetik aurrera, segidaren gai guztiek limitearen zeinua dute.


limitea ez bada zero, $l$-ren inguruan egongo dira gai gehienak, beraz, gai gehienak positibo izango dira

24 Prop $\{a_n\}$ eta $\{b_n\}$ segidaren limitea $l$ bada, orduan $\forall n \geq n_0$ $a_n \leq c_n \leq b_n$ bada, $\{c_n\}$ segida konbergentea izango da, eta bere limitea $l$ izango da

## 3.5 Segida konbergenteak

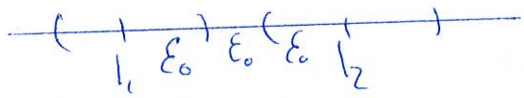
2o Prop $\{a_n\}$ Segida konbergentea bada, limite bakarra du

Froga Demagun $l_1$ eta $l_2$ direla $\{a_n\}$ segidaren limite desberdin, $l_1 \neq l_2$; beraz, $|l_1 - l_2| > 0$ da; har dezagun $\varepsilon_0 = \frac{|l_1 - l_2|}{3}$ erradioa:

$\varepsilon(l_1, \varepsilon_0)$

$a_1, a_2, \dots, a_{n_1}, a_{n_1+1}, \dots, \boxed{a_{n_2}}, a_{n_2+1}$

$\underbrace{\quad}_{n_0} \rightarrow \varepsilon(l_2, \varepsilon_0)$



Osa ditzagun $\varepsilon(l_1, \varepsilon_0)$ eta $\varepsilon(l_2, \varepsilon_0)$ inguruneak; ingurune horiek disjuntoak dira, hau da, $\varepsilon(l_1, \varepsilon_0) \cap \varepsilon(l_2, \varepsilon_0) = \emptyset$

$l_1$ bada $\{a_n\}$ segidaren limitea, $\varepsilon_0 > 0$ emanik ~~$\exists n(\varepsilon_0) \in \mathbb{N}/$~~

$\exists n(\varepsilon_0) \in \mathbb{N} / \forall n \geq n_1(\varepsilon_0) \quad a_n \in \varepsilon(l_1, \varepsilon_0)$

$l_2$ bada $\{a_n\}$ segidaren limitea, $\varepsilon_0 > 0$ emanik, $\exists n_2(\varepsilon_0) \in \mathbb{N} / \forall n \geq n_2(\varepsilon_0)$

$a_n \in \varepsilon(l_2, \varepsilon_0)$

Hortik, $n_0(\varepsilon_0) = \max\{n_1(\varepsilon_0), n_2(\varepsilon_0)\}$ bada, $\forall n \geq n_0(\varepsilon_0) \quad a_n \in \varepsilon(l_1, l_0)$

$n \geq n_0$ izanik n horrek betetzen du lehen propietatea, hau da, $l_1$-en ingurunean egongo da

eta $n \geq n_0$ izanik bigarren propietatea ere betetzen du, ~~ber~~ $n_2$ baino handiagoa baita. Orduan, $a_n \in \varepsilon(l_1, \varepsilon_0) \cap \varepsilon(l_2, \varepsilon_0)$

Baina hori ezinezkoa da, zeren eta, bi ~~limite~~ ~~en~~ inguruneak disjuntoak dira.

Ondorioz, bi limiteak ezin dira desberdinak izan.

$$\lim_{n\to\infty} \frac{1^2+2^2+\dots+n^2}{1+2^2+\dots+n}\tan\frac{1}{n} = \lim_{n\to\infty}\frac{1^2+2^2+\dots+n^2}{(1+2+\dots+n)n} \underset{\text{Stolz}}{=} \lim_{n\to\infty}\frac{(n+1)^2}{(1+2+\dots+n+(n+1))(n+1)-(1+2\dots n)n} =$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{\frac{(n+2)(n+1)(n+1)}{2}-\frac{(n+1)n^2}{2}} = \lim_{n\to\infty}\frac{(n+1)^2}{(n+1)\left(\frac{(n+2)(n+1)-n^2}{2}\right)} =$$

$$= \lim_{n\to\infty}\frac{n+1}{\left(\frac{\cancel{n^2}+n+2n+2-\cancel{n^2}}{2}\right)} = \lim_{n\to\infty}\frac{n+1}{\frac{3n+2}{2}} =$$

$$= \lim_{n\to\infty}\frac{2n+2}{3n+2} = \boxed{\frac{2}{3}}$$

$$(1+2+3\dots+n) = \frac{(n+1)n}{2}$$

24

$$\sqrt[3]{n+\sqrt{n}}-\sqrt[3]{n}$$

$$\lim_{n\to\infty}\sqrt[3]{n+\sqrt{n}}-\sqrt[3]{n}=\cancel{\lim_{n\to\infty}}\sqrt[3]{n+\sqrt{n}-n}$$

$$=\lim_{n\to\infty}\sqrt[3]{n}\left(\frac{\sqrt[3]{n+\sqrt{n}}}{\sqrt[3]{n}}-1\right)=\lim_{n\to\infty}\sqrt[3]{n}\left(\sqrt[3]{\frac{n+\sqrt{n}}{n}}-1\right)=\lim_{n\to\infty}\sqrt[3]{n}\ln\sqrt[3]{\frac{n+\sqrt{n}}{n}}=$$

$$\boxed{\left\{\sqrt[3]{\frac{n+\sqrt{n}}{n}}-1\right\}\sim\left\{\ln\sqrt[3]{\frac{n+\sqrt{n}}{n}}\right\}}$$

$$=\lim_{n\to\infty}\sqrt[3]{n}\ln\left(\frac{n+\sqrt{n}}{n}\right)^{1/3}=\lim_{n\to\infty}\frac{\sqrt[3]{n}}{3}\ln\left(\frac{n+\sqrt{n}}{n}\right)=\lim_{n\to\infty}\frac{\sqrt[3]{n}}{3}\ln\left(1+\frac{\sqrt{n}}{n}\right)=$$

$$=\lim_{n\to\infty}\frac{\sqrt[3]{n}}{3}\cdot\frac{\sqrt{n}}{n}=\lim_{n\to\infty}\frac{n^{\frac{1}{3}+\frac{1}{2}}}{3n}=\lim_{n\to\infty}\frac{n^{\frac{5}{6}}}{3n}=0$$

$$\boxed{\left\{\ln\left(1+\frac{\sqrt{n}}{n}\right)\right\}\sim\left\{\frac{\sqrt{n}}{n}\right\}}$$

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$$\frac{1^2+2^2\sin\frac{1}{2}+\dots+n^2\sin\frac{1}{n}}{n^2}$$

$$\lim_{n\to\infty}\frac{1^2+2^2\sin\frac{1}{2}+\dots+n^2\sin\frac{1}{n}}{n^2}\underset{\boxed{\text{Stolz}}}{=}\lim_{n\to\infty}\frac{\left(\cancel{1^2+2^2\sin\frac{1}{2}+\dots+n^2\sin\frac{1}{n}}+(n+1)^2\sin\frac{1}{n+1}\right)-\left(\cancel{1^2+\dots+n^2\sin\frac{1}{n}}\right)}{(n+1)^2-n^2}=$$

1. $\{b_n\}=n^2$ kesrilli monotonoe de

3. 2 $\{\lim_{n\to\infty}\{b_n=\infty$

$$=\lim_{n\to\infty}\frac{(n+1)^2\sin\frac{1}{n+1}}{(n+1)^2-n^2}=\frac{(n+1)^2\sin\frac{1}{n+1}}{\cancel{n^2}+2n+1-\cancel{n^2}}=\frac{(n+1)^2\cdot\frac{1}{n+1}}{2n+1}=\boxed{\frac{1}{2}}$$

$$\boxed{\left\{\sin\frac{1}{n+1}\right\}\sim\left\{\frac{1}{n+1}\right\}}$$

22)

$$n\left(\sqrt{\frac{n+1}{n}}-1\right)$$

$$\lim_{n\to\infty} n\left(\sqrt{\frac{n+1}{n}}-1\right) \underset{\{a_n-1\}\sim\{\ln a_n\}}{\Rightarrow} \lim_{n\to\infty} n\left(\ln\sqrt{\frac{n+1}{n}}\right) \Rightarrow \lim_{n\to\infty} n\left\{\ln\left(\frac{n+1}{n}\right)^{1/2}\right\} \underset{\text{ln-ren proprietatea ercbiliz}}{\Rightarrow} \lim_{n\to\infty} n\frac{1}{2}\ln\frac{n+1}{n} \Rightarrow$$

$$\underset{\{a_n-1\}\sim\{\ln a_n\}}{\Rightarrow} \lim_{n\to\infty} n\frac{1}{2}\left(\frac{n+1}{n}-1\right) \Rightarrow \lim_{n\to\infty} \frac{n}{2}\left(\frac{n+1}{n}-\frac{n}{n}\right) \Rightarrow \lim_{n\to\infty}\frac{\cancel{n}}{2\cancel{n}} \Rightarrow \frac{1}{2}$$

6)

$$\frac{1}{\ln n}\sum_{k=1}^{n}\frac{\pi}{k}\cos\frac{\pi}{k}$$

$a_n =$

$b_n = \ln n$

$$\lim_{n\to\infty}\frac{\pi\cos\pi+\frac{\pi}{2}\cos\frac{\pi}{2}+\dots+\frac{\pi}{n}\cos\frac{\pi}{n}}{\ln n} \underset{\text{Stolz}}{=}$$

$(a_{n+1})$

$$= \lim_{n\to\infty}\frac{\left(\pi\cos\pi+\frac{\pi}{2}\cos\frac{\pi}{2}+\dots+\frac{\pi}{n}\cos\frac{\pi}{n}+\frac{\pi}{n+1}\cos\frac{\pi}{n+1}\right)-\underbrace{\left(\pi\cos\pi+\frac{\pi}{2}\cos\frac{\pi}{2}+\dots+\frac{\pi}{n}\cos\frac{\pi}{n}\right)}^{a_n}}{\ln(n+1)-\ln n} =$$

$$= \lim_{n\to\infty}\frac{\frac{\pi}{n+1}\cos\frac{\pi}{n+1}}{\ln(n+1)-\ln n} = \lim_{n\to\infty}\frac{\frac{\pi}{n+1}\cos\frac{\pi}{n+1}}{\ln\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty}\frac{\frac{\pi}{n+1}\cos\frac{\pi}{n+1}}{\ln\left(\underbrace{\frac{n}{n}}_{1}+\frac{1}{n}\right)} \underset{\left\{\ln\left(1+\frac{1}{n}\right)\sim\left(\frac{1}{n}\right)\right\}}{=} \lim_{n\to\infty}\frac{\frac{\pi}{n+1}\cos\frac{\pi}{n+1}}{\frac{1}{n}} =$$

$$= \lim_{n\to\infty} n\left(\frac{\pi}{n+1}\cos\frac{\pi}{n+1}\right) = \lim_{n\to\infty}\frac{n\pi}{n+1}\cos\frac{\pi}{n+1} = \lim_{n\to\infty}\pi\underbrace{\cos 0}_{1} = \boxed{\pi}$$

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$$\lim_{n\to\infty} \frac{1^2+2^2+3^2+\dots+n^2}{n^3} \underset{\text{Stolz}}{=} \lim_{n\to\infty} \frac{(1^2+2^2+3^2+\dots+n^2+(n+1)^2)-(1^2+2^2+\dots+n^2)}{(n+1)^3-n^3} =$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{(n+1)^3-n^3} = \lim_{n\to\infty} \frac{(n+1)^2}{\cancel{n^3}-\cancel{n^3}+3n^2+3n+1} = \lim \frac{\overbrace{\cancel{(n+1)}^2}^{n^2+2n+1}}{3\underbrace{\cancel{(n+1)}^2}_{3n^2+3n+1}} = \boxed{\frac{1}{3}}$$

Stolz-en baldintzak $\to$ 3.2 $\lim_{n\to\infty} = n^3 = +\infty$

1. $n^3$ monotono gorakorra

3. $\boxed{\frac{1}{3}}$

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$$\lim_{n\to\infty}\left(\cos\sqrt{\frac{2\ln 5}{n}}\right)^n = 1^\infty = \ell \to \text{Indeterminazioa}$$

$$\ln \ell \Rightarrow \lim_{n\to\infty} \ln\left(\cos\sqrt{\frac{2\ln 5}{n}}\right)^n = \lim_{n\to\infty} n\ln\left(\cos\sqrt{\frac{2\ln 5}{n}}\right) =$$

$$= \lim_{n\to\infty} n\cdot\ln 1 = \lim_{n\to\infty} \boxed{\infty\cdot 0} \to \text{Indeterminazioa} \to \text{Gisinosen baliokidetasunak erabili}$$

$$= \lim_{n\to\infty} n\left[\cos\sqrt{\frac{2\ln 5}{n}} - 1\right] = \lim_{n\to\infty} n\left[\frac{-\left(\sqrt{\frac{2\ln 5}{n}}\right)^2}{2}\right] = \lim_{n\to\infty} \frac{-\cancel{n}\cdot\frac{2\ln 5}{\cancel{n}}}{2} =$$

$$\boxed{1-\cos a_n \sim \frac{a_n^2}{2}}$$

$$= \lim_{n\to\infty} \frac{-2\ln 5}{2} = \lim_{n\to\infty} -\ln 5 = -\ln 5$$

$$\ln \ell = -\ln 5 \Rightarrow \ell = 5^{-1} \Rightarrow \ell = \frac{1}{5}$$