Egitura Aljebraikoak

B, Barne-eragiketa: AxA eragiketa egin eta ~~A~~ emaitza A multzoan baldin badago.

$$* : A \times A \longrightarrow A$$
$$(a,b) \longrightarrow a*b$$
$$a,b \in A$$

Propietateak:

- elkarkorra: $(a*b)*c = a*(b*c)$
- trukakorra: $a*b = b*a$
- elementu neutroa: $a*e = a$
- elementu simetrikoa: $a*a' = a'*a = e$
  - Biderketa idazkera: $a' = a^{-1}$ alderantzizkoa
  - A " " : $a' = -a$ aurkakoa.
- Elementu Sinplifikagarria: $a*x = y*a \Longrightarrow x = \cancel{y}$
- Elementu idepotentea: Bere buruarekin biderketu eta zenbaki bera eman.

Caylyren taula, Z_3 ko kongruentziak.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

∃ Elementu neutroa, trukatze-lege

a	b	a+b	0	1	2
0	0	0	0	1	2
	1	1	1	2	0
	2	2	2	0	1
1	0	1	1	2	0
	1	2	2	0	1
	2	0	0	1	2
2	0	2	2	0	1
	1	0	0	1	2
	2	1	1	2	0

a	b	0: 012	1: 012	2: 012
0		012	120	201
1				
2				

Ariketak

① a)

$*: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, Barne-eragiketa? $(\mathbb{Z},+)$ $(\mathbb{Z}-\{0\},\cdot)$

$x,y\in\mathbb{Z} \Rightarrow x*y\in\mathbb{Z}$?

$x,y,\ 3,2\in\mathbb{Z} \Rightarrow 3x,\ 2y\in\mathbb{Z} \Rightarrow$

$(\mathbb{Z},+)$

$\Rightarrow 3x+2y\in\mathbb{Z}$

Elkartze-legea betetzen du?

$x,y,z\in\mathbb{Z}$ $\quad (x*y)*z = x*(y*z)$

$(x*y)*z = (2x+3y)*z = \cancel{2x+3y+z}\ 3(3x+2y)+2z = 9x+6y+2z \neq$

$(\mathbb{Z},+),\ (\mathbb{Z}-\{0\},\cdot)$

$x*(y*z) = x*(3y+2z) = 3x+2(3y+2z) = 3x+6y+4z$

Elementu neutroa?

$\forall x\in\mathbb{Z} \quad \exists e\in\mathbb{Z} \quad x*e = e*x = x$

$x*e = \boxed{3x+2e = x}$

$3x+2e=x \Rightarrow 2x+2e=0 \Rightarrow \boxed{e=-x}$

Trukatze legea?

$x * y = 3x + 2y$

$y * x = 3y + 2x$ $\neq$

b)

$\mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}$

$x, y \in \mathbb{Q} \quad x \oplus y \in \mathbb{Q}$

~~$x \oplus y$~~ $= x, y, 1 \in \mathbb{Q} \Rightarrow x+y+1 \in \mathbb{Q}$

$(\mathbb{Q}, +)$

$(\mathbb{Q} - \{0\}, \cdot)$

Elkarkorra? → Elkartze-legea

$\forall x, y, z \in \mathbb{Q} \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z$

$x \oplus (y \oplus z) = x \oplus (yz + 1) = \cancel{xyz} \; x(yz+1) + 1 = xyz + x + 1$

$(x \oplus y) \oplus z = (xy+1) \oplus z = (xy+1) \cdot z + 1 = xyz + z + 1$

Elemento neutro?

$x \oplus e = xe + 1 = x$

$xe + 1 = x \Rightarrow e = \frac{x-1}{x} \Rightarrow x \neq 0$

Ez dago. elemento neutrorik

Egitura Aljebraikoak

Barne-eragiketak

$(A, *)$ taldea

$(A, *, \cdot)$ $\begin{cases} \text{Eraztuna} \\ \text{Gorputza} \end{cases}$

Kanpoko eragiketak

$(V, \oplus, *)$ Bektore-espazioa

V, K

$* : K \times V \to V$

$(k, v) \mapsto k * v \in V$

1. $(V, \oplus)$ talde trukakorra
2. $(K, +, \cdot)$ Gorputza

$V(, \oplus, *)$ Bektore-espazioa

1. $\forall k \in K \quad \forall \bar{v}_1, \bar{v}_2 \in V$

$$k * (\bar{v}_1 \oplus \bar{v}_2) = \underbrace{k * \bar{v}_1}_{\in V} \oplus \underbrace{k * \bar{v}_2}_{\in V}$$

2. $\forall k_1, k_2 \in K \quad \forall \bar{v} \in V \quad (\underset{\uparrow k}{k_1 + k_2}) * v = \underset{\in V}{k_1 * \bar{v}_1} \oplus \underset{\in V}{k_2 * \bar{v}}$

3. $k_1, k_2 \in K \quad \forall \bar{v} \in V$

$$(k_1 \cdot k_2) * \bar{v} = \underbrace{k_1 * \underbrace{(k_2 * \bar{v})}_{\in V}}_{\in V}$$

4. $1 \in K$ (unitatea) $\forall \bar{v} \in V \quad 1 * \bar{v} = \bar{v}$

1. $0 \in K$ (zeroa) $\forall \bar{v} = V \quad 0 * \bar{v} = \bar{0}$

$0 * \bar{v} = (0+0) * v = 0 * \bar{v} \oplus 0 * \bar{v} \Rightarrow$ Bektore espazioaren definizioaren propietatea

$\uparrow$ $0 \in K$ - zero

$\bar{0} \in V$

$\Rightarrow 0 * \bar{v} \oplus \bar{0} = 0 * \bar{v} \oplus 0 * \bar{v} \Rightarrow \bar{0} = 0 * \bar{v}$

sinplifikagarriak

2. $\vec{0} \in V$ (Zero Bektorea) $\forall k \in K \quad k * \vec{0} = \vec{0}$

$$k * \vec{0} = (k \times k) * \vec{0} = k * \vec{0} \oplus k * \vec{0}$$

$$k * \cancel{\vec{0} * \vec{0}} = k * \vec{0} \oplus \cancel{k*\vec{0}}\ k * \vec{0} \Rightarrow \vec{0} = k * \vec{0}$$

3.

$$\forall k \in K \quad \forall \vec{v} \in V \quad k * \vec{v} = \vec{0} \Leftrightarrow k = 0 \vee \vec{v} = \vec{0}$$

$\boxed{\Leftarrow}$

$$k = 0 \vee \vec{v} = \vec{0} \Rightarrow k * \vec{v} = \vec{0} \quad \boxed{1. \text{ eta } 2.}$$

$\boxed{\Rightarrow}$

Demagun $k \neq 0$ eta $k * \vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$

$(K, +, \cdot)$ Gorputza $\Rightarrow \exists k^{-1} \in K$

$$(k * \vec{v}) = \vec{0} \Rightarrow k^{-1} * (k * \vec{v}) = k^{-1} * \vec{0} \Rightarrow (k^{-1} \cdot k) * \vec{v} = k^{-1} * \vec{0}$$

Definizioaren 3. propietatea

$$\Rightarrow 1 * \vec{v} = k^{-1} * \vec{0} \Rightarrow \vec{v} = k^{-1} * \vec{0} \Rightarrow \vec{v} = \vec{0}$$

K Gorputza

Definizioaren 4. propietatea

4. $\forall \vec{v} \in V \quad (-1) * \vec{v} = -\vec{v}$

$(V, \oplus)$ Aurkakoa

$$(-1) * \vec{v} = -\vec{v} \Rightarrow (-1) * \vec{v} \oplus \vec{v} = -\vec{v} \oplus \vec{v} \Rightarrow \cancel{(-1) * \vec{v}}$$

$$(-1) * \vec{v} \oplus \vec{v} \Rightarrow -\vec{v} = (-1) * \vec{v} \quad \text{Aurkakoa}$$

Bektore-espazioen adibideak

$\mathbb{R}: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad$ $\mathbb{R}: \mathbb{R} \times M \to M$

$(k, \vec{v}) \to \underset{\in \mathbb{R}}{k \cdot \vec{v}}$ $\qquad$ $(k, \vec{v}) \to k \underset{\mathbb{R}}{\cdot} \vec{v}$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2 \quad k = 3 \in \mathbb{R}$ $\qquad$ $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \in M \quad k = 3 \in \mathbb{R}$

$3 \underset{\mathbb{R}}{\cdot} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ $\qquad$ $3 \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 6 & 9 \end{pmatrix} \in M$

$(V, \oplus, *)$

$W = \{(x,y) \in \mathbb{R}^2 / y = x\}$ $\quad W \leq \mathbb{R}^2$?

$\mathbb{R}$ $\forall w_1, w \in W$ $\quad \forall k_1, k_2 \in \mathbb{R}$

$k_1 w_1 + k_2 w_2 \in W$

$w_1, w_2 \in W \Rightarrow \bar{w}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \; w^2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$k_1 w_1 + k_2 w_2 = k_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + k_2 \begin{pmatrix} \cancel{x_1} \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 x_1 \\ k_2 x_2 \end{pmatrix} + \begin{pmatrix} k_2 x_1 \\ k_1 x_2 \end{pmatrix} = \begin{pmatrix} k_1 x_1 + k_2 x_2 \\ k_2 x_1 + k_2 x_2 \end{pmatrix} \in W$

1

a)

$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 / x_1 = 0 \right\}$

$\forall \bar{w}_1, \bar{w}_2 \in W$ $\forall k_1, k_2 \in \mathbb{R}$ $\quad k_1 \cdot \bar{w}_1 + k_2 \cdot \bar{w}_2 \in W$

$k_1 \cdot \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ k_1 x_2 \\ k_2 x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ k_2 y_2 \\ k_2 y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ k_1 x_2 + k_2 y_2 \\ k_1 x_3 + k_2 y_3 \end{pmatrix} \in W$

b)

$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 / x_1 = 1 \right\}$

$\forall \bar{w}_1, \bar{w}_2 \in W$ $\quad \forall k_1, k_2 \in \mathbb{R}$ $\quad k_1 \cdot w_1 + k_2 \bar{w}_2 \in W$

$k_1 \cdot w_1 + k_2 \bar{w}_2 = k_1 \begin{pmatrix} 1 \\ x_2 \\ x_3 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_1 x_2 \\ k_1 x_3 \end{pmatrix} + \begin{pmatrix} k_2 \\ k_2 y_2 \\ k_2 y_3 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 \\ k_1 x_2 + k_2 y_2 \\ k_1 x_3 + k_2 y_3 \end{pmatrix} \notin W$

Ez da betetzen.

c)

$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 / x_1 = 0 \lor x_2 = 0 \right\} \Rightarrow w_1 = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \quad w_2 = \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$

$\begin{pmatrix} 0 \\ k_1 x_2 \\ k_1 x_3 \end{pmatrix} \begin{pmatrix} k_2 y_1 \\ 0 \\ k_2 y_3 \end{pmatrix} = \begin{pmatrix} k_2 y_1 \\ k_1 x_2 \\ k_1 x_3 + k_2 y_3 \end{pmatrix} \notin W$

e)

$\bar{x} = (1,1,0)$ etc $\bar{y} = (2,0,1)$

$\bar{w}_1, w_2 \in W$

$\bar{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\bar{w}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\cancel{k_1 \cdot w_1 + k_2 \cdot w_2 = k_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2k_2 \\ 0 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_1 + 2k_2 \\ k_1 \\ k_2 \end{pmatrix}}$

$\alpha(k_1 \cdot \bar{x} + k_2 \cdot \bar{y}) + \beta(k_1' \cdot \bar{x} + k_2' \bar{y}) = \alpha k_1 \bar{x} + \beta k_1' \bar{x} + \alpha k_2 \bar{y} + \beta k_2' \bar{y} =$

$= \underbrace{(\alpha k_1 + \beta k_1')}_{\in \mathbb{R}} \bar{x} + \underbrace{(\alpha k_2 + \beta k_2')}_{\in \mathbb{R}} \bar{y} \in W$

$V = \left\{ \overset{\bar{v}_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \overset{\bar{v}_2}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \right\}$

$N_{(A)} = \{ \bar{x} \in \mathbb{R}^2, A\bar{x} = \bar{0} \} \subset \mathbb{R}^2$

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$r(A) = 2$

$\begin{cases} k_1 + k_2 = 0 \\ -k_2 = 0 \end{cases}$

$\bar{b} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \in S$ $\boxed{A\bar{x} = \bar{b}}$

$\bar{b} = k_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \Rightarrow \bar{S} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

$r(A) = 2 < 3 = n$

Sistema határozott indeterminata

$$\begin{cases} k_1 + \quad + k_3 = x_1 \\ \quad k_2 + k_3 = x_2 \end{cases}$$

$\bar{S}_0 = \bar{S}_p + \bar{S}_H$

$\underline{\bar{S}_H}$

$$\begin{cases} k_1 + \ldots + k_3 = 0 \\ \quad k_2 + k_3 = 0 \end{cases}$$

$S_H = k_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ $\Rightarrow$ Girerrizzo

Aldegesek: k_1, k_2

Aldegei asllec: k_3

$$\begin{cases} k_1 = -k_3 \\ k_2 = -k_3 \end{cases}$$

$\bar{S}_p = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$ $k_1 = x_1$ $k_2 = x_2$ $\bar{S}_0 = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

$\bar{b} \notin \langle \bar{v}_1, \bar{v}_2 \rangle$

$$\overset{A}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}} \overset{\bar{x}}{\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}} = \overset{\bar{b}}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$\left(\begin{array}{cc|c} \multicolumn{2}{c}{A} & \bar{b} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$ Betesezino

③

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$$

$$c_1\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + c_4\begin{pmatrix}0\\1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow$$

$$\begin{pmatrix}1&1&0&0\\1&0&0&1\\0&1&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0&0\\1&0&0&1\\0&1&1&0\\0&0&1&1\end{pmatrix} \xrightarrow[E_{21}(-1)]{} \begin{pmatrix}1&1&0&0\\0&-1&0&1\\0&1&1&0\\0&0&1&1\end{pmatrix} \xrightarrow[E_{32}(1)]{} \begin{pmatrix}1&1&0&0\\0&-1&0&1\\0&0&1&1\\0&0&1&1\end{pmatrix} \xrightarrow[E_{43}(-1)]{} \begin{pmatrix}1&1&0&0\\0&-1&0&1\\0&0&1&1\\0&0&0&0\end{pmatrix} \quad r(A)=3$$

$r(A) = 3 < 4$

Sistema Bateragarri indeterminatuak

④

a) $A \cdot \vec{x} = \vec{0}$

$$\begin{pmatrix}1&1&3\\1&2&1\\2&1&1\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1&1&3\\1&2&1\\2&1&1\end{pmatrix} \xrightarrow[E_{21}(-1)]{} \begin{pmatrix}1&1&3\\0&1&-2\\2&1&1\end{pmatrix} \xrightarrow[E_{31}(-2)]{} \begin{pmatrix}1&1&3\\0&1&-2\\0&-1&-5\end{pmatrix} \xrightarrow[E_{23}(1)]{} \begin{pmatrix}1&1&3\\0&1&-2\\0&0&-7\end{pmatrix} \quad r(A)=3=n$$

$k_1 = k_2 = k_3 = k_4 = 0$

S.B.D

$\forall M_a \in M \quad \exists \mathcal{E} \in M \quad M_a + \mathcal{E} = \mathcal{E} + M_a = M_a$

$M_a = xI + yA \qquad x, y \in \mathbb{Q}$

$\mathcal{E} = e_1 I + e_2 A \Rightarrow \mathcal{E} = 0I + 0A = 0$

$M_a + \mathcal{E} = (xI + yA) + (e_1 I + e_2 A) =$

$= (x+e_1)I + (y+e_2)A = M$

$\Rightarrow \begin{cases} x + e_1 = x \Rightarrow e_1 = 0 \\ y + e_2 = y \Rightarrow e_2 = 0 \end{cases}$

$M_a' = x'I + y'A$

$M_a + M_a' = M_a' + M_a = 0$

$M_a + M_a' = (xI + yA) + (x'I + y'A) =$

$= (x+x')I + (y+y')A = 0$

$\Rightarrow \begin{cases} x + x' = 0 \Rightarrow x' = -x' \\ y + y' = 0 \Rightarrow y = -y' \end{cases}$

$M_a' = (-x)I + (-y)A$

$M_a + M_a' = (xI + yA) + ((-x)I + (-y)A) = (x-x)I + (y-y)A = 0I + 0A = 0$

trullefze-legea

$\forall M_1, M_2 \in M \quad M_1 + M_2 = M_2 + M_1$

$M_1 = x_1 I + y_1 A$

$M_2 = x_2 I + y_2 A \qquad x_1, x_2, y_1, y_2 \in \mathbb{Q}$

$M_1 + M_2 = (x_1 I + y_1 A) + (x_2 I + y_2 A) = (x_1 + x_2)I + (y_1 + y_2)A = (x_2 + x_1)I + (y_2 + y_1)A =$

$= (x_2 I + y_2 A) + (x_1 I + y_1 A) = M_2 + M_1$

Aljebra

(10)

$x, y \in \mathbb{Q}$

$$M = \left\{ \begin{pmatrix} x-3y & 4y \\ -4y & x+3y \end{pmatrix} / x, y \in \mathbb{Q} \right\}$$

$(M,+,\cdot)$ Gorputza?

$(M,+)$ talde abeldarra?

$(M,\cdot)$ talde abeldarra?
0 Matrizea kenduta.

Banallorra Betulletarchillo

$$+ : M \times M \longrightarrow M$$
$$(M_1, M_2) \longrightarrow M_1 + M_2 \in M$$

$$M_1 = \begin{pmatrix} x-3y & 4y \\ -4y & x+3y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} -3y & 4y \\ -4y & 3y \end{pmatrix} = x \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{I} + y \overbrace{\begin{pmatrix} -3 & 4 \\ -4 & 3 \end{pmatrix}}^{A} = x_1 I_1 + y_1 A$$

$$M_2 = x_2 I + y_2 A$$

$x_1, y_1, x_2, y_2 \in \mathbb{Q}$ $(\mathbb{Q},+,\cdot)$ Eraztuna da. Ondorioz, $x_1+x_2 \in \mathbb{Q}$

$$M_1 + M_2 = (x_1 I + y_1 A) + (x_1 I + (I x_2 + y_2 A) = (x_1+x_2) I + (y_1+y_2) A$$

Elkartze-legea

$$\forall M_1, M_2, M_3 \in M \quad M_1 + (M_2+M_3) = (M_1+M_2)+M_3$$

$$M_1 + (M_2+M_3) = (x_1 I + y_1 A) + [(x_2 I + A y_2) + (x_3 I + y_3 A)] =$$
$$= (x_1 I + y_1 A) + [(x_2+x_3) I + (y_2+y_3) A] =$$
$$= (x_1 + (x_2+x_3)) I + (y_1 + (y_2+y_3)) A =$$
$$= (x_1+x_2+x_3) I + (y_1+y_2+y_3) A =$$

$$(M_1+M_2)+M_3 = [(x_1 I + A y_1) + (x_2 I + y_2 A)] + (x_3 I + y_3$$
$$= (x_3 I + y_3 A) + [(x_1+x_2) I + (y_1+y_2) A] =$$
$$= (x_3 + (x_1+x_2)) I + (y_3 + (y_1+y_2)) A =$$
$$= (x_3+x_1+x_2) I + (y_3+y_1+y_2) A$$

$$= x_1x_2\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}-3(x_1y_2+x_2y_1)-7y_1y_2 & 4(x_1y_2+x_2y_1)\\ -4(x_2y_1+x_1y_2) & 3(x_1y_2+x_2y_1)+7y_1y_2\end{pmatrix} =$$

$$= x_1x_2\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}\cancel{-3(x_1}-7y_1y_2 & 0\\ 0 & -7y_1y_2\end{pmatrix} + \begin{pmatrix}-3(x_1y_2+x_2y_1) & 4(x_1x_2+x_2y_1)\\ -4(x_2y_1+\cancel{x_1}x_2 & 3(x_1y_2+x_2y_1)\end{pmatrix} =$$

$$= x_1x_2 I \cancel{+} - 7I_{(y_1y_2)} + (x_1x_2+x_2\cancel{x_1}) A$$

$$M_1M_2 = (x_1x_2) I + (x_1y_2 I A) + (y_1x_2 A\cdot I) + y_1y_2 A^2$$

$$A^2 = \begin{pmatrix}-3&4\\-4&3\end{pmatrix}\begin{pmatrix}-3&4\\-4&3\end{pmatrix} = \begin{pmatrix}-7&0\\0&-7\end{pmatrix} \Rightarrow A^2 = -7(I)$$

$$M_1M_2 = (x_1x_2) I + (x_1y_2) A + (y_1x_2) A + (y_1y_2) - 7I = ((x_1x_2) - 7(y_1y_2)) I + (x_1y_2 + y_1x_2) A$$

Ellkartoe-leges

$$\forall M_1, M_2, M_3 \in M$$

$$M_1\cdot(M_2\cdot M_3) = \cancel{M_3(M_1}M_2\,(M_1\cdot M_2)\cdot M_3$$

$$M_1 = x_1 I + y_1 A$$

$$M_2 = x_2 I + y_2 A$$

$$M_3 = x_3 I + y_3 A$$

$$M_1\cdot M_2 = (x_1x_2 - 7y_1y_2) I + (x_1y_2 + y_1x_2) A$$

$$M_2\cdot M_3 = (x_2\cdot x_3 - 7y_2y_3) I + (x_2y_3 + y_2x_3) A$$

$$M_1(M_2\cdot M_3) = (x_1 I + y_1 A)[(x_2x_3 - 7y_2y_3) I + (x_2y_3 + y_2x_3) A] =$$

$$= x_1[x_2x_3 - 7y_2y_3] I + x_1(x_2y_3 + y_2x_3) IA + y_1(x_2x_3 - 7y_2y_3) AI + y_1(y_2y_3 - 7x_2\cancel{y_3})(-7I) =$$

$$= [x_1(x_2x_3 - 7y_2y_3) - 7y_1(x_2y_3 + y_2x_3)] I + [x_1(x_2y_3 + y_2x_3) + y_1(x_2x_3 - 7y_2y_3)] A$$

Aljebra

10)

$M \times M \longrightarrow M$

$M_1 \times M_2 \longrightarrow M_1 \cdot M_2 \in M$?

$M_1 = x_1 I + y_1 A \quad x_1, y_1 \in \mathbb{Q}$

$M_2 = x_2 I + y_2 A \quad x_2, y_2 \in \mathbb{Q}$

$$M_1 = \begin{pmatrix} x_1 - 3y_1 & 4y_1 \\ -4y_1 & x_1 + 3y_1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} x_2 - 3y_2 & 4y_2 \\ -4y_2 & x_2 + 3y_2 \end{pmatrix} \qquad M_1 \cdot M_2 = \begin{pmatrix} (x_1-3y_1)\cdot(x_2-3y_2) + (-4y_2 \cdot 4y_1) & (x_1-3y_1)(4y_2) + 4y_1(x_2+3y_2) \\ (-4y_1)(x_2-3y_2) + (x_1+3y_1)(-4y_2) & (-4y_1)(4y_2) + (x_1+3y_1)(x_2+3y_2) \end{pmatrix} =$$

$(M_1M_2)_{11} = (x_1-3y_1)(x_2-3y_2) + 4y_1(-4y_2) = x_1x_2 - 3y_2x_1 - 3y_1x_2 - 16y_1y_2 + 9y_1y_2 =$

$= x_1x_2 - 3y_2x_1 - 3y_1x_2 - 7y_1y_2$

$(M_1M_2)_{12} = (x_1-3y_1)(4y_2) + 4y_1(x_2+3y_2) = 4y_2x_1 - 12x_1y_2 + 4y_1x_2 + 12y_1y_2 = 4y_2x_1 + 4y_1x_2$

$(M_1M_2)_{21} = (-4y_1)(x_2-3y_2) + (x_1+3y_1)(-4y_2) = -4y_1x_2 + 12y_1y_2 - 4y_2x_1 - 12y_1y_2 = -(4y_1x_2 + 4y_2x_1)$

$(M_1M_2)_{22} = -4y_1(4y_2) + (x_1+3y_1)(x_2+3y_2) = -16y_1y_2 + x_1x_2 + 3y_2x_1 + 3y_1x_2 + 9y_1y_2 =$

$= x_1x_2 + 3y_2x_1 + 3y_1x_2 - 7y_1y_2$

$$= \begin{pmatrix} x_1x_2 - 3y_2x_1 - 3y_1y_2 - 7y_1y_2 & 4y_2x_1 + 4y_1x_2 \\ -(4y_1x_2 + 4y_2x_1) & x_1x_2 + 3y_2x_1 + 3y_1x_2 - 7y_1y_2 \end{pmatrix}$$

$$M_1M_2 = \begin{pmatrix} x_1x_2 & 0 \\ 0 & x_1x_2 \end{pmatrix} + \begin{pmatrix} -3x_1y_2 - 3y_1x_2 - 7y_1y_2 & 4x_1y_2 + 4x_2y_1 \\ -4x_1y_2 - 4x_2y_1 & 3x_1y_2 + 3y_1x_2 + 7y_1y_2 \end{pmatrix} =$$