Kalkulua
Ariketa ebatziak

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Aurkibidea

1 Gaia
Aldagai anitzeko funtzioak

6.1 - Kalkulatu ezazu funtzio honen limite bikoitza jatorrian:

$$f(x,y) = \frac{x}{x+y}$$

Ebazpena.

Funtzio honek jatorrian limite bikoitzik duen edo ez aztertzeko bere limite berrituak eta norebideko limiteak aztertuko ditugu. Limite berrituak eta norabideko limiteak berdinak ez badira, funtzioak ez du limite bikoitzik izango.
Berdinak badira, aldiz, funtzioak limite bikoitza izan dezake eta beste limiten balioa izango du.
Beraz, ikus dezagun zeintzuk diren $f(x,y)$ funtzioaren limiteak jatorrian:

Dimentsio bakarreko limiteak:

$$\lim_{x\to 0} f(x,y) = g(y) = \frac{0}{0+y} = 0$$

$$\lim_{y\to 0} f(x,y) = h(x) = \frac{x}{x+0} = 1$$

Limite berrituak:

$$\lim_{y\to 0} g(y) = 0 \quad ; \quad \lim_{x\to 0} h(x) = 1.$$

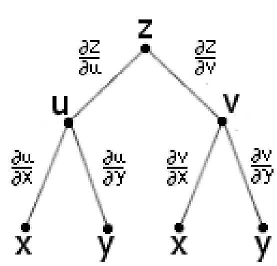
Lortutako limite berrituak bat ez datozenez, ondorioztatu dezakegu $f(x,y)$ funtzioak ez duela limite bikoitzik izango jatorrian.

2 Gaia
Aldagai anitzeko funtzioen diferentziagarritasuna

22 - Egiazta ezazu $x^2\dfrac{\partial^2 z}{\partial x^2} - y^2\dfrac{\partial^2 z}{\partial y^2} = 0$ berdintza, $z = f(u) + \sqrt{u}g(v)$, $u = xy$ eta $v = \dfrac{y}{x}$ izanik.

Ebazpena.

Ondorengo diagramak z-ren konposaketa adierazten du, katearen erregelaren arabera:



Hemendik ondorengo berdintzak atera ditzakegu:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad ; \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \quad ;$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right) \quad ;$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2} \quad ;$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial u}{\partial x}\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right) + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} +$$
$$\frac{\partial v}{\partial x}\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right) + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2} \quad ;$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 z}{\partial v \partial u}\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2} \quad ;$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right) \quad ;$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2} \quad ;$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial u}{\partial y}\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right) + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} +$$
$$\frac{\partial v}{\partial y}\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right) + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2} \quad ;$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^2 z}{\partial v \partial u}\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 z}{\partial v \partial u}\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2} \quad ;$$

Kalkulatu ditzagun aurreko berdintzetan agertzen diren aldagaiak:

$$z = f(u) + \sqrt{u}g(v) \quad ; \quad u = xy \quad ; \quad v = \frac{y}{x} \quad ;$$

$$\frac{\partial z}{\partial u} = f'_{(u)} + \frac{g_{(v)}}{2\sqrt{u}} \quad ; \quad \frac{\partial z}{\partial v} = \sqrt{u}g'_{(v)} \quad ;$$

$$\frac{\partial^2 z}{\partial u^2} = f''_{(u)} - \frac{g_{(v)}}{4\sqrt{u^3}} \quad ; \quad \frac{\partial^2 z}{\partial v^2} = \sqrt{u}g''_{(v)} \quad ; \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{g'_{(v)}}{2\sqrt{u}} \quad ;$$

$$\frac{\partial u}{\partial x} = y \quad ; \quad \frac{\partial u}{\partial y} = x \quad ; \quad \frac{\partial v}{\partial x} = \frac{-y}{x^2} \quad ; \quad \frac{\partial v}{\partial y} = \frac{1}{x} \quad ;$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad ; \quad \frac{\partial^2 v}{\partial x^2} = \frac{2y}{x^3} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad ; \quad \frac{\partial^2 v}{\partial y^2} = 0 \quad ;$$

Beraz, ondorengoak dira $\frac{\partial^2 z}{\partial x^2}$ eta $\frac{\partial^2 z}{\partial y^2}$-ren balioak:

$$\frac{\partial^2 z}{\partial x^2} = \left(f''(u) - \frac{g(v)}{4\sqrt{u^3}}\right)\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{g'(v)}{2\sqrt{u}}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right) + \left(\sqrt{u}g''(v)\right)\left(\frac{\partial v}{\partial x}\right)^2 +$$
$$\left(f'(u) + \frac{g(v)}{2\sqrt{u}}\right)\frac{\partial^2 u}{\partial x^2} + \left(\sqrt{u}g'(v)\right)\frac{\partial^2 v}{\partial x^2} \quad ;$$

$$\frac{\partial^2 z}{\partial y^2} = \left(f''(u) - \frac{g(v)}{4\sqrt{u^3}}\right)\left(\frac{\partial u}{\partial y}\right)^2 + 2\left(\frac{g'(v)}{2\sqrt{u}}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\sqrt{u}g''(v)\right)\left(\frac{\partial v}{\partial y}\right)^2 +$$
$$\left(f'(u) + \frac{g(v)}{2\sqrt{u}}\right)\frac{\partial^2 u}{\partial y^2} + \left(\sqrt{u}g'(v)\right)\frac{\partial^2 v}{\partial y^2} \quad ;$$

u, v, $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial v}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial y}$, $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 v}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial y^2}$ eta $\dfrac{\partial^2 v}{\partial y^2}$ ordezkatuz:

$$\frac{\partial^2 z}{\partial x^2} = \left(f^{''}(u) - \frac{g(v)}{4\sqrt{(xy)^3}}\right) y^2 + \left(\frac{g^{'}(v)}{\sqrt{xy}} y \frac{-y}{x^2}\right) + \left(\sqrt{xy} g^{''}(v)\right) \left(\frac{-y}{x^2}\right)^2 + \left(f^{'}(u) + \frac{g(v)}{2\sqrt{xy}}\right) \cdot 0 + \left(\sqrt{xy} g^{'}(v)\right) \frac{2y}{x^3} \quad ;$$

$$\frac{\partial^2 z}{\partial y^2} = \left(f^{''}(u) - \frac{g(v)}{4\sqrt{(xy)^3}}\right) x^2 + \left(\frac{g^{'}(v)}{\sqrt{xy}} x \frac{1}{x}\right) + \left(\sqrt{xy} g^{''}(v)\right) \left(\frac{1}{x^2}\right) + \left(f^{'}(u) + \frac{g(v)}{2\sqrt{xy}}\right) \cdot 0 + \left(\sqrt{xy} g^{'}(v)\right) \cdot 0 \quad ;$$

Beraz, ordezkatu ditzagun deribatuak berdintzaren ezkerreko atalean:

$$x^2 \left(\left(f^{''}(u) - \frac{g(v)}{4\sqrt{(xy)^3}}\right) y^2 - \frac{y^2 g^{'}(v)}{x^2\sqrt{xy}} + \left(\sqrt{xy} g^{''}(v)\right) \frac{y^2}{x^4} + \left(\sqrt{xy} g^{'}(v)\right) \frac{2y}{x^3} \right) - y^2 \left(\left(f^{''}(u) - \frac{g(v)}{4\sqrt{(xy)^3}}\right) x^2 + \frac{g^{'}(x,y)}{\sqrt{xy}} + \frac{\sqrt{xy} g^{''}(v)}{x^2} \right) =$$

$$-\frac{y^2 g^{'}(v)}{\sqrt{xy}} + \frac{2y}{x} \left(\sqrt{xy} g^{'}(v)\right) - \frac{y^2 g^{'}(v)}{\sqrt{xy}} =$$

$$\frac{2y\sqrt{xy}\sqrt{xy} g^{'}(v)}{x\sqrt{xy}} - \frac{2y^2 g(v)^{'}}{\sqrt{xy}} = \frac{2y^2 g^{'}(v)}{\sqrt{xy}} - \frac{2y^2 g^{'}(v)}{\sqrt{xy}} = 0 \quad .$$

Beraz, berdintza betetzen dela frogatuta geratzen da.

3 Gaia
Aldagai anitzeko funtzioen analisi lokala

19 - Zer luze dira azalera txikiena duen v bolumeneko paralelepipedoaren ertzak? Eta azalera handienekoak?

Geometrian, paralelepipedo bat hiru paralelogramo bikotez osatutako gorputz bat da, izanik paralelogramo bat bi ertz bikotedun plano bat.

Paralelogramoen ardatz kartesiarrekiko duten inklinazio angeluak ezagutzen ez ditugunez, paralelepipedoa paralelepipedo mota berezi bat dela suposatuko dugu; ortoedro bat, X0, Y0 eta Z0 ardatzak jarraitzen dituen paralelepipedo bat. Gero ortoedroaren planoak nahi bezala mugitzeko aukera izango dugu (mugitu, eta ez biratu, lehenengoarekin paralelepipedo bat izaten jarraituko lukeelako, baina bigarrenarekin ez).

Beraz, ortoedroaren aldeak kalkulatu eta gero, nahi dugun paralelepipedoaren ertzak kalkulatzeko, ondorengo eraldaketa egingo diogu ortoedroaren hertz bakoitzari:

$$x^* = \frac{x}{\cos\alpha\cos\beta} \quad ; \quad y^* = \frac{y}{\cos\beta\cos\delta} \quad ; \quad z^* = \frac{z}{\cos\alpha\cos\delta}$$

Izanik α, β eta δ ertzen arteko angeluak.

Ortoedro baten bolumena kalkulatzeko ondorengo funtzioa erabiltzen da:

$b(x, y, z) = xyz$

Izanik x, y eta z ortoedroaren aldeak.

Ortoedroaren bolumenaren balioa v izatea nahi dugunez, izanik v balio erreal positibo bat, ondorengo murriztapenarekin geratuko gara:

$xyz = v$

Ortoedroaren azalera kalkulatzeko, bere aldetik, ondorengo funtzioa erabiltzen da:

$f(x, y, z) = 2xy + 2yz + 2xz$

Beraz, azalera maximo zein minimoko v bolumeneko ortoedroaren aldeak kalkulatzeko, $f(x, y, z)$ funtzioaren mutur erlatiboak kalkulatuko ditugu $xyz = v$ murriztapenarekiko:

Hau da $F(x, y, z)$ funtzio laguntzailea:

$F(x, y, z) = 2xy + 2xz + 2yz + \lambda(xyz - v)$

Beraz, hauek dira bere lehenengo mailako deribatuak:

$D_1F(x, y, z) = 2y + 2z + \lambda yz$
$D_2F(x, y, z) = 2x + 2z + \lambda xz$
$D_1F(x, y, z) = 2x + 2y + \lambda xy$

Kalkulatu ditzagun, beraz, puntu kritikoak:

$$\begin{cases} 2y + 2z + \lambda yz = 0 \\ 2x + 2z + \lambda xz = 0 \\ 2x + 2y + \lambda xy = 0 \\ xyz = v \end{cases}$$

Lambda islatu dezagun:

$$\begin{cases} \lambda = \dfrac{-2y - 2z}{yz} \\ \lambda = \dfrac{-2x - 2z}{xz} \\ \lambda = \dfrac{-2x - 2y}{xy} \end{cases}$$

Lambdaren ekuazioak berdintzen baditugu:

$$\frac{-2y - 2z}{yz} = \frac{-2x - 2z}{xz} \Rightarrow \frac{-2}{z} - \frac{2}{y} = \frac{-2}{z} - \frac{2}{x} \Rightarrow -\frac{2}{y} = -\frac{2}{x} \Rightarrow y = x$$

$$\frac{-2x - 2z}{xz} = \frac{-2x - 2y}{xy} \Rightarrow \frac{-2}{z} - \frac{2}{x} = \frac{-2}{y} - \frac{2}{x} \Rightarrow -\frac{2}{z} = -\frac{2}{y} \Rightarrow z = y$$

Beraz, x, y eta z balio berdina dute. Ordezkatu ditzagun baldintzan:

$xyz = v \Rightarrow x^3 = v \Rightarrow x = \sqrt[3]{v}$

Hortaz, ondorengoa da puntu kritiko bakarra:

$x = y = z = \sqrt[3]{v}$

Beraz, aztertu dezagun $(\sqrt[3]{v}, \sqrt[3]{v}, \sqrt[3]{v})$ puntu kritikoaren izaera:

Puntu kritikoaren izaera aztertzeko, hessetarrak kalkulatzeko $g(x, y)$ funtzioaz baliatuko gara:

$f(x, y, z) = 2xy + 2xz + 2yz$ **;** $xyz = v \Rightarrow z = \dfrac{v}{xy}$ **;**

$$g(x, y) = 2xy + \frac{2xv}{xy} + \frac{2yv}{xy} \Rightarrow g(x, y) = 2xy + \frac{2v}{x} + \frac{2v}{y}$$

$$D_1g(x, y) = 2y - \frac{2v}{x^2} \qquad D_{11}g(x, y) = \frac{4v}{x^3}$$

$$D_2g(x, y) = 2x - \frac{2v}{y^2} \qquad D_{12}g(x, y) = 2$$

$$D_{21}g(x, y) = 2$$

$$D_{22}g(x, y) = \frac{4v}{y^3}$$

Beraz, ondorengoak dira hessetarrak:

$$1 \;\; ; \;\; D_{11}g_{(x,y)} = \frac{4v}{x^3} \;\; ; \;\; \Delta = \begin{vmatrix} D_{11}g_{(x,y)} & D_{12}g_{(x,y)} \\ D_{21}g_{(x,y)} & D_{22}g_{(x,y)} \end{vmatrix} = \begin{vmatrix} \frac{4v}{x^3} & 2 \\ 2 & \frac{4v}{y^3} \end{vmatrix} = \frac{16v^2}{x^3y^3}$$

Beraz, puntu kritiko honi dagozkien hessetarren balioak ondorengoak dira:

Puntua	1	$D_{11}g_{(x,y)}$	Δ
$\left(\sqrt[3]{2v}, \sqrt[3]{2v}, \frac{\sqrt[3]{2v}}{2}\right)$	1	2	2

Beraz, minimo erlatibo bat dugu puntu kritiko hau.

Beraz, v bolumeneko paralelepipedo baten azalera minimoa $2(\sqrt[3]{v})^2 + 2(\sqrt[3]{v})^2 + 2(\sqrt[3]{v})^2 = 6v^{2/3}$ izango da. Bere aldetik, maximorik lortu ez denez, ondorioztatu dezakegu v bolumeneko paralelepipedo batek izan dezakeen azalera maximoa infinitua dela. Beste era batera esanda, beti izango da azalera handiagoa duen v bolumeneko paralelepipedo bat.

4 Gaia
Integral mugagabeak

7.5 - Kalkulatu ezazu ondoko integrala:

$\int \frac{dx}{\sqrt{e^x - 1}}$

Ebazpena.

Integral hau kalkulatzeko, egin dezagun ondorengo aldagai aldaketa:

$t = \sqrt{e^x - 1}$

Berdintza honetatik ondorengoa ondorioztatzen dugu:

$$dt = \frac{e^x dx}{2\sqrt{e^x - 1}} \quad ; \quad dx = \frac{2tdt}{t^2 + 1}$$

Beraz, aldagai berriarekin integralaren itxura ondorengoa izango da:

$$\int \frac{dx}{\sqrt{e^x - 1}} = \int \frac{\frac{2tdt}{t^2+1}}{t} = \int \frac{2dt}{t^2+1}$$

Kalkulatu dezagun integrala:

$$\int \frac{2dt}{t^2+1} = 2 \arctan t$$

Beraz, t aldagaia berriro ordezkatzen badugu, bilatzen ari garen emaitza lortuko dugu:

$$2 \arctan t = 2 \arctan(\sqrt{e^x - 1})$$

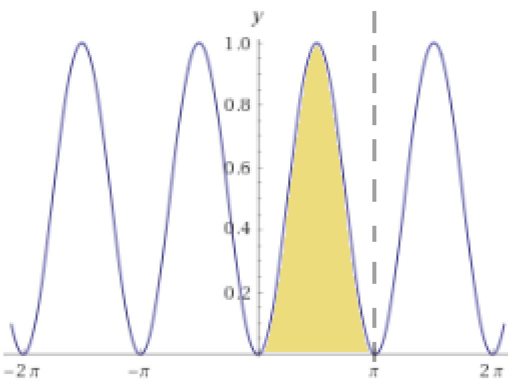
Frogatu dezagun lortutako integralaren deribatua $\frac{dx}{\sqrt{e^x - 1}}$ dela:

$$(2 \arctan(\sqrt{e^x - 1}))' = \frac{\frac{2e^x dx}{2\sqrt{e^x - 1}}}{e^x} = \frac{\frac{e^x dx}{\sqrt{e^x - 1}}}{e^x} = \frac{dx}{\sqrt{e^x - 1}}$$ Beraz, frogatuta geratzen da.

5 Gaia
Integral mugatuak

13.5 - Kalkula itzazu ondoko funtzioak biratzean osatzen duen gorputzaren bolumena:

$y = \sin^2 x$, $[0, \pi]$ tartean.



Hau da biratu beharko dugun barrutia.

Ebazpena.

Funtzio batek ardatz baten inguruan biratzean sortzen duen gorputzaren bolumena kalkulatzako, aski da $\pi f(x)^2$-ren integrala kalkulatzea, izanik $f(x)$ aipatutako funtzioa. Beraz, ondorengo integrala kalkulatu beharko dugu:

$\pi \int_0^\pi (\sin^2 x)^2 dx = \pi \int_0^\pi \sin^4 x dx$

Burutu dezagun lehenengo integral mugagabearen kalkulua, zatika:

$u = \sin^2 x \qquad du = 2 \sin x \cos x dx \qquad dv = \sin^2 x dx \qquad v = \int sin^2 x dx$

v zatikako integral bat denez, kalkulatu dezagun lehenengo integral hori:

$p = \sin x \qquad dp = \cos x dx \qquad dq = \sin x dx \qquad q = -\cos x$

$I_{dv} = -\cos x \sin x - \int -\cos x \cos x dx = -\cos x \sin x + \int \cos^2 x dx$

$I_{dv} = -\cos x \sin x + \int \left(1 - \sin^2 x\right) dx = -\cos x \sin x + \int dx - \int \sin^2 x dx$

$2I_{dv} = x - \cos x \sin x \Rightarrow I_{dv} = \dfrac{x - \cos x \sin x}{2}$

Beraz, jarrai dezagun integral nagusia:

$$\int \sin^4 x dx = \frac{(x - \cos x \sin x)\sin^2 x}{2} - \int \frac{(x - \cos x \sin x) 2 \sin x \cos x dx}{2} =$$
$$\frac{(x - \cos x \sin x)\sin^2 x}{2} - \int x \sin x \cos x dx + \int \sin^2 x \cos^2 x dx =$$
$$\frac{(x - \cos x \sin x)\sin^2 x}{2} - \int x \sin x \cos x dx + \int \sin^2 x (1 - \sin^2 x) dx =$$
$$\frac{(x - \cos x \sin x)\sin^2 x}{2} - \int x \sin x \cos x dx + \int \sin^2 x dx - \int \sin^4 x) dx$$

Jatorrizko integrala eta jada integratu dugun gai bat agertzen direnez, ordezkatu ditzagun:

$\int \sin^4 x dx = I$ bada,

$$\int \sin^4 x dx = \frac{(x - \cos x \sin x)\sin^2 x}{2} - \int x \sin x \cos x dx + \int \sin^2 x dx - \int \sin^4 x) dx \Rightarrow$$
$$I = \frac{(x - \cos x \sin x)\sin^2 x}{2} - \int x \sin x \cos x dx + \frac{x - \sin x \cos x}{2} - I \Rightarrow$$
$$2I = \frac{(x - \cos x \sin x)\sin^2 x + x - \sin x \cos x}{2} - \int x \sin x \cos x dx$$

$x \sin x \cos x dx$-ren integrala zatikakoa denez, burutu dezagun:

$$\int x \sin x \cos x dx = \frac{1}{2} \int x 2 \sin x \cos x dx$$

$n = x \qquad dn = dx \qquad dm = 2 \sin x \cos x dx \qquad m = \sin^2 x$

$$\frac{1}{2} \int x 2 \sin x \cos x dx = \frac{1}{2}\left(x \sin^2 x - \frac{x - \sin x \cos x}{2}\right) = \frac{2x \sin^2 x + \sin x \cos x - x}{4}$$

Beraz, ordezkatu dezagun:

$$2I = \frac{(x - \cos x \sin x)\sin^2 x + x - \sin x \cos x}{2} - \frac{2x \sin^2 x + \sin x \cos x - x}{4} =$$

$$\frac{2(x - \cos x \sin x)\sin^2 x + 2x - 2\sin x \cos x - 2x\sin^2 x - \sin x \cos x + x}{4} \Rightarrow$$

$$I = \frac{-2\sin^3 x \cos x - 3\sin x \cos x + 3x}{8} = \frac{-(2\sin^2 x + 3)\sin x \cos x + 3x}{8}$$

Hortaz, ondorengoa da biraketan lortutako gorputzaren bolumena:

$$B = \pi \left[\frac{-(2\sin^2 x + 3)\sin x \cos x + 3x}{8} \right]_0^{\pi} =$$

$$\pi \left(\frac{-(2\sin^2 \pi + 3)\sin \pi \cos \pi + 3\pi}{8} - \frac{-(2\sin^2 0 + 3)\sin 0 \cos 0 + 3 \cdot 0}{8} \right) = \frac{3\pi^2}{8}$$