1 Congruence with Isomorphic Binary Expression Nodes

Theorem 1.1. *The inferred type of a binary expression node from any valid abstract syntax tree is unaffected by the ordering of its two children nodes.*

Proof. Let Σ be the set of all nodes from a valid abstract syntax tree. We can then define Γ to be the set of all possible data types in our language, and similarly define F to be the set of all valid data types in our program, such that $F \subset \Gamma$.

Let us then describe the process of inferring a data type from any node $\delta \in \Sigma$ as the transformation function $\lambda(\delta) \rightarrow \gamma, \ \forall \delta \in \Sigma$ where $\gamma \in \Gamma$.

Let us also define the binary operation of comparing two data types as $\cap : \Gamma \times \Gamma \rightarrow \Gamma$ such that $\gamma_i \cap \gamma_j \rightarrow \gamma_k$ where $\gamma_k \in F \leftrightarrow \gamma_i \equiv \gamma_j$ otherwise $\gamma \notin F$.

To infer the data type of a binary expression, we can further describe that particular function as $\lambda(\hat{\delta}) = \lambda(\alpha) \cap \lambda(\beta)$ where α and β represent the left and right children respectfully of the binary node $\hat{\delta}$.

To show that the inferred type of a node is unaffected by the ordering of its two children nodes, we need to show that the binary operation $\cap$ is commutative such that $\gamma_i \cap \gamma_j \equiv \gamma_j \cap \gamma_i$.

It shows from our definition of $\cap$ that for $\gamma_i \cap \gamma_j$ to produce a data type namely γ_k such that $\gamma_k \in F$, it is required that both γ_i and γ_j must be the equivalent. Therefore $\gamma_i \equiv \gamma_j \equiv \gamma_{\prime}$ which implies that it is equivalent to say that $\gamma_{\prime} \cap \gamma_{\prime} \rightarrow \gamma_k$ for some $\gamma_k \in F$.

Therefore, it must also be true to say that $\gamma_j \cap \gamma_i \rightarrow \gamma_k$ for some $\gamma_k \in F$ since it is empirically true that $\gamma_i \cap \gamma_j \equiv \gamma_{\prime} \cap \gamma_{\prime} \equiv \gamma_j \cap \gamma_i$.

Since $\gamma_j \cap \gamma_i \equiv \gamma_i \cap \gamma_j$, we can conclude that $\lambda(\hat{\delta}) = \lambda(\alpha) \cap \lambda(\beta)$ where α and β represent the left and right children respectfully of the binary node $\hat{\delta}$ is equivalent to $\lambda(\hat{\delta}) = \lambda(\beta) \cap \lambda(\alpha)$.

Therefore, the proposition that the inferred type of a binary expression node from any valid abstract syntax tree is unaffected by the ordering of its two children nodes. □

Corollary 1.1.1. *Two binary expression nodes from any valid abstract syntax trees are said to have the same type if both nodes are isomorphic to one another.*

Proof. Let us assume that we have two binary expression nodes namely α and β from some abstract syntax tree which are isomorphic to one another such that there exists a bijection $\phi : \alpha \rightarrow \beta$ which preseves the integrety of each node through their transformations.

Therefore, if one node α has some inferred type γ, then it is known from 1.1 that all isomorphic variations of α will have the same inferred type γ. □