J. Soc. Indust. Appl. Math.
Vol. 4, No. 1, March, 1956
Printed in U.S.A.

A VARIATIONAL PROBLEM WITH CONSTRAINTS IN DYNAMIC PROGRAMMING*

RICHARD BELLMAN

1. Introduction. We have, in another place, [5], considered the problem of maximizing the functional

$$J(y) = \int_0^T F(x, y)\, dt \tag{1.1}$$

subject to the constraints

$$\begin{aligned} &\text{a.} \quad dx/dt = G(x, y),\ x(0) = c, \\ &\text{b.} \quad 0 \leqq y \leqq x, \end{aligned} \tag{1.2}$$

using the classical techniques of the calculus of variations, with modifications imposed by the constraint (1.2b).

We now wish to treat the problem using the approach of the theory of dynamic programming. Expositions of the techniques we shall employ may be found in [1], [2], or [3], and we refer to these works for more detailed discussion.

The basic idea is to consider a variational problem as a multi-stage decision process of continuous type. The emphasis will not be upon determining $y(0)$ as a function of c and T above, rather than determining y as a function of t for $0 \leqq t \leqq T$.

We shall show how we may obtain a functional equation for

$$U(c, T) = \mathrm{Max}_y\, J(y),$$

and how a limiting form of this functional equation may be used to determine the structure of the extremal curve given certain simple structural properties of $F(x, y)$ and $G(x, y)$.

In order to establish these results rigorously, we consider first the finite analogue of the variational problem above, namely, the problem of maximizing

$$J(\{y_k\}) = \sum_{k=0}^{N} F(x_k, y_k), \tag{1.3}$$

where the y_k are subject to the constraints

$$\begin{aligned} &\text{a.} \quad x_{k+1} - x_k = G(x_k, y_k) \qquad (k = 0, 1, 2, \cdots, N-1), \\ &\text{b.} \quad 0 \leqq y_k \leqq x_k . \end{aligned} \tag{1.4}$$

* Received by the editors March 24, 1955.

This we again attack by means of the functional equation approach, employing a technique we have used repeatedly, cf. [1], [2], [3]. This approach is particularly suitable for a machine computation of the problem.

2. The basic functional equation. Let us make the assumption that the solution to the variational problem described in (1.1) and (1.2) exists and that $y(t)$ is a continuous function of t in $[0, T]$ for every $T \geqq 0$.

We then define the function

$$U(c, T) = \operatorname{Max}_y J(y).$$

Let us now proceed formally to obtain a partial differential equation for U on the assumption that U has continuous partials with respect to c and T.

We have, along an extremal,

$$U(c, S + T) = \int_0^S F(x, y)\, dt + \int_S^{S+T} F(x, y)\, dt.$$

Hence, employing the "principle of optimality", the section of the function $y(t)$ in $[0, S]$ is determined by the equation

$$U(c, S + T) = \operatorname*{Max}_{y[0,S]} \left[\int_0^S F(x, y)\, dt + \int_S^{S+T} F(x, y)\, dt \right]. \tag{2.1}$$

Consider the second integral over $[S, S + T]$. At $t = S$, the value of x is $x(S)$ as determined by the differential equation of (1.2a). Since the problem is independent of the starting time, we see that

$$\int_S^{S+T} F(x, y)\, dt = U(c(S), T).$$

Hence, (2.1) becomes

$$U(c, S + T) = \operatorname*{Max}_{y[0,S]} \left[\int_0^S F(x, y)\, dt + U(c(S), T) \right].$$

If we now let $S \to +0$ and assume the continuity of all functions appearing, we obtain in the limit the partial differential equation

$$U_T = \operatorname*{Max}_{0 \leqq v \leqq c} [F(c, v) + G(c, v) U_c], \tag{2.2}$$

where we set $v = y(0)$ for typographical convenience. The condition $0 \leqq v \leqq c$ is the consequence of the restriction $0 \leqq y \leqq x$.

3. Heuristic considerations. Let us now see what we can deduce from (2.2) under the assumption that U_c is a positive, increasing function of T. This is a very natural condition to expect to be satisfied in a number of problems of engineering and economic origin.

We first of all impose the conditions that $F(x, y)$ and $G(x, y)$ be concave functions of y for all $x \geqq y \geqq 0$. It follows that the function

$$K(v) = F(c, v) + G(c, v)U_c$$

for any fixed values of c and U_c will be a concave function and possess a single maximum, which may be at $v = 0$, $v = c$, or in between.

If we assume that U_c is a continuous function of T, as we have every right to expect, we obtain the important result that as T varies, a region where $v = 0$ can never abut a region where $v = c$. There must always be a transition region where $0 < v < c$, a region we call the Euler region since the Euler equation must be valid there, a region of free variation.

The critical value of v is the point where $K'(v) = 0$, or

$$F_v + G_v U_c = 0.$$

Let us make the further assumption that $G_v > 0$ for all v in the range $0 \leqq v \leqq c$ for any $c > 0$. If we assume that $U_c \to \infty$ as $T \to \infty$, again a physically plausible result in many of these problems, it follows that, for sufficiently large T, $K'(v) = F_v + G_v U_c$ will be positive for $0 \leqq v \leqq c$. The maximum will then occur at $v = c$.

It is easy to show that $U_c \to 0$ as $T \to 0$. If we assume that $F_v < 0$ for all $0 \leqq v \leqq c$, (e.g., $F(x, y) = x - y$, see below), we see that $K'(v) < 0$ for small T and hence the maximum is at $v = 0$.

Under the assumption that U_c is positive and monotone increasing in T, we see that the solution will be the following:

$$\begin{aligned} v &= 0 && (0 \leqq c \leqq T_1), \\ 0 < v &< c && (T_1 < T < T_2), \\ v &= c && (T_2 \leqq T), \end{aligned}$$

where the transition points T_1 and T_2, in general, will be functions of c, and may degenerate, i.e., $T_1 = 0$, $T_1 = T_2$ or $T_2 = T$.

We see then the utility of the functional equation as a heuristic device for obtaining information concerning the structure of the solution.

To prove these results, we have two avenues open to us. We may first of all use the method of successive approximations on (2.2), obtaining a sequence of functions $U_n(c, T)$ in the following way. A choice of $v_0(c, T)$ subject to the inequality $0 \leqq v_0(c, T) \leqq c$, yields a function $U_0(c, T)$ determined by the partial differential equation

$$\begin{aligned} U_{0T} &= F(c, v_0) + G(c, v_0)U_{0c}, \\ U_0(c, 0) &= 0 \qquad (c \geqq 0). \end{aligned}$$

Having obtained U_0, we choose a new $v_1(c, T)$ as the function yielding the maximum of

$$K(v, U_0) = F(c, v) + G(c, v)U_{0c}$$

subject to $0 \leqq v \leqq c$. The choice of v_1 yields in turn a function U_1 as the solution of

$$U_{1T} = K(v_1, U_1),$$
$$U_1(c, 0) = 0 \qquad (c \geqq 0).$$

We now continue in this fashion.

We shall in a subsequent paper, in a series of which [4] is the first, consider in detail the existence and uniqueness of solutions of functional equations of the form of (2.2) and the convergence of the successive approximations above.

Here we shall follow a different path and consider the discrete multi-stage process corresponding to the variational problem. As we noted above, a derivation of these results employing classical techniques is contained in [5].

4. Finite version. Replacing the integral by a Riemann sum and the differential equation by a difference equation, we are led in a natural way to consider the problem of determining the maximum of

$$J(\{y_k\}) = \sum_{k=0}^{N} F(x_k, y_k),$$

where x_k and y_k are related by the equations

$$x_{k+1} - x_k = G(x_k, y_k) \qquad (k = 0, 1, 2, \cdots, N - 1),$$
$$x_0 = c,$$

and y_k is restricted by the condition,

$$0 \leqq y_k \leqq x_k \qquad (k = 0, 1, 2, \cdots, N).$$

We assume that F and G are continuous functions of x and y, and set

$$U_N(c) = \operatorname*{Max}_{\{y_k\}} J(\{y_k\}).$$

The basic functional equation is now

$$U_{N+1}(c) = \operatorname*{Max}_{0 \leqq v \leqq c} [F(c, v) + U_N(c + G(c, v))] \quad (N = 0, 1, 2, \cdots),$$

using the notation $v = y_0$, with

$$U_0(c) = \operatorname*{Max}_{0 \leqq v \leqq c} F(c, v).$$

In order to simplify the analysis we shall consider only the case where

$$F(c, v) = c - v,$$

a case of some interest in itself.

It corresponds to a multi-stage allocation process where we have a single resource, measured by x_k at time k. A certain quantity of this resource y_k may be used to increase x_k, yielding

$$x_{k+1} = x_k + G(x_k, y_k)$$

at time $k + 1$. The profit on the other hand will be measured by $x_k - y_k$. The problem is to determine the allocation policy which maximizes the total N-stage profit (cf. [1], [2] for a discussion of similar problems).

5. A simple case. Let us begin with a simple case, where the analysis will not overshadow the ideas. The discrete version is the problem of maximizing

$$J(\{y_k\}) = \sum_{k=0}^{N} (x_k - y_k),$$

where

$$\begin{aligned} &\text{(a)} \qquad x_{k+1} = x_k + b(y_k), \\ &\text{(b)} \qquad 0 \leqq y_k \leqq x_k \qquad (k = 0, 1, 2, \cdots, N). \end{aligned}$$

Setting

$$U_N(c) = \underset{\{y_k\}}{\text{Max}}\, J(\{y_k\}),$$

we clearly have the functional equation

$$\begin{aligned} U_{N+1}(c) &= \underset{0 \leqq v \leqq c}{\text{Max}} [c - v + U_N(c + b(v))] \quad (N = 1, 2, \cdots), \\ U_0(c) &= c. \end{aligned}$$

Our aim is to determine the structure of the optimal policy under appropriate assumptions concerning $b(y)$. We shall assume

$$\begin{aligned} &b(0) = 0, \qquad b'(0) = \infty, \\ &b'(y) > 0, \qquad b'(y) \to 0 \qquad (y \to \infty), \\ &b''(y) < 0. \end{aligned}$$

A simple function satisfying these conditions is $b(y) = y^{1/2}$.

We shall show that for each N, the optimal first allocation $v_N(c)$ has the following form as a function of c,

$$\begin{aligned} v_N(c) &= c \qquad (0 \leqq c \leqq c_N), \\ 0 < v_N(c) &< c \qquad (c_N < c), \end{aligned}$$

where $\{c_N\}$ is a sequence we shall determine inductively below. In this case, there is only one transition point.

The proof will be inductive. Let us begin with the case $N = 1$. We have

$$U_1(c) = \underset{0 \leqq v \leqq c}{\text{Max}} [c - v + U_0(c + b(v))] = \underset{0 \leqq v \leqq c}{\text{Max}} [G_0(v, c)]. \tag{5.1}$$

If an internal maximum occurs, it occurs at a point where $\partial G_0/\partial v = 0$. This equation is

$$\frac{1}{b'(v)} = U_0'(c + b(v)). \tag{5.2}$$

Since $U_0(c) = c$, $U_0'(c) = 1$, equation (5.2) by virtue of the assumptions concerning b has precisely one positive root. If this root is less than c, there is an internal maximum. The critical value of c is then given by the root of

$$\frac{1}{b'(c)} = 1;$$

call this value c_1.

If $c < c_1$, the maximum in (5.1) will occur at $v = c$. If $c > c_1$, there will be an internal maximum at v_1.

Hence, we have

$$\begin{aligned} U_1(c) &= U_0(c + b(c)) && (c \leqq c_1, v = c) \\ &= c - v_1 + U_0(c + b(v_1)) && (c > c, v = v_1 < c). \end{aligned}$$

The function $U_1(c)$ is clearly differentiable for $c < c_1$ and for $c > c_1$. We have

$$\begin{aligned} U_1'(c) &= (1 + b'(c))U_0'(c + b(c)) = 1 + b'(c) && (c < c_1) \\ &= 1 + U_0'(c + b(c)) = 2 && (c > c_1). \end{aligned}$$

At $c = c_1$, $b'(c) = 1$, and we have equality. Hence, $U_1(c)$ has a continuous derivative. Furthermore, $U_1'(c) > U_0'(c)$ for $c \geqq 0$.

Let us now investigate convexity. We have

$$\begin{aligned} U_1''(c) &= b''(c) && (c < c_1) \\ &= 0 && (c > c_1). \end{aligned}$$

Hence $U_1''(c) \leqq 0$ for all $c > 0$.

We now turn to the case $N = 2$. We have

$$U_2(c) = \underset{0 \leqq v \leqq c}{\text{Max}} [c - v + U_1(c + b(v))].$$

If an internal maximum occurs, it occurs at the point where

$$\frac{1}{b'(v)} = U_1'(c + b(v)). \tag{5.3}$$

Since $U_1''(c + b(v)) \leqq 0$ and $(1/b'(v))' > 0$, there is precisely one root of this equation, which we call $v_2 = v_2(c)$. Since $U_1' > U_0'$, it follows that $v_2(c) > v_1(c)$.

As above, the critical value of c is the root of

$$\frac{1}{b'(c)} = U_1'(c + b(c)). \tag{5.4}$$

Call this root c_2. Since $U_1' > U_0'$ and $(1/b')' > 0$, $c_2 > c_1$. We thus have

$$\begin{aligned} U_2(c) &= U_1(c + b(c)) && (\text{for } c \leqq c_2) \\ &= c - v_2 + U_1(c + b(v_2)) && (\text{for } c \geqq c_2). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} U_2'(c) &= [1 + b'(c)]U_1'(c + b(c)) && (\text{for} \quad c < c_2) \\ &= 1 + U_1'(c + b(v_2)) + \frac{dv^2}{dc}[-1 + b'(v_2)U_1'(c + b(v_2))] \\ &= 1 + U_1'(c + b(v_2)) = 1 + 1/b'(v_2) && (\text{for} \quad c > c_2). \end{aligned}$$

At $c = c_2$, $U_2'(c)$ is continuous, recalling.

Let us now examine the convexity of $U_2(c)$.

$$\begin{aligned} U_2''(c) &= b''(c)U_1'(c + b(c)) + [1 + b'(c)]^2 U_1''(c + b(c)) && (\text{for } c < c_2) \\ &= U_1''(c + b(v_2))\left[1 + b'(v_2)\frac{dv_2}{dc}\right] && (\text{for } c > c_2). \end{aligned} \tag{5.5}$$

Using (5.3),

$$-\frac{b''(v_2)}{[b'(v_2)]^2}\frac{dv_2}{dc} = U_1''(c + b(v_2))\left[1 + b'(v_2)\frac{dv_2}{dc}\right], \tag{5.6}$$

or

$$\frac{dv_2}{dc}\left[b'(v_2)U_1''(c + b(v_2)) + \frac{b''(v_2)}{[b'(v_2)]^2}\right] = -U_1''(c + b(v_2)).$$

Hence, $dv_2/dc < 0$. Returning to (5.6) this yields $1 + b'(v_2)(dv_2/dc) > 0$. Using this in (5.5), $U_2''(c) < 0$ for $c > c_2$. Since (5.5) shows that $U_2''(c) < 0$ for $0 \leqq c < c_2$, we see that $U_2''(c) < 0$ for all $c \geqq 0$, but is not continuous at $c = c_2$.

Now to the final step that

$$U_2'(c) > U_1'(c).$$

Having established this, we have all the ingredients of an inductive proof.

To establish this inequality, we must consider three distinct intervals $[0, c_1]$, $[c_1, c_2]$, and $[c_2, \infty]$. In $[0, c_1]$, we have

$$U(c)_2' = [1 + b'(c)]U_1'(c + b(c)) > [1 + b'(c)]U_0'(c + b(c)) = U_1'(c).$$

In $[c_2, \infty]$, we have

$$U_2'(c) = [1 + 1/b'(v_2)] > [1 + 1/b'(v_1)] = U_1'(c),$$

since $v_2 > v_1$. The remaining interval to consider is $[c_1, c_2]$. In $[c_1, c_2]$ we have

$$U_1'(c) = 1 + 1/b'(v_1),$$

$$U_2'(c) = [1 + b'(c)]U_1'(c + b(c)).$$

Since $v_2 = c$ in $[c_1, c_2]$,

$$-1 + b'(v)U_1'(c + b(v)) \geqq 0$$

for $0 \leqq v \leqq c_2$. In particular,

$$b'(c)U_1'(c + b(c)) \geqq 1. \tag{5.7}$$

We now wish to show that

$$U_1'(c + b(c)) \geqq 1/b'(v_1)$$

for $c_1 \leqq c \leqq c_2$. Since $1/b'(v)$ is increasing and $v_1(c) \leqq c$, $1/b'(v_1) \leqq 1/b'(c)$. By (5.7) it follows that, for $c \leqq c_2$,

$$U_1'(c + b(c)) \geqq 1/b'(c) \geqq 1/b'(v_1).$$

We now have all the material required for an inductive proof of the following

THEOREM. *For each N there exists a function $v_N(c)$ with the following properties*:

(a) *$v_N(c)$ is monotone decreasing as c increases*;

(b) $$v_{N+1}(c) > v_N(c) \qquad (N = 1, 2, \cdots);$$

(c) *There is a unique solution of $v_N(c) = c$ which we call c_N, and $c_{N+1} > c_N$*;

(d) *For $0 \leqq c \leqq c_N$, we have $U_N(c) = U_{N-1}(c + b(c))$*;

(e) *For $c_N \leqq c$, we have $U_N(c) = c - v_N(c) + U_{N-1}[c + b(v_N(c))]$*;

(f) $$U_N'(c) \geqq U_{n-1}'(c) \qquad (N = 1, 2, \cdots, c \geqq 0);$$

(g) $$U_N''(c) \leqq 0 \qquad (N = 1, 2, \cdots, c \geqq 0).$$

6. A more general problem. Let us now consider the problem of maximizing

$$J(y) = \int_0^T (x - y)\, dt$$

subject to the relation

$$\frac{dx}{dt} = G(x, y) \qquad (x(0) = c),$$

and the constraint

$$0 \leqq y \leqq x.$$

As above, we begin by considering the discrete version of the problem, where

$$J[\{y_k\}] = \sum_{k=0}^{N} (x_k - y_k),$$

and

(a) $$x_{k+1} = x_k + G(x_k, y_k) \qquad (k = 0, 1, 2, \cdots, N - 1),$$

(b) $$0 \leqq y_k \leqq x_k.$$

Setting

$$U_N(c) = \text{Max}_y\, J[\{y_k\}],$$

we clearly have

$$U_1(c) = c,$$

$$U_{N+1}(c) = \max_{0 \leqq v \leqq c} [c - v + U_N(c + G(c, v))] \qquad (N = 1, 2, \cdots).$$

Our aim is to determine the structure of the optimal policy under appropriate assumptions concerning $G(c, v)$. This is equivalent to determining the structure of $v = v_N(c)$ as a function of c and N.

We shall assume that $G(c, v)$ satisfies the following conditions:

(a) $G_v(c, v) > 0$, $G_v(c, v) \to 0$ as $v \to \infty$, $G_v(c, v) \to \infty$ as $v \to 0$, uniformly in c.
(b) $G_c > 0$, $G_{cv} \leqq 0$.
(c) $[1/G_v(c, v)]v = c$ is monotone increasing in c.
(d) $r^2 G_{cc} + 2rsG_{cv} + s^2 G_{vv}$ is a negative definite form.
(e) $[1 + G_c(c, v)]/G_v(c, v)$ is monotone increasing in v.*

As above, we shall employ an inductive approach.

---

* In the limit of the continuous case, this requires that $1/G_v(c, v)$ be monotone increasing in v, which is a consequence of (d).

7. The case $N = 2$. We have

$$U_2(c) = \operatorname*{Max}_{0 \leqq v \leqq c} [c - v + U_1(c + G(c, v))]. \tag{7.1}$$

Let us set

$$F_1(v, c) = c - v + U_1(c + G(c, v)).$$

If an internal maximum occurs in (7.1), it occurs at a point where

$$\partial F_1/\partial v = 1,$$

or

$$1/G_v(c, v) = U_1'(c + G(c, v) = 1. \tag{7.2}$$

By virtue of our assumptions concerning G, this equation has precisely one root for any value of c. If this root is less than c, there is an internal maximum.

The critical value of c is then given by the root of

$$[1/G_v(c, v)]_{v=c} = 1.$$

Let us call this value c_1. Our assumption ensures that this equation has precisely one root.

If $c < c_1$, there will be no internal maximum and the maximum will occur at $v = c$. If $c > c_1$, there will be an internal maximum at a value $v < c$ which we call $v_1 = v_1(c)$.

Hence we have

$$\begin{aligned} U_2(c) &= U_1(c + G(c, c)) && (c \leqq c_1) \\ &= c - v_1 + U_1(c + G(c, v_1)) && (c \geqq c_1). \end{aligned}$$

The function $f_2(c)$ is clearly differentiable for $c < c_1$ and $c > c_1$. We have

$$\begin{aligned} U_2'(c) &= 1 + (G_c(c, c) + [G_v(c, v)]_{v=c}) && (c < c_1), \\ &= 1 + (1 + G_c(c, v_1)) && (c > c_1). \end{aligned} \tag{7.3}$$

For further use, we note that (7.2) shows that we may write

$$U_2'(c) = 1 + (1 + G_c(c, v_1))/G_v(c, v_1) \qquad (c > c_1).$$

At $c = c_1$, $v_1(c) = c$ and $[G_v(c, v)]_{v=c} = 1$. Hence we have equality of right and left hand derivatives of $U_2(c)$ at $c = c_1$. Thus $U_2'(c)$ is continuous for all values of c.

Since $U_1'(c) = 1$, (7.3) shows that

$$U_2'(c) > U_1'(c)$$

for all $c \geqq 0$.

We now wish to show that $U_2'(c)$ is monotone increasing, which is to say that $U_2(c)$ is concave. To accomplish this, we shall prove a useful lemma.

8. A lemma on concavity. Let us prove

LEMMA 1. *Let $\phi(x, y)$ be a concave function of x, y for $x, y \geqq 0$,*

$$\phi(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \geqq \lambda_1\phi(x_1, y_1) + (1 - \lambda)\phi(x_2, y_2)$$

for $0 \leqq \lambda \leqq 1$.

Then

$$f(x) = \operatorname*{Max}_{0 \leqq y \leqq x} \phi(x, y)$$

is concave.

PROOF. If $y = x$, we have $f(x) = \phi(x, x)$, clearly concave. If $y = 0$, $f(x) = \phi(x, 0)$, also clearly concave. If $0 < y < x$, then y is determined by

$$\phi_y(x, y) = 0. \tag{8.1}$$

Then

$$f'(x) = \phi_x + \phi_y \frac{dy}{dx} = \phi_x .$$

Thus

$$\begin{aligned} f''(x) &= \phi_{xx} + \phi_{xy} \frac{dy}{dx} \\ &= \frac{\phi_{xx}\phi_{yy} - \phi_{xy}^2}{\phi_{yy}}, \end{aligned} \tag{8.2}$$

using (8.1) which yields $\phi_{xy} + \phi_{yy}\, dy/dx = 0$.

The condition that $\phi(x, y)$ be concave is equivalent to

$$r^2\phi_{xx} + 2rs\phi_{xy} + s^2\phi_{yy}$$

being negative definite. Hence the right side of the equation in (8.2) is negative.

Let us now show that

$$F(c, v) = c - v + U(G(c, v))$$

is concave if U is concave and monotone increasing and G is concave. We have

$$\begin{aligned} F_{cc} &= G_{cc}U'(G) + G_c^2U''(G), \\ F_{vv} &= G_{vv}U'(G) + G_v^2U''(G), \\ F_{cv} &= G_{cv}U'(G) + G_cG_vU''(G). \end{aligned}$$

From this it is clear that $F(c, v)$ is concave.

Combining these results we see that $f_2(c)$ is concave.

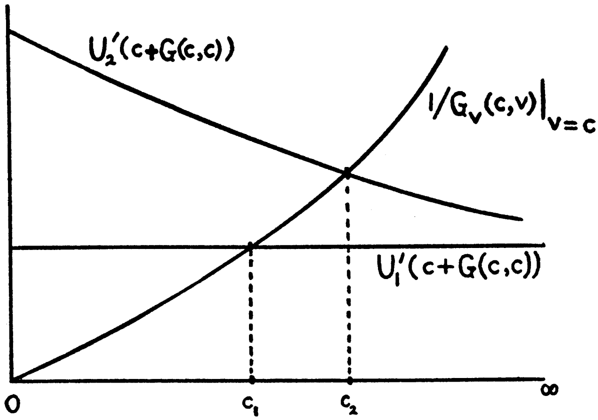

FIG. 1

9. The case $N = 3$. We have

$$U_3(c) = \underset{0 \leq v \leq c}{\text{Max}} [c - v + U_2(c + G(c, v))]$$
$$= \underset{0 \leq v \leq c}{\text{Max}} [F_2(v, c)].$$

If an internal maximum occurs, it occurs at a point where

$$1/G_v(c, v) = U_2'(c + G(c, v)). \tag{9.1}$$

Since $1/G_v$ is monotone increasing in v, and $U_2'(c + G(c, v))$ is monotone decreasing in v, there is exactly one root of this equation which we call $v_2 = v_2(c)$. Since $U_2' > U_1'$ it is clear that $v_2(c) > v_1(c)$.

The critical value of c is the root of

$$[1/G_v(c, v)]_{v=c} = U_2'(c + G(c, c)).$$

Call this root c_2. It is clear that $c_2 > c_1$.

From (9.1) we see that dv/dc is negative. Referring to Figure 1, it is clear that $c_2 > c_1$. Hence

$$U_3(c) = U_2(c + G(c, c)) \qquad (c < c_2)$$
$$= c - v_2 + U_2(c + G(c, v_2)) \qquad (c > c_2).$$

Thus

$$U_3'(c) = \{1 + [G_c(c, c) + [G_v(c, v)]_{v=c}]\} U_2'(c + G(c, c)) \quad (c < c_2)$$
$$= 1 + (1 + G_c(c, v_2)U_2'(c + G(c, v_2)) \qquad (c > c_2)$$
$$= 1 + \frac{1 + G_c(c, v_2)}{G_v(c, v_2)}.$$

The concavity of $U_3(c)$ follows as before.

That $U_3'(c) > U_2'(c)$ is clear for $c > c_2$ and $0 \leqq c \leqq c_1$. It remains to prove that the inequality holds in $[c_1, c_2]$. After having established this, we have all the material for an inductive proof of the structure of $v_N(c)$.

In $[c_1, c_2]$ we have

$$U_2'(c) = 1 + (1 + G_c(c, v_1))U_1'(c + G(c, v_1)),$$
$$U_3'(c) = [1 + (G_c(c, c) + [G_v(c, v)]_{v=c})]U_2'(c + G(c, c)).$$

The fact that $v_2 = c$ in $[c_1, c_2]$ implies that

$$-1 + G_v U_2'(c + G) \geqq 0$$

for $0 \leqq v \leqq c$ and $c_1 \leqq c \leqq c_2$. Hence

$$U_3'(c) \geqq \frac{[1 + G_c(c, c) + [G_v(c, v)]_{v=c}]}{[G_v(c, v)]_{v=c}}$$
$$\geqq 1 + \frac{1 + G_c(c, c)}{[G_v(c, v)]_{v=c}}.$$

On the other hand,

$$U_2'(c) = 1 + \frac{1 + G_c(c, v_1)}{G_v(c, v_1)}.$$

Since $v_1(c) \leqq c$ for $c_1 \leqq c < \infty$, and $[1 + G_c(c, v)]/G_v(c, v)$ qua function of v is monotone increasing, by assumption, we see that $U_3'(c) > U_2'(c)$. We now have all the material required for an inductive proof.

Summarizing our results we see that we have a monotone increasing sequence, $\{c_N\}$, possessing the property that

$$\begin{aligned} v_N &= c && (0 \leqq c \leqq c_N) \\ &= v_N(c) < c && (c_N < c < \infty). \end{aligned}$$

Each function $U_N(c)$ is monotone increasing in c and concave and the sequence $\{U_N'(c)\}$ is monotone increasing in N.

This property carries over in the limit as the discrete process goes over to the continuous and enables us to use the functional equation to determine the properties of the solution. It may be proved by a straightforward argument that the limit of the discrete case is indeed the continuous. We shall omit the proof here.

REFERENCES

1. R. Bellman, *The theory of dynamic programming*, Bull. Amer. Math. Soc., Vol. 60 (1954), pp. 503–515.

2. ———, *An introduction to the theory of dynamic programming*, RAND Monograph No. R-245, 1952.
3. ———, *Dynamic programming of continuous processes*, RAND Monograph R-271, 1954.
4. ———, *Dynamic programming and a new formalism in the calculus of variations*— Rivista di Parma, (to appear).
5. R. Bellman, W. Fleming and D. V. Widder, *Variational problems with constraints*, Annali di Matematica, (to appear).

Rand Corporation