First-order logic

Chapter 8

Outline

- ◇ Why FOL?
- ◇ Syntax and semantics of FOL
- ◇ Fun with sentences
- ◇ Wumpus world in FOL

Pros and cons of propositional logic

- 😊 Propositional logic is **declarative**: pieces of syntax correspond to facts
- 😊 Propositional logic allows partial/disjunctive/negated information
  (unlike most data structures and databases)
- 😊 Propositional logic is **compositional**:
  meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- 😊 Meaning in propositional logic is **context-independent**
  (unlike natural language, where meaning depends on context)
- ☹ Propositional logic has very limited expressive power
  (unlike natural language)
  E.g., cannot say "pits cause breezes in adjacent squares"
  except by writing one sentence for each square

First-order logic

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .
- Relations: red, round, bogus, prime, multistoried . . .,
  brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
- Functions: father of, best friend, third inning of, one more than, end of . . .

Logics in general

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Syntax of FOL: Basic elements

Constants	$KingJohn,\ 2,\ UCB, \ldots$
Predicates	$Brother,\ >, \ldots$
Functions	$Sqrt,\ LeftLegOf, \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\land \ \lor \ \neg \ \Rightarrow \ \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

Atomic sentences

Atomic sentence $=$ $predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term $=$ $function(term_1, \ldots, term_n)$
or $constant$ or $variable$

E.g., $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$
$>(1,2) \vee \leq(1,2)$
$>(1,2) \wedge \neg >(1,2)$

Truth in first-order logic

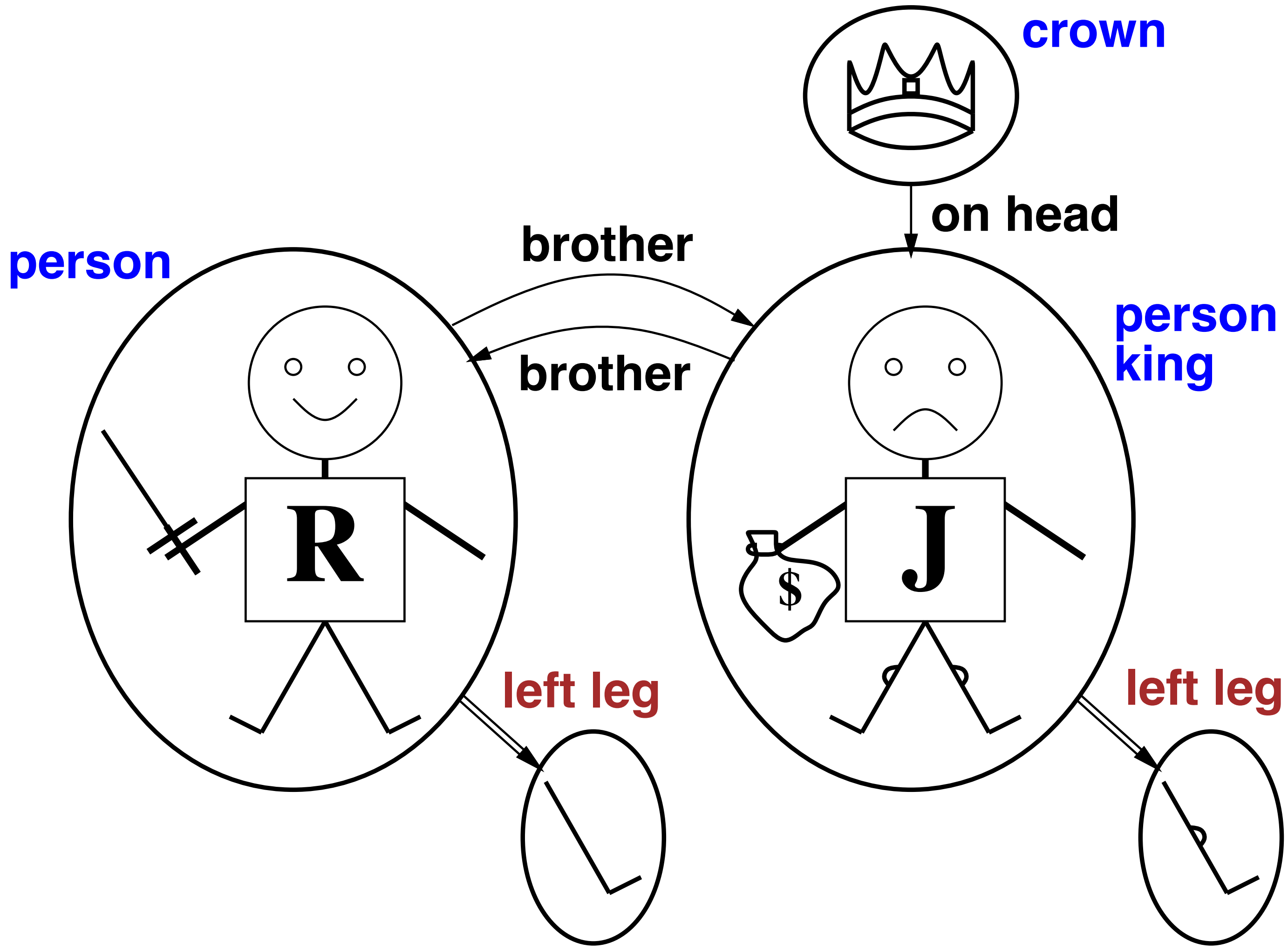
Sentences are true with respect to a model and an interpretation

Model contains ≥ 1 objects (domain elements) and relations among them

Interpretation specifies referents for
- constant symbols $\rightarrow$ objects
- predicate symbols $\rightarrow$ relations
- function symbols $\rightarrow$ functional relations

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
iff the objects referred to by $term_1, \ldots, term_n$
are in the relation referred to by $predicate$

Models for FOL: Example



Truth example

Consider the interpretation in which
$Richard$ $\rightarrow$ Richard the Lionheart
$John$ $\rightarrow$ the evil King John
$Brother$ $\rightarrow$ the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞
  For each k-ary predicate P_k in the vocabulary
    For each possible k-ary relation on n objects
      For each constant symbol C in the vocabulary
        For each choice of referent for C from n objects . . .

Computing entailment by enumerating FOL models is not easy!

Universal quantification

$\forall \langle variables \rangle \; \langle sentence \rangle$

Everyone at Berkeley is smart:
$\forall x \;\; At(x, Berkeley) \;\Rightarrow\; Smart(x)$

$\forall x \;\; P$ is true in a model m iff P is true with x being **each** possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$\begin{aligned} & (At(KingJohn, Berkeley) \;\Rightarrow\; Smart(KingJohn)) \\ \wedge \; & (At(Richard, Berkeley) \;\Rightarrow\; Smart(Richard)) \\ \wedge \; & (At(Berkeley, Berkeley) \;\Rightarrow\; Smart(Berkeley)) \\ \wedge \; & \ldots \end{aligned}$$

A common mistake to avoid

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\land$ as the main connective with $\forall$:

$$\forall\, x \;\; At(x, Berkeley) \land Smart(x)$$

means "Everyone is at Berkeley and everyone is smart"

Existential quantification

$\exists \langle variables \rangle \;\; \langle sentence \rangle$

Someone at Stanford is smart:
$\exists\, x \;\; At(x, Stanford) \land Smart(x)$

$\exists\, x \;\; P$ is true in a model m iff P is true with x being **some** possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, Stanford) \land Smart(KingJohn)) \\
\lor\; & (At(Richard, Stanford) \land Smart(Richard)) \\
\lor\; & (At(Stanford, Stanford) \land Smart(Stanford)) \\
\lor\; & \ldots
\end{aligned}
$$

Another common mistake to avoid

Typically, $\land$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists\, x \;\; At(x, Stanford) \Rightarrow Smart(x)$$

is true if there is anyone who is not at Stanford!

Properties of quantifiers

$\forall\, x\ \ \forall\, y$ is the same as $\forall\, y\ \ \forall\, x$ (why??)

$\exists\, x\ \ \exists\, y$ is the same as $\exists\, y\ \ \exists\, x$ (why??)

$\exists\, x\ \ \forall\, y$ is **not** the same as $\forall\, y\ \ \exists\, x$

$\exists\, x\ \ \forall\, y\ \ Loves(x, y)$
"There is a person who loves everyone in the world"

$\forall\, y\ \ \exists\, x\ \ Loves(x, y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall\, x\ \ Likes(x, IceCream) \qquad \neg\exists\, x\ \ \neg Likes(x, IceCream)$

$\exists\, x\ \ Likes(x, Broccoli) \qquad \neg\forall\, x\ \ \neg Likes(x, Broccoli)$

Fun with sentences

Brothers are siblings

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$\forall\, x, y\ \ Brother(x, y)\ \Rightarrow\ Sibling(x, y)$.

"Sibling" is symmetric

Fun with sentences

Brothers are siblings

$\forall x, y \ \ Brother(x, y) \ \Rightarrow \ Sibling(x, y)$.

"Sibling" is symmetric

$\forall x, y \ \ Sibling(x, y) \ \Leftrightarrow \ Sibling(y, x)$.

One's mother is one's female parent

Fun with sentences

Brothers are siblings

$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$.

"Sibling" is symmetric

$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$.

One's mother is one's female parent

$\forall\, x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$.

A first cousin is a child of a parent's sibling

Fun with sentences

Brothers are siblings

$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$.

"Sibling" is symmetric

$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$.

One's mother is one's female parent

$\forall\, x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \land Parent(x, y))$.

A first cousin is a child of a parent's sibling

$\forall\, x, y \;\; FirstCousin(x, y) \;\Leftrightarrow\; \exists\, p, ps \;\; Parent(p, x) \land Sibling(ps, p) \land Parent(ps, y)$

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall\, x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
$2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; [\neg(x = y) \wedge \exists\, m, f \;\; \neg(m = f) \wedge$$
$$Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

Summary

First-order logic:
- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:
- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB