RSA - The keys to security

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Motivation

- dealt with encryption (f.i. https protocol, ssh)
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$\rightarrow$ How does RSA work again?

What to expect?

- hopefully some better understanding of RSA
- how RSA works with example and implementation
- insight in mathematical detail with idea for proof
- evaluation of security level

Definition

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→ What is a public key encryption technique?
→ How is data transmitted?

Symmetric and asymmetric cryptography

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2. Asymmetric cryptography (= public key cryptography): There are two keys - public key and private key. If locked with public key $\rightarrow$ unlock with private key. If locked with private key $\rightarrow$ unlock with public key.

$\rightarrow$ Why public and private? And still: how is data transmitted?

Symmetric cryptography

Alice wants to send Bob a message.

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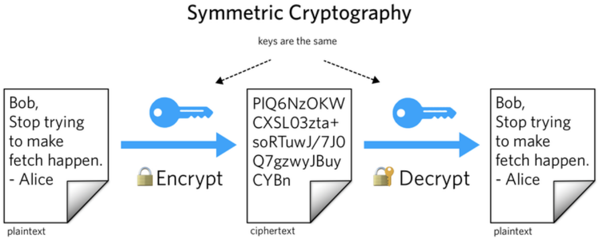

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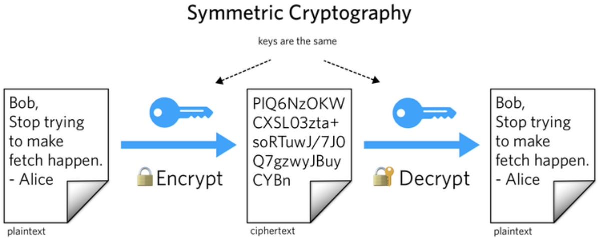

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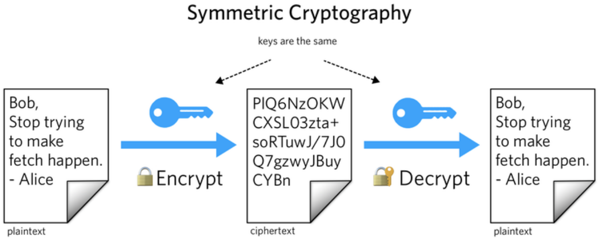

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Good Questions!

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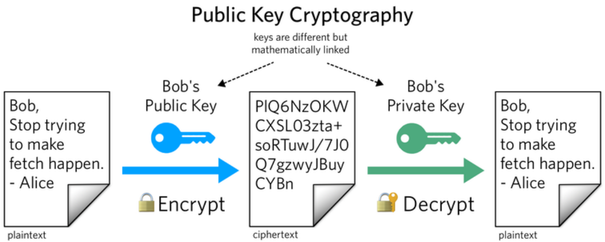

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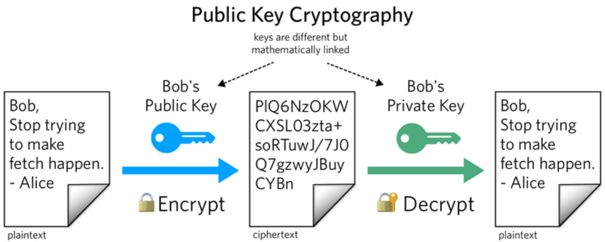

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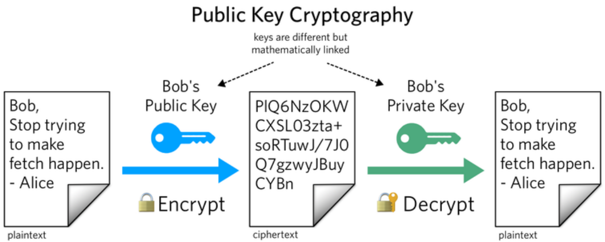

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$\rightarrow$ That's nice! How are the keys generated?

$\rightarrow$ How are they „connected“?

Key generation

Short summary, we need:

1. two keys
2. minimum requirement: one key decrypts messages sent by the other
3. there is no other key with feature (2)
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Prerequisites:

- **prime number** (Primzahl): number with two (natural) factors: 1 and itself
- **modulo**: system of arithmetic, where numbers „wrap around“ after certain value, f.e. $18 \equiv 6 \mod 12$
- **coprime** (teilerfremd): two integers are called coprime, if they have no common factor/divisor (except for the obvious 1)

Algorithm

1. Choose (big) prime numbers p and q, $p \neq q$
2. $n \rightarrow p \cdot q$
3. Compute $\phi(n) = \phi(p \cdot q) = (p-1) \cdot (q-1)$
4. Choose e with $1 < e < \phi(n)$, such that $\phi(n), e$ coprime
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$\rightarrow$ How do we encrypt?
$\rightarrow$ How do we find p, q?
$\rightarrow$ How do we find e?
$\rightarrow$ How do we find d?
$\rightarrow$ Most important: Why does this work?

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   $e \to 23$, (e is prime number, e is not divisor of 120)
5. (Compute d, such that $d \cdot e \equiv 1 \mod \phi(n)$)
   Extended Euclidean Algorithm: $d \to 47$
   $(23 \cdot 47 = 1081 = 9 \cdot 120 + 1)$

Encryption

1. We have prime numbers p, q, their product n, coprime numbers e, $\phi(n)$ and d
2. **Pairs** (e, n) **and** (d, n) **are our keys! (In our case** $(23, 143), (47, 143)$**)**

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That's the original message! $\rightarrow$ Again, why does this work?

Mathematical Details

Main question: given keys (e, n) and (d, n), message m: Why is

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Idea of proof (complete proof in the sources):

- $e \cdot d \equiv 1 \mod \phi(n)$ means there is a number k with $e \cdot d = 1 + k \cdot \phi(n)$
- Use *Fermat-Euler theorem* to show $m = m^{1+k\cdot\phi(n)} \mod p$ and $m = m^{1+k\cdot\phi(n)} \mod q$
- Use *Chinese Remainder theorem* to show $m \cdot m^{k\cdot\phi(n)} \equiv m \mod p \cdot q$

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- The complexity lies either in factorizing n to get p and q or in computing the message through encrypted message and public key
- For both problems no polynomial-time algorithm is known
- Quantum computers **could** crack the encryption in less than a day

Implementation

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Problem: slow, workaround: combine with symmetric cryptography

Summary

- generates two keys: public and private
- uses modulo arithmetic with big prime numbers
- key prime numbers are modular inverse to each other
- hard to crack, because no **practical** algorithm for factorization is known
- implementation in Go relies on big integers (math/big) and crypto packages (crypto/rand)
- can be combined with symmetric cryptography algorithms

Sources

- RSA proof
- RSA general information
- Computing modular inverse: example
- Breaking RSA Encryption
- What is a rune?
- math/big docs
- crypto/rand docs