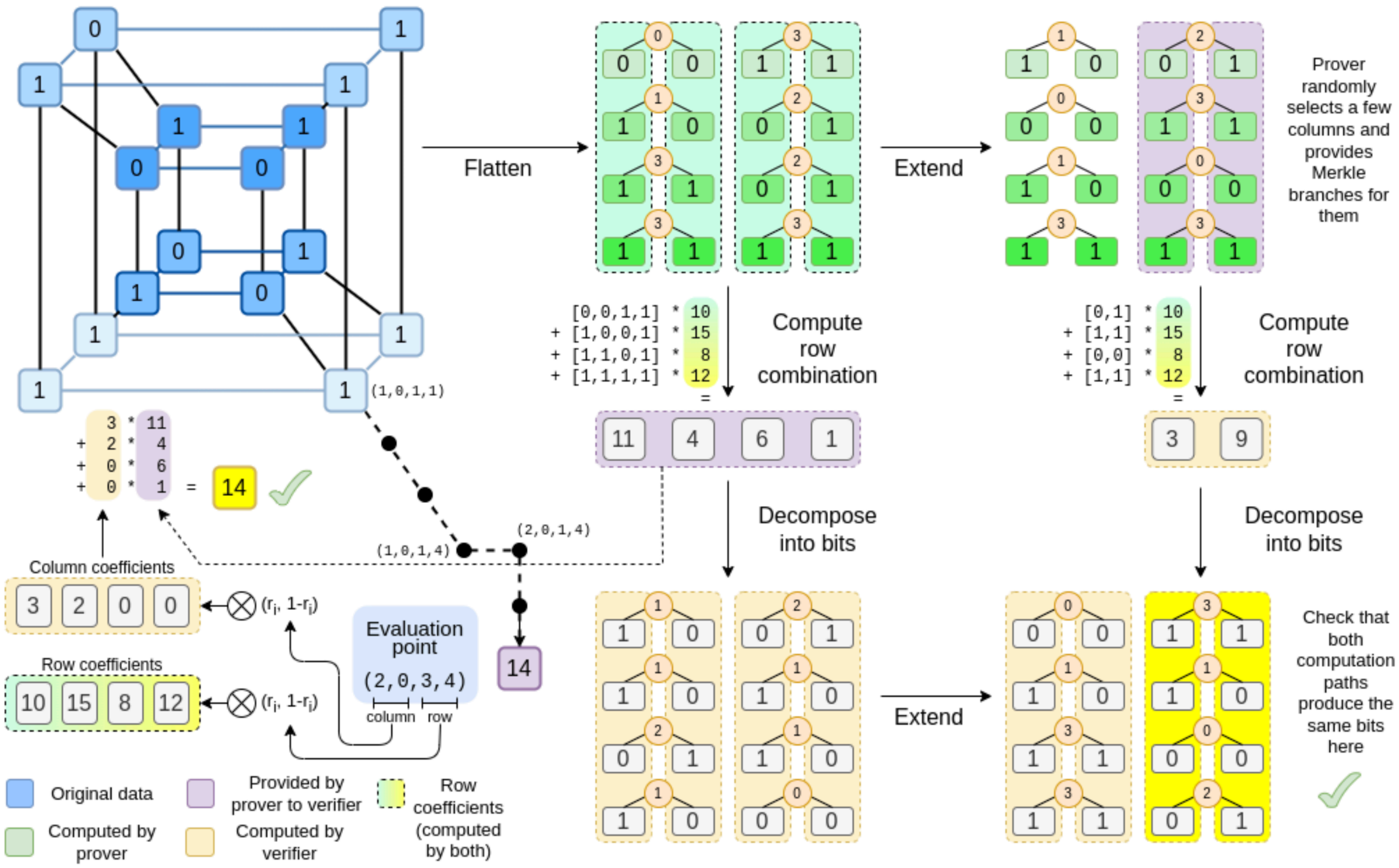
Binius

Dana Ben Porath



Based on [Vitalik's blog on Binius]

Intro - calculating $F(r_1, r_2, r_3, r_4)$

Using tensor products

F(0,0,0,0)	F(1,0,0,0)	F(0,1,0,0)	F(1,1,0,0)
3	1	4	1
F(0,0,1,0)	F(1,0,1,0)	F(0,1,1,0)	F(1,1,1,0)
5	9	2	6
F(0,0,0,1)	F(1,0,0,1)	F(0,1,0,1)	F(1,1,0,1)
5	3	5	8
F(0,0,1,1)	F(1,0,1,1)	F(0,1,1,1)	F(1,1,1,1)
9	7	9	3

Intro - calculating $F(r_1, r_2, r_3, r_4)$

Using tensor products

$$F(r_1, r_2, r_3, r_4) = F(0{,}0{,}0{,}0)(1 - r_0)(1 - r_1)(1 - r_2)(1 - r_3)$$
$$+F(1{,}0{,}0{,}0)r_0(1 - r_1)(1 - r_2)(1 - r_3) + \ldots$$

F(0,0,0,0)	F(1,0,0,0)	F(0,1,0,0)	F(1,1,0,0)	
3	1	4	1	$(1 - r_2)(1 - r_3)$
F(0,0,1,0)	F(1,0,1,0)	F(0,1,1,0)	F(1,1,1,0)	
5	9	2	6	$r_2(1 - r_3)$
F(0,0,0,1)	F(1,0,0,1)	F(0,1,0,1)	F(1,1,0,1)	
5	3	5	8	$(1 - r_2)r_3$
F(0,0,1,1)	F(1,0,1,1)	F(0,1,1,1)	F(1,1,1,1)	
9	7	9	3	$r_2 r_3$
$(1 - r_0)(1 - r_1)$	$r_0(1 - r_1)$	$(1 - r_0)r_1$	$r_0 r_1$	

Intro - calculating $F(r_1, r_2, r_3, r_4)$

Using tensor products

$$F(r_1, r_2, r_3, r_4) = F(0,0,0,0)(1-r_0)(1-r_1)(1-r_2)(1-r_3)$$
$$+F(1,0,0,0)r_0(1-r_1)(1-r_2)(1-r_3) + \dots$$

F(0,0,0,0) 3	F(1,0,0,0) 1	F(0,1,0,0) 4	F(1,1,0,0) 1	$(1-r_2)(1-r_3)$
F(0,0,1,0) 5	F(1,0,1,0) 9	F(0,1,1,0) 2	F(1,1,1,0) 6	$r_2(1-r_3)$
F(0,0,0,1) 5	F(1,0,0,1) 3	F(0,1,0,1) 5	F(1,1,0,1) 8	$(1-r_2)r_3$
F(0,0,1,1) 9	F(1,0,1,1) 7	F(0,1,1,1) 9	F(1,1,1,1) 3	r_2r_3
$(1-r_0)(1-r_1)$	$r_0(1-r_1)$	$(1-r_0)r_1$	r_0r_1	

Intro - calculating $F(r_1, r_2, r_3, r_4)$

Using tensor products

$$F(r_1, r_2, r_3, r_4) = F(0{,}0{,}0{,}0)(1-r_0)(1-r_1)(1-r_2)(1-r_3)$$
$$+F(1{,}0{,}0{,}0)r_0(1-r_1)(1-r_2)(1-r_3) + \dots$$

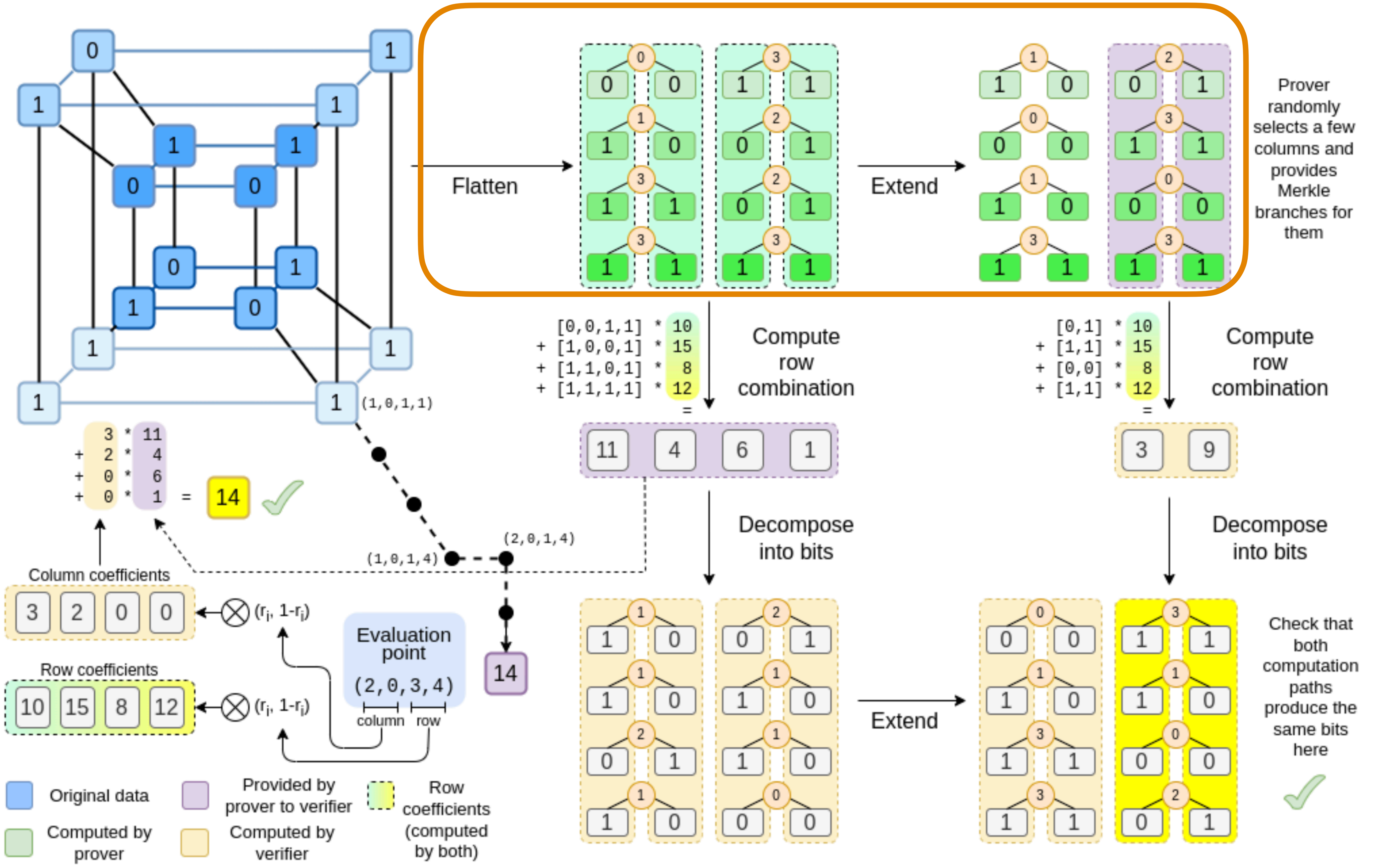
F(0,0,0,0) 3	F(1,0,0,0) 1	F(0,1,0,0) 4	F(1,1,0,0) 1	$(1-r_2)(1-r_3)$
F(0,0,1,0) 5	F(1,0,1,0) 9	F(0,1,1,0) 2	F(1,1,1,0) 6	$r_2(1-r_3)$
F(0,0,0,1) 5	F(1,0,0,1) 3	F(0,1,0,1) 5	F(1,1,0,1) 8	$(1-r_2)r_3$
F(0,0,1,1) 9	F(1,0,1,1) 7	F(0,1,1,1) 9	F(1,1,1,1) 3	$r_2 r_3$
$(1-r_0)(1-r_1)$	$r_0(1-r_1)$	$(1-r_0)r_1$	$r_0 r_1$	

Exemple - calculating F(1,2,3,4)

3	1	4	1	$\cdot\ 6$
+ 5	9	2	6	$\cdot\ (-9)$
+ 5	3	5	8	$\cdot\ (-8)$
+ 9	7	9	3	$\cdot\ 12$
41	−15	74	−76	

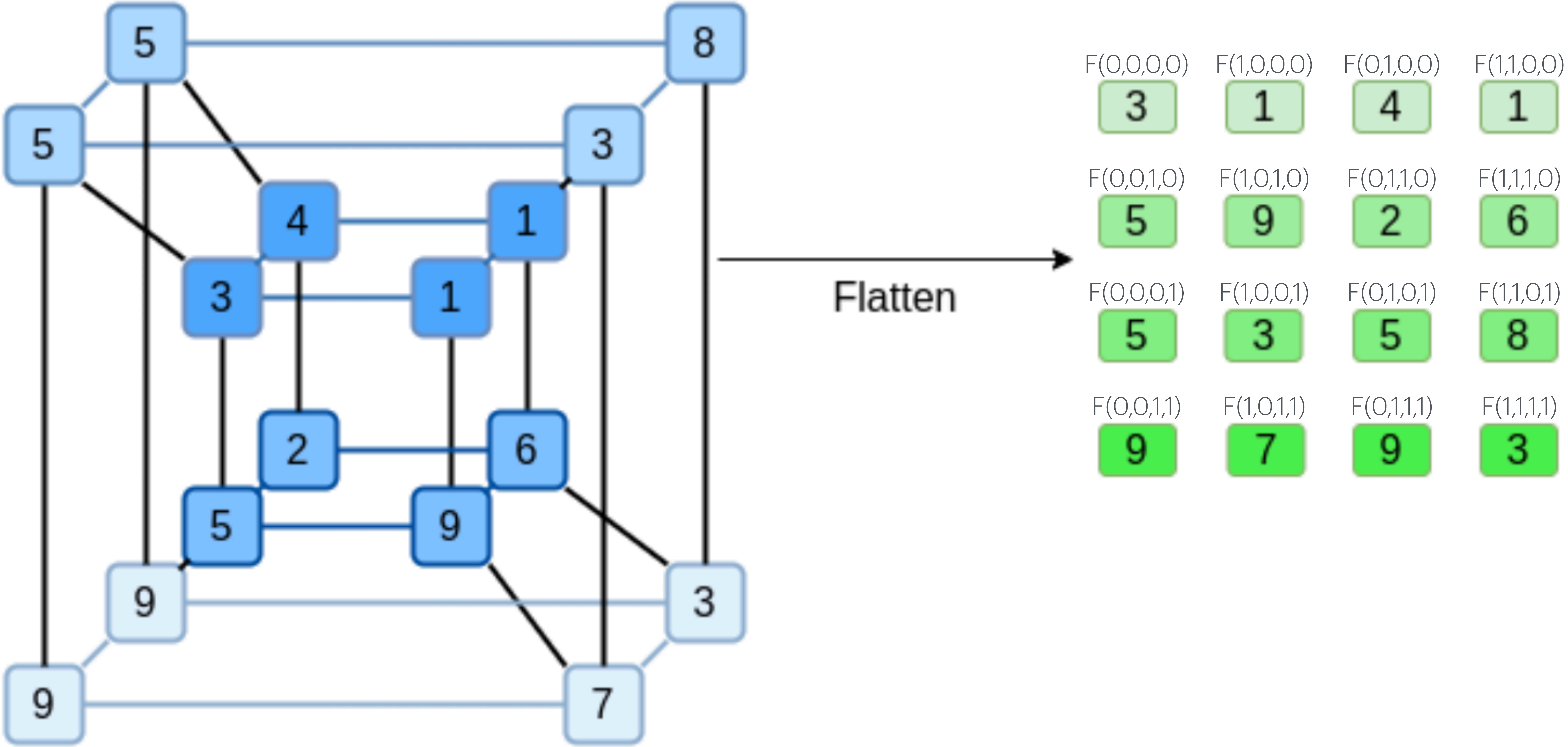
$$F(1{,}2{,}3{,}4) = [0, -1{,}0{,}2] \cdot [\ 41\ ,\ -15\ ,\ 74\ ,\ -76\ ] = -137$$

Binius



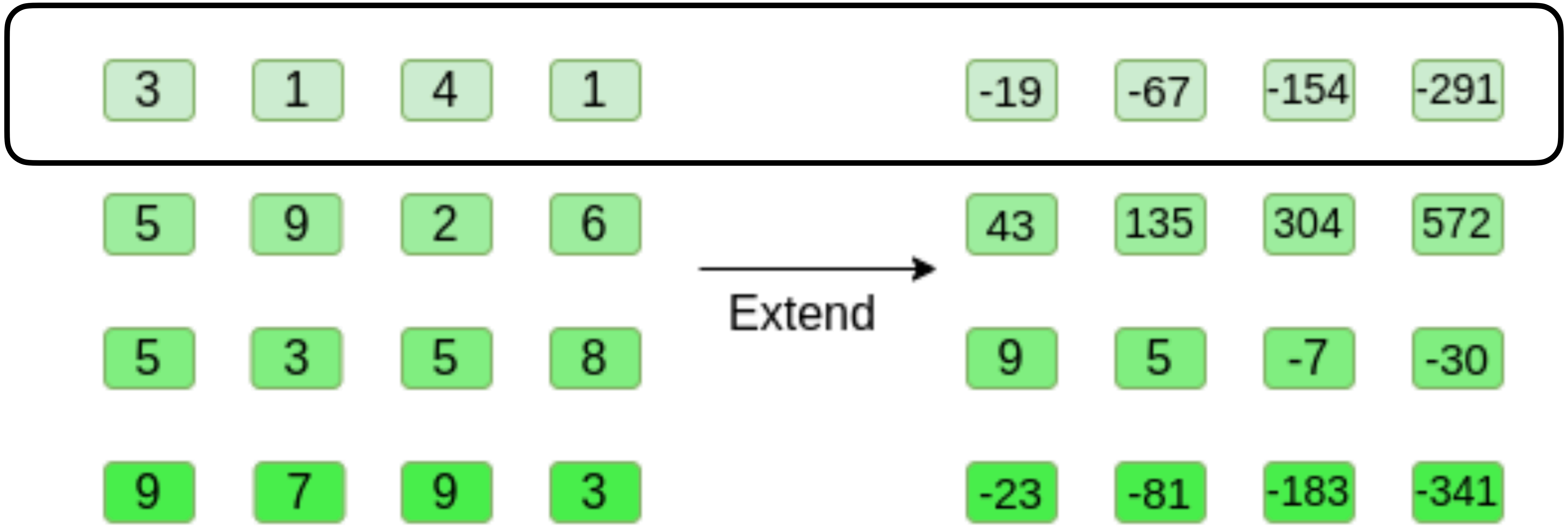
Flatten
Extend
Prover randomly selects a few columns and provides Merkle branches for them
Compute row combination
Compute row combination
Decompose into bits
Decompose into bits
Extend
Check that both computation paths produce the same bits here
(1,0,1,1)
(1,0,1,4)
(2,0,1,4)
Column coefficients
Row coefficients
$(r_i, 1-r_i)$
$(r_i, 1-r_i)$
Evaluation point
(2,0,3,4)
column
row
Original data
Computed by prover
Provided by prover to verifier
Computed by verifier
Row coefficients (computed by both)

Flatten


5
8
5
3
4
1
3
1
2
6
5
9
9
3
9
7
Flatten
F(0,0,0,0)
3
F(1,0,0,0)
1
F(0,1,0,0)
4
F(1,1,0,0)
1
F(0,0,1,0)
5
F(1,0,1,0)
9
F(0,1,1,0)
2
F(1,1,1,0)
6
F(0,0,0,1)
5
F(1,0,0,1)
3
F(0,1,0,1)
5
F(1,1,0,1)
8
F(0,0,1,1)
9
F(1,0,1,1)
7
F(0,1,1,1)
9
F(1,1,1,1)
3

Extend

Lagrange polynomials / Reed-Solomon extension



$$(0,3), (1,1), (2,4), (3,1) ----> (4, -19), (5, -67), (6, -154), (7, -291)$$

Extend

Lagrange polynomials / Reed-Solomon extension

$$(0,3), (1,1), (2,4), (3,1) ----> (4,-19), (5,-67), (6,-154), (7,-291)$$

The commitment is the root of the Merkle tree of the columns

Extend

Lagrange polynomials / Reed-Solomon extension

Given $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, the Lagrange interpolating polynomial is:

$L(x) = \sum_{j=0}^{n} y_j l_j(x)$, where $l_j(x) = \prod_{\substack{0 \leqslant m \leqslant n \\ m \neq j}} \frac{x - x_m}{x_j - x_m}$

Extend

Lagrange polynomials / Reed-Solomon extension

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Barycentric form:

$$L(x) = l(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} y_j, \text{ where}$$

$$l(x) = \prod_{0 \leqslant m \leqslant n} (x - x_m) \ \& \ w_j(x) = \prod_{\substack{0 \leqslant m \leqslant n \\ m \neq j}} \frac{1}{x_j - x_m}$$

Extend

Lagrange polynomials / Reed-Solomon extension

Given $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, the Lagrange interpolating polynomial is:

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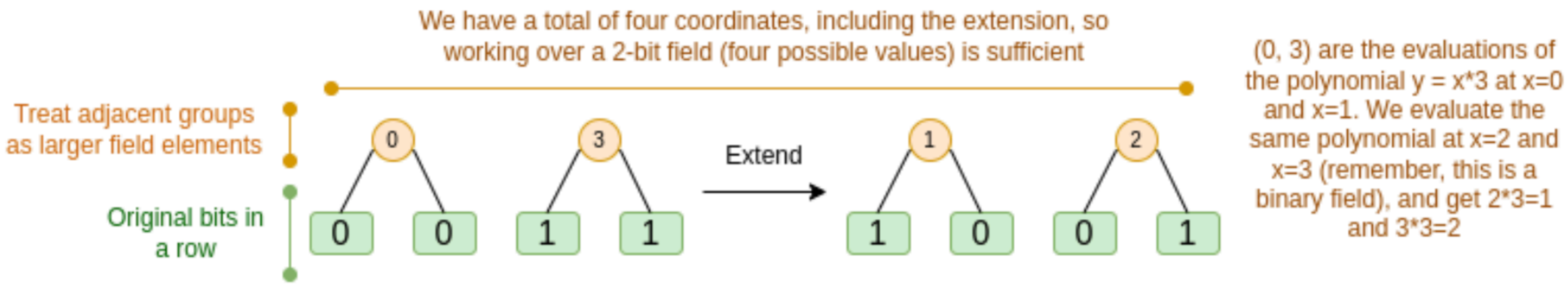
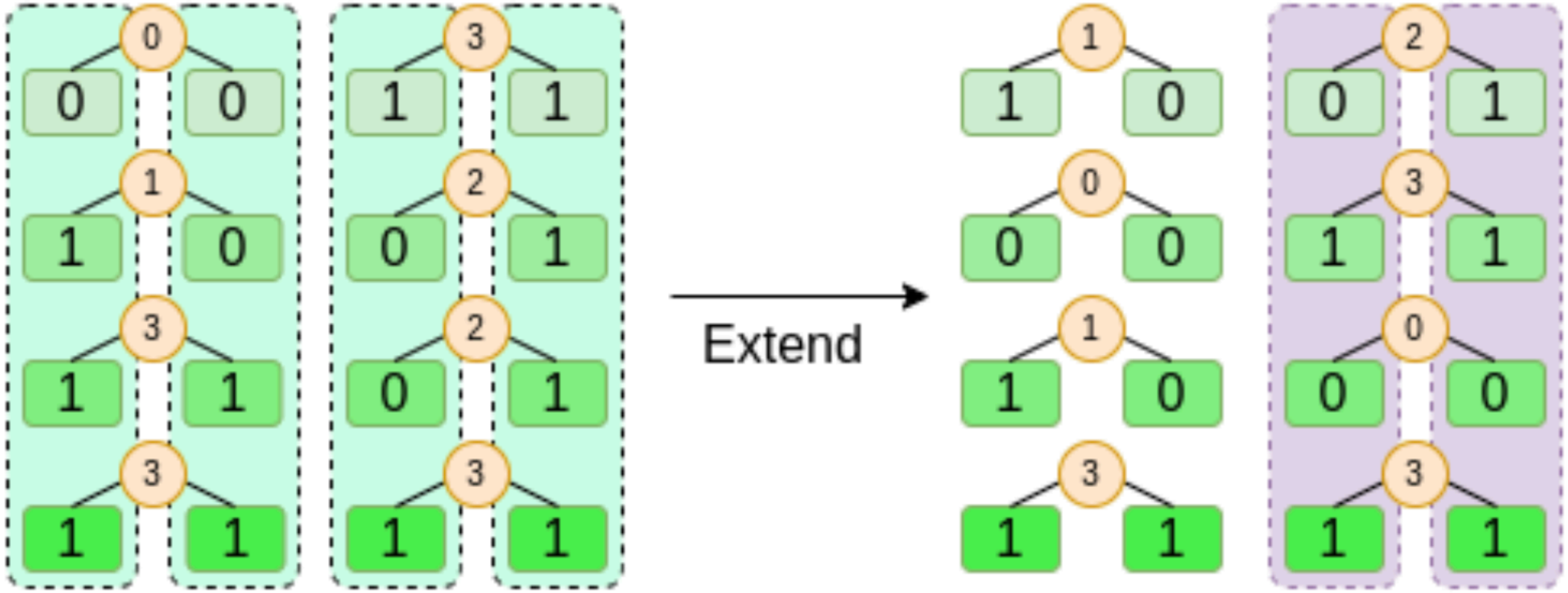
A limitation - if you are extending n values to kn values, you need to be working in a field that has kn different values that you can use as coordinates.

Extend - binary fields

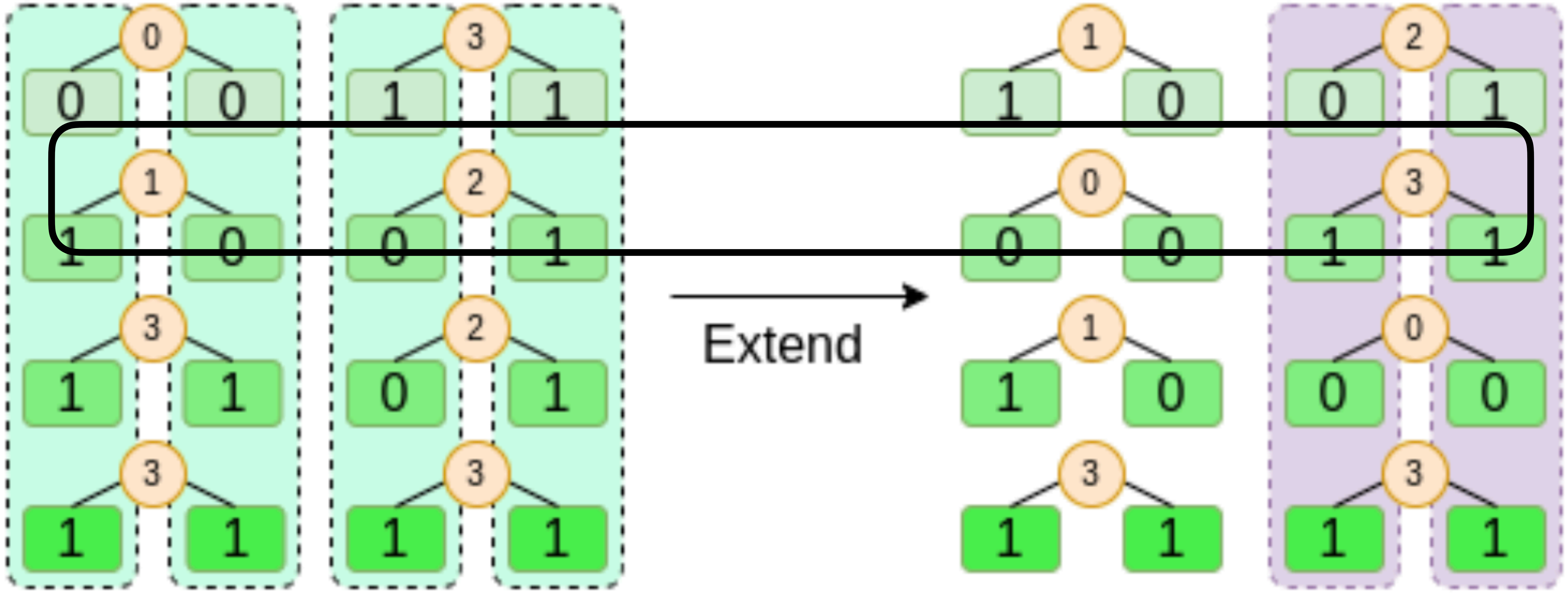


Extend - binary fields



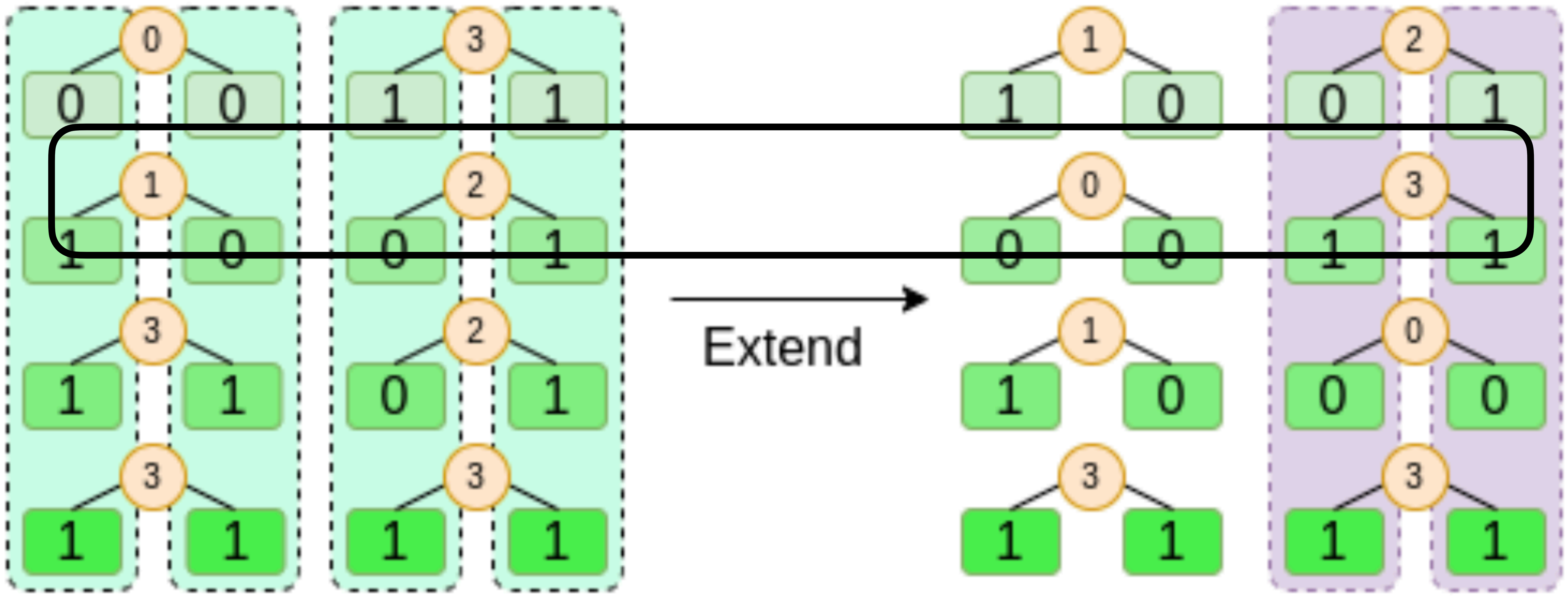
Extend - binary fields

(0,1), (1,2) – – – – > (2,0), (3,3)



Extend - binary fields

(0,1), (1,2) – – – – – > (2,0), (3,3)



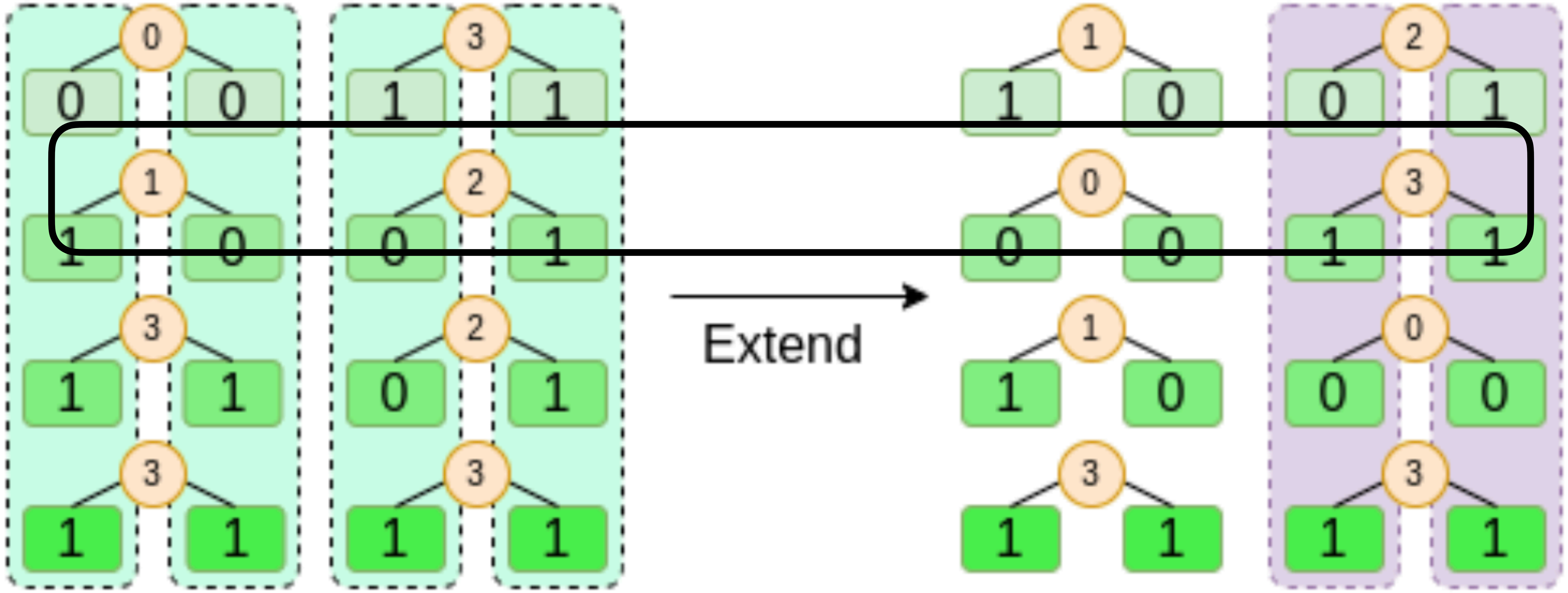
+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

$$L(x) = l(x)\sum_{j=0}^{n}\frac{w_j}{x-x_j}y_j, \text{ where } l(x) = \prod_{0\leqslant m\leqslant n}(x-x_m) \ \& \ w_j(x) = \prod_{\substack{0\leqslant m\leqslant n\\ m\neq j}}\frac{1}{x_j-x_m}$$

Extend - binary fields

(0,1), (1,2) – – – – – > (2,0), (3,3)



+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

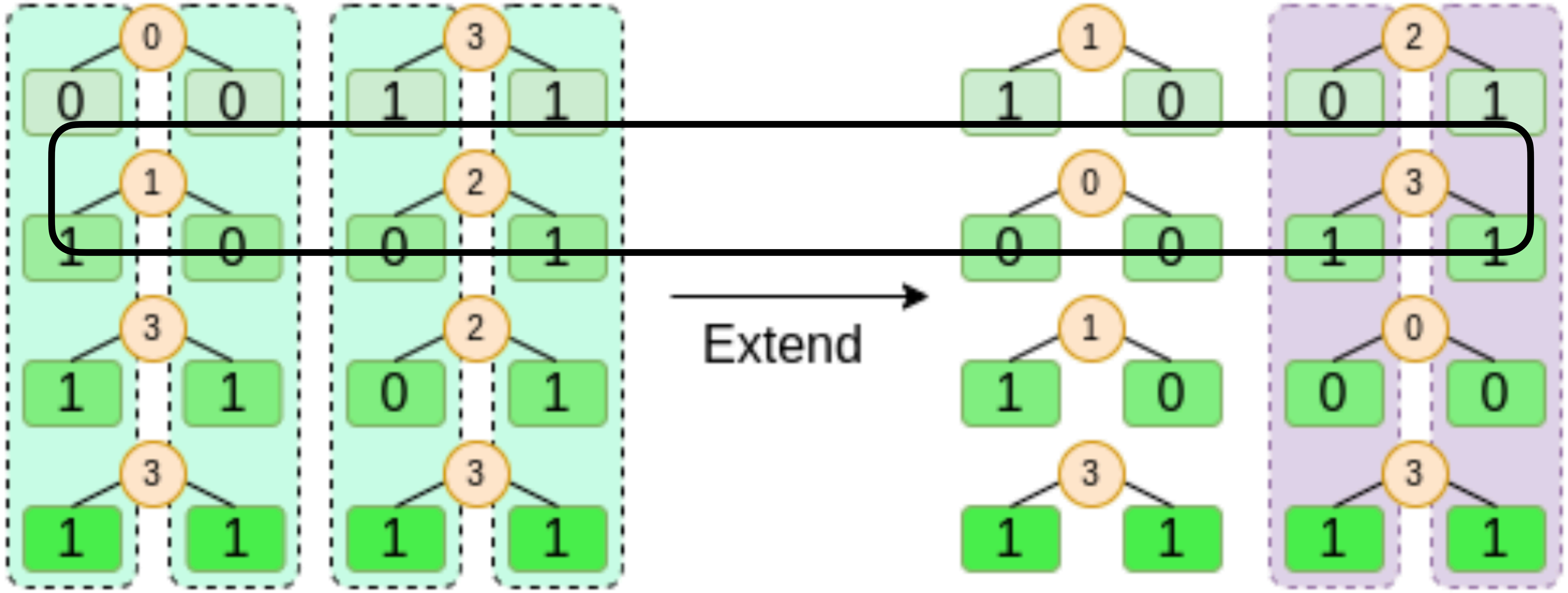
*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

$$L(x) = l(x)\sum_{j=0}^{n}\frac{w_j}{x-x_j}y_j, \text{ where } l(x) = \prod_{0\leqslant m\leqslant n}(x-x_m) \;\&\; w_j(x) = \prod_{\substack{0\leqslant m\leqslant n\\ m\neq j}}\frac{1}{x_j-x_m}$$

$$\Rightarrow \quad l(x) = x(x-1), w_0 = \frac{1}{0-1} = 1, w_1 = \frac{1}{1-0} = 1$$

Extend - binary fields

(0,1), (1,2) – – – – > (2,0), (3,3)



+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

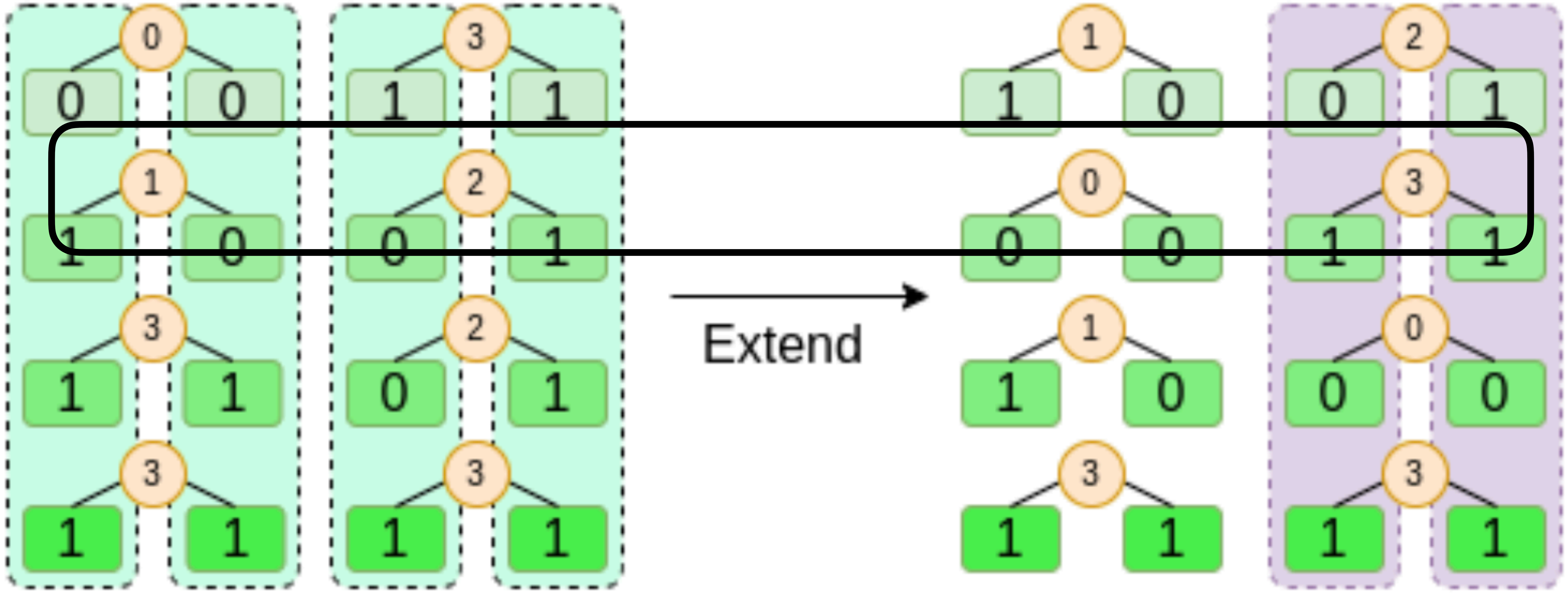
$$L(x) = l(x)\sum_{j=0}^{n}\frac{w_j}{x-x_j}y_j, \text{ where } l(x) = \prod_{0\leqslant m\leqslant n}(x-x_m) \ \&\ w_j(x) = \prod_{\substack{0\leqslant m\leqslant n\\ m\neq j}}\frac{1}{x_j-x_m}$$

➡ $l(x) = x(x-1), w_0 = \frac{1}{0-1} = 1, w_1 = \frac{1}{1-0} = 1$

➡ $L(x) = x(x-1)\left[\frac{1\cdot 1}{x} + \frac{1\cdot 2}{x-1}\right] = (x-1) + 2x = -1 + (1+2)x = 1 + 3x$

Extend - binary fields

$(0,1),(1,2) - - - - > (2,0),(3,3)$



+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

$$L(x) = l(x)\sum_{j=0}^{n} \frac{w_j}{x-x_j} y_j, \text{ where } l(x) = \prod_{0\leqslant m\leqslant n}(x-x_m) \;\&\; w_j(x) = \prod_{\substack{0 \leqslant m \leqslant n \\ m \neq j}} \frac{1}{x_j - x_m}$$

$$l(x) = x(x-1), w_0 = \frac{1}{0-1} = 1, w_1 = \frac{1}{1-0} = 1$$

$$L(x) = x(x-1)\left[\frac{1\cdot 1}{x} + \frac{1\cdot 2}{x-1}\right] = (x-1) + 2x = -1 + (1+2)x = 1 + 3x$$

$$L(2) = 0, L(3) = 3$$

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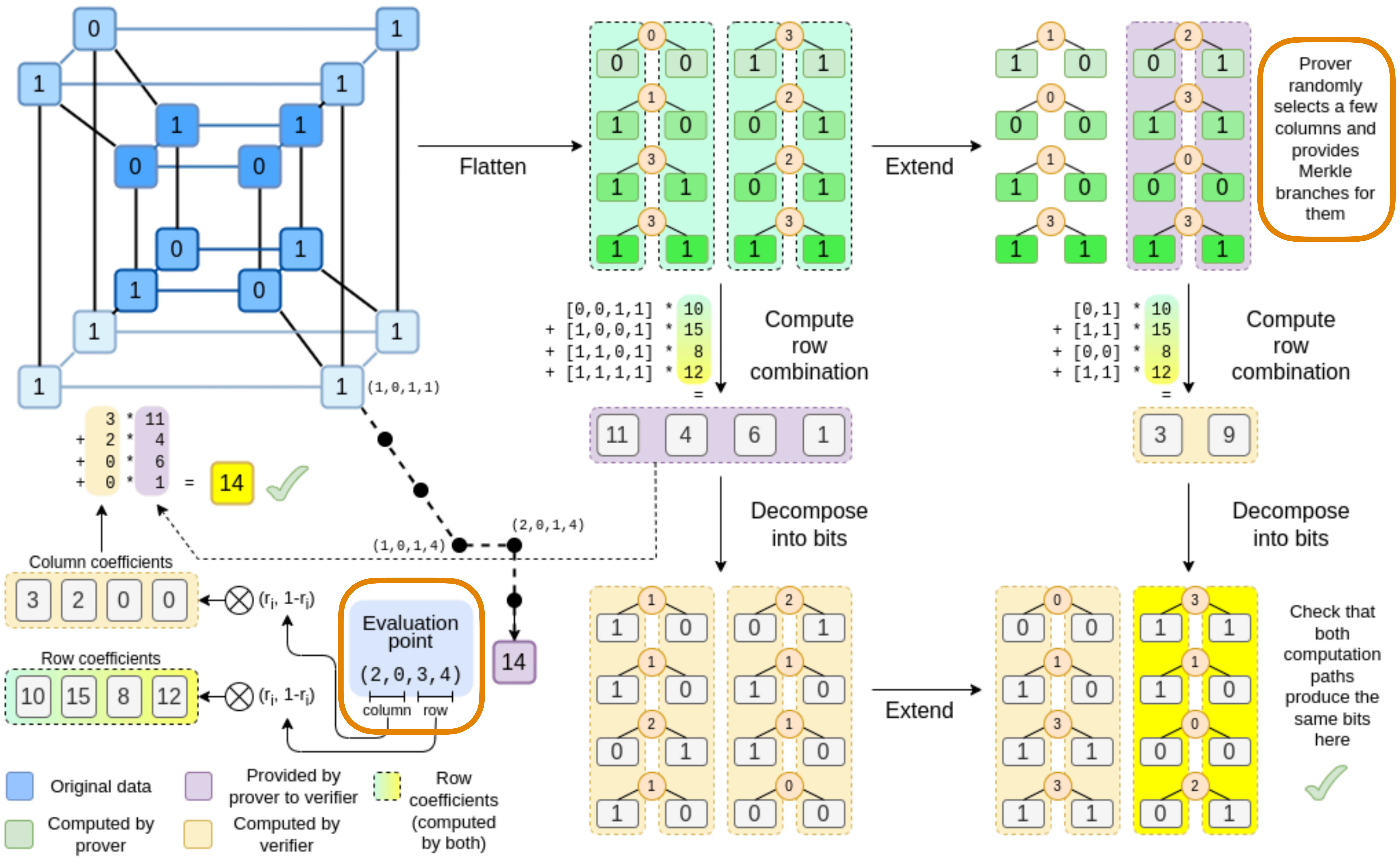
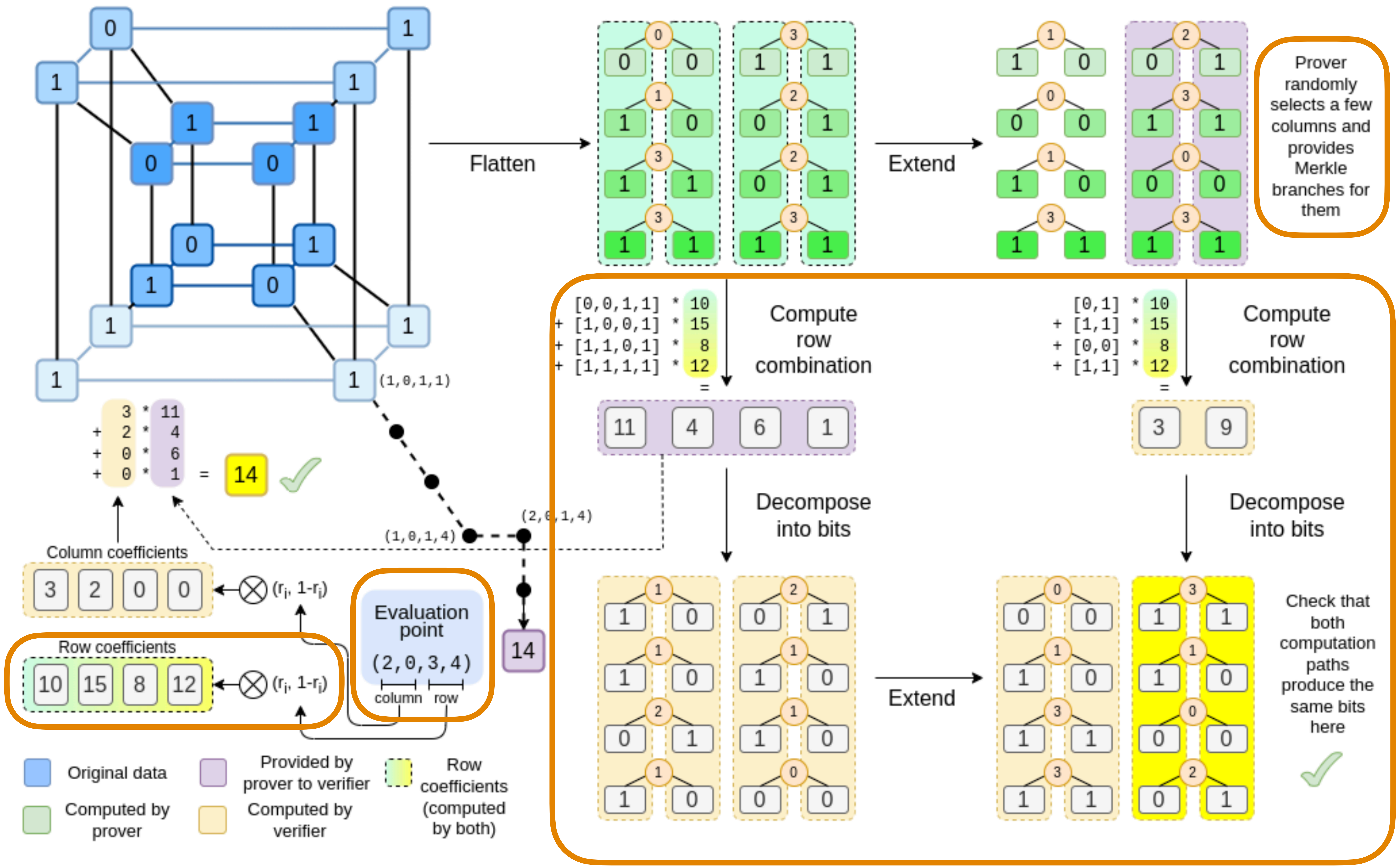

Flatten
Extend
Prover randomly selects a few columns and provides Merkle branches for them
Compute row combination
Compute row combination
Decompose into bits
Decompose into bits
Extend
Check that both computation paths produce the same bits here
Column coefficients
Row coefficients
Evaluation point
(2,0,3,4)
column
row
(1,0,1,1)
(1,0,1,4)
(2,0,1,4)
Original data
Computed by prover
Provided by prover to verifier
Computed by verifier
Row coefficients (computed by both)

Binius



Flatten
Extend
Prover randomly selects a few columns and provides Merkle branches for them
[0,0,1,1] * 10
+ [1,0,0,1] * 15
+ [1,1,0,1] * 8
+ [1,1,1,1] * 12
=
Compute row combination
[0,1] * 10
+ [1,1] * 15
+ [0,0] * 8
+ [1,1] * 12
=
Compute row combination
11 4 6 1
3 9
Decompose into bits
Decompose into bits
Extend
Check that both computation paths produce the same bits here
(1,0,1,1)
3 * 11
+ 2 * 4
+ 0 * 6
+ 0 * 1
= 14
(1,0,1,4)
(2,0,1,4)
14
Column coefficients
3 2 0 0
(r_i, 1-r_i)
Row coefficients
10 15 8 12
(r_i, 1-r_i)
Evaluation point
(2,0,3,4)
column row
Original data
Computed by prover
Provided by prover to verifier
Computed by verifier
Row coefficients (computed by both)

Compute row combination

In the evaluation point $(r_o, r_1, r_2, r_3) = (2,0,3,4)$

$$\otimes_{i=2,3}(1 - r_i, r_i) = [(1 - r_2) \cdot (1 - r_3), r_2 \cdot (1 - r_3), (1 - r_2) \cdot r_3, r_2 \cdot r_3] = [(1 - 3) \cdot (1 - 4), 3 \cdot (1 - 4), (1 - 3) \cdot 4, 3 \cdot 4]$$

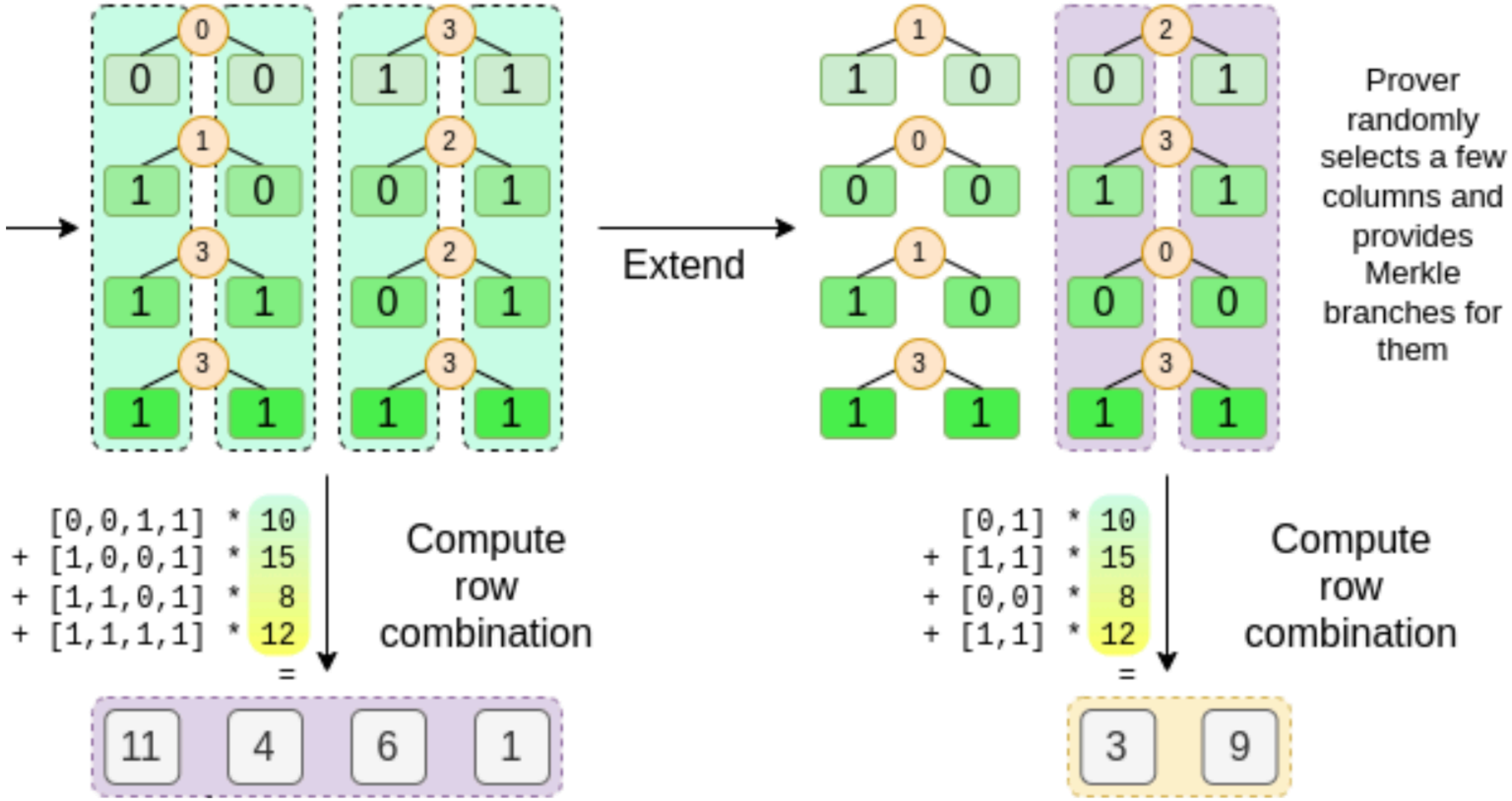
$$= [2 \cdot 5, 3 \cdot 5, 2 \cdot 4, 3 \cdot 4] = [10, 15, 8, 12]$$

Compute row combination

In the evaluation point $(r_o, r_1, r_2, r_3) = (2,0,3,4)$

$$\otimes_{i=2,3}(1 - r_i, r_i) = [(1 - r_2)\cdot(1 - r_3), r_2\cdot(1 - r_3), (1 - r_2)\cdot r_3, r_2\cdot r_3] = [(1-3)\cdot(1-4), 3\cdot(1-4), (1-3)\cdot 4, 3\cdot 4]$$

$$= [2\cdot 5, 3\cdot 5, 2\cdot 4, 3\cdot 4] = [10,15,8,12]$$



The linearity of the extension

A linear combination of the extension = the extension of a linear combination

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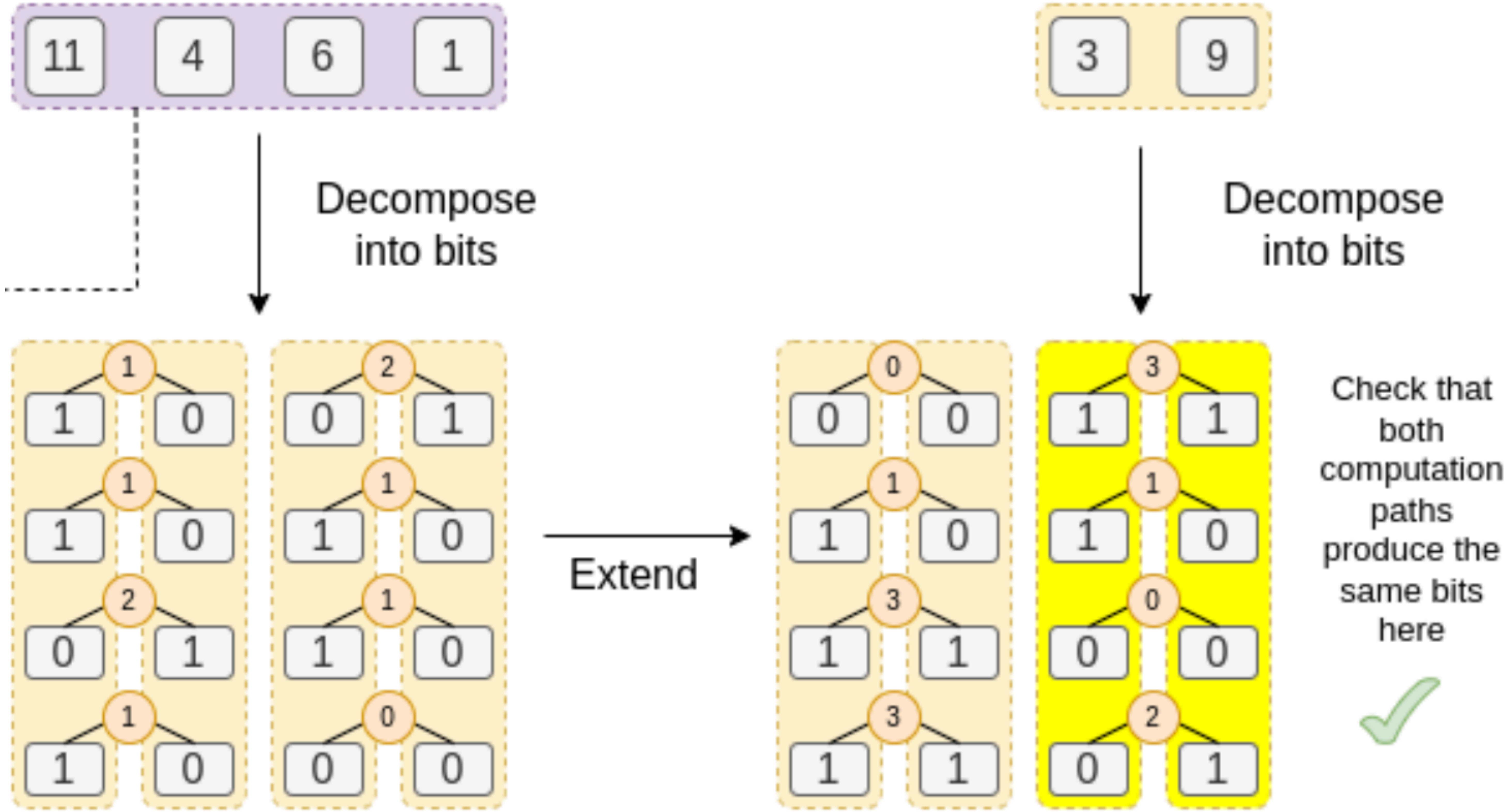

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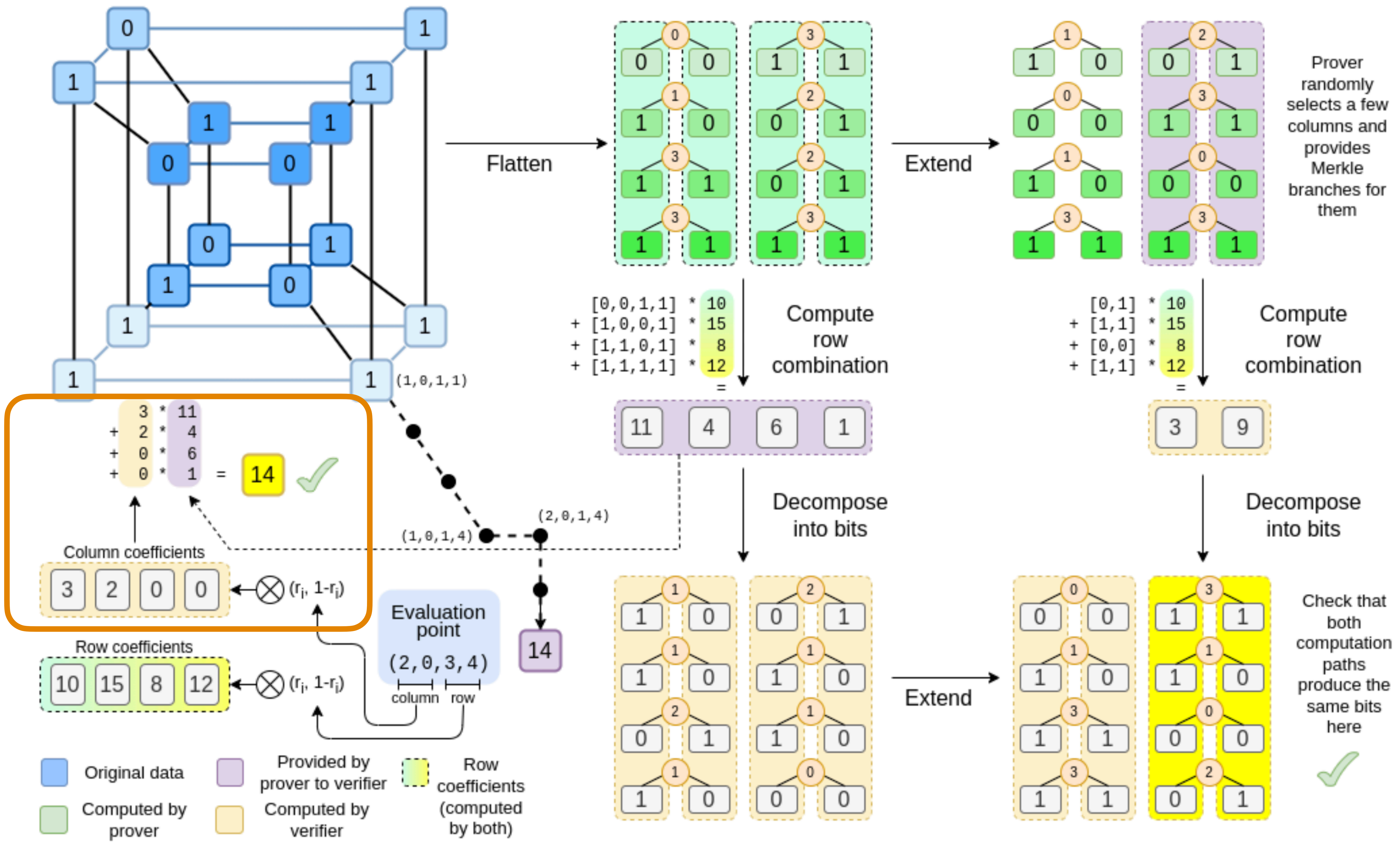
Verify that the answer is 14



Binius

Thank you!