## Ingo Rohlfing: Multiple non-exclusive hypotheses in Bayesianism

In this post, I briefly illustrate for the discrete version of Bayes' theorem the implications of working with multiple non-exclusive hypotheses. I will explain what the specific issue with non-exclusive hypotheses is, how to take it into account when specifying the theorem and what the scope of the problem is.

The post is only a minor extension of what Sheryl Zaks has written about in two articles(). Based on an email exchange I just had with someone about this, I thought it might be a good idea to put up a short post on this.

### The issue with non-exclusive hypotheses

The toy example of Bayes theorem in the *process tracing* literature uses two exclusive hypotheses and one piece of evidence or a single body of evidence $E$. This is totally fine for introducing the theorem to someone who is not familiar with the theorem (it was for me). However, the toy example is as far as away from actual process tracing because we usually have more than two hypotheses; the hypotheses are not exclusive, at least not all of them; we have many pieces of evidence.

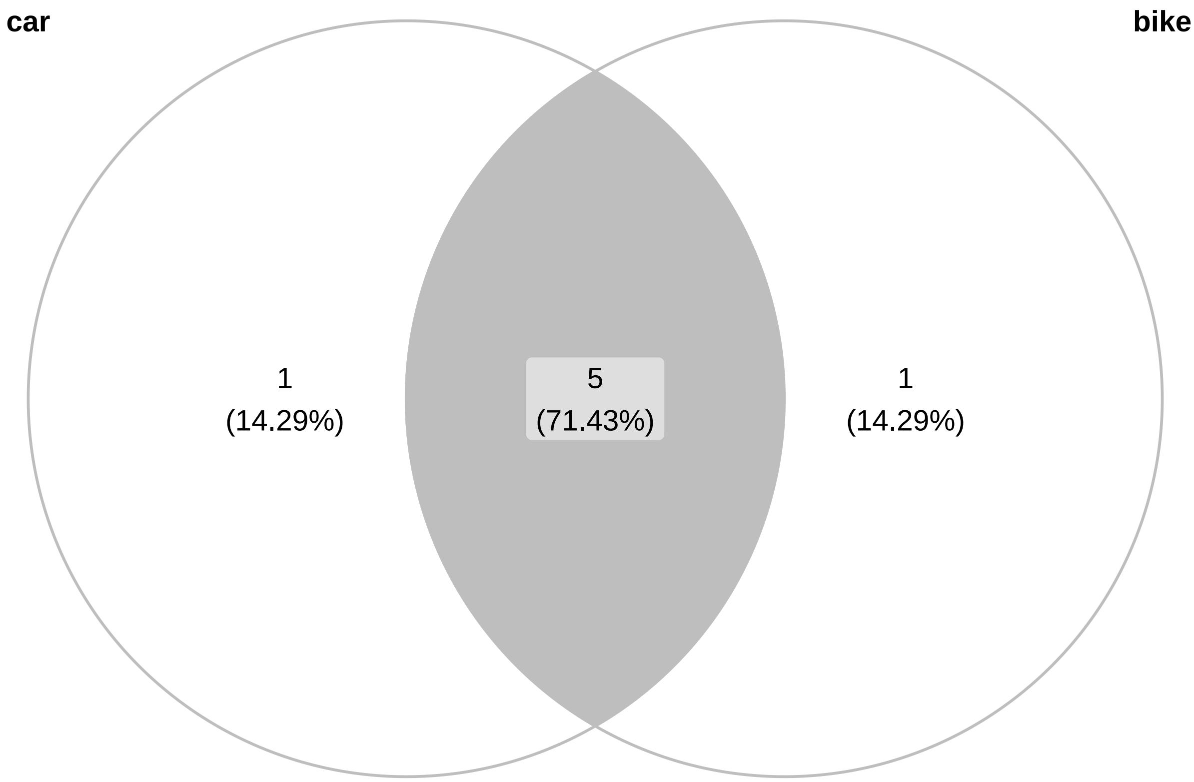
Concerning the hypotheses, the *priors* for the hypotheses need to add up to 1. With two hypotheses $H_A$ and $H_0$, this is a piece of cake because $p(H_A) = 1 - p(H_0)$. (Of course, coming up with the priors in an empirical analysis is hard.) In practice, you can always only specify two hypotheses. Imagine the outcome is the position change of a party. $H_A$ could be that the party responds to changes in the position of the median voter. $A$ is the median voter here and $H_A$ the corresponding hypothesis. $H_0$ would simply be the negation and say that it is not the median voter who makes a party move. This is a possible way of theorizing, but neither very informative nor elegant. $H_0$ includes the possibility that it is neither the median voter nor anything else that explains party behavior, or that it is not the median voter, but another cause $B$ that makes parties move. This is way it has been recommended to break up $H_0$ and put forward multiple, substantively informative hypotheses.

What happens now when we add a second variable $B$ that is position change of the *party supporters*? In this case, it makes most sense to assume that an effect the median voter and supporters are not exclusive. It could be that both have an effect, or only one of the two or neither of both, which means that the null hypothesis $H_0$ is also in the game here.

I want to illustrate the implications of two non-exclusive hypotheses plus the null hypothesis with a simple example that might be more intuitive than priors that we attach to hypotheses. Suppose you have data for 10 people and want to know how many own a car ($A$) or a bike ($B$). To make the example closer to the priors you need for Bayes, we can assume you want to know the probability that a randomly sampled person is a car owner or a bike owner.

The data is as follows: 5 people own a car and a bike; 1 person owns a car; 1 person owns a bike; 3 people own neither a car nor a bike. In a, say, naive analysis, you would once count the number of car owners (6) and the number of bike owners (6). This gives a sum of 12, meaning the probability of randomly choosing a car owner or a bike owner is an incredible 120%! The problem obviously is that you count the people who own a car and a bike twice. The Venn diagram below illustrates the issue and also ways how it can be addressed. The three people who neither have a car nor a bike are explicitly displayed, but can be imagined as being located outside of the 'car' set and the 'bike' set.

```r
library(ggVennDiagram) # available on CRAN
library(ggplot2)
carbike <- list(car = 1:6, bike = 2:7)
ggVennDiagram(carbike) +
  theme(legend.position = "none") + # remove legend
  scale_fill_gradient(low = "white", high = "gray") # reconfigure coloring
```




**Specifying non-exclusive hypotheses**

There are two ways to specify the priors when two hypotheses are non- exclusive. First, you can specify the total probability for each hypothesis and then subtract the joint probability of the non-exclusive hypothesis: $p(H_A) + p(H_B) - p(H_A \cap H_B)$. If you add $p(H_0) to this, you would get 1 as a result. With regard to the car-bike example, this means your calculation would be 6+6-5 = 7. A second possibility is to "exclusivy" each prior. The problem with the naive approach is to ignore that $p(H_A) = p(H_A \cap H_B) + p(H_A \cap \neg H_B)$ and that $p(H_B) = p(H_B \cap H_A) + p(H_B \cap \neg H_A)$ because of which $p(H_A \cap H_B)$ is counted twice. To "exclusify" the priors requires it to decompose the total probabilities $p(H_A)$ and $p(H_B)$ into the exclusive components:

- $p(H_A \cap \neg H_B)$ (corresponds to the 1 car owner)
- $p(H_B \cap \neg H_A)$ (corresponds to the 1 bike owner)
- $p(H_A \cap H_B)$ (corresponds to the 5 car and bike owners)
- $p(\neg H_A \cap \neg H_B)$ (corresponds to the 3 people without a car and a bike)

I agree with Sheryl Zaks that the specification of priors is more challenging than one might believe because most examples (methodological and substantive) use the toy example with two exclusive hypotheses. (And I also wouldn't accept "Priors do wash out." as a counterargument.) For me, exclusifying the priors might be relatively easier because they represent the different theoretical arguments one can make. $p(H_A \cap \neg H_B)$ means that only hypothesis A is expected to be correct; $p(H_B \cap \neg H_A)$ that only hypothesis B is correct; $p(H_A \cap H_B)$ that both hypotheses are correct and both variables (or conditions or mechanisms) play a role. $H_0$ completes the setting and is the same as $p(\neg H_A \cap \neg H_B)$,

**The scale of the problem.**

The simple car-bike example indicate that the challenges of prior specification can escalate quickly. Moving from the toy example to two non-exclusive hypotheses means that we have to specify four priors instead of two; if we add a third non-exclusive hypothesis, we have eight priors. You might see already where this is going: The number of priors increases *exponentially* with the number of non-exclusive hypotheses. If some, but not all hypotheses are exclusive, the number is somewhere between the number of hypotheses and 2ˆH, $H$ being the number of hypotheses.

All this does not invalidate Bayesian process tracing as an approach and its use for empirical research. I only mean to show for one selected element what it requires to do process tracing that complies with Bayes' theorem. This quickly becomes demanding when we work with non-exclusive hypotheses that are, in my view, the rule in the social sciences.