# FYS4560 Project 2

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## 1 Di-lepton production in $e^+e^-$ in the Standard Model

Possible feynman diagrams for the process

$$e^-e^+ \to l^-l^+$$

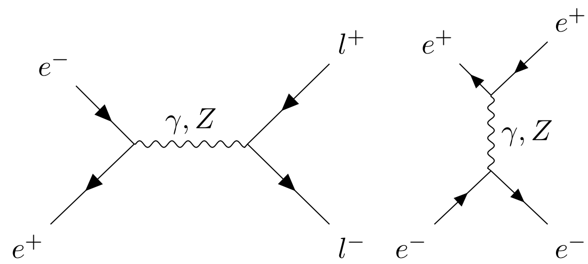


Figure 1: Possible diagrams for di-lepton production.

### 1.1 $e^-e^+ \to \mu^+\mu^-$ through QED

We find the expression for the matrix element from the Feynman diagram in Fig. (2).

$$\begin{aligned} i\mathcal{M} &= \bar{v}^s(p_2)(-ie\gamma^\mu)u^{s'}(p_1)\Big(\frac{-g_{\mu\nu}}{k^2}\Big)\bar{u}^r(k_1)(-ie\gamma^\nu)v^{r'}(k_2) \\ &= \frac{ie^2}{s}\Big(\bar{v}^s(p_2)\gamma^\mu u^{s'}(p_1)\Big)\Big(\bar{u}^r(k_1)\gamma_\mu v^{r'}(k_2)\Big) \end{aligned}$$

Which gives for the matrix element squared

$$|\mathcal{M}|^2 = \frac{e^4}{k^4}\Big(\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu v(p_2)\Big)\Big(\bar{u}(k_1)\gamma_\mu v(k_2)\bar{v}(k_2)\gamma_\nu u(k_1)\Big)$$

Averaging over spins and taking the trace we find

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{k^4}\text{tr}\Big[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\Big]\text{tr}\Big[(\not{k}_1 + m_m)\gamma_\mu(\not{k}_2 - m_m)\gamma_\nu\Big].$$

Since $m_e << m_m$, we can set $m_e = 0$. Traces of odd numbers of gamma matrices are zero, so we can reduce the number of terms to

$$\begin{aligned} \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{e^4}{4k^4}\text{tr}\Big[p_{2\rho}p_{1\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\Big]\text{tr}\Big[k_1^{\rho'}k_2^{\sigma'}\gamma_{\rho'}\gamma_\mu\gamma_{\sigma'}\gamma_\nu - m_m^2\gamma_\mu\gamma_\nu\Big] \\ &= \frac{e^4}{k^4}4\Bigg[\Big(p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu\Big)\Big(k_{1\mu}k_{2\nu} - (k_2\cdot k_1)g_{\mu\nu} + k_{1\nu}k_{2\mu} - m_m^2 g_{\mu\nu}\Big) \\ &= \frac{e^4}{k^4}8\Bigg[(p_1\cdot k_2)(p_2\cdot k_1) + (p_1\cdot k_1)(p_2\cdot k_2) - m_m^2(p_1\cdot p_2)\Bigg] \end{aligned}$$

#### 1.1.1 Kinematics

Assume the electron and muon momenta make an angle $\theta$ between them. Using the Mandelstam variables, and assuming $m_e$ we then get

$$\begin{aligned} k^2 = (p_1+p_2)^2 = 4E^2 = s \\ p_1\cdot k_1 = p_2\cdot k_2 = E^2 - E|\mathbf{k}|\cos\theta \\ p_1\cdot k_2 = p_2\cdot k_1 = E^2 + E|\mathbf{k}|\cos\theta \end{aligned}$$

Putting this into our expression we get

$$\begin{aligned} \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{8e^4}{16s^2}\Big[E^2(E - \mathbf{k}\cos\theta)^2 + E^2(E + \mathbf{k}\cos\theta)^2 + 2m_m^2E^2\Big] \\ &= e^4\Big[\Big(1 + \frac{m_m^2}{s}\Big) + \Big(1 - \frac{m_m^2}{s}\cos^2\theta\Big)\Big] \end{aligned}$$

### 1.1.2 Differential cross section

We can find the differential cross section for this process. The formula for differential cross section of two final-state particles in the center of mass frame is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}|v_{\mathcal{A}} - v_{\mathcal{B}}|} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{cm}} |\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to p_1, p_2)|^2.$$

For this specific calculation $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 2$, and $E_{\mathcal{A}} = E_{\mathcal{B}} = E_{cm}/2$, so we get

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{2E_{cm}^2} \frac{|\mathbf{k}|}{16\pi^2 E_{cm}} \cdot \frac{1}{4} \sum_{spins} |\mathcal{M}|^2 \\
&= \frac{\alpha}{4s}\sqrt{1 - \frac{m_m^2}{s}}\left[\left(1 + \frac{m_m^2}{s}\right) + \left(1 - \frac{m_m^2}{s}\right)\cos^2\theta\right]
\end{aligned}$$

We can rewrite the differential solid angle as $d\Omega = \sin\theta d\theta d\phi = -d\cos\theta d\phi$, so we get

$$\frac{d\sigma}{d\cos\theta} = -\frac{\alpha}{4s}\sqrt{1 - \frac{m_m^2}{s}}\left[\left(1 + \frac{m_m^2}{s}\right) + \left(1 - \frac{m_m^2}{s}\right)\cos^2\theta\right] d\phi$$

### 1.1.3 Cross section

We can now find the total cross section by integrating over the angles

$$\begin{aligned}
\sigma &= -\frac{\alpha}{4s}\sqrt{1 - \frac{m_m^2}{s}} \int_{\Omega} \left[\left(1 + \frac{m_m^2}{s}\right) + \left(1 - \frac{m_m^2}{s}\right)\cos^2\theta\right] d\phi d\cos\theta \\
&= 2\pi\frac{\alpha}{4s}\sqrt{1 - \frac{m_m^2}{s}}\left[\left(1 + \frac{m_m^2}{s}\right)\cos\theta + \frac{1}{3}\left(1 - \frac{m_m^2}{s}\right)\cos^3\theta\right]_{-1}^{1} \\
&= 4\pi\frac{\alpha}{4s}\sqrt{1 - \frac{m_m^2}{s}}\left[\left(1 + \frac{m_m^2}{s}\right) + \frac{1}{3}\left(1 - \frac{m_m^2}{s}\right)\right] \\
\sigma &= \frac{4\pi\alpha}{3s}\sqrt{1 - \frac{m_m^2}{s}}\left[1 + \frac{1}{2}\frac{m_m^2}{s}\right]
\end{aligned}$$

## 1.2 $e^+e^- \to \mu^+\mu^-$

The feynman diagrams that contribute in the electroweak interaction are shown in Fig. (2).

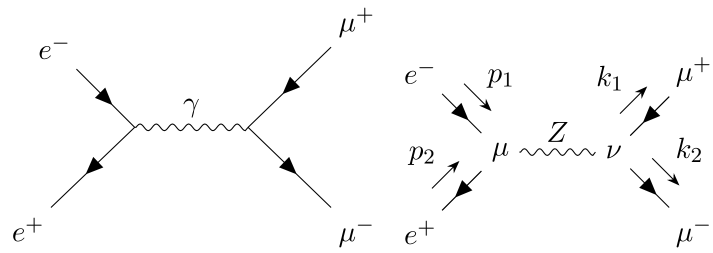


Figure 2: Feynman diagrams for $e^+e^- \to \mu^+\mu^-$.

The cross section for this process gets contributions from the purely electromagnetic ($\gamma$), the purely electroweak ($Z$) and the cross terms between them. We will calulate the terms separately.

## 1.3 Purely electromagnetic interaction

Here we can use the expression found earlier, but since we are working with a much larger energy scale (assume $E_e >> m_e$), we can set the muon mass to zero as well, $m_m = 0$, with yields the expression

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = e^4\big(1+\cos^2\theta\big),$$

which we can rewrite using $e^4 = (e^2)^2 = (4\pi\alpha)^2$ to get

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 16\pi^2\alpha^2\big(1+\cos^2\theta\big).$$

## 1.4 Purely electroweak contribution

We will here use the expressions

$$g_v^f = \frac{1}{2}T_f^3 - Q_f\sin^2\theta_W, \tag{1}$$

$$g_A^f = \frac{1}{2}T_f^3. \tag{2}$$

The expression for the matrix element from the feynman diagram in Fig. (2) is

$$\begin{aligned}
i\mathcal{M} &= \bar{v}^s(p_2)\big[i\frac{g}{\cos\theta_W}\gamma_\mu(g_v^f - g_A^f\gamma_5)\big]u^s(p_1)\Big(-\frac{ig_{\mu\nu}}{k^2-M_Z^2+i\epsilon}\Big)\bar{u}^s(k_2)\big[i\frac{g}{\cos\theta_W}\gamma_\nu(g_v^f - g_A^f\gamma_5)\big]v^s(k_1) \\
&= \frac{ig^2}{(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\bar{v}^s(p_2)\gamma_\mu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(k_2)\gamma^\mu(g_v^f - g_A^f\gamma_5)v^s(k_1)
\end{aligned}$$

$$\begin{aligned}
|\mathcal{M}|^2 = &\frac{g^4}{(k^2-M_Z^2)^2\cos^4\theta_W}\Big(\bar{v}^s(p_2)\gamma_\mu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(p_1)\gamma_\nu(g_v^f - g_A^f\gamma_5)v^s(p_2)\Big) \\
&\times\Big(\bar{u}^s(k_2)\gamma^\mu(g_v^f - g_A^f\gamma_5)v^s(k_1)\bar{v}^s(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)u^s(k_2)\Big)
\end{aligned}$$

Take the trace and get (where $A^2 = \frac{g^4}{(k^2-M_Z^2)^2\cos^4\theta_W}$), and since $M_z >> m_e, m_\mu$, set $m_e = m_\mu \simeq 0$

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{1}{4}A^2\mathrm{tr}\Big[(\not{p}_2 - m_e)\gamma_\mu(g_v^e - g_A^e\gamma_5)(\not{p}_1 + m_e)\gamma_\nu(g_v^e - g_A^e\gamma_5)\Big] \\
&\times\mathrm{tr}\Big[(\not{k}_2 + m_\mu)\gamma^\mu(g_v^l - g_A^l\gamma_5)(\not{k}_1 - m_\mu)\gamma^\nu(g_v^l - g_A^l\gamma_5)\Big] \\
\text{(set mass to zero)} &= \frac{1}{4}A^2\mathrm{tr}\Big[\not{p}_2\gamma_\mu(g_v^e - g_A^e\gamma_5)\not{p}_1\gamma_\nu(g_v^e - g_A^e\gamma_5)\Big]\mathrm{tr}\Big[\not{k}_2\gamma^\mu(g_v^l - g_A^l\gamma_5)\not{k}_1\gamma^\nu(g_v^l - g_A^l\gamma_5)\Big] \\
&= A^2\mathrm{tr}\Big[(g_v^e)^2\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu + (g_A^e)^2\not{p}_2\gamma_\mu\gamma_5\not{p}_1\gamma_\nu\gamma_5 - g_v^e g_A^e\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\gamma_5 - g_v^e g_A^e\not{p}_2\gamma_\mu\gamma_5\not{p}_1\gamma_\nu g_v^e)\Big] \\
&\times\mathrm{tr}\Big[(g_v^l)^2\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu + (g_A^l)^2\not{k}_2\gamma^\mu\gamma_5\not{k}_1\gamma^\nu\gamma_5 - g_v^l g_A^l\not{k}_2\gamma_\mu\not{k}_1\gamma_\nu\gamma_5 - g_v^l g_A^l\not{k}_2\gamma_\mu\gamma_5\not{k}_1\gamma_\nu\gamma_5)\Big] \\
&= \frac{1}{4}A^2\mathrm{tr}\Big[\big((g_A^e)^2 + (g_v^e)^2\big)\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu - 2g_v^e g_A^e\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\gamma_5\Big] \\
&\times\mathrm{tr}\Big[\big((g_v^l)^2 + (g_A^l)^2\big)\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu - 2g_v^l g_A^l\not{k}_2\gamma_\mu\not{k}_1\gamma_\nu\gamma_5\Big] \\
&= \frac{1}{4}A^2\mathrm{tr}\Big[\big((g_v^e)^2 + (g_A^e)^2\big)p_2^\rho p_1^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu - 2g_v^e g_A^e p_2^\rho p_1^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_5\Big] \\
&\times\mathrm{tr}\Big[\big((g_v^l)^2 + (g_A^l)^2\big)k_{2\rho}k_{1\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - 2g_v^l g_A^l k_2^\rho k_1^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_5\Big]
\end{aligned}$$

Now use the trace identities for the gamma matrices

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{3}$$
$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma} \tag{4}$$

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{1}{4}16A^2\Big[((g_v^e)^2+(g_A^e)^2)p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\big) - 2g_v^e g_A^e p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\nu}\Big] \\
&\times\Big[((g_v^l)^2+(g_A^l)^2)k_{2\rho}k_{1\sigma}\big(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\big) - 2g_v^l g_A^l k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu}\Big] \\
&= \frac{1}{4}16A^2\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}\cdot p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\big)k_{2\rho}k_{1\sigma}\big(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\big) \\
&- \{2((g_v^e)^2+(g_A^e)^2)g_v^l g_A^l\}\cdot p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\big)k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu}) \\
&- \{2((g_v^l)^2+(g_A^l)^2)g_v^e g_A^e\}\cdot p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\mu}k_{2\rho}k_{1\sigma}\big(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\big) \\
&+ \{4g_v^e g_A^e g_v^l g_A^l\}\cdot p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\nu}k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu}\Big] \\
&= \frac{1}{4}16A^2\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}\cdot\big(p_{2\mu}p_{1\nu} - p_{2\sigma}p_1^\sigma g_{\mu\nu} + p_{2\nu}p_{1\mu}\big)\Big(k_2^\mu k_1^\nu - k_2^\sigma k_{1\sigma}g^{\mu\nu} + k_2^\nu k_1^\mu\Big) \\
&- \{2((g_v^e)^2+(g_A^e)^2)g_v^l g_A^l\}\cdot\big(p_{2\mu}p_{1\nu} - p_{2\sigma}p_1^\sigma g_{\mu\nu} + p_{2\nu}p_{1\mu}\big)k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu}) \\
&- \{2((g_v^l)^2+(g_A^l)^2)g_v^e g_A^e\}\cdot p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\mu}\big(k_2^\mu k_1^\nu - k_2^\sigma k_{1\sigma}g^{\mu\nu} + k_2^\nu k_1^\mu\big) + 4!\{4g_v^e g_A^e g_v^l g_A^l\}\cdot p_2^\rho k_{2\rho}p_1^\sigma k_{1\sigma}\Big]
\end{aligned}$$

Where we've used

$$\epsilon_{\rho\mu\sigma\mu}\epsilon^{\rho\mu\sigma\mu} = n! = 4! = 24.$$

$$\begin{aligned}
&= \frac{1}{4}16A^2\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}\cdot\Big[2(p_2\cdot k_2)(p_1\cdot k_1) + 2(p_1\cdot k_2)(p_2\cdot k_1)\Big] \\
&- \{2((g_v^e)^2+(g_A^e)^2)g_v^l g_A^l\}\cdot\Big(\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu} - p_{2\sigma}p_1^\sigma g_{\mu\nu} + p_{2\nu}p_{1\mu}\big)k_{2\rho}k_{1\sigma} + \epsilon^{\rho\mu\sigma\nu}p_{2\rho}p_{1\sigma}\big(k_{2\mu}k_{1\nu} - k_{2\sigma}k_1^\sigma g_{\mu\nu} + k_{2\nu}k_{1\mu}\big)\Big) \\
&+ 4!\{4g_v^e g_A^e g_v^l g_A^l\}\cdot(p_2\cdot k_2)(p_1\cdot k_1)\Big] \\
&= I + II + III
\end{aligned}$$

### 1.4.1 Second term ($II$)

Take a look at the second term, and notice that $\epsilon^{\mu\nu\rho\sigma}$ is zero if any two indices are equal, so $\epsilon^{\mu\nu\rho\sigma}g_{\mu\nu} = 0$, so we get

$$\begin{aligned}
&\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu} - p_{2\sigma}p_1^\sigma g_{\mu\nu} + p_{2\nu}p_{1\mu}\big)k_{2\rho}k_{1\sigma} + \epsilon^{\rho\mu\sigma\nu}p_{2\rho}p_{1\sigma}\big(k_{2\mu}k_{1\nu} - k_{2\sigma}k_1^\sigma g_{\mu\nu} + k_{2\nu}k_{1\mu}\big) \\
&= \epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma} + p_{2\nu}p_{1\mu}k_{2\rho}k_{1\sigma}\big) + \epsilon^{\rho\mu\sigma\nu}\big(p_{2\rho}p_{1\sigma}k_{2\mu}k_{1\nu} + p_{2\rho}p_{1\sigma}k_{2\nu}k_{1\mu}\big) \\
&= \epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma} + p_{2\nu}p_{1\mu}k_{2\rho}k_{1\sigma}\big) + \epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma} + p_{2\mu}p_{1\nu}k_{2\sigma}k_{1\rho}\big) \\
&= 2\big(\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma} + \epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\sigma}k_{1\rho}\big) \\
&= 2\big(\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma} - \epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}\big) = \underline{0}
\end{aligned}$$

### 1.4.2 First term ($I$)

$$\begin{aligned}
p_2^\rho p_1^\sigma 4\big(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\big)4k_{2\rho}k_{1\sigma}&\Big(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\Big) \\
&= \Big(p_{2\mu}p_{1\nu} - p_{2\sigma}p_1^\sigma g_{\mu\nu} + p_{2\nu}p_{1\mu}\Big)\Big(k_2^\mu k_1^\nu - k_2^\sigma k_{1\sigma}g^{\mu\nu} + k_2^\nu k_1^\mu\Big) \\
&= \Big[(p_2\cdot k_2)(p_1\cdot k_1) - (k_2\cdot k_1)(p_2\cdot p_1) + (p_1\cdot k_2)(p_2\cdot k_1) \\
&- (p_2\cdot p_1)(k_2\cdot k_1) + 4(k_2\cdot k_1)(p_2\cdot p_1) - (p_2\cdot p_1)(k_2\cdot k_1) \\
&+ (k_2\cdot p_1)(p_2\cdot k_1) - (k_2\cdot k_1)(p_2\cdot p_1) + (p_2\cdot k_2)(p_1\cdot k_1)\Big] \\
&= \Big[2(p_2\cdot k_2)(p_1\cdot k_1) + 2(p_1\cdot k_2)(p_2\cdot k_1)\Big]
\end{aligned}$$

**Back to $|\mathcal{M}|^2$**

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}16A^2\Big[\{2((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)+4\cdot 4!\cdot g_v^e g_A^e g_v^l g_A^l\}\cdot(p_2\cdot k_2)(p_1\cdot k_1)$$
$$+2\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}(p_1\cdot k_2)(p_2\cdot k_1)\Big]$$
$$=\frac{1}{4}32\frac{g^4}{(s-M_Z^2)^2\cos^4\theta_W}\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)+48\cdot g_v^e g_A^e g_v^l g_A^l\}\cdot(p_2\cdot k_2)(p_1\cdot k_1)$$
$$+\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}(p_1\cdot k_2)(p_2\cdot k_1)\Big]$$

## Kinematics

We work in the center of mass-frame. Since $|\mathbf{k}| = \sqrt{E^2-m_\mu^2}\simeq\sqrt{E^2}=E$, we get

$$s = k^2 = (p_1+p_2)^2 = (k_1+k_2)^2 = 4E^2$$
$$(p_1\cdot k_1) = (p_2\cdot k_2) = E^2 - E|\mathbf{k}|\cos\theta \simeq E^2(1-\cos\theta) = \frac{1}{4}s(1-\cos\theta)$$
$$(p_1\cdot k_2) = (p_2\cdot k_1) = E^2 + E|\mathbf{k}|\cos\theta \simeq E^2(1+\cos\theta) = \frac{1}{4}s(1+\cos\theta)$$

Putting this into the expression we get

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}32\frac{g^4}{(s-M_Z^2)^2\cos^4\theta_W}\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)+48\cdot g_v^e g_A^e g_v^l g_A^l\}\cdot(\frac{1}{4}s(1-\cos\theta))^2$$
$$+\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}(\frac{1}{4}s(1+\cos\theta))^2\Big]$$
$$=\frac{1}{4}4\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}\cdot(1+\cos^2\theta)+24g_v^e g_A^e g_v^l g_A^l(1-\cos\theta)^2\Big]$$

We know that muons and electrons behave in the exact same way, only their mass is different, so we set $g_v^e = g_v^l$, $g_A^e = g_A^l$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}4\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}\Big[((g_v^e)^2+(g_A^e)^2)^2\cdot(1+\cos^2\theta)+24(g_v^e)^2(g_A^e)^2(1-\cos\theta)^2\Big]$$
$$=\frac{1}{4}4\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}\Big[((g_v^e)^2+(g_A^e)^2)^2\cdot(1+\cos^2\theta)+24(g_v^e)^2(g_A^e)^2(1-2\cos\theta+\cos^2\theta)\Big]$$
$$=\frac{1}{4}4\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}\Big[[((g_v^e)^2+(g_A^e)^2)^2+24(g_v^e)^2(g_A^e)^2]\cdot(1+\cos^2\theta)-48(g_v^e)^2(g_A^e)^2 2\cos\theta\Big]$$

Here we can use that
$g^4/\cos^4\theta_w = (e/\sin\theta_W)^4/\cos^4\theta_W = e^4/(\sin\theta_W\cos_W)^4 = e^4/(1/2\sin 2\theta_W)^4 = e^4(M_Z^2)^2G^2 2/(\alpha^2\pi^2)$ to rewrite this expression

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 8\frac{1}{4}\frac{e^4G^2s^2}{\alpha^2\pi^2}\Big(\frac{M_Z^2}{s-M_Z^2}\Big)^2\Big[[((g_v^e)^2+(g_A^e)^2)^2+24(g_v^e)^2(g_A^e)^2]\cdot(1+\cos^2\theta)-48(g_v^e)^2(g_A^e)^2 2\cos\theta\Big]$$
$$=32G^2s^2\Big(\frac{M_Z^2}{s-M_Z^2}\Big)^2\Big[[((g_v^e)^2+(g_A^e)^2)^2+24(g_v^e)^2(g_A^e)^2]\cdot(1+\cos^2\theta)-96(g_v^e)^2(g_A^e)^2\cos\theta\Big]$$

### 1.5 Cross terms

We get the cross terms from the expression

$$
\begin{aligned}
|\mathcal{M}|^2 &= (i\mathcal{M}_\gamma + i\mathcal{M}_Z)(-i\mathcal{M}_\gamma^* - i\mathcal{M}_Z^*) \\
&= |\mathcal{M}_\gamma|^2 + \mathcal{M}_\gamma\mathcal{M}_Z^* + \mathcal{M}_Z\mathcal{M}_\gamma^* + |\mathcal{M}_Z|^2.
\end{aligned}
$$

We use the expressions we've found for $\mathcal{M}_{Z/\gamma}$, and write out the cross terms, which we will call $\mathcal{M}_\times = \mathcal{M}_\gamma\mathcal{M}_Z^* + \mathcal{M}_Z\mathcal{M}_\gamma^*$

$$
\begin{aligned}
\mathcal{M}_\times = &\left(\frac{ie^2}{s}\left(\bar{v}^s(p_2)\gamma^\mu u^{s'}(p_1)\right)\left(\bar{u}^r(k_1)\gamma_\mu v^{r'}(k_2)\right)\right) \\
&\times\left(\frac{ig^2}{(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\bar{v}^s(p_2)\gamma_\nu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)v^s(k_2)\right)^* \\
&+\left(\frac{ig^2}{(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\bar{v}^s(p_2)\gamma_\nu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)v^s(k_2)\right) \\
&\times\left(\frac{ie^2}{s}\left(\bar{v}^s(p_2)\gamma^\mu u^{s'}(p_1)\right)\left(\bar{u}^r(k_1)\gamma_\mu v^{r'}(k_2)\right)\right)^* \\
&= \frac{e^2g^2}{s(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\Bigg[\left(\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma_\nu(g_v^f - g_A^f\gamma_5)v(p_2)\right)\left(\bar{u}(k_1)\gamma_\mu v(k_2)\bar{v}(k_2)\gamma^\nu(g_v^f - g_A^f\gamma_5)u(k_1)\right) \\
&+\left(\bar{v}(p_2)\gamma_\nu(g_v^f - g_A^f\gamma_5)u(p_1)\bar{u}(p_1)\gamma^\mu v(p_2)\right)\left(\bar{u}(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)v(k_2)\bar{v}(k_2)\gamma_\mu u(k_1)\right)\Bigg]
\end{aligned}
$$

Introduce the coefficient $B = \frac{e^2g^2}{s(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}$ and take the trace and average over spins, also note that we can move $\gamma^5\not{p}\gamma^\mu = -\not{p}\gamma^5\gamma^\mu = \not{p}\gamma^\mu\gamma^5$

$$
\begin{aligned}
\frac{1}{4B}\sum_{spins}\mathcal{M}_\times &= \frac{1}{4}\mathrm{tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma_\nu(g_v^f - g_A^f\gamma_5)\right]\mathrm{tr}\left[(\not{k}_1 + m_m)\gamma_\mu(\not{k}_2 - m_m)\gamma^\nu(g_v^f - g_A^f\gamma_5)\right] \\
&+\mathrm{tr}\left[(\not{p}_2 - m_e)\gamma_\nu(\not{p}_1 + m_e)\gamma^\mu(g_v^f - g_A^f\gamma_5)\right]\mathrm{tr}\left[(\not{k}_1 + m_m)\gamma^\nu(\not{k}_2 - m_m)\gamma_\mu(g_v^f - g_A^f\gamma_5)\right] \\
&= \frac{1}{4}2\mathrm{tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma_\nu(g_v^f - g_A^f\gamma_5)\right]\mathrm{tr}\left[(\not{k}_1 + m_m)\gamma_\mu(\not{k}_2 - m_m)\gamma^\nu(g_v^f - g_A^f\gamma_5)\right]
\end{aligned}
$$

We can now set the muon- and electron masses to zero

$$
\begin{aligned}
\frac{1}{8B}\sum_{spins}\mathcal{M}_\times &= \frac{1}{4}\mathrm{tr}\left[p_{2\rho}p_{1\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu(g_v^f - g_A^f\gamma_5)\right]\mathrm{tr}\left[k_1^\rho k_2^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu(g_v^f - g_A^f\gamma_5)\right] \\
&= \frac{1}{4}\left(p_{2\rho}p_{1\sigma}\left[g_v^f(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}) - g_A^f\epsilon^{\rho\mu\sigma\nu}\right]\right)\left(k_1^\rho k_2^\sigma\left[g_v^f(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}) - g_A^f\epsilon_{\rho\mu\sigma\nu}\right]\right) \\
&= \frac{1}{4}\left(g_v^f(p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu) - g_A^f p_{2\rho}p_{1\sigma}\epsilon^{\rho\mu\sigma\nu}\right)\left(g_v^f(k_{1\mu}k_{2\nu} - (k_1\cdot k_2)g_{\mu\nu} + k_{1\nu}k_{2\mu}) - g_A^f k_1^\rho k_2^\sigma\epsilon_{\rho\mu\sigma\nu}\right) \\
&= \frac{1}{4}g_v^f(p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu)g_v^f(k_{1\mu}k_{2\nu} - (k_1\cdot k_2)g_{\mu\nu} + k_{1\nu}k_{2\mu}) \\
&- g_A^f g_v^f\left(p_{2\rho}p_{1\sigma}\epsilon^{\rho\mu\sigma\nu}(k_{1\mu}k_{2\nu} - (k_1\cdot k_2)g_{\mu\nu} + k_{1\nu}k_{2\mu}) + (p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu)k_1^\rho k_2^\sigma\epsilon_{\rho\mu\sigma\nu}\right) \\
&+ g_A^f p_{2\rho}p_{1\sigma}\epsilon^{\rho\mu\sigma\nu}g_A^f k_1^\rho k_2^\sigma\epsilon_{\rho\mu\sigma\nu} \\
&= \frac{1}{4}2(g_v^f)^2\left[(p_2\cdot k_2)(p_1\cdot k_1) + (p_1\cdot k_2)(p_2\cdot k_1)\right] + 4!(g_A^f)^2(p_1\cdot k_2)(p_2\cdot k_1)
\end{aligned}
$$

**Kinematics**

Using the kinematics derived in the previous calculation, we can simplify this expression

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|_\times &= \frac{1}{4}4B\Big((g_v^f)^2\Big[\frac{1}{16}s^2(1-\cos\theta)^2+\frac{1}{16}s^2(1+\cos\theta)^2\Big]+12(g_A^f)^2\frac{1}{16}s^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{4}\frac{1}{2}Bs^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big]+6(g_A^f)^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{4}\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big]+6(g_A^f)^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{4}\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\Big([(g_v^f)^2+6(g_A^f)^2]\Big[1+\cos^2\theta\Big]+12(g_A^f)^2\cos\theta\Big)\\
&= \frac{1}{4}\frac{1}{2}\frac{e^4}{s(s-M_Z^2)\cos^2\theta_W\sin^2\theta_W}s^2\Big([(g_v^f)^2+6(g_A^f)^2]\Big[1+\cos^2\theta\Big]+12(g_A^f)^2\cos\theta\Big)
\end{aligned}
$$

We can now use that $\cos^{-2}\theta_W\sin^{-2}\theta_W=(\cos\theta_W\sin\theta_W)^{-2}=(1/2\sin 2\theta_W)^{-2}=M_Z^2G\sqrt{2}/(\alpha\pi)$, along with $e^4=(e^2)^2=(4\pi\alpha)^2=16\pi^2\alpha^2$, to get

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|_\times &= \frac{2\pi^2\alpha^2M_Z^2G\sqrt{2}}{s(s-M_Z^2)\alpha\pi}s^2\Big([(g_v^f)^2+6(g_A^f)^2]\Big[1+\cos^2\theta\Big]+12(g_A^f)^2\cos\theta\Big)\\
&= \frac{4}{\sqrt{2}}\pi\alpha Gs\frac{M_Z^2}{(s-M_Z^2)}\Big([(g_v^f)^2+6(g_A^f)^2]\Big[1+\cos^2\theta\Big]+12(g_A^f)^2\cos\theta\Big)
\end{aligned}
$$

### 1.6 Combining the terms

We can now combine the matrix element terms

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 =& 16\pi^2\alpha^2\big(1+\cos^2\theta\big) \\
&+32G^2s^2\Big(\frac{M_Z^2}{s-M_Z^2}\Big)^2\Big[[((g_v^e)^2+(g_A^e)^2)^2+24(g_v^e)^2(g_A^e)^2]\cdot(1+\cos^2\theta)-96(g_v^e)^2(g_A^e)^2\cos\theta\Big] \\
&+\frac{4}{\sqrt{2}}\pi\alpha Gs\frac{M_Z^2}{(s-M_Z^2)}\Big([(g_v^f)^2+6(g_A^f)^2]\Big[1+\cos^2\theta\Big]+12(g_A^f)^2\cos\theta\Big) \\
=&\Bigg[16\pi^2\alpha^2+32G^2s^2\Big(\frac{M_Z^2}{s-M_Z^2}\Big)^2[((g_v^e)^2+(g_A^e)^2)^2+24(g_v^e)^2(g_A^e)^2] \\
&+\frac{4}{\sqrt{2}}\pi\alpha Gs\frac{M_Z^2}{(s-M_Z^2)}[(g_v^f)^2+6(g_A^f)^2]\Bigg](1+\cos^2\theta) \\
&+\Bigg[32G^2s^2\Big(\frac{M_Z^2}{s-M_Z^2}\Big)^2(-96(g_v^e)^2(g_A^e)^2)+\frac{4}{\sqrt{2}}\pi\alpha Gs\frac{M_Z^2}{(s-M_Z^2)}12(g_A^f)^2\Bigg]\cos\theta \\
=&4\Bigg[4\pi^2\alpha^2+\frac{1}{\sqrt{2}}\pi\alpha Gs\frac{M_Z^2}{(s-M_Z^2)}[(g_v^f)^2+6(g_A^f)^2] \\
&+8G^2s^2\Big(\frac{M_Z^2}{s-M_Z^2}\Big)^2[((g_v^e)^2+(g_A^e)^2)^2+24(g_v^e)^2(g_A^e)^2]\Bigg](1+\cos^2\theta) \\
&+24\Bigg[\sqrt{2}(g_A^f)^2\pi s\frac{M_Z^2}{(s-M_Z^2)}\alpha G+(-4)8G^2s^2\Big(\frac{M_Z^2}{s-M_Z^2}\Big)^2(g_v^e)^2(g_A^e)^2\Bigg]\cos\theta
\end{aligned}
$$

Now look at the coefficients separately and try to 'clean up'. The $1+\cos^2\theta$-coefficient is

$$
\begin{aligned}
\text{First } =&4\alpha^2\Bigg[4\pi^2+\frac{1}{\sqrt{2}}\pi\frac{M_Z^2}{(s-M_Z^2)}[g_v^2+6g_A^2]\Big(\frac{sG}{\alpha}\Big) \\
&+8\Big(\frac{M_Z^2}{s-M_Z^2}\Big)^2[(g_v^2+g_A^2)^2+24g_v^2g_A^2]\Big(\frac{sG}{\alpha}\Big)^2\Bigg]
\end{aligned}
$$

The $\cos\theta$-coefficient is (we will now write $g_A=g_A^e$ and $g_v=g_v^e$)

$$
\text{Second } =24\alpha^2\pi^2\Bigg[\frac{\sqrt{2}g_A^2}{\pi}\frac{M_Z^2}{(s-M_Z^2)}\Big(\frac{sG}{\alpha}\Big)+(-4)8\Big(\frac{sG}{\alpha}\Big)^2\Big(\frac{M_Z^2}{s-M_Z^2}\Big)^2g_v^2g_A^2\Bigg]
$$

#### 1.6.1 Differential cross section

Again, the differential cross section is given by

$$
\Big(\frac{d\sigma}{d\Omega}\Big)_{CM}=\frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}|v_{\mathcal{A}}-v_{\mathcal{B}}|}\frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{cm}}|\mathcal{M}(p_{\mathcal{A}},p_{\mathcal{B}}\to p_1,p_2)|^2.
$$

### 1.7 Some identities

The identities used to simplify the expressions in these calculations are as follows [1]

$$
\alpha=\frac{e^2}{4\pi} \tag{5}
$$

$$
M_Z=\Big(\frac{\alpha\pi}{G\sqrt{2}}\Big)^{1/2}\frac{2}{\sin 2\theta_W} \tag{6}
$$

$$
g\sin\theta_W=e \tag{7}
$$

## References


[1] Franz Mandl and Graham Shaw. *Quantum field theory.* John Wiley & Sons, 2010.