FYS4560 Project 2 - Appendices

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1 Appendix A

These are the calculations for the purely electroweak contribution to $\mathcal{M}_{e^-e^+\to\mu^+\mu^-}$.

1.1 Feynman rules

$$vertex = \frac{-ig}{2\cos\theta_W}\gamma^\mu(g_v - \gamma_5 g_A),$$

where

$$g_v = T_f^3 - 2Q_f\sin^2\theta_W, \tag{1}$$
$$g_A = T_f^3. \tag{2}$$

1.2 Matrix element $i\mathcal{M}$

The expression for the matrix element is

$$\begin{aligned}
i\mathcal{M} &= \bar{v}^s(p_2)\big[i\frac{g}{2\cos\theta_W}\gamma_\mu(g_v - g_A\gamma_5)\big]u^s(p_1)\Big(-\frac{ig_{\mu\nu}}{k^2 - M_Z^2 + i\epsilon}\Big)\bar{u}^s(k_2)\big[i\frac{g}{2\cos\theta_W}\gamma_\nu(g_v - g_A\gamma_5)\big]v^s(k_1)\\
&= \frac{1}{4}\frac{ig^2}{(k^2 - M_Z^2 + i\epsilon)\cos^2\theta_W}\Big(\bar{v}^s(p_2)\gamma_\mu(g_v - g_A\gamma_5)u^s(p_1)\Big)\Big(\bar{u}^s(k_2)\gamma^\mu(g_v - g_A\gamma_5)v^s(k_1)\Big)
\end{aligned}$$

1.3 Matrix element squared $|\mathcal{M}|^2$

$$\begin{aligned}
|\mathcal{M}|^2 = \frac{1}{16}\frac{g^4}{(k^2 - M_Z^2)^2\cos^4\theta_W}&\Big(\bar{v}^s(p_2)\gamma_\mu(g_v - g_A\gamma_5)u^s(p_1)\bar{u}^s(p_1)\gamma_\nu(g_v - g_A\gamma_5)v^s(p_2)\Big)\\
&\times\Big(\bar{u}^s(k_2)\gamma^\mu(g_v - g_A\gamma_5)v^s(k_1)\bar{v}^s(k_1)\gamma^\nu(g_v - g_A\gamma_5)u^s(k_2)\Big)
\end{aligned}$$

1.4 Spin average and trace

Take the trace and get (where $A^2 = \frac{g^4}{(k^2-M_Z^2)^2\cos^4\theta_W}$), and since $M_z >> m_e, m_\mu$, set $m_e = m_\mu \simeq 0$. We will use trace identities for the gamma matrices.

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 =& \frac{1}{4}\frac{1}{16}A^2\mathrm{tr}\Big[(\not{p}_2 - m_e)\gamma_\mu(g_v - g_A\gamma_5)(\not{p}_1 + m_e)\gamma_\nu(g_v - g_A\gamma_5)\Big]\\
&\times\mathrm{tr}\Big[(\not{k}_2 + m_\mu)\gamma^\mu(g_v - g_A\gamma_5)(\not{k}_1 - m_\mu)\gamma^\nu(g_v - g_A\gamma_5)\Big]\\
\text{(set mass to zero)} =& \frac{1}{4}\frac{1}{16}A^2\mathrm{tr}\Big[\not{p}_2\gamma_\mu(g_v - g_A\gamma_5)\not{p}_1\gamma_\nu(g_v - g_A\gamma_5)\Big]\mathrm{tr}\Big[\not{k}_2\gamma^\mu(g_v - g_A\gamma_5)\not{k}_1\gamma^\nu(g_v - g_A\gamma_5)\Big]\\
=& \frac{1}{4}\frac{1}{16}A^2\mathrm{tr}\Big[g_v^2\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu + g_A^2\not{p}_2\gamma_\mu\gamma_5\not{p}_1\gamma_\nu\gamma_5 - g_vg_A\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\gamma_5 - g_vg_A\not{p}_2\gamma_\mu\gamma_5\not{p}_1\gamma_\nu)\Big]\\
&\times\mathrm{tr}\Big[g_v^2\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu + g_A^2\not{k}_2\gamma^\mu\gamma_5\not{k}_1\gamma^\nu\gamma_5 - g_vg_A\not{k}_2\gamma_\mu\not{k}_1\gamma_\nu\gamma_5 - g_vg_A\not{k}_2\gamma_\mu\gamma_5\not{k}_1\gamma_\nu)\Big]
\end{aligned}$$

Use that $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^5)^2 = 1$, and find

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 =& \frac{1}{4}\frac{1}{16}A^2\mathrm{tr}\Big[(g_A^2 + g_v^2)\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu - 2g_vg_A\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\gamma_5\Big]\mathrm{tr}\Big[(g_v^2 + g_A^2)\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu - 2g_vg_A\not{k}_2\gamma_\mu\not{k}_1\gamma_\nu\gamma_5\Big]\\
=& \frac{1}{4}\frac{1}{16}A^2\mathrm{tr}\Big[(g_v^2 + g_A^2)p_2^\rho p_1^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu - 2g_vg_Ap_2^\rho p_1^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_5\Big]\\
&\times\mathrm{tr}\Big[(g_v^2 + g_A^2)k_{2\rho}k_{1\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - 2g_vg_Ak_2^\rho k_1^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_5\Big]
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{1}{4}\frac{1}{16}16A^2\Big[(g_v^2+g_A^2)p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\sigma}g_{\mu\nu}+g_{\rho\nu}g_{\mu\sigma}\big)-2g_vg_Ap_2^\rho p_1^\sigma(-i\epsilon_{\rho\mu\sigma\nu})\Big]\\
&\times\Big[(g_v^2+g_A^2)k_{2\rho}k_{1\sigma}\big(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}\big)-2g_vg_Ak_{2\rho}k_{1\sigma}(-i\epsilon^{\rho\mu\sigma\nu})\Big]\\
&= \frac{1}{4}A^2\Big[(g_v^2+g_A^2)^2\cdot p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\sigma}g_{\mu\nu}+g_{\rho\nu}g_{\mu\sigma}\big)k_{2\rho}k_{1\sigma}\big(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}\big)\\
&+i2(g_v^2+g_A^2)g_vg_A\cdot p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\sigma}g_{\mu\nu}+g_{\rho\nu}g_{\mu\sigma}\big)k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu})\\
&+i2(g_v^2+g_A^2)g_vg_A\cdot p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\mu}k_{2\rho}k_{1\sigma}\big(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}\big)\\
&-4g_v^2g_A^2\cdot p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\nu}k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu}\Big]\\
&= \frac{1}{4}A^2\Big[(g_v^2+g_A^2)^2\cdot\Big(p_{2\mu}p_{1\nu}-p_{2\sigma}p_1^\sigma g_{\mu\nu}+p_{2\nu}p_{1\mu}\Big)\Big(k_2^\mu k_1^\nu-k_2^\sigma k_{1\sigma}g^{\mu\nu}+k_2^\nu k_1^\mu\Big)-4!4g_v^2g_A^2\cdot p_2^\rho k_{2\rho}p_1^\sigma k_{1\sigma}\\
&-2(g_v^2+g_A^2)g_vg_A\cdot\Big(\big(p_{2\mu}p_{1\nu}-p_{2\sigma}p_1^\sigma g_{\mu\nu}+p_{2\nu}p_{1\mu}\big)k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu}+p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\mu}\big(k_2^\mu k_1^\nu-k_2^\sigma k_{1\sigma}g^{\mu\nu}+k_2^\nu k_1^\mu\big)\Big)\Big]\\
&= \frac{1}{4}A^2(I+II+III)
\end{aligned}
$$

Where we've used

$$
\begin{aligned}
\epsilon_{\rho\mu\sigma\mu}\epsilon^{\rho\mu\sigma\mu} &= n! = 4! = 24,\\
g_{\mu\nu}g^{\mu\nu} &= 4\\
\gamma^\mu\gamma_\mu &= 4
\end{aligned}
$$

We now calculate and simplify the terms separately.

1.4.1 First term (I)

$$
\begin{aligned}
p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\sigma}g_{\mu\nu}+g_{\rho\nu}g_{\mu\sigma}\Big)&k_{2\rho}k_{1\sigma}\Big(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}\Big)\\
&= \big(p_{2\mu}p_{1\nu}-p_{2\sigma}p_1^\sigma g_{\mu\nu}+p_{2\nu}p_{1\mu}\Big)\Big(k_2^\mu k_1^\nu-k_2^\sigma k_{1\sigma}g^{\mu\nu}+k_2^\nu k_1^\mu\Big)\\
&= \Big[(p_2\cdot k_2)(p_1\cdot k_1)-(k_2\cdot k_1)(p_2\cdot p_1)+(p_1\cdot k_2)(p_2\cdot k_1)\\
&-(p_2\cdot p_1)(k_2\cdot k_1)+4(k_2\cdot k_1)(p_2\cdot p_1)-(p_2\cdot p_1)(k_2\cdot k_1)\\
&+(k_2\cdot p_1)(p_2\cdot k_1)-(k_2\cdot k_1)(p_2\cdot p_1)+(p_2\cdot k_2)(p_1\cdot k_1)\Big]\\
&= 2\Big[(p_2\cdot k_2)(p_1\cdot k_1)+(p_1\cdot k_2)(p_2\cdot k_1)\Big]
\end{aligned}
$$

1.4.2 Second term II

This is a straight-forward contraction

$$
II = -4!4g_v^2g_A^2\cdot p_2^\rho k_{2\rho}p_1^\sigma k_{1\sigma} = -24\cdot 4g_v^2g_A^2(p_2\cdot k_2)(p_1\cdot k_1)
$$

1.4.3 Third term (III)

Take a look at the second term, and notice that $\epsilon^{\mu\nu\rho\sigma}$ is zero if any two indices are equal, so $\epsilon^{\mu\nu\rho\sigma}g_{\mu\nu}=0$, so we get

$$
\begin{aligned}
&\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}-p_{2\sigma}p_1^\sigma g_{\mu\nu}+p_{2\nu}p_{1\mu}\big)k_{2\rho}k_{1\sigma}+\epsilon^{\rho\mu\sigma\nu}p_{2\rho}p_{1\sigma}\big(k_{2\mu}k_{1\nu}-k_{2\sigma}k_1^\sigma g_{\mu\nu}+k_{2\nu}k_{1\mu}\big)\\
&=\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}+p_{2\nu}p_{1\mu}k_{2\rho}k_{1\sigma}\big)+\epsilon^{\rho\mu\sigma\nu}\big(p_{2\rho}p_{1\sigma}k_{2\mu}k_{1\nu}+p_{2\rho}p_{1\sigma}k_{2\nu}k_{1\mu}\big)\\
&=\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}+p_{2\nu}p_{1\mu}k_{2\rho}k_{1\sigma}\big)+\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}+p_{2\mu}p_{1\nu}k_{2\sigma}k_{1\rho}\big)\\
&=2\big(\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}+\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\sigma}k_{1\rho}\big)=2\big(\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}-\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}\big)=\underline{0}
\end{aligned}
$$

Back to $|\mathcal{M}|^2$

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{1}{4}A^2\Big[2(g_v^2+g_A^2)^2\Big[(p_2\cdot k_2)(p_1\cdot k_1)+(p_1\cdot k_2)(p_2\cdot k_1)\Big]-24\cdot 4g_v^2g_A^2(p_2\cdot k_2)(p_1\cdot k_1)\Big]\\
&= \frac{1}{2}\frac{g^4}{(s-M_Z^2)^2\cos^4\theta_W}\Big[(g_v^2+g_A^2)^2\Big[(p_2\cdot k_2)(p_1\cdot k_1)+(p_1\cdot k_2)(p_2\cdot k_1)\Big]-24\cdot 2g_v^2g_A^2(p_2\cdot k_2)(p_1\cdot k_1)\Big]
\end{aligned}
$$

Kinematics

We work in the center of mass-frame. Since $|\mathbf{k}| = \sqrt{E^2 - m_\mu^2} \simeq \sqrt{E^2} = E$, we get

$$
\begin{aligned}
s = k^2 = (p_1 + p_2)^2 &= (k_1 + k_2)^2 = 4E^2 \\
(p_1 \cdot k_1) = (p_2 \cdot k_2) &= E^2 - E|\mathbf{k}|\cos\theta \simeq E^2(1 - \cos\theta) = \frac{1}{4}s(1 - \cos\theta) \\
(p_1 \cdot k_2) = (p_2 \cdot k_1) &= E^2 + E|\mathbf{k}|\cos\theta \simeq E^2(1 + \cos\theta) = \frac{1}{4}s(1 + \cos\theta)
\end{aligned}
$$

Putting this into the expression we get

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = &= \frac{1}{2}\frac{g^4}{(s - M_Z^2)^2\cos^4\theta_W}\Big[(g_v^2 + g_A^2)^2\Big[\frac{1}{16}s^2(1-\cos\theta)^2 + \frac{1}{16}s^2(1+\cos\theta)^2\Big] - 48g_v^2g_A^2\frac{1}{16}s^2(1-\cos\theta)^2\Big] \\
&= \frac{1}{16}\frac{g^4 s^2}{(s - M_Z^2)^2\cos^4\theta_W}\Big[\big((g_v^2 + g_A^2)^2 - 24g_v^2g_A^2\big)(1 + \cos^2\theta) + 48g_v^2g_A^2\cos\theta\Big]
\end{aligned}
$$

Here we can use that
$g^4/\cos^4\theta_w = (e/\sin\theta_W)^4/\cos^4\theta_W = e^4/(\sin\theta_W\cos_W)^4 = e^4/(1/2\sin 2\theta_W)^4 = e^4(M_Z^2)^2G^22/(\alpha^2\pi^2)$ to rewrite this expression

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{1}{16}\frac{(\alpha 4\pi)^2(M_Z^2)^2G^22s^2}{(s - M_Z^2)^2\alpha^2\pi^2}\Big[\big((g_v^2 + g_A^2)^2 - 24g_v^2g_A^2\big)(1 + \cos^2\theta) + 48g_v^2g_A^2\cos\theta\Big] \\
&= 2\Big(\frac{M_Z^2}{s - M_Z^2}\Big)^2(sG)^2\Big[\big((g_v^2 + g_A^2)^2 - 24g_v^2g_A^2\big)(1 + \cos^2\theta) + 48g_v^2g_A^2\cos\theta\Big]
\end{aligned}
$$

2 Appendix B

These are the calculations for the cross term matrix elements.

2.1 Cross terms

We get the cross terms from the expression

$$\begin{aligned}|\mathcal{M}|^2 &= (i\mathcal{M}_\gamma + i\mathcal{M}_Z)(-i\mathcal{M}_\gamma^* - i\mathcal{M}_Z^*)\\ &= |\mathcal{M}_\gamma|^2 + \mathcal{M}_\gamma\mathcal{M}_Z^* + \mathcal{M}_Z\mathcal{M}_\gamma^* + |\mathcal{M}_Z|^2.\end{aligned}$$

We use the expressions we've found for $\mathcal{M}_{Z/\gamma}$, and write out the cross terms, which we will call $\mathcal{M}_\times = \mathcal{M}_\gamma\mathcal{M}_Z^* + \mathcal{M}_Z\mathcal{M}_\gamma^*$

$$\begin{aligned}\mathcal{M}_\times = &\left(\frac{ie^2}{s}\left(\bar{v}^s(p_2)\gamma^\mu u^{s'}(p_1)\right)\left(\bar{u}^r(k_1)\gamma_\mu v^{r'}(k_2)\right)\right)\\ &\times\left(\frac{ig^2}{(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\bar{v}^s(p_2)\gamma_\nu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)v^s(k_2)\right)^*\\ &+\left(\frac{ig^2}{(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\bar{v}^s(p_2)\gamma_\nu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)v^s(k_2)\right)\\ &\times\left(\frac{ie^2}{s}\left(\bar{v}^s(p_2)\gamma^\mu u^{s'}(p_1)\right)\left(\bar{u}^r(k_1)\gamma_\mu v^{r'}(k_2)\right)\right)^*\\ &= \frac{e^2g^2}{s(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\Bigg[\left(\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma_\nu(g_v^f - g_A^f\gamma_5)v(p_2)\right)\left(\bar{u}(k_1)\gamma_\mu v(k_2)\bar{v}(k_2)\gamma^\nu(g_v^f - g_A^f\gamma_5)u(k_1)\right)\\ &+\left(\bar{v}(p_2)\gamma_\nu(g_v^f - g_A^f\gamma_5)u(p_1)\bar{u}(p_1)\gamma^\mu v(p_2)\right)\left(\bar{u}(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)v(k_2)\bar{v}(k_2)\gamma_\mu u(k_1)\right)\Bigg]\end{aligned}$$

Introduce the coefficient $B = \frac{e^2g^2}{s(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}$ and take the trace and average over spins, also note that we can move $\gamma^5\not{p}\gamma^\mu = -\not{p}\gamma^5\gamma^\mu = \not{p}\gamma^\mu\gamma^5$

$$\begin{aligned}\frac{1}{4B}\sum_{spins}\mathcal{M}_\times &= \frac{1}{4}\mathrm{tr}\Big[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma_\nu(g_v^f - g_A^f\gamma_5)\Big]\mathrm{tr}\Big[(\not{k}_1 + m_m)\gamma_\mu(\not{k}_2 - m_m)\gamma^\nu(g_v^f - g_A^f\gamma_5)\Big]\\ &+\mathrm{tr}\Big[(\not{p}_2 - m_e)\gamma_\nu(\not{p}_1 + m_e)\gamma^\mu(g_v^f - g_A^f\gamma_5)\Big]\mathrm{tr}\Big[(\not{k}_1 + m_m)\gamma^\nu(\not{k}_2 - m_m)\gamma_\mu(g_v^f - g_A^f\gamma_5)\Big]\\ &= \frac{1}{4}2\mathrm{tr}\Big[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma_\nu(g_v^f - g_A^f\gamma_5)\Big]\mathrm{tr}\Big[(\not{k}_1 + m_m)\gamma_\mu(\not{k}_2 - m_m)\gamma^\nu(g_v^f - g_A^f\gamma_5)\Big]\end{aligned}$$

We can now set the muon- and electron masses to zero

$$\begin{aligned}\frac{1}{8B}\sum_{spins}\mathcal{M}_\times &= \frac{1}{4}\mathrm{tr}\Big[p_{2\rho}p_{1\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu(g_v^f - g_A^f\gamma_5)\Big]\mathrm{tr}\Big[k_1^\rho k_2^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu(g_v^f - g_A^f\gamma_5)\Big]\\ &= \frac{1}{4}\Big(p_{2\rho}p_{1\sigma}\big[g_v^f(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}) - g_A^f\epsilon^{\rho\mu\sigma\nu}\big]\Big)\Big(k_1^\rho k_2^\sigma\big[g_v^f(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}) - g_A^f\epsilon_{\rho\mu\sigma\nu}\big]\Big)\\ &= \frac{1}{4}\Big(g_v^f(p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu) - g_A^f p_{2\rho}p_{1\sigma}\epsilon^{\rho\mu\sigma\nu}\Big)\Big(g_v^f(k_{1\mu}k_{2\nu} - (k_1\cdot k_2)g_{\mu\nu} + k_{1\nu}k_{2\mu}) - g_A^f k_1^\rho k_2^\sigma\epsilon_{\rho\mu\sigma\nu}\Big)\\ &= \frac{1}{4}g_v^f(p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu)g_v^f(k_{1\mu}k_{2\nu} - (k_1\cdot k_2)g_{\mu\nu} + k_{1\nu}k_{2\mu})\\ &- g_A^f g_v^f\Big(p_{2\rho}p_{1\sigma}\epsilon^{\rho\mu\sigma\nu}(k_{1\mu}k_{2\nu} - (k_1\cdot k_2)g_{\mu\nu} + k_{1\nu}k_{2\mu}) + (p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu)k_1^\rho k_2^\sigma\epsilon_{\rho\mu\sigma\nu}\Big)\\ &+ g_A^f p_{2\rho}p_{1\sigma}\epsilon^{\rho\mu\sigma\nu}g_A^f k_1^\rho k_2^\sigma\epsilon_{\rho\mu\sigma\nu}\\ &= \frac{1}{4}2(g_v^f)^2\Big[(p_2\cdot k_2)(p_1\cdot k_1) + (p_1\cdot k_2)(p_2\cdot k_1)\Big] + 4!(g_A^f)^2(p_1\cdot k_2)(p_2\cdot k_1)\end{aligned}$$

Kinematics

Using the kinematics derived in the previous calculation, we can simplify this expression

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|_\times &= \frac{1}{4}4B\Big((g_v^f)^2\Big[\frac{1}{16}s^2(1-\cos\theta)^2+\frac{1}{16}s^2(1+\cos\theta)^2\Big]+12(g_A^f)^2\frac{1}{16}s^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{4}\frac{1}{2}Bs^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big]+6(g_A^f)^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{4}\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big]+6(g_A^f)^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{4}\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\Big([(g_v^f)^2+6(g_A^f)^2]\Big[1+\cos^2\theta\Big]+12(g_A^f)^2\cos\theta\Big)\\
&= \frac{1}{4}\frac{1}{2}\frac{e^4}{s(s-M_Z^2)\cos^2\theta_W\sin^2\theta_W}s^2\Big([(g_v^f)^2+6(g_A^f)^2]\Big[1+\cos^2\theta\Big]+12(g_A^f)^2\cos\theta\Big)
\end{aligned}
$$

We can now use that $\cos^{-2}\theta_W\sin^{-2}\theta_W=(\cos\theta_W\sin\theta_W)^{-2}=(1/2\sin2\theta_W)^{-2}=M_Z^2G\sqrt{2}/(\alpha\pi)$, along with $e^4=(e^2)^2=(4\pi\alpha)^2=16\pi^2\alpha^2$, to get

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|_\times &= \frac{2\pi^2\alpha^2M_Z^2G\sqrt{2}}{s(s-M_Z^2)\alpha\pi}s^2\Big([(g_v^f)^2+6(g_A^f)^2]\Big[1+\cos^2\theta\Big]+12(g_A^f)^2\cos\theta\Big)\\
&= \frac{4}{\sqrt{2}}\pi\alpha Gs\frac{M_Z^2}{(s-M_Z^2)}\Big([(g_v^f)^2+6(g_A^f)^2]\Big[1+\cos^2\theta\Big]+12(g_A^f)^2\cos\theta\Big)
\end{aligned}
$$

3 Appendix C

3.1 Trace identities

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{3}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma} \tag{4}$$