FYS4560 Project 2

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1 Di-lepton production in e^+e^- in the Standard Model

Possible feynman diagrams for the process

$$e^-e^+ \to l^-l^+$$

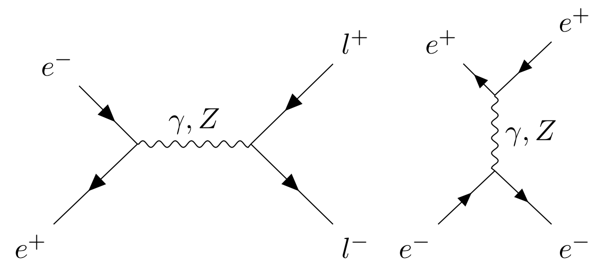

Figure 1: Possible diagrams for di-lepton production.

1.1 $e^-e^+ \to \mu^+\mu^-$ through QED

We find the expression for the matrix element from the Feynman diagram in Fig. (2).

$$\begin{aligned} i\mathcal{M} &= \bar{v}^s(p_2)(-ie\gamma^\mu)u^{s'}(p_1)\big(\frac{-g_{\mu\nu}}{k^2}\big)\bar{u}^r(k_1)(-ie\gamma^\nu)v^{r'}(k_2) \\ &= \frac{ie^2}{s}\Big(\bar{v}^s(p_2)\gamma^\mu u^{s'}(p_1)\Big)\Big(\bar{u}^r(k_1)\gamma_\mu v^{r'}(k_2)\Big) \end{aligned}$$

Which gives for the matrix element squared

$$|\mathcal{M}|^2 = \frac{e^4}{k^4}\Big(\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu v(p_2)\Big)\Big(\bar{u}(k_1)\gamma_\mu v(k_2)\bar{v}(k_2)\gamma_\nu u(k_1)\Big)$$

Averaging over spins and taking the trace we find

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{k^4}\text{tr}\Big[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\Big]\text{tr}\Big[(\not{k}_1 + m_m)\gamma_\mu(\not{k}_2 - m_m)\gamma_\nu\Big].$$

Since $m_e << m_m$, we can set $m_e = 0$. Traces of odd numbers of gamma matrices are zero, so we can reduce the number of terms to

$$\begin{aligned} \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{e^4}{4k^4}\text{tr}\Big[p_{2\rho}p_{1\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\Big]\text{tr}\Big[k_1^{\rho'}k_2^{\sigma'}\gamma_{\rho'}\gamma_\mu\gamma_{\sigma'}\gamma_\nu - m_m^2\gamma_\mu\gamma_\nu\Big] \\ &= \frac{e^4}{k^4}4\Bigg[\Big(p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu\Big)\Big(k_{1\mu}k_{2\nu} - (k_2\cdot k_1)g_{\mu\nu} + k_{1\nu}k_{2\mu} - m_m^2 g_{\mu\nu}\Big) \\ &= \frac{e^4}{k^4}8\Bigg[(p_1\cdot k_2)(p_2\cdot k_1) + (p_1\cdot k_1)(p_2\cdot k_2) - m_m^2(p_1\cdot p_2)\Bigg] \end{aligned}$$

1.1.1 Kinematics

Assume the electron and muon momenta make an angle θ between them. Using the Mandelstam variables, and assuming m_e we then get

$$\begin{aligned} k^2 = (p_1+p_2)^2 = 4E^2 = s \\ p_1\cdot k_1 = p_2\cdot k_2 = E^2 - E|\mathbf{k}|\cos\theta \\ p_1\cdot k_2 = p_2\cdot k_1 = E^2 + E|\mathbf{k}|\cos\theta \end{aligned}$$

Putting this into our expression we get

$$\begin{aligned} \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{8e^4}{16s^2}\Big[E^2(E-\mathbf{k}\cos\theta)^2 + E^2(E+\mathbf{k}\cos\theta)^2 + 2m_m^2E^2\Big] \\ &= e^4\Big[\Big(1+\frac{m_m^2}{s}\Big) + \Big(1 - \frac{m_m^2}{s}\cos^2\theta\Big)\Big] \end{aligned}$$

1.1.2 Differential cross section

We can find the differential cross section for this process. The formula for differential cross section of two final-state particles in the center of mass frame is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} \frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}|v_{\mathcal{A}} - v_{\mathcal{B}}|} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{cm}} |\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to p_1, p_2)|^2.$$

For this specific calculation $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 2$, and $E_{\mathcal{A}} = E_{\mathcal{B}} = E_{cm}/2$, so we get

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{2E_{cm}^2} \frac{|\mathbf{k}|}{16\pi^2 E_{cm}} \cdot \frac{1}{4} \sum_{spins} |\mathcal{M}|^2 \\
&= \frac{\alpha}{4s} \sqrt{1 - \frac{m_m^2}{s}} \left[\left(1 + \frac{m_m^2}{s}\right) + \left(1 - \frac{m_m^2}{s}\right) \cos^2\theta\right]
\end{aligned}$$

We can rewrite the differential solid angle as $d\Omega = \sin\theta d\theta d\phi = -d\cos\theta d\phi$, so we get

$$\frac{d\sigma}{d\cos\theta} = -\frac{\alpha}{4s} \sqrt{1 - \frac{m_m^2}{s}} \left[\left(1 + \frac{m_m^2}{s}\right) + \left(1 - \frac{m_m^2}{s}\right) \cos^2\theta\right] d\phi$$

1.1.3 Cross section

We can now find the total cross section by integrating over the angles

$$\begin{aligned}
\sigma &= -\frac{\alpha}{4s} \sqrt{1 - \frac{m_m^2}{s}} \int_\Omega \left[\left(1 + \frac{m_m^2}{s}\right) + \left(1 - \frac{m_m^2}{s}\right) \cos^2\theta\right] d\phi d\cos\theta \\
&= 2\pi \frac{\alpha}{4s} \sqrt{1 - \frac{m_m^2}{s}} \left[\left(1 + \frac{m_m^2}{s}\right) \cos\theta + \frac{1}{3}\left(1 - \frac{m_m^2}{s}\right) \cos^3\theta\right]_{-1}^{1} \\
&= 4\pi \frac{\alpha}{4s} \sqrt{1 - \frac{m_m^2}{s}} \left[\left(1 + \frac{m_m^2}{s}\right) + \frac{1}{3}\left(1 - \frac{m_m^2}{s}\right)\right] \\
\sigma &= \frac{4\pi\alpha}{3s} \sqrt{1 - \frac{m_m^2}{s}} \left[1 + \frac{1}{2}\frac{m_m^2}{s}\right]
\end{aligned}$$

1.2 $e^+e^- \to \mu^+\mu^-$

The feynman diagrams that contribute in the electroweak interaction are shown in Fig. (2).

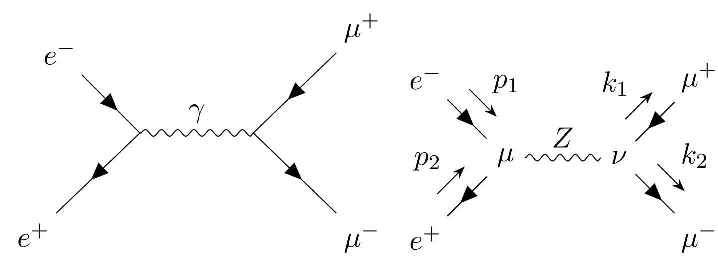

Figure 2: Feynman diagrams for $e^+e^- \to \mu^+\mu^-$.

The cross section for this process gets contributions from the purely electromagnetic (γ), the purely electroweak (Z) and the cross terms between them. We will calulate the terms separately.

1.3 Purely electromagnetic interaction

Here we can use the expression found earlier, but since we are working with a much larger energy scale (assume $E_e >> m_e$), we can set the muon mass to zero as well, $m_m = 0$, with yields the expression

$$\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = e^4\big(1+\cos^2\theta\big)$$

1.4 Purely electroweak contribution

We will here use the expressions

$$g_v^f = \frac{1}{2}T_f^3 - Q_f \sin^2\theta_W, \tag{1}$$

$$g_A^f = \frac{1}{2}T_f^3. \tag{2}$$

The expression for the matrix element from the feynman diagram in Fig. (2) is

$$\begin{aligned}
i\mathcal{M} &= \bar{v}^s(p_2)\big[i\frac{g}{\cos\theta_W}\gamma_\mu(g_v^f - g_A^f\gamma_5)\big]u^s(p_1)\Big(-\frac{ig_{\mu\nu}}{k^2 - M_Z^2 + i\epsilon}\Big)\bar{u}^s(k_2)\big[i\frac{g}{\cos\theta_W}\gamma_\nu(g_v^f - g_A^f\gamma_5)\big]v^s(k_1) \\
&= \frac{ig^2}{(k^2 - M_Z^2 + i\epsilon)\cos^2\theta_W}\bar{v}^s(p_2)\gamma_\mu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(k_2)\gamma^\mu(g_v^f - g_A^f\gamma_5)v^s(k_1)
\end{aligned}$$

$$\begin{aligned}
|\mathcal{M}|^2 = &\frac{g^4}{(k^2 - M_Z^2)^2\cos^4\theta_W}\Big(\bar{v}^s(p_2)\gamma_\mu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(p_1)\gamma_\nu(g_v^f - g_A^f\gamma_5)v^s(p_2)\Big) \\
&\times\Big(\bar{u}^s(k_2)\gamma^\mu(g_v^f - g_A^f\gamma_5)v^s(k_1)\bar{v}^s(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)u^s(k_2)\Big)
\end{aligned}$$

Take the trace and get (where $A^2 = \frac{g^4}{(k^2-M_Z^2)^2\cos^4\theta_W}$), and since $M_z >> m_e, m_\mu$, set $m_e = m_\mu \simeq 0$

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= A^2\mathrm{tr}\Big[(\not{p}_2 - m_e)\gamma_\mu(g_v^e - g_A^e\gamma_5)(\not{p}_1 + m_e)\gamma_\nu(g_v^e - g_A^e\gamma_5)\Big] \\
&\times \mathrm{tr}\Big[(\not{k}_2 + m_\mu)\gamma^\mu(g_v^l - g_A^l\gamma_5)(\not{k}_1 - m_\mu)\gamma^\nu(g_v^l - g_A^l\gamma_5)\Big] \\
\text{(set mass to zero)} &= A^2\mathrm{tr}\Big[\not{p}_2\gamma_\mu(g_v^e - g_A^e\gamma_5)\not{p}_1\gamma_\nu(g_v^e - g_A^e\gamma_5)\Big]\mathrm{tr}\Big[\not{k}_2\gamma^\mu(g_v^l - g_A^l\gamma_5)\not{k}_1\gamma^\nu(g_v^l - g_A^l\gamma_5)\Big] \\
&= A^2\mathrm{tr}\Big[(g_v^e)^2\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu + (g_A^e)^2\not{p}_2\gamma_\mu\gamma_5\not{p}_1\gamma_\nu\gamma_5 - g_v^e g_A^e\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\gamma_5 - g_v^e g_A^e\not{p}_2\gamma_\mu\gamma_5\not{p}_1\gamma_\nu g_v^e)\Big] \\
&\times \mathrm{tr}\Big[(g_v^l)^2\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu + (g_A^l)^2\not{k}_2\gamma^\mu\gamma_5\not{k}_1\gamma^\nu\gamma_5 - g_v^l g_A^l\not{k}_2\gamma_\mu\not{k}_1\gamma_\nu\gamma_5 - g_v^l g_A^l\not{k}_2\gamma_\mu\gamma_5\not{k}_1\gamma_\nu\gamma_5)\Big] \\
&= A^2\mathrm{tr}\Big[\big((g_A^e)^2 + (g_v^e)^2\big)\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu - 2g_v^e g_A^e\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\gamma_5\Big] \\
&\times \mathrm{tr}\Big[\big((g_v^l)^2 + (g_A^l)^2\big)\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu - 2g_v^l g_A^l\not{k}_2\gamma_\mu\not{k}_1\gamma_\nu\gamma_5\Big] \\
&= A^2\mathrm{tr}\Big[\big((g_v^e)^2 + (g_A^e)^2\big)p_2^\rho p_1^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu - 2g_v^e g_A^e p_2^\rho p_1^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_5\Big] \\
&\times \mathrm{tr}\Big[\big((g_v^l)^2 + (g_A^l)^2\big)k_{2\rho}k_{1\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - 2g_v^l g_A^l k_2^\rho k_1^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_5\Big]
\end{aligned}$$

Now use the trace identities for the gamma matrices

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{3}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma} \tag{4}$$

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= 16A^2\Big[((g_v^e)^2+(g_A^e)^2)p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\sigma}g_{\mu\nu}+g_{\rho\nu}g_{\mu\sigma}\big)-2g_v^eg_A^ep_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\nu}\Big]\\
&\times\Big[((g_v^l)^2+(g_A^l)^2)k_{2\rho}k_{1\sigma}\big(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}\big)-2g_v^lg_A^lk_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu}\Big]\\
&= 16A^2\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}\cdot p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\sigma}g_{\mu\nu}+g_{\rho\nu}g_{\mu\sigma}\big)k_{2\rho}k_{1\sigma}\big(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}\big)\\
&-\{2((g_v^e)^2+(g_A^e)^2)g_v^lg_A^l\}\cdot p_2^\rho p_1^\sigma\big(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\sigma}g_{\mu\nu}+g_{\rho\nu}g_{\mu\sigma}\big)k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu})\\
&-\{2((g_v^l)^2+(g_A^l)^2)g_v^eg_A^e\}\cdot p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\mu}k_{2\rho}k_{1\sigma}\big(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}\big)\\
&+\{4g_v^eg_A^eg_v^lg_A^l\}\cdot p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\nu}k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu}\Big]\\
&= 16A^2\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}\cdot\big(p_{2\mu}p_{1\nu}-p_{2\sigma}p_1^\sigma g_{\mu\nu}+p_{2\nu}p_{1\mu}\big)\Big(k_2^\mu k_1^\nu-k_2^\sigma k_{1\sigma}g^{\mu\nu}+k_2^\nu k_1^\mu\Big)\\
&-\{2((g_v^e)^2+(g_A^e)^2)g_v^lg_A^l\}\cdot\big(p_{2\mu}p_{1\nu}-p_{2\sigma}p_1^\sigma g_{\mu\nu}+p_{2\nu}p_{1\mu}\big)k_{2\rho}k_{1\sigma}\epsilon^{\rho\mu\sigma\nu})\\
&-\{2((g_v^l)^2+(g_A^l)^2)g_v^eg_A^e\}\cdot p_2^\rho p_1^\sigma\epsilon_{\rho\mu\sigma\mu}\big(k_2^\mu k_1^\nu-k_2^\sigma k_{1\sigma}g^{\mu\nu}+k_2^\nu k_1^\mu\big)+4!\{4g_v^eg_A^eg_v^lg_A^l\}\cdot p_2^\rho k_{2\rho}p_1^\sigma k_{1\sigma}\Big]
\end{aligned}
$$

Where we've used

$$\epsilon_{\rho\mu\sigma\mu}\epsilon^{\rho\mu\sigma\mu}=n!=4!=24.$$

$$
\begin{aligned}
&= 16A^2\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}\cdot\Big[2(p_2\cdot k_2)(p_1\cdot k_1)+2(p_1\cdot k_2)(p_2\cdot k_1)\Big]\\
&-\{2((g_v^e)^2+(g_A^e)^2)g_v^lg_A^l\}\cdot\Big(\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}-p_{2\sigma}p_1^\sigma g_{\mu\nu}+p_{2\nu}p_{1\mu}\big)k_{2\rho}k_{1\sigma}+\epsilon^{\rho\mu\sigma\nu}p_{2\rho}p_{1\sigma}\big(k_{2\mu}k_{1\nu}-k_{2\sigma}k_1^\sigma g_{\mu\nu}+k_{2\nu}k_{1\mu}\big)\Big)\\
&+4!\{4g_v^eg_A^eg_v^lg_A^l\}\cdot(p_2\cdot k_2)(p_1\cdot k_1)\Big]\\
&= I+II+III
\end{aligned}
$$

1.4.1 Second term (II)

Take a look at the second term, and notice that $\epsilon^{\mu\nu\rho\sigma}$ is zero if any two indices are equal, so $\epsilon^{\mu\nu\rho\sigma}g_{\mu\nu}=0$, so we get

$$
\begin{aligned}
&\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}-p_{2\sigma}p_1^\sigma g_{\mu\nu}+p_{2\nu}p_{1\mu}\big)k_{2\rho}k_{1\sigma}+\epsilon^{\rho\mu\sigma\nu}p_{2\rho}p_{1\sigma}\big(k_{2\mu}k_{1\nu}-k_{2\sigma}k_1^\sigma g_{\mu\nu}+k_{2\nu}k_{1\mu}\big)\\
&=\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}+p_{2\nu}p_{1\mu}k_{2\rho}k_{1\sigma}\big)+\epsilon^{\rho\mu\sigma\nu}\big(p_{2\rho}p_{1\sigma}k_{2\mu}k_{1\nu}+p_{2\rho}p_{1\sigma}k_{2\nu}k_{1\mu}\big)\\
&=\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}+p_{2\nu}p_{1\mu}k_{2\rho}k_{1\sigma}\big)+\epsilon^{\rho\mu\sigma\nu}\big(p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}+p_{2\mu}p_{1\nu}k_{2\sigma}k_{1\rho}\big)\\
&=2\big(\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}+\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\sigma}k_{1\rho}\big)\\
&=2\big(\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}-\epsilon^{\rho\mu\sigma\nu}p_{2\mu}p_{1\nu}k_{2\rho}k_{1\sigma}\big)=\underline{0}
\end{aligned}
$$

1.4.2 First term (I)

$$
\begin{aligned}
&p_2^\rho p_1^\sigma 4\big(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\sigma}g_{\mu\nu}+g_{\rho\nu}g_{\mu\sigma}\big)4k_{2\rho}k_{1\sigma}\Big(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}\Big)\\
&\qquad=\Big(p_{2\mu}p_{1\nu}-p_{2\sigma}p_1^\sigma g_{\mu\nu}+p_{2\nu}p_{1\mu}\Big)\Big(k_2^\mu k_1^\nu-k_2^\sigma k_{1\sigma}g^{\mu\nu}+k_2^\nu k_1^\mu\Big)\\
&\qquad=\Big[(p_2\cdot k_2)(p_1\cdot k_1)-(k_2\cdot k_1)(p_2\cdot p_1)+(p_1\cdot k_2)(p_2\cdot k_1)\\
&\qquad-(p_2\cdot p_1)(k_2\cdot k_1)+4(k_2\cdot k_1)(p_2\cdot p_1)-(p_2\cdot p_1)(k_2\cdot k_1)\\
&\qquad+(k_2\cdot p_1)(p_2\cdot k_1)-(k_2\cdot k_1)(p_2\cdot p_1)+(p_2\cdot k_2)(p_1\cdot k_1)\Big]\\
&\qquad=\Big[2(p_2\cdot k_2)(p_1\cdot k_1)+2(p_1\cdot k_2)(p_2\cdot k_1)\Big]
\end{aligned}
$$

Back to $|\mathcal{M}|^2$

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= 16A^2\Big[\{2((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)+4\cdot 4!\cdot g_v^eg_A^eg_v^lg_A^l\}\cdot(p_2\cdot k_2)(p_1\cdot k_1)\\
&+2\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}(p_1\cdot k_2)(p_2\cdot k_1)\Big]\\
&=32\frac{g^4}{(s-M_Z^2)^2\cos^4\theta_W}\Big[\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)+48\cdot g_v^eg_A^eg_v^lg_A^l\}\cdot(p_2\cdot k_2)(p_1\cdot k_1)\\
&+\{((g_v^e)^2+(g_A^e)^2)((g_v^l)^2+(g_A^l)^2)\}(p_1\cdot k_2)(p_2\cdot k_1)\Big]
\end{aligned}
$$

Kinematics

We work in the center of mass-frame. Since $|\mathbf{k}| = \sqrt{E^2 - m_\mu^2} \simeq \sqrt{E^2} = E$, we get

$$s = k^2 = (p_1 + p_2)^2 = (k_1 + k_2)^2 = 4E^2$$
$$(p_1 \cdot k_1) = (p_2 \cdot k_2) = E^2 - E|\mathbf{k}|\cos\theta \simeq E^2(1 - \cos\theta) = \frac{1}{4}s(1 - \cos\theta)$$
$$(p_1 \cdot k_2) = (p_2 \cdot k_1) = E^2 + E|\mathbf{k}|\cos\theta \simeq E^2(1 + \cos\theta) = \frac{1}{4}s(1 + \cos\theta)$$

Putting this into the expression we get

$$\begin{aligned} \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 32\frac{g^4}{(s-M_Z^2)^2\cos^4\theta_W}&\Big[\{((g_v^e)^2 + (g_A^e)^2)((g_v^l)^2 + (g_A^l)^2) + 48 \cdot g_v^e g_A^e g_v^l g_A^l\} \cdot (\frac{1}{4}s(1-\cos\theta))^2 \\ &+ \{((g_v^e)^2 + (g_A^e)^2)((g_v^l)^2 + (g_A^l)^2)\}(\frac{1}{4}s(1+\cos\theta))^2\Big] \\ = 4\frac{g^4 s^2}{(s-M_Z^2)^2\cos^4\theta_W}&\Big[\{((g_v^e)^2 + (g_A^e)^2)((g_v^l)^2 + (g_A^l)^2)\} \cdot (1+\cos^2\theta) + 24 g_v^e g_A^e g_v^l g_A^l(1-\cos\theta)^2\Big] \end{aligned}$$

We know that muons and electrons behave in the exact same way, only their mass is different, so we set $g_v^e = g_v^l$, $g_A^e = g_A^l$

$$\begin{aligned} \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 4\frac{g^4 s^2}{(s-M_Z^2)^2\cos^4\theta_W}&\Big[((g_v^e)^2 + (g_A^e)^2)^2 \cdot (1+\cos^2\theta) + 24(g_v^e)^2(g_A^e)^2(1-\cos\theta)^2\Big] \\ = 4\frac{g^4 s^2}{(s-M_Z^2)^2\cos^4\theta_W}&\Big[((g_v^e)^2 + (g_A^e)^2)^2 \cdot (1+\cos^2\theta) + 24(g_v^e)^2(g_A^e)^2(1-2\cos\theta+\cos^2\theta)\Big] \end{aligned}$$

1.5 Cross terms

We get the cross terms from the expression

$$\begin{aligned}|\mathcal{M}|^2 &= (i\mathcal{M}_\gamma + i\mathcal{M}_Z)(-i\mathcal{M}_\gamma^* - i\mathcal{M}_Z^*)\\ &= |\mathcal{M}_\gamma|^2 + \mathcal{M}_\gamma\mathcal{M}_Z^* + \mathcal{M}_Z\mathcal{M}_\gamma^* + |\mathcal{M}_Z|^2.\end{aligned}$$

We use the expressions we've found for $\mathcal{M}_{Z/\gamma}$, and write out the cross terms, which we will call $\mathcal{M}_\times = \mathcal{M}_\gamma\mathcal{M}_Z^* + \mathcal{M}_Z\mathcal{M}_\gamma^*$

$$\begin{aligned}\mathcal{M}_\times = &\left(\frac{ie^2}{s}\left(\bar{v}^s(p_2)\gamma^\mu u^{s'}(p_1)\right)\left(\bar{u}^r(k_1)\gamma_\mu v^{r'}(k_2)\right)\right)\\
&\times\left(\frac{ig^2}{(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\bar{v}^s(p_2)\gamma_\nu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)v^s(k_2)\right)^*\\
&+\left(\frac{ig^2}{(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\bar{v}^s(p_2)\gamma_\nu(g_v^f - g_A^f\gamma_5)u^s(p_1)\bar{u}^s(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)v^s(k_2)\right)\\
&\times\left(\frac{ie^2}{s}\left(\bar{v}^s(p_2)\gamma^\mu u^{s'}(p_1)\right)\left(\bar{u}^r(k_1)\gamma_\mu v^{r'}(k_2)\right)\right)^*\\
&= \frac{e^2g^2}{s(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}\Bigg[\left(\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma_\nu(g_v^f - g_A^f\gamma_5)v(p_2)\right)\left(\bar{u}(k_1)\gamma_\mu v(k_2)\bar{v}(k_2)\gamma^\nu(g_v^f - g_A^f\gamma_5)u(k_1)\right)\\
&+\left(\bar{v}(p_2)\gamma_\nu(g_v^f - g_A^f\gamma_5)u(p_1)\bar{u}(p_1)\gamma^\mu v(p_2)\right)\left(\bar{u}(k_1)\gamma^\nu(g_v^f - g_A^f\gamma_5)v(k_2)\bar{v}(k_2)\gamma_\mu u(k_1)\right)\Bigg]\end{aligned}$$

Introduce the coefficient $B = \frac{e^2g^2}{s(k^2-M_Z^2+i\epsilon)\cos^2\theta_W}$ and take the trace and average over spins, also note that we can move $\gamma^5\not{p}\gamma^\mu = -\not{p}\gamma^5\gamma^\mu = \not{p}\gamma^\mu\gamma^5$

$$\begin{aligned}\frac{1}{4B}\sum_{spins}\mathcal{M}_\times &= \mathrm{tr}\Big[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma_\nu(g_v^f - g_A^f\gamma_5)\Big]\mathrm{tr}\Big[(\not{k}_1 + m_m)\gamma_\mu(\not{k}_2 - m_m)\gamma^\nu(g_v^f - g_A^f\gamma_5)\Big]\\
&+\mathrm{tr}\Big[(\not{p}_2 - m_e)\gamma_\nu(\not{p}_1 + m_e)\gamma^\mu(g_v^f - g_A^f\gamma_5)\Big]\mathrm{tr}\Big[(\not{k}_1 + m_m)\gamma^\nu(\not{k}_2 - m_m)\gamma_\mu(g_v^f - g_A^f\gamma_5)\Big]\\
&= 2\mathrm{tr}\Big[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma_\nu(g_v^f - g_A^f\gamma_5)\Big]\mathrm{tr}\Big[(\not{k}_1 + m_m)\gamma_\mu(\not{k}_2 - m_m)\gamma^\nu(g_v^f - g_A^f\gamma_5)\Big]\end{aligned}$$

We can now set the muon- and electron masses to zero

$$\begin{aligned}\frac{1}{8B}\sum_{spins}\mathcal{M}_\times &= \mathrm{tr}\Big[p_{2\rho}p_{1\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu(g_v^f - g_A^f\gamma_5)\Big]\mathrm{tr}\Big[k_1^\rho k_2^\sigma\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu(g_v^f - g_A^f\gamma_5)\Big]\\
&= \Big(p_{2\rho}p_{1\sigma}\left[g_v^f(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}) - g_A^f\epsilon^{\rho\mu\sigma\nu}\right]\Big)\Big(k_1^\rho k_2^\sigma\left[g_v^f(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}) - g_A^f\epsilon_{\rho\mu\sigma\nu}\right]\Big)\\
&= \Big(g_v^f(p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu) - g_A^f p_{2\rho}p_{1\sigma}\epsilon^{\rho\mu\sigma\nu}\Big)\Big(g_v^f(k_{1\mu}k_{2\nu} - (k_1\cdot k_2)g_{\mu\nu} + k_{1\nu}k_{2\mu}) - g_A^f k_1^\rho k_2^\sigma\epsilon_{\rho\mu\sigma\nu}\Big)\\
&= g_v^f(p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu)g_v^f(k_{1\mu}k_{2\nu} - (k_1\cdot k_2)g_{\mu\nu} + k_{1\nu}k_{2\mu})\\
&- g_A^f g_v^f\Big(p_{2\rho}p_{1\sigma}\epsilon^{\rho\mu\sigma\nu}(k_{1\mu}k_{2\nu} - (k_1\cdot k_2)g_{\mu\nu} + k_{1\nu}k_{2\mu}) + (p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu} + p_2^\nu p_1^\mu)k_1^\rho k_2^\sigma\epsilon_{\rho\mu\sigma\nu}\Big)\\
&+ g_A^f p_{2\rho}p_{1\sigma}\epsilon^{\rho\mu\sigma\nu}g_A^f k_1^\rho k_2^\sigma\epsilon_{\rho\mu\sigma\nu}\\
&= 2(g_v^f)^2\Big[(p_2\cdot k_2)(p_1\cdot k_1) + (p_1\cdot k_2)(p_2\cdot k_1)\Big] + 4!(g_A^f)^2(p_1\cdot k_2)(p_2\cdot k_1)\end{aligned}$$

Kinematics

Using the kinematics derived in the previous calculation, we can simplify this expression

$$\begin{aligned}\frac{1}{4}\sum_{spins}|\mathcal{M}|_\times &= 4B\Big((g_v^f)^2\Big[\frac{1}{16}s^2(1-\cos\theta)^2 + \frac{1}{16}s^2(1+\cos\theta)^2\Big] + 12(g_A^f)^2\frac{1}{16}s^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{2}Bs^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big] + 6(g_A^f)^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big] + 6(g_A^f)^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{2}\frac{\alpha^4 4^4\pi^4}{s(s-M_Z^2)}\frac{1}{\sin^2\theta_W\cos^2\theta_W}s^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big] + 6(g_A^f)^2(1+\cos\theta)^2\Big)\\
&= \frac{1}{2}\frac{\alpha^4 4^4\pi^4}{s(s-M_Z^2)}\frac{M_Z^2G\sqrt{2}}{\alpha\pi}s^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big] + 6(g_A^f)^2(1+\cos\theta)^2\Big)\\
&= 128\frac{\alpha^3\pi^3M_Z^2G}{s(s-M_Z^2)}\sqrt{2}s^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big] + 6(g_A^f)^2(1+\cos\theta)^2\Big)\end{aligned}$$

1.6 Combining the terms

We can now combine the matrix element terms

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}| &= e^4\big(1+\cos^2\theta\big)\\
&+4\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}\Big[((g_v^e)^2+(g_A^e)^2)^2\cdot(1+\cos^2\theta)+24(g_v^e)^2(g_A^e)^2(1-\cos\theta)^2\Big]\\
&+\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big]+6(g_A^f)^2(1+2\cos\theta+\cos^2\theta)\Big)\\
&=e^4\big(1+\cos^2\theta\big)\\
&+4\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}\Big[((g_v^e)^2+(g_A^e)^2)^2\cdot(1+\cos^2\theta)+24(g_v^e)^2(g_A^e)^2(1-2\cos\theta+\cos^2\theta)\Big]\\
&+\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\Big((g_v^f)^2\Big[1+\cos^2\theta\Big]+6(g_A^f)^2(1+2\cos\theta+\cos^2\theta)\Big)\\
&=\left[e^4+4\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}\big(((g_v^e)^2+(g_A^e)^2)^2+24(g_v^e)^2(g_A^e)^2\big)+\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2((g_v^f)^2+6(g_A^f)^2)\right](1+c\\
&+\left[4\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}(-2\cdot 24(g_v^e)^2(g_A^e)^2)+\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\cdot 2\cdot g(g_A^f)^2\right]\cos\theta\\
&=\Bigg[16\pi^2\alpha+4\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}\big(((g_v^e)^2+(g_A^e)^2)^2+24(g_v^e)^2(g_A^e)^2\big)\\
&+\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2((g_v^f)^2+6(g_A^f)^2)\Bigg](1+\cos^2\theta)\\
&+\left[-192\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}(g_v^e)^2(g_A^e)^2+\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\cdot g(g_A^f)^2\right]\cos\theta\\
&=\Bigg[16\pi^2\alpha+4\frac{e^4s^2}{(s-M_Z^2)^2}\frac{1}{\cos^4\theta_W\sin^4\theta_W}\big(((g_v^e)^2+(g_A^e)^2)^2+24(g_v^e)^2(g_A^e)^2\big)\\
&+\frac{1}{2}\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2((g_v^f)^2+6(g_A^f)^2)\Bigg](1+\cos^2\theta)\\
&+\left[-192\frac{g^4s^2}{(s-M_Z^2)^2\cos^4\theta_W}(g_v^e)^2(g_A^e)^2+\frac{e^2g^2}{s(s-M_Z^2)\cos^2\theta_W}s^2\cdot g(g_A^f)^2\right]\cos\theta
\end{aligned}
$$

Note:

$$
\begin{aligned}
\alpha &= \frac{e^2}{4\pi}\\
m_Z = (\frac{\alpha\pi}{G\sqrt{2}})^{1/2}\frac{2}{\sin 2\theta_W} &= (\frac{\alpha\pi}{G\sqrt{2}})^{1/2}\frac{2}{2\cos\theta_W\sin\theta_W}\\
g\sin\theta_W &= e\\
m_z = \frac{m_w}{\cos\theta} &=
\end{aligned}
$$