

FYS4150: Project 1

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a)

We want to solve the one-dimensional Poisson equation with Dirichlet boundary conditions:

$$-u''(x) = f(x), \quad x \in (0, 1), \quad u(0) = u(1) = 0 \quad (1)$$

We approximate the second derivative using the 3 point formula:

$$-\frac{v_{i+1} + v_{i-1} - 2v_i}{h^2} = f_i \quad (2)$$

For $i = 1, 2, \dots, n$, where $f_i = f(x_i)$. We rewrite this equation:

$$-v_{i-1} + 2v_i - v_{i+1} = h^2 f_i$$

Setting in $i = 1, 2, 3$ we start to see a pattern:

$$-v_0 + 2v_1 - v_2 = h^2 f_1$$

Since $v_0 = 0$:

$$2v_1 - v_2 = h^2 f_1$$

$$-v_1 + 2v_2 - v_3 = h^2 f_2$$

$$-v_2 + 2v_3 - v_4 = h^2 f_3$$

We can write this as a matrix working on a vector:

$$\begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots \\ 0 & -1 & 2 & -1 & 2 & \dots \\ & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & & -1 & 2 & -1 \\ 0 & \dots & & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ \dots \\ \dots \\ v_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \dots \\ \dots \\ \dots \\ \dots \\ \tilde{b}_n \end{pmatrix}$$

$$\mathbf{A}\mathbf{v} = \mathbf{b}$$

We assume a source term and closed-form solution:

$$f(x) = 100e^{-10x} \quad (3)$$

$$u(x) = 1 - (1 - e^{-10})x - e^{-10x} \quad (4)$$

Check that this satisfies the Poisson equation:

$$u'(x) = -1 + e^{-10} + 10e^{-10x}$$

$$u''(x) = -100e^{-10x} = -f(x)$$

b)

We rewrite the set of equations in terms of a tridiagonal matrix:

$$\begin{pmatrix} b_1 & c_1 & 0 & \dots & \dots & \dots \\ a_2 & b_2 & c_2 & \dots & \dots & \dots \\ 0 & a_3 & b_3 & c_3 & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ \dots \\ \dots \\ v_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \dots \\ \dots \\ \dots \\ \tilde{b}_n \end{pmatrix}$$

In index notation this becomes:

$$a_i v_{i-1} + b_i v_i + c_i v_{i+1} = \tilde{b}_i \quad (5)$$

With $a_1 = c_n = 0$. The algorithm for solving this set of equations consists of two steps; a decomposition and forward substitution followed by a backward substitution:

Step 1, forward substitution:

k

Step 1:

$$\beta = b_1$$

$$\gamma = vector[n]$$

$$v_1 = \frac{\tilde{b}_1}{b_1}$$

for $(j = 2; j \leq n; j = j + 1) :$

$$\gamma_j = \frac{c_{j-1}}{\beta}$$

$$\beta = b_j - a_j * \gamma_j$$

$$v_j = \frac{(\tilde{b}_j - a_j \cdot v_{j-1})}{\beta}$$

Step 2:

for $(j = n - 1; j \geq 1; j = j - 1) :$

$$v_j = \gamma_{j+1} \cdot v_{j+1}$$

References

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