Introduction to Statistics - Young Researchers Fellowship Program

Lecture 3 - Introduction to probability

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Why probability?

- Probability is a foundational concept in inferential statistics.
- It is related to the idea that because it is hard to access the entire population, we can use a sample to make inferences about the population.
- We then take a sample, which is thought to be representative of the population, and use it to make inferences about the population.

Why probability?

- However, even if we ensure the sample is similar to the population based on theoretical knowledge, we would never expect that it is exactly the same.
  - There will always be some difference, potentially due to random variation, which make the sample different from the population.
- Probability is the mathematical tool that allows us to quantify the uncertainty associated with the sample.
  - How far are we from the population?
  - If there are significant differences, are we looking for an unrepresentative sample?
  - Is a sample mean fundamentally different from the population mean?
- All the questions above can be answered using the concepts of probability.

Probability

- A measure of the likelihood through which an event will occur.
- For example: when seeing gray clouds, we might think that it will rain. The probability of rain is higher when we see gray clouds.
    - However, it is not certain that it will rain. It might rain, or it might not (it is uncertain).

Approaches to probability calculation

- There are two main approaches to calculate probability:
  - Frequentist approach
  - Bayesian approach
- The frequentist approach is based on the idea that probability is the long-run frequency of an event.
  - For example, the probability of getting a head when flipping a coin is 0.5 because in the long run, half of the flips will be heads.
- This means that frequentist probability will calculate a probability based on past data

$$\text{Probability} = \frac{\text{Number of times an event occurred}}{\text{Total number of trials}}$$

Approaches to probability calculation

- Another major approach to probability is the Bayesian approach (pronounced “bay-zee-an”).
- The Bayesian approach is based on the idea that probability is a a measure of the degree of belief that an event will occur.
  - Hence, it is like a measure of how “good” our information is.
- We won’t focus Bayesian approaches on this module, but it is good to know that it exists, and researchers have devoted a lot of time to it.
  - Typically, we will want to understand frequentist approaches first and gain mathematical intuition before we move to Bayesian approaches.

Basic probability rules

- Two major rules govern probability:

1. All probabilities are between 0 and 1.
    - A probability of 0 means that the event will never happen (0% chance).
    - A probability of 1 means that the event will always happen (100% chance).
    - This means probabilities are always in proportion form, but can be easily converted to percentages.
2. The sum of the probabilities for all possible outcomes of an event must sum to 1.
    - In an exam, you'd have two outcomes: pass or fail (assuming no other "weird" outcomes).
    - The probability of passing plus the probability of failing must sum to 1.
    - This means 100% of outcomes are covered in the "possible outcomes" set, each with a probability.

Probability: the world of board games

- Two typical examples often emerge when discussing probability:
  - Flipping a “fair” coin
  - Rolling a “fair” die
- A fair coin is a coin that has an equal probability of landing on heads or tails.
- So, because there is 1 head and 1 tail, the probability of getting a head is 1/2, and the probability of getting a tail is 1/2.

$$(P(H)) = \frac{Number of heads}{Total number of outcomes} = \frac{1}{2}$$

$$(P(T)) = \frac{Number of tails}{Total number of outcomes} = \frac{1}{2}$$

Probability: the world of board games

- The other common example is rolling a die.
- A die has 6 faces, each with a number from 1 to 6.
- The probability of getting a 1 is $1/6$, the probability of getting a 2 is $1/6$, and so on.

Probability: the world of board games

- It is common to also see drawing a card from a deck of cards as an example of probability.
- Decks of cards have 52 cards, with 4 suits (hearts, diamonds, clubs, and spades) and 13 cards in each suit.
- Special cards include the King, Queen, the Jack, and the Ace.
  - There are 4 of each of these cards in the deck.
  - The probability of drawing a King is $4/52$, the probability of drawing a Queen is $4/52$, and so on.

The lingo: experiments, events, outcomes and sample spaces

- There is specific terminology and definitions in probability that must be understood before we move forward.
- An **experiment** is a process that generates well-defined outcomes, and we're interested in probabilities associated with these outcomes.
  - For example, flipping a coin, rolling a die, or drawing a card from a deck.
- An **event** is a subset of the outcomes of an experiment.
  - For example, in flipping a coin, the event of getting a head is a subset of the outcomes of the experiment.

The lingo: experiments, events, outcomes and sample spaces

- In simple experiments, we often see "events" covering one outcome, and not more.
    - However, events can "group" outcomes in a specific way.
    - E.g. the event of getting an even number when rolling a die. This event covers outcomes 2, 4, and 6.
- The **sample space** is the set of all possible outcomes of an experiment.
    - For example, when flipping a coin, the sample space is $\{H, T\}$.
    - When rolling a die, the sample space is $\{1, 2, 3, 4, 5, 6\}$.

Set theory and probability

- Probability is closely related to set theory.
  - Sets are collections of objects, and in probability, we often think of events as sets of outcomes.
- Thus, it is common to see notation from set theory in probability.
  - A set is denoted by curly braces, e.g. $\{1, 2, 3, 4, 5, 6\}$.
  - We use letters to denote sets, e.g. A, B, C, etc.
  - We use set operations to calculate probabilities.
  - We use Venn diagrams to visualize probabilities.
- Set theory operators, such as belongs to ($\in$), union ($\cup$), intersection ($\cap$), and complement ($'$) are used in probability.
  - These often also come in terms of logical operators, such as "and" and "or".

Basic operations in probability

- There are three basic operations in probability:
  - Union
  - Intersection
  - Complement

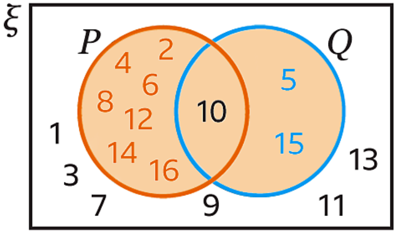
Union

- The union of two events A and B is the event that *either* A or B or both occur.
  - Meaning that any two of them can occur, but not necessarily both.
  - The union of A and B is denoted by $A \cup B$ (set theory notation).
  - The probability of the union of A and B is denoted by $P(A \cup B)$.
- Example: what is the probability of getting one or two when rolling a die?
  - Notice the use of "or" in the question. Union is often associated with the "OR" logical operator.
  - This can be written as $P(A \cup B)$, where A is the event of getting a 1 and B is the event of getting a 2.

Venn Diagrams

- A Venn diagram is a visual representation of sets and their relationships.
- In probability, we often use Venn diagrams to visualize the union of two events.
- The union of two events is the shaded area in the Venn diagram that covers both events.

Union in a Venn diagram



$P \cup Q = \{2, 4, 6, 8, 12, 14, 16, 10, 5, 15\}$

Figure 1: Union in a Venn diagram. Source: BBC Bitesize

Intersection

- The intersection of two events A and B is the event that *both* A and B occur.
  - Meaning that both events must occur.
  - The intersection of A and B is denoted by $A \cap B$ (set theory notation).
  - The probability of the intersection of A and B is denoted by $P(A \cap B)$.
- Example: what is the probability of getting an even number **and** a number less than 4 when rolling a die?
  - Intersection is often associated with the "AND" logical operator.

Intersection in a Venn diagram

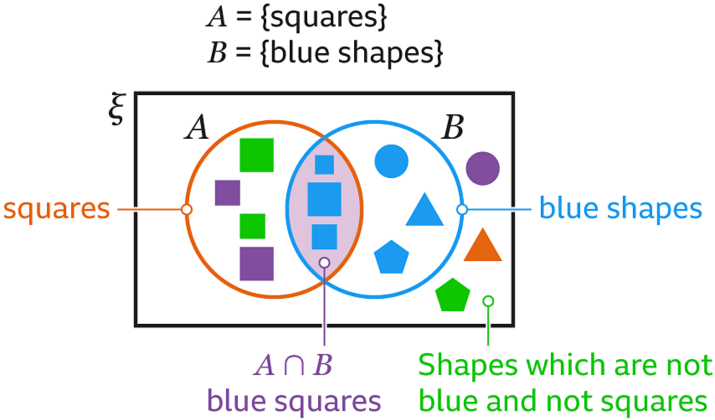


Figure 2: Intersection in a Venn diagram. Source: BBC Bitesize

Complement

- The complement of an event A is the event that A does *not* occur.
  - Meaning that the event does not happen.
  - The complement of A is denoted by A' (set theory notation).
  - The probability of the complement of A is denoted by $P(A')$.
- Example: what is the probability of not getting a 1 when rolling a die?
  - The complement of getting a 1 is not getting a 1.
  - The complement of an event is often associated with the "NOT" logical operator.
- It is often very easy to calculate the complement of an event, as it is the *whole* minus the probability of the event.

$$P(A') = 1 - P(A)$$

Complement in a Venn diagram

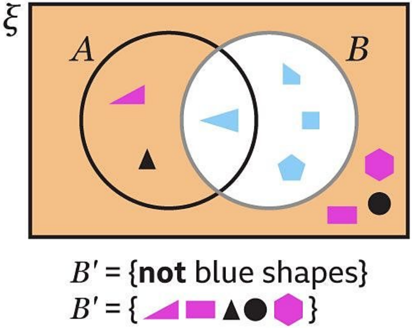


Figure 3: Complement in a Venn diagram. Source: BBC Bitesize

Computing the probability of union: mutually exclusive events and the addition rule

- To actually calculate a probability associated with the union of two events, we need to know a little bit more about the nature of the events.
- For instance, in the die example, the events of getting a 1 and getting a 2 are **mutually exclusive**
  - This means that the events *cannot happen at the same time*.
  - The probability of the union of mutually exclusive events is the sum of the probabilities of the events.

Union under mutually exclusive events

$$P(A \cup B) = P(A) + P(B)$$

where A and B are mutually exclusive events.

Computing the probability of union: non-mutually exclusive events and the addition rule

- However, if the events are not mutually exclusive, we need to consider the probability of the intersection of the events.
- Events that might not be mutually exclusive happen very often. For instance, what is the probability of going to the beach and getting a sunburn?
    - These events are not mutually exclusive, as you can go to the beach and get a sunburn at the same time.

Union under non-mutually exclusive events

- If we apply the "formula" above, we will overcount the probability of the intersection of the events.
- Example: the probability of finding a company that is both in Pichincha, and is listed as active.
  - If we know the probability of finding companies in Pichincha, it will include those listed as active and otherwise.
  - If we know the probability of finding companies listed as active, it will include those in Pichincha and otherwise.
- Summing these two without doing anything else will overcount the probability of finding a company that is both in Pichincha and is listed as active.
  - Solution: subtract the probability of the intersection of the events.

Union under non-mutually exclusive events: the addition rule

- The probability of the union of two events is given by the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- This is a broad formula that can be applied to any two events, whether they are mutually exclusive or not.
  - The intersection of two mutually exclusive events is 0, so the formula simplifies to the one we saw before.

Computing probabilities of intersection

- There is no direct way to calculate the probability of the intersection of two events unless we know more.
- Some cases emerge:
  - If we know the union of two events and their associated probabilities, we can calculate the probability of the intersection by rearranging the addition rule.
  - If we know the events are mutually exclusive, the probability of the intersection is 0.
  - If we know events are *independent*, there is a direct way to calculate the probability of the intersection.

Independence

- Two events are independent if the occurrence of one event does not affect the occurrence of the other event.
- For example, the probability of getting a head when flipping a coin is independent of the probability of getting a 1 when rolling a die.
  - Basically means the events are not related.
- If two events are independent, the probability of the intersection of the events is the product of the probabilities of the events.
  - This is called the multiplication rule.

Independence: the multiplication rule

- If two events are independent, the probability of the intersection of the events is given by the multiplication rule:

$$P(A \cap B) = P(A) \times P(B)$$

when A and B are independent events.

Conditional probability

- Conditional probability is the probability of an event given that another event has occurred.
- We are interested in this because once an event has occurred, the probability of another event might change.
  - This is somewhat of a Bayesian idea: once we obtain new information, our beliefs might change.
- For instance, we are interested in knowing the probability of finding an active company, *given* that the company is in Pichincha.
  - This is different from the probability of finding an active company in general.
- This is a conditional probability: the condition is that the company is in Pichincha.
  - This is denoted by $P(A|B)$, where A is the event of finding an active company and B is the event of finding a company in Pichincha.

Conditional probability formula

- The conditional probability of event A given event B is denoted by $P(A|B)$.
- The formula for conditional probability is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Following our above example:
  - A is the event of finding an active company.
  - B is the event of finding a company in Pichincha.
- We find active companies in Pichincha (the intersection of the events) and divide by the probability of finding a company in Pichincha.
- Conditionality is the theoretical probability concept of “subsetting” a sample space based on the occurrence of another event.

Conditional probability: independence

- Conditional probability is a key concept for independence: if two events are truly independent, the conditional probability of one event given the other is the same as the probability of the event (without the condition).
  - This is since conditioning on the other event does not change the probability of the event.
- Example: if we know that a die roll gives an even number, what is the probability of getting a 2 in a second roll?
  - Two die rolls are independent events (from our logical understanding of the experiment)
  - So, the probability of getting a 2 in the second roll is the same as the probability of getting a 2 any time we roll a die.

Conditional probability: independence

- The general rule is that for any two independent events A and B:

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

The multiplication rule for independent events

- If two events are independent, the probability of the intersection of the events is the product of the probabilities of the events.
  - This is the multiplication rule for independent events.
  - It will only apply if the events are independent!

$$P(A \cap B) = P(A) \times P(B)$$

The multiplication rule for independent events

- For instance, if we know the probability of getting a head when flipping a coin is 0.5, and the probability of getting a 1 when rolling a die is 1/6, we can calculate the probability of getting a head and a 1.

$$P(\text{Head} \cap 1) = P(\text{Head}) \times P(1) = 0.5 \times \frac{1}{6} = \frac{1}{12}$$

- This is a very powerful rule that can be used to calculate the probability of the intersection of independent events.
- Notice that I don't use my companies' example here, as the events are likely not independent.
    - Active companies may tend to be in Pichincha as it is the capital
    - This underscores the need to actually understand the nature of the events, or the *data generating process*.

Combinatorics or the field of counting