# Introduction to Statistics - Young Researchers Fellowship Program

## Lecture 3 - Introduction to probability

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# Why probability?

# Why probability?

- Probability is a foundational concept in inferential statistics.
- It is related to the idea that because it is hard to access the entire population, we can use a sample to make inferences about the population.
- We then take a sample, which is thought to be representative of the population, and use it to make inferences about the population.

# Why probability?

- However, even if we ensure the sample is similar to the population based on theoretical knowledge, we would never expect that it is exactly the same.
  - There will always be some difference, potentially due to random variation, which make the sample different from the population.
- Probability is the mathematical tool that allows us to quantify the uncertainty associated with the sample.
  - How far are we from the population?
  - If there are significant differences, are we looking for an unrepresentative sample?
  - Is a sample mean fundamentally different from the population mean?
- All the questions above can be answered using the concepts of probability.

# Basic probability concepts

# Probability

- A measure of the likelihood through which an event will occur.
- For example: when seeing gray clouds, we might think that it will rain. The probability of rain is higher when we see gray clouds.
    - However, it is not certain that it will rain. It might rain, or it might not (it is uncertain).

# Approaches to probability calculation

- There are two main approaches to calculate probability:
    - Frequentist approach
    - Bayesian approach
- The frequentist approach is based on the idea that probability is the long-run frequency of an event.
    - For example, the probability of getting a head when flipping a coin is 0.5 because in the long run, half of the flips will be heads.
- This means that frequentist probability will calculate a probability based on past data

$$\text{Probability} = \frac{\text{Number of times an event occurred}}{\text{Total number of trials}}$$

# Approaches to probability calculation

- Another major approach to probability is the Bayesian approach (pronounced "bay-zee-an").
- The Bayesian approach is based on the idea that probability is a a measure of the degree of belief that an event will occur.
  - Hence, it is like a measure of how "good" our information is.
- We won't focus Bayesian approaches on this module, but it is good to know that it exists, and researchers have devoted a lot of time to it.
  - Typically, we will want to understand frequentist approaches first and gain mathematical intuition before we move to Bayesian approaches.

# Basic probability rules

- Two major rules govern probability:

1. All probabilities are between 0 and 1.
   - A probability of 0 means that the event will never happen (0% chance).
   - A probability of 1 means that the event will always happen (100% chance).
   - This means probabilities are always in proportion form, but can be easily converted to percentages.
2. The sum of the probabilities for all possible outcomes of an event must sum to 1.
   - In an exam, you'd have two outcomes: pass or fail (assuming no other "weird" outcomes).
   - The probability of passing plus the probability of failing must sum to 1.
   - This means 100% of outcomes are covered in the "possible outcomes" set, each with a probability.

# Probability: the world of board games

- Two typical examples often emerge when discussing probability:
  - Flipping a "fair" coin
  - Rolling a "fair" die
- A fair coin is a coin that has an equal probability of landing on heads or tails.
- So, because there is 1 head and 1 tail, the probability of getting a head is 1/2, and the probability of getting a tail is 1/2.

$$(P(H)) = \frac{Numberofheads}{Totalnumberofoutcomes} = \frac{1}{2}$$

$$(P(T)) = \frac{Numberoftails}{Totalnumberofoutcomes} = \frac{1}{2}$$

# Probability: the world of board games

- The other common example is rolling a die.
- A die has 6 faces, each with a number from 1 to 6.
- The probability of getting a 1 is $1/6$, the probability of getting a 2 is $1/6$, and so on.

# Probability: the world of board games

- It is common to also see drawing a card from a deck of cards as an example of probability.
- Decks of cards have 52 cards, with 4 suits (hearts, diamonds, clubs, and spades) and 13 cards in each suit.
- Special cards include the King, Queen, the Jack, and the Ace.
  - There are 4 of each of these cards in the deck.
  - The probability of drawing a King is $4/52$, the probability of drawing a Queen is $4/52$, and so on.

# The lingo: experiments, events, outcomes and sample spaces

- There is specific terminology and definitions in probability that must be understood before we move forward.
- An **experiment** is a process that generates well-defined outcomes, and we're interested in probabilities associated with these outcomes.
  - For example, flipping a coin, rolling a die, or drawing a card from a deck.
- An **event** is a subset of the outcomes of an experiment.
  - For example, in flipping a coin, the event of getting a head is a subset of the outcomes of the experiment.

# The lingo: experiments, events, outcomes and sample spaces

- In simple experiments, we often see "events" covering one outcome, and not more.
  - However, events can "group" outcomes in a specific way.
  - E.g. the event of getting an even number when rolling a die. This event covers outcomes 2, 4, and 6.
- The **sample space** is the set of all possible outcomes of an experiment.
  - For example, when flipping a coin, the sample space is $\{H, T\}$.
  - When rolling a die, the sample space is $\{1, 2, 3, 4, 5, 6\}$.

# Set theory and probability

- Probability is closely related to set theory.
  - Sets are collections of objects, and in probability, we often think of events as sets of outcomes.
- Thus, it is common to see notation from set theory in probability.
  - A set is denoted by curly braces, e.g. $\{1, 2, 3, 4, 5, 6\}$.
  - We use letters to denote sets, e.g. A, B, C, etc.
  - We use set operations to calculate probabilities.
  - We use Venn diagrams to visualize probabilities.
- Set theory operators, such as belongs to ($\in$), union ($\cup$), intersection ($\cap$), and complement ($'$) are used in probability.
  - These often also come in terms of logical operators, such as "and" and "or".

# Basic operations in probability

- There are three basic operations in probability:
    - Union
    - Intersection
    - Complement

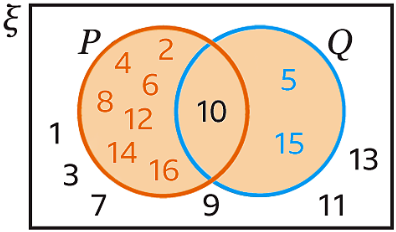
# Union

- The union of two events A and B is the event that *either* A or B or both occur.
  - Meaning that any two of them can occur, but not necessarily both.
  - The union of A and B is denoted by $A \cup B$ (set theory notation).
  - The probability of the union of A and B is denoted by $P(A \cup B)$.
- Example: what is the probability of getting one or two when rolling a die?
  - Notice the use of "or" in the question. Union is often associated with the "OR" logical operator.
  - This can be written as $P(A \cup B)$, where A is the event of getting a 1 and B is the event of getting a 2.

# Venn Diagrams

- A Venn diagram is a visual representation of sets and their relationships.
- In probability, we often use Venn diagrams to visualize the union of two events.
- The union of two events is the shaded area in the Venn diagram that covers both events.

# Union in a Venn diagram




$$P \cup Q = \{2, 4, 6, 8, 12, 14, 16, 10, 5, 15\}$$

Figure 1: Union in a Venn diagram. Source: BBC Bitesize

# Intersection

- The intersection of two events A and B is the event that *both* A and B occur.
  - Meaning that both events must occur.
  - The intersection of A and B is denoted by $A \cap B$ (set theory notation).
  - The probability of the intersection of A and B is denoted by $P(A \cap B)$.
- Example: what is the probability of getting an even number **and** a number less than 4 when rolling a die?
  - Intersection is often associated with the "AND" logical operator.

# Intersection in a Venn diagram

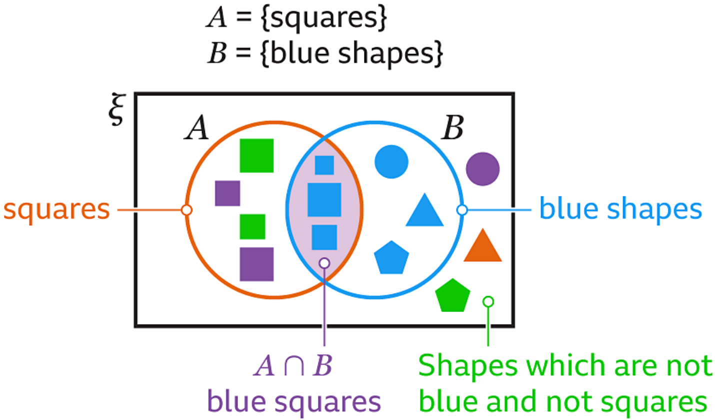



Figure 2: Intersection in a Venn diagram. Source: BBC Bitesize

# Complement

- The complement of an event A is the event that A does *not* occur.
  - Meaning that the event does not happen.
  - The complement of A is denoted by $A'$ (set theory notation).
  - The probability of the complement of A is denoted by $P(A')$.
- Example: what is the probability of not getting a 1 when rolling a die?
  - The complement of getting a 1 is not getting a 1.
  - The complement of an event is often associated with the "NOT" logical operator.
- It is often very easy to calculate the complement of an event, as it is the *whole* minus the probability of the event.

$$P(A') = 1 - P(A)$$

# Complement in a Venn diagram

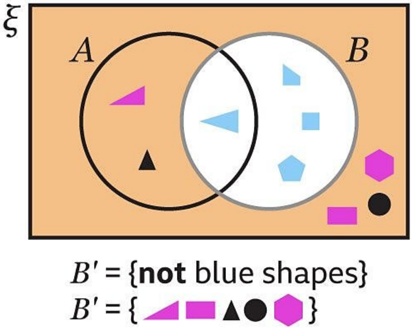



Figure 3: Complement in a Venn diagram. Source: BBC Bitesize

# Computing the probability of union: mutually exclusive events and the addition rule

- To actually calculate a probability associated with the union of two events, we need to know a little bit more about the nature of the events.
- For instance, in the die example, the events of getting a 1 and getting a 2 are **mutually exclusive**
  - This means that the events *cannot happen at the same time*.
  - The probability of the union of mutually exclusive events is the sum of the probabilities of the events.

# Union under mutually exclusive events

$$P(A \cup B) = P(A) + P(B)$$

where A and B are mutually exclusive events.

# Computing the probability of union: non-mutually exclusive events and the addition rule

- However, if the events are not mutually exclusive, we need to consider the probability of the intersection of the events.
- Events that might not be mutually exclusive happen very often. For instance, what is the probability of going to the beach and getting a sunburn?
    - These events are not mutually exclusive, as you can go to the beach and get a sunburn at the same time.

# Union under non-mutually exclusive events

- If we apply the "formula" above, we will overcount the probability of the intersection of the events.
- Example: the probability of finding a company that is both in Pichincha, and is listed as active.
  - If we know the probability of finding companies in Pichincha, it will include those listed as active and otherwise.
  - If we know the probability of finding companies listed as active, it will include those in Pichincha and otherwise.
- Summing these two without doing anything else will overcount the probability of finding a company that is both in Pichincha and is listed as active.
  - Solution: subtract the probability of the intersection of the events.

# Union under non-mutually exclusive events: the addition rule

- The probability of the union of two events is given by the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- This is a broad formula that can be applied to any two events, whether they are mutually exclusive or not.
  - The intersection of two mutually exclusive events is 0, so the formula simplifies to the one we saw before.

# Computing probabilities of intersection

- There is no direct way to calculate the probability of the intersection of two events unless we know more.
- Some cases emerge:
    - If we know the union of two events and their associated probabilities, we can calculate the probability of the intersection by rearranging the addition rule.
    - If we know the events are mutually exclusive, the probability of the intersection is 0.
    - If we know events are *independent*, there is a direct way to calculate the probability of the intersection.

# Independence

- Two events are independent if the occurrence of one event does not affect the occurrence of the other event.
- For example, the probability of getting a head when flipping a coin is independent of the probability of getting a 1 when rolling a die.
  - Basically means the events are not related.
- If two events are independent, the probability of the intersection of the events is the product of the probabilities of the events.
  - This is called the multiplication rule.

# Independence: the multiplication rule

- If two events are independent, the probability of the intersection of the events is given by the multiplication rule:

$$P(A \cap B) = P(A) \times P(B)$$

when A and B are independent events.

# Conditional probability

- Conditional probability is the probability of an event given that another event has occurred.
- We are interested in this because once an event has occurred, the probability of another event might change.
  - This is somewhat of a Bayesian idea: once we obtain new information, our beliefs might change.

# Conditional probability

- For instance, we are interested in knowing the probability of finding an active company, *given* that the company is in Pichincha.
    - This is different from the probability of finding an active company in general.
- This is a conditional probability: the condition is that the company is in Pichincha.
    - This is denoted by $P(A|B)$, where A is the event of finding an active company and B is the event of finding a company in Pichincha.

# Conditional probability formula

- The conditional probability of event A given event B is denoted by $P(A|B)$.
- The formula for conditional probability is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

# Conditional probability example

- Following our above example:
  - A is the event of finding an active company.
  - B is the event of finding a company in Pichincha.
- We find active companies in Pichincha (the intersection of the events) and divide by the probability of finding a company in Pichincha.
- Conditionality is the theoretical probability concept of "subsetting" a sample space based on the occurrence of another event.
  - So, we are theorizing our use of `filter()` in R.

# Conditional probability: independence

- Conditional probability is a key concept for independence: if two events are truly independent, the conditional probability of one event given the other is the same as the probability of the event (without the condition).
  - This is since conditioning on the other event does not change the probability of the event.
- Example: if we know that a die roll gives an even number, what is the probability of getting a 2 in a second roll?
  - Two die rolls are independent events (from our logical understanding of the experiment)
  - So, the probability of getting a 2 in the second roll is the same as the probability of getting a 2 any time we roll a die.

# Conditional probability: independence

- The general rule is that for any two independent events A and B:

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

# The multiplication rule for independent events

- If two events are independent, the probability of the intersection of the events is the product of the probabilities of the events.
  - This is the multiplication rule for independent events.
  - It will only apply if the events are independent!

$$P(A \cap B) = P(A) \times P(B)$$

# The multiplication rule for independent events

- For instance, if we know the probability of getting a head when flipping a coin is 0.5, and the probability of getting a 1 when rolling a die is $1/6$, we can calculate the probability of getting a head and a 1.

$$P(\textsf{Head} \cap 1) = P(\textsf{Head}) \times P(1) = 0.5 \times \frac{1}{6} = \frac{1}{12}$$

- This is a very powerful rule that can be used to calculate the probability of the intersection of independent events.
- Notice that I don't use my companies' example here, as the events are likely not independent.
  - Active companies may tend to be in Pichincha as it is the capital
  - This underscores the need to actually understand the nature of the events, or the *data generating process*.

# Random variables and probability distributions

# Probability distributions

- Probability distributions are a way to describe the likelihood of different outcomes in an experiment.
- This means that we can use our laws of probability to construct more sophisticated models.
  - These models will describe the likelihood of different outcomes.

## Why even use them?

- In statistical inference, we will move away from simple experiments like flipping coins and rolling dice.
- Our experiments will be related to the process of sampling.
    - Sampling complies with laws of probability, as every event (one specific sample) is a subset of the sample space (the population), and such sample will look like the population, but subject to random variation.
- Probability distributions will help us understand the likelihood of different outcomes in our samples, by constructing a model of the process of sampling (i.e. the probability distribution).
    - These probability distributions are based on guesses about how the population behaves.
    - If our guess is good, the sample will behave according to the probability distribution. Otherwise, it won't and we will have to revise our guess (another probability distribution).

# Random variables

- Random variables (RVs) numerically describe the outcomes of an experiment.
    - Because they are random, the numerical value of the random variable is uncertain, hence the name.
- Two major types of random variables:
    - Discrete random variables (e.g. the number of heads when flipping a coin)
    - Continuous random variables (e.g. the height of a person)
- Random variables are denoted by capital letters, e.g. $X$, $Y$, $Z$.

# Probability distributions of RVs

- The fact that these are random does not mean that we cannot describe them or predict their behavior.
- RVs will have a probability distribution, since numerical outcomes that they might take have associated probabilities.
- The probability distribution of a random variable is a *function* that describes the likelihood of different outcomes.
    - This function will assign a probability to each possible outcome of the random variable.

# Discrete random variables

- Discrete random variables are random variables that can take on a finite number of values.
    - It is finite because we can count the number of values the random variable can take (even if it is a large number).
- For example, the number of heads when flipping a coin is a discrete random variable.
    - It can take on values 0 or 1.

# The PDF of RVs (love the jargon or perish!)

- A probability density function (PDF) is a function that describes the likelihood of different outcomes of a random variable.
- Denoted by $f(x)$, where $x$ is the value of the random variable, and the output of the function $f$ is the probability of the random variable taking on that value.
- For discrete random variables, the PDF is often called the probability mass function (PMF), but the idea is the same.
- We won't always have a function that takes an equation form.
  - Discrete RVs will often have a table of probabilities which is called the PDF ($f$)
  - Continuous RVs will often have a function that describes the likelihood of different outcomes.

# Building the PDF of a discrete RV: the example of a fair coin

- The fair coin example (getting a head) has a simple enough PDF. See below:

| Value of the random variable | Probability |
|---|---|
| 0 | 0.5 |
| 1 | 0.5 |

- Notice that the value of the random variable $X# is the number of heads when flipping a coin.
- The probability of getting a head is 0.5, and the probability of getting a tail is 0.5.
    - Getting 0 heads in this experiment is the same as getting a tail.

# Building the PDF of a discrete RV: the example of a fair die

- The fair die example (rolling a die) has a more complex PDF. See below:

| Value of the random variable | Probability |
|---|---|
| 1 | $1/6$ |
| 2 | $1/6$ |
| 3 | $1/6$ |
| 4 | $1/6$ |
| 5 | $1/6$ |
| 6 | $1/6$ |

- In this case, the value of the random variable $X$ is the number that comes up when rolling a die.
- The probability of getting a 1 is $1/6$, the probability of getting a 2 is

# The Cumulative Distribution Function (CDF)

- The cumulative distribution function (CDF) is a function that describes the probability that a random variable is **less than or equal** to a certain value.
- Denoted by $F(x)$, where $x$ is the value of the random variable, and the output of the function $F$ is the probability that the random variable is less than or equal to $x$.
- The CDF is a very useful function that can be used to calculate probabilities of ranges of values of the random variable.

# Building the CDF of a discrete RV: coin example

- The CDF of a discrete random variable is built by summing the probabilities of the random variable being less than or equal to a certain value.
- For the coin example, the CDF is as follows:

| Value of the random variable | Probability | Cumulative Probability |
|---|---|---|
| 0 | 0.5 | 0.5 |
| 1 | 0.5 | 1.0 |

- The probability of getting 0 heads or less is simply 0.5.
  - This is the probability of getting a tail.
- Notice how the probability of getting one head OR less is 1.
  - This is a union of the events of getting 0 heads and getting 1 head.
  - $P(X \leq 1) = P(X = 0) + P(X = 1) = 0.5 + 0.5 = 1.0$

# Expected value of a discrete RV

- The expected value of a random variable is something like the "average" value of the random variable.
- To obtain averages for RVs which are discrete, we multiply each possible value of the random variable by its probability, and sum these products.

$$E(X) = \sum_i x_i \cdot P(X = x_i)$$

# Example: expected value of a coin

- For the coin example, the expected value is:

$$E(X) = 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5$$

- This means that the expected value of the number of heads when flipping a coin is 0.5.
- Does this make sense? It will often happen that the expected value is not a possible value of the random variable.
  - This is because the expected value is a measure of central tendency, and not necessarily a possible value of the random variable.
- In an intuitive sense, the expected value is the value we would expect to get if we repeated the experiment many, many times, and took the average of the outcomes.

# Variance of a discrete RV

- The variance of a random variable is a measure of how spread out the values of the random variable are.
- To calculate the variance of a discrete random variable, we calculate the expected value of the squared difference between the random variable and its expected value.

$$\mathsf{Var}(X) = E((X - E(X))^2) = \sum_i (x_i - E(X))^2 \cdot P(X = x_i)$$

# Example: variance of a coin

- For the coin example, the variance is:

$$\text{Var}(X) = (0 - 0.5)^2 \cdot 0.5 + (1 - 0.5)^2 \cdot 0.5 = 0.25$$

- This means that the variance of the number of heads when flipping a coin is 0.25.
- Once again, higher variance means that the values of the random variable are more spread out.

# Laws of expected value and variance

- When working with expected values and variances, there are some laws that can be useful as shorthands.
- For any constants $a$ and $b$:

$$E(aX + b) = aE(X) + b$$

$$\mathsf{Var}(aX + b) = a^2\mathsf{Var}(X)$$

## Continuous random variables

- There is a specific detail about the PDF of continuous random variables: the probability of the random variable taking on a specific value is 0.
- This is because continuous random variables can take on an infinite number of values, and the probability of any one value is 0.
    - Why infinite? Because we can always find a value between two values (i.e. with a lot of decimals).
- Intuitively, think that whenever we have a continuous random variable, it is extremely unlikely that we will get a specific value.
    - Measurements are never perfect. We can never measure a person's height to the exact millimeter, hence, we can't even know the exact value of the random variable.

# The PDF of continuous RVs

- The PDF of a continuous random variable is a function that describes the likelihood of different outcomes of the random variable.
- This PDF is often a complicated equation in function form.
- We don't really work with PDFs, rather, we work with the CDF of the random variable.

# The CDF of continuous RVs

- The CDF of a continuous random variable is a function that describes the probability that the random variable is less than or equal to a certain value.
- In this context, this can be understood as a function $F(x)$ that always gives you the area under the curve of the PDF up to a certain value $x$.
- Hence, the CDF is an integral of the PDF.

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

where $f(t)$ is the PDF of the random variable.

# Expected value of a continuous RV

- The expected value of a continuous random variable is calculated in the same way as for discrete random variables.
- We multiply each possible value of the random variable by its probability, and sum these products. However, this time, we use an integral instead of a sum.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

- Don't worry about these! We need not calculate these by hand. In an applied context, we just need to know what PDFs and CDFs are.

# Variance of a continuous RV

- The variance of a continuous random variable is calculated in the same way as for discrete random variables.
- We calculate the expected value of the squared difference between the random variable and its expected value. However, this time, we use an integral instead of a sum.

$$\mathsf{Var}(X) = \int_{-\infty}^{\infty} (x - E(X))^2 \cdot f(x) dx$$

# The normal distribution