Electromagnetic theory - Numerical Project

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1 Introduction

By using the dipole model one can numerically simulate the magnetic field around the Earth using the dipole model equation directly. Furthermore, by then solving the Lorentz equation one can model how a particle will move in Earth's magnetic field. By simulating the movement of particles in the magnetic field one can explain real life phenomena such as the northern light.

In this assignment the mentioned formulas have been implemented in a python program to simulate Earth's magnetic field and how particles move in it. The Runge-Kutta method of the 4th order has been used to solve the differential equations, and finally a validity test was conducted in order to test the validity of the method.

2 Theory

The magnetic field around Earth can be approximated by the dipole model, given by Eq. 1.

$$\mathbf{B}_{dip} = \frac{\mu_0}{4\pi r^3}(3(\mathbf{m}\cdot\hat{r})\hat{r} - \mathbf{m}) \tag{1}$$

The constant in this equation can be modified by considering the strength of the magnetic field at the equator. At the equator, the dot product between the magnetic moment $\mathbf{m}$ and the direction of $\mathbf{r}$ will be 0. In addition, since $\mathbf{m} = \hat{m}m_0$, m_0 can be pulled out of the parentheses and into the constant [1]. The resulting equation is Eq. 2.

$$\mathbf{B}_{dip} = B_0(3(\hat{m}\cdot\hat{r})\hat{r} - \hat{m}) \tag{2}$$

The value B_0 can be found online, however for this assignment I have chosen to set it equal to 1.

The x axis is defined to be from the Sun and to Earth, the y axis is the elliptic, and the z axis is then perpendicular to the elliptic. In addition, the Earth is tilted away from the Sun. In order to implement this, the direction of the dipole moment, which will decide the direction of the magnetic field, was calculated using the $\hat{r}$ components of spherical coordinates. These are given by Eq. 3.

$$\begin{aligned}
\hat{x} &= sin(\theta)cos(\phi)\hat{r} \\
\hat{y} &= sin(\theta)sin(\phi)\hat{r} \\
\hat{z} &= cos(\theta)\hat{r}
\end{aligned} \tag{3}$$

Since the magnetic moment will be in the xz plane, ϕ will be 0. θ is 23.4°, which is the angle by which the Earth is tilted [2]. Hence, $\hat{m}$ is directly defined by $\hat{x}$, $\hat{y}$ and $\hat{z}$ with these angles.

In order to simulate a particle moving in the magnetic field, the Lorentz equation has to be solved.

$$\mathbf{F}_{mag} = q(\mathbf{v}\times\mathbf{B}) \tag{4}$$

On differential form this is given by Eq. 5.

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{d\mathbf{v}}{dt} = \frac{q}{m}(\frac{d\mathbf{x}}{dt} \times \mathbf{B}) \tag{5}$$

The constant $\frac{q}{m}$ was set to 1 in the code. In order to solve this differential equation numerically the Runge-Kutta method of the 4th order was implemented.

3 Results

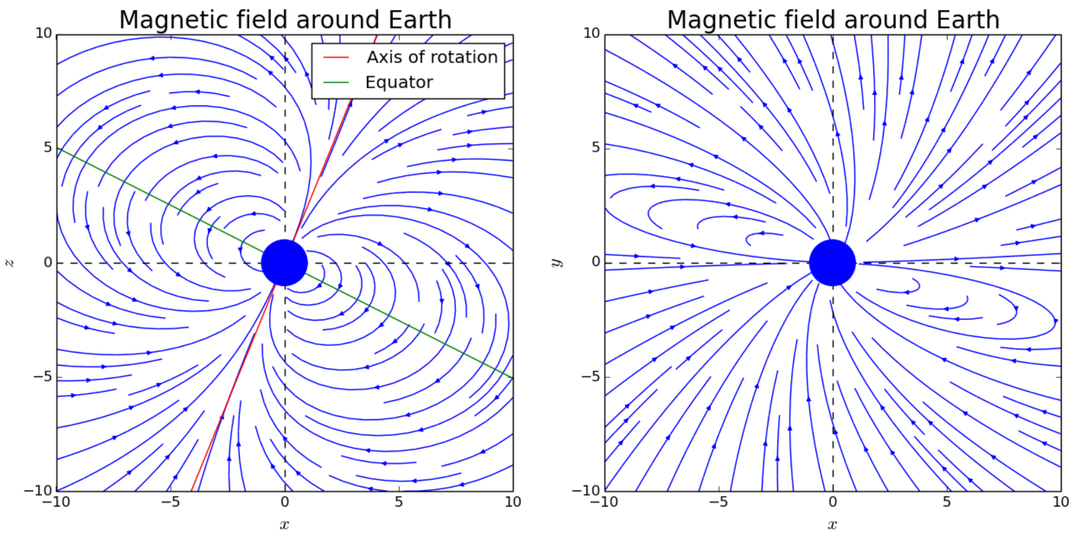

Figure 1: Numerical simulation of the magnetic field around Earth in the xz and xy plane.

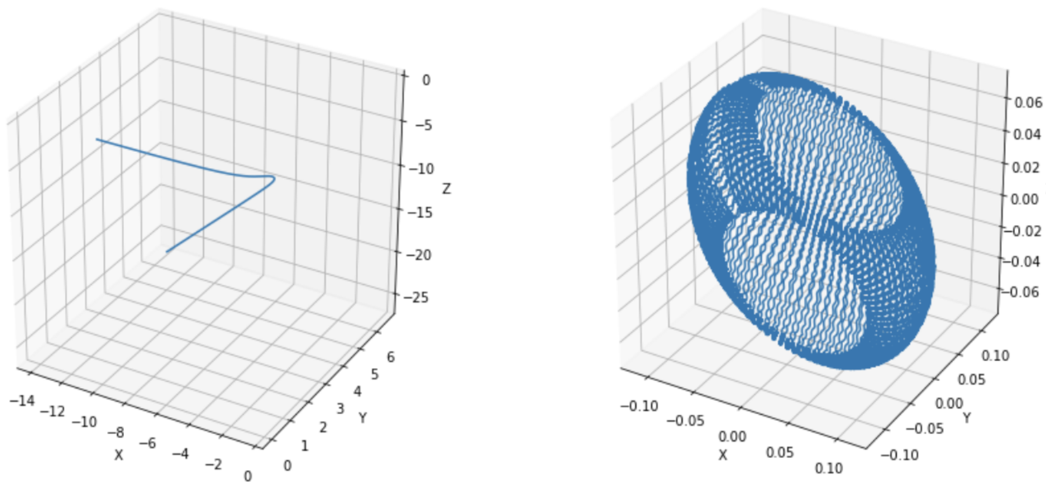

Figure 2: Movement of a particle in Earth's magnetic field. The first plot is of a particle with initial position far awar from Earth, and the second one is a particle with initial position close to Earth.

In Figure 1 the magnetic field in the xz and xy plane can be seen. The axial tilt is mainly visible in the xz plane, however as the tilt will be out of the xy plane, we can see that the magnetic field is not as circular, but skewed.

In Figure 2, the movement of two particles is plotted in 3D. The first plot represents a particle with an initial position far away from Earth. One can clearly see how the particle is deflected by the magnetic field. In the second plot, the particle oscillates between the poles of the Earth. Additionally one can note that the particle does not move in a straight line between the two poles, but in a helical motion. This is the motion expected in a uniform magnetic field.

4 Discussion of results

The results regarding the movement of a particle in Earth's magnetic field corresponds well with the expected behaviour. The deflection of the particle which was situated far away from Earth represents clearly how charged particles coming from the Sun are deflected when interfering with the magnetic field around Earth. In the plot showing the particles movement close to the Earth, it can both be seen how the particle moves between the poles, and how the particle move in a helical movement between the poles. This movement explain why the northern lights can be seen only at the poles.

5 Numerical validity

Since Lorentz law, Eq. 4, is perpendicular to the velocity of the particle, it will do no work on it. Thus, in order to test the numerical validity of the method used to solve equation, we just have to consider the kinetic energy. This has been done by finding the initial kinetic energy of a particle situated close to the Earth (see Figure 2), and then running the simulation for 50 different timesteps ranging from 10^{-7} to 10^{-1}. After the simulation, the kinetic energy of the last timestep was subtracted from the initial kinetic energy. The difference was added to an array which then was plotted against the different timesteps previously defined. The result can be seen in Figure 3 and in Figure 4 with a logarithmic scale. [3]

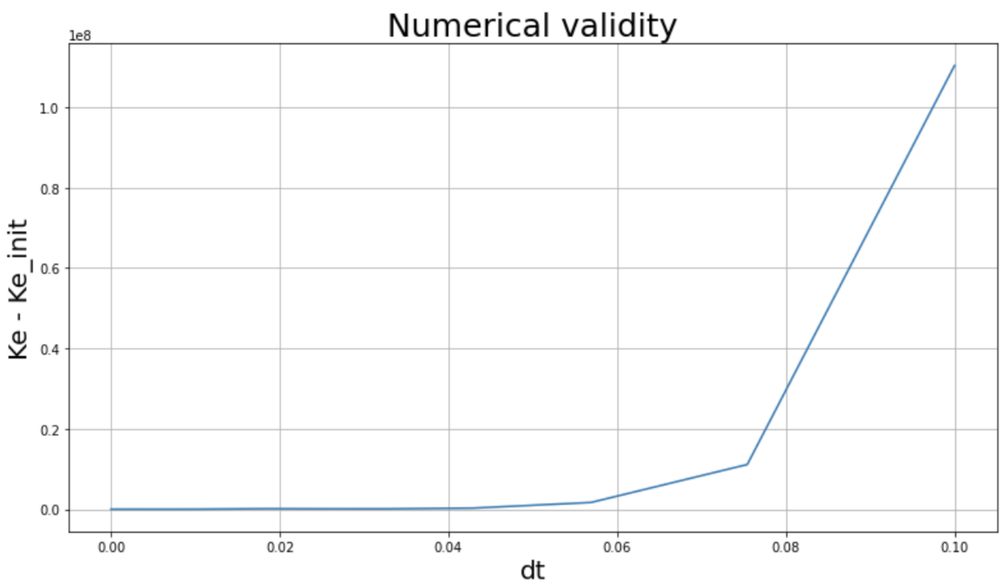

Figure 3: Timestep plotted against the deviation in kinetic energy.

Here it can clearly be seen that with a low enough timestep, the deviation is small enough to conclude that the Runge-Kutta method of the 4th order is a good method to solve the Lorentz equation.

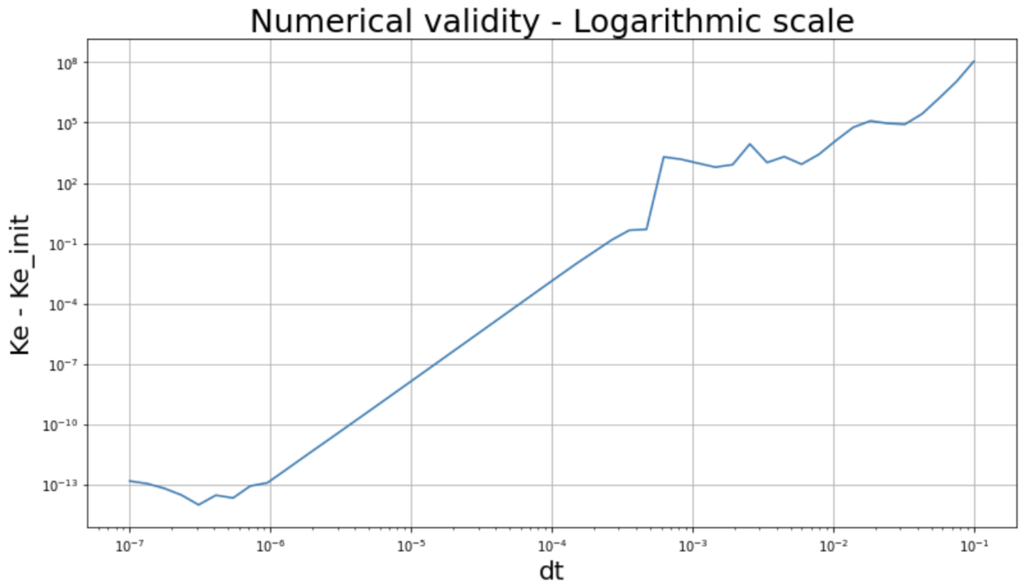

Figure 4: Timestep plotted against the deviation in kinetic energy with a logarithmic scale.

The logarithmic scale gives a better plot to see what happens at low timesteps, as the plot in Figure 3 quickly goes to 0 both for the timestep and the deviation in kinetic energy. This plot is further proof that the numerical model is a good one as the deviations in kinetic energy goes down to an order of 10^{13}.

6 Conclusion

By using the dipole model for the magnetic field of the Earth we get a resulting field which behaves as one would expect concerning both direction and shape. The movement of a particle in this field also yields results that correspond well to what we observe in real life.

7 References

References

[1] D.J Griffiths, *Introduction to Electrodynamics, 4th edition.* Cambridge University Press, 2017

[2] Wikipedia, *Axial tilt.* Available: $https : //en.wikipedia.org/wiki/Axial_{t}ilt$

[3] E.S Øyre, N.H Aase, T. Ballsetad, J.A. Støvneng, *Uniform Magnetic Field.* Available: $https : //nbviewer.org/urls/www.numfys.net/media/notebooks/uniform_{m}agnetic_{f}ield_{f}inal.ipynb$