TTK4190 Guidance and Control of Vehicles

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Problem 1 - Attitude Control of Satellite

Problem 1.1

The equilibrium point, $\mathbf{x}_0$, for the system:

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{T}_q(\mathbf{q})\boldsymbol{\omega} \\ \mathbf{I}_{CG}\dot{\boldsymbol{\omega}} - \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} &= \boldsymbol{\tau}\end{aligned} \tag{1}$$

with $\mathbf{x} = \begin{bmatrix}\varepsilon \\ \omega\end{bmatrix}$, can be found by setting $\dot{\mathbf{q}}$ and $\dot{\boldsymbol{\omega}}$ to zero. Using (2.77) from the textbook [1], the equations become:

$$\begin{aligned}\mathbf{0} &= \frac{1}{2}\begin{bmatrix}-\boldsymbol{\eta}^T \\ \boldsymbol{\eta}\mathbf{I_3} + \boldsymbol{S}(\boldsymbol{\varepsilon})\end{bmatrix}\boldsymbol{\omega} \\ \boldsymbol{\tau} &= -\mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega}\end{aligned} \tag{2}$$

Using the values $\mathbf{q} = [1,0,0,0]^T$ and $\boldsymbol{\tau} = 0$ and solving for $\boldsymbol{\omega}$ we get equilibrium point $[0,0,0,0,0,0]^T$.

The linearized system can then be found by differentiating (1) with respect to $\mathbf{x}$ and evaluating the Jacobian in the equilibrium point. The equation for $\dot{\boldsymbol{\eta}}$ is disregarded as the value of η at any time is given by $\boldsymbol{\varepsilon}$ in $\boldsymbol{\eta} = \sqrt{1-\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}}$.

Linearizing with respect to $\mathbf{x} = [\boldsymbol{\varepsilon}, \boldsymbol{\omega}]^T$ and using the product rule we get the following equations:

$$\begin{aligned}\frac{\partial}{\partial\boldsymbol{\varepsilon}}\dot{\boldsymbol{\varepsilon}} &= \frac{1}{2}\left(\frac{\partial}{\partial\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}\times\boldsymbol{\omega} + \boldsymbol{\varepsilon}\times\frac{\partial}{\partial\boldsymbol{\varepsilon}}\boldsymbol{\omega}\right) \quad = -\frac{1}{2}S(j\boldsymbol{\omega}) \\ \frac{\partial}{\partial\boldsymbol{\omega}}\dot{\boldsymbol{\varepsilon}} &= \frac{1}{2}(\eta\boldsymbol{I}_3 + S(\boldsymbol{\varepsilon}) \\ \frac{\partial}{\partial\boldsymbol{\varepsilon}}\dot{\boldsymbol{\omega}} &= \mathbf{0} \\ \frac{\partial}{\partial\boldsymbol{\omega}}\dot{\boldsymbol{\omega}} &= \mathbf{0}\end{aligned} \tag{3}$$

And then differentiating with respect to the input $\mathbf{u} = \boldsymbol{\tau}$ and using that $\mathbf{I_g} = mR_{33}^2$:

$$\frac{\partial}{\partial\boldsymbol{\tau}}\dot{\boldsymbol{\omega}} = \frac{1}{mR_{33}^2}\mathbf{I}_3 \tag{4}$$

Evaluating the expressions in the equlibrium point we get the following linearized system matrices:

$$\mathbf{A} = \begin{bmatrix}0&0&0&\frac{1}{2}&0&0\\0&0&0&0&\frac{1}{2}&0\\0&0&0&0&0&\frac{1}{2}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix} \quad \mathbf{B} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\\\frac{1}{mR_{33}^2}&0&0\\0&\frac{1}{mR_{33}^2}&0\\0&0&\frac{1}{mR_{33}^2}\end{bmatrix}$$

Problem 1.2

The attitude control law is given by:

$$\boldsymbol{\tau} = -\mathbf{K}_d\boldsymbol{\omega} - k_p\boldsymbol{\varepsilon} = \begin{bmatrix}-k_p\mathbf{I}_3 & -\mathbf{K}_d\end{bmatrix}\begin{bmatrix}\boldsymbol{\varepsilon} \\ \boldsymbol{\omega}\end{bmatrix} = -\mathbf{K}\mathbf{x} \tag{5}$$

To evaluate the stability of the closed-loop system, we calculate the eigenvalues. The closed loop system equations become:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \tag{6}$$

We use MATLAB to calculate the eigenvalues of the matrix $\mathbf{A} - \mathbf{BK}$:

$$\boldsymbol{\lambda}_{\mathbf{1,2}} = -0,0278 \pm 0,0248i \tag{7}$$

The real parts of the poles are negative, so the linearized system is stable.

For a satellite, we would want to use as little energy as possible, and we don't mind if the system is slow since we see no cases were a quick response is necessary. Therefore, we would want complex poles and a small radius to use as little energy as possible. Complex poles lead to a larger overshoot, so the angle should not be too wide. Another reason to keep the radius small is to make the system less noise sensitive.

Problem 1.3

We use the control law (5) with $k_p = 2$ and $k_d = 40$ and simulate the system with initial conditions:

$$\Theta_0 = \begin{bmatrix} -5^\circ \\ 10^\circ \\ -20^\circ \end{bmatrix} \tag{8}$$

We have small radius for the imaginary poles and therefore expect small control input. The angles are expected to approach zero. In figures 1-3, we see that these expectations are somewhat met. As seen in Figure 1, the control input is small, and we see in Figure 2 that the Euler angles approach zero, although slowly. This coincides with the fact that the angular velocities are small, as seen in Figure 3.The slow response in the Euler angles could be due to small k_d and k_p values and because the reference for the angular velocities is implicitly zero.

To make the system follow nonzero reference signals, add a reference-feedforward term to the control law.

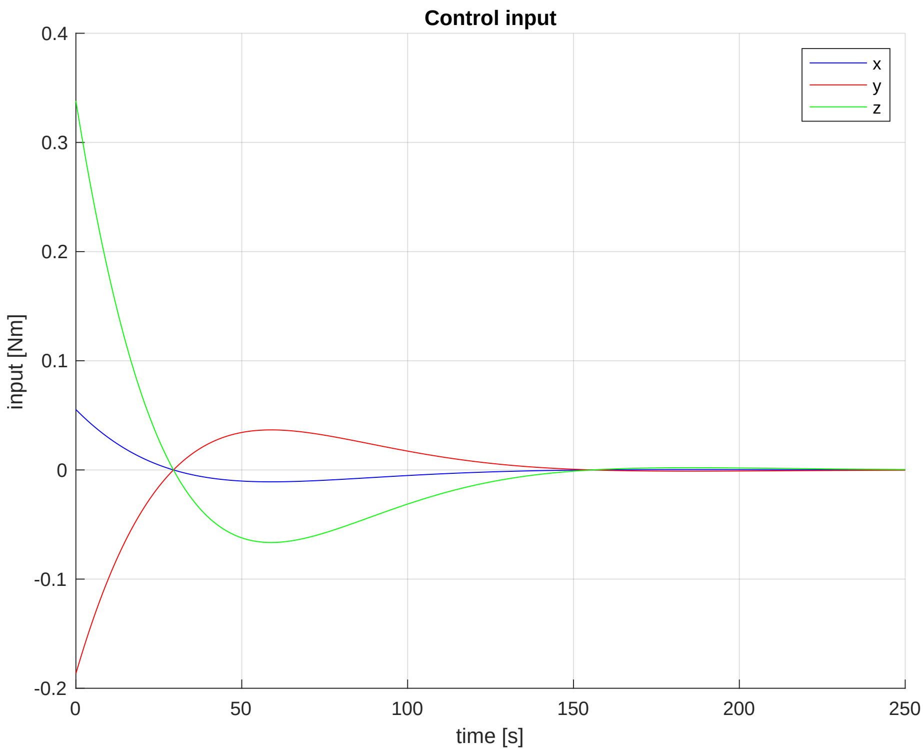

Figure 1: Control Input using control law (5)

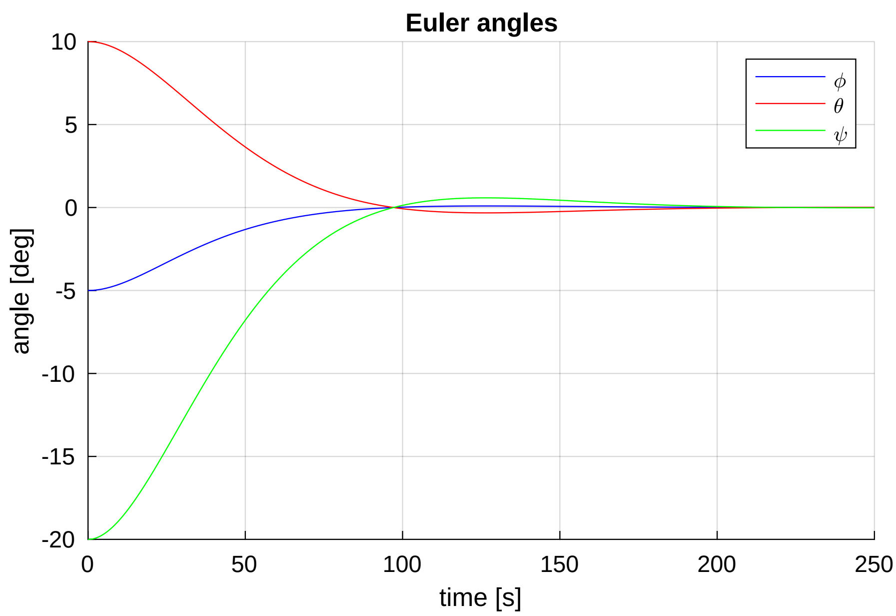

Figure 2: Euler Angles with control law (5)

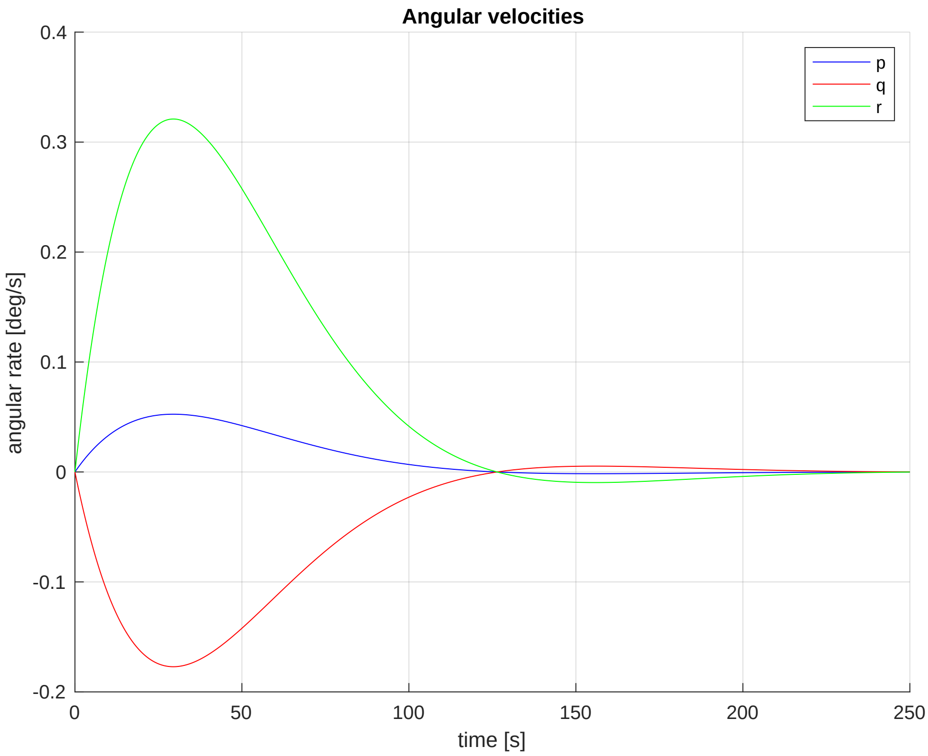

Figure 3: Euler Angular Velocities with (5)

Problem 1.4

The quaternion error can be written as

$$\tilde{\mathbf{q}} := \begin{bmatrix} \tilde{\eta} \\ \tilde{\varepsilon} \end{bmatrix} = \bar{\mathbf{q}}_d \otimes \mathbf{q} = \begin{bmatrix} \bar{\eta}_d \eta - \bar{\varepsilon}_d^T \varepsilon \\ \bar{\eta}_d \varepsilon + \eta \bar{\varepsilon}_d + S(\bar{\varepsilon}_d)\varepsilon \end{bmatrix} \tag{9}$$

After convergence, when $\mathbf{q} = \mathbf{q_d}$, $\tilde{\mathbf{q}}$ becomes $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$ as shown in (10). We observe that the error converge to zero, which is what we want.

$$\tilde{\mathbf{q}} = \begin{bmatrix} \eta\eta + \varepsilon^T \varepsilon \\ \eta\varepsilon - \eta\varepsilon - S(\varepsilon)\varepsilon \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \tag{10}$$

Problem 1.5

For this problem, we simulate the system with the modified control law,

$$\boldsymbol{\tau} = -\mathbf{K}_d \boldsymbol{\omega} - k_p \tilde{\boldsymbol{\varepsilon}} \tag{11}$$

, with $k_p = 20$ and $k_d = 400$, initial conditions (8) and the time variying reference signal $\mathbf{q}_d$ corresponding to

$$\boldsymbol{\Theta}_d = \begin{bmatrix} 0 \\ 15\cos(0,1t) \\ 10\sin(0,05t) \end{bmatrix} \tag{12}$$

.

With this control law, we see that the attitude, figure Figure 5, attempts to follow the non-zero reference signal as expected. Although, we observe that the angle error, Figure 7, is quite large. This is due to the same issue as in Problem 1.3, with the reference value for the angular velocities, Figure 6 being implicitly set to zero. In addition, we observe that the input values, Figure 4, are about 10 times larger, which coincides with the controller values being 10 times larger.

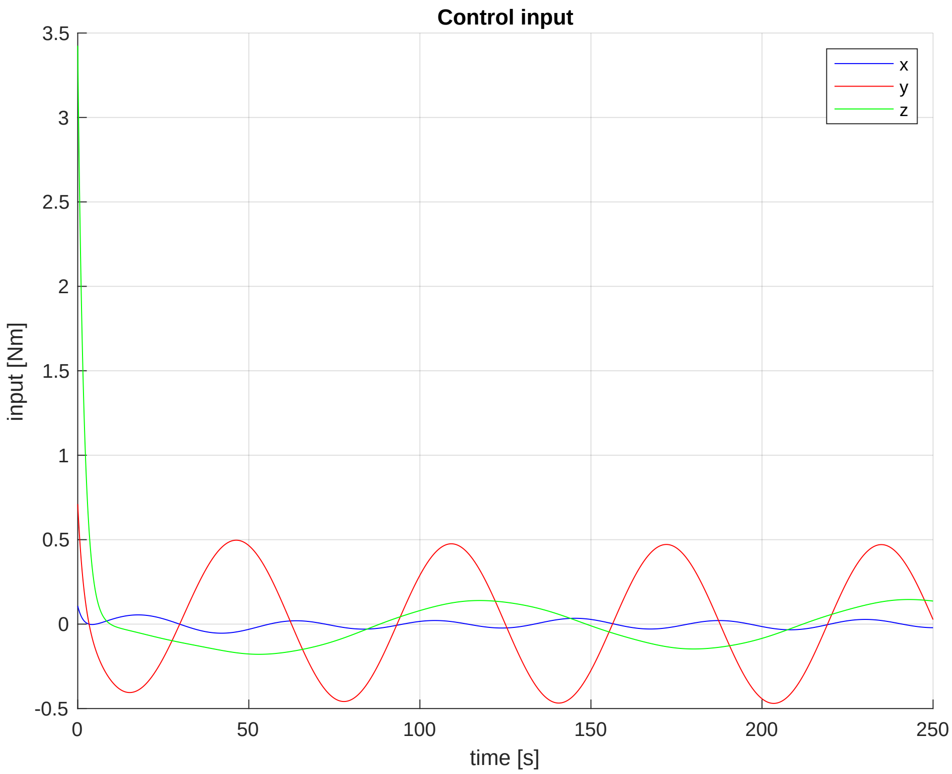

Figure 4: Control Input using control law (11)

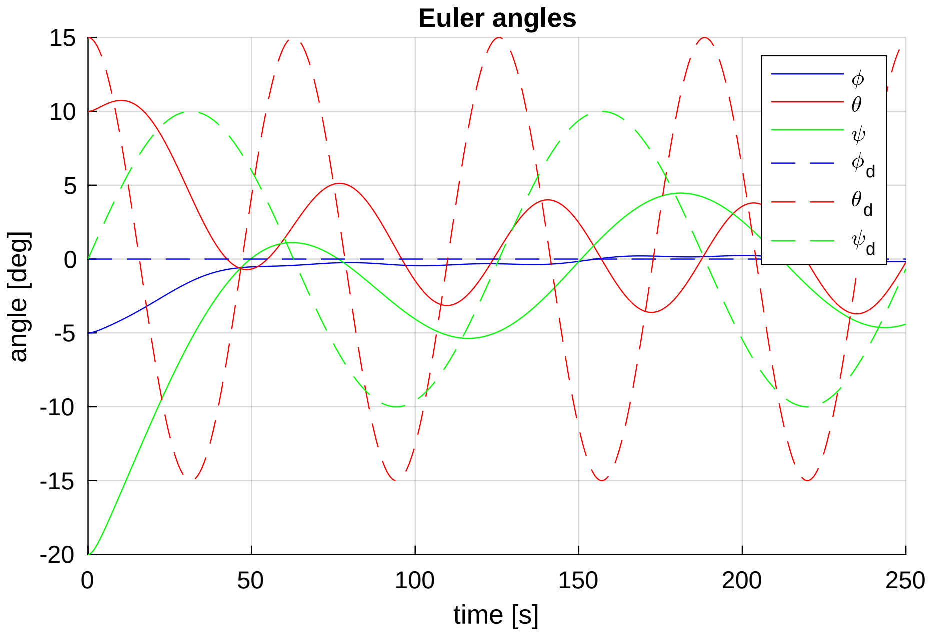

Figure 5: Euler Angles with control law (11)

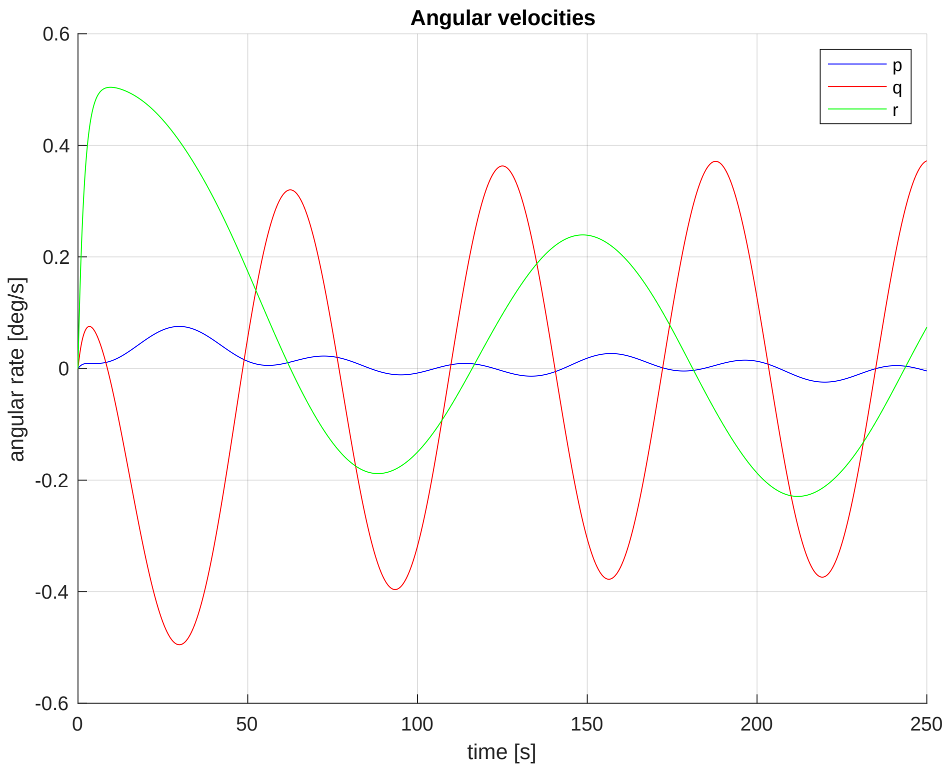

Figure 6: Euler Angular Velocities with (11)

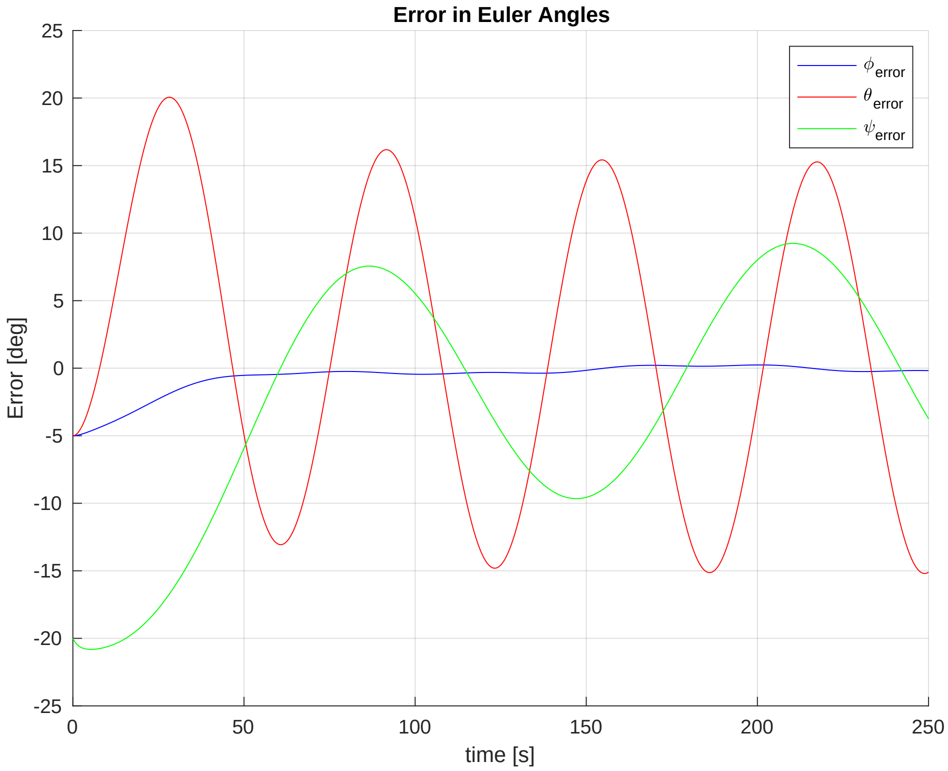

Figure 7: Error euler angles with control law (11)

Problem 1.6

The control law in this problem can be written as

$$\boldsymbol{\tau} = -\mathbf{K}_d\tilde{\boldsymbol{\omega}} - k_p\tilde{\boldsymbol{\varepsilon}} \tag{13}$$

and the desired angular velocity as

$$\boldsymbol{\omega}_d = \mathbf{T}^{-1}_{\Theta_d}(\Theta_d)\dot{\Theta}_d \tag{14}$$

$$\dot{\boldsymbol{\Theta}}_d = \begin{bmatrix} 0 \\ -1,5\sin(0,1t) \\ 0,5\cos(0,05t) \end{bmatrix} \tag{15}$$

We see that the attitude, Figure 9, follows its trajectory much better than in the last task. This is probably due to the new element of (13) that gives the angular velocities non-zero references to follow. The angular velocities in Figure 10 are much larger than in Figure 6 and this allows the system to move more freely and follow the attitude trajectories better. The control input, Figure 8, is larger than in the previous task, probably due to the non-zero reference for the angular velocity. Still, we see that the angle error, Figure 11, is non-zero. A way to improve the control law is to add a reference feedforward term for the angular acceleration as well.

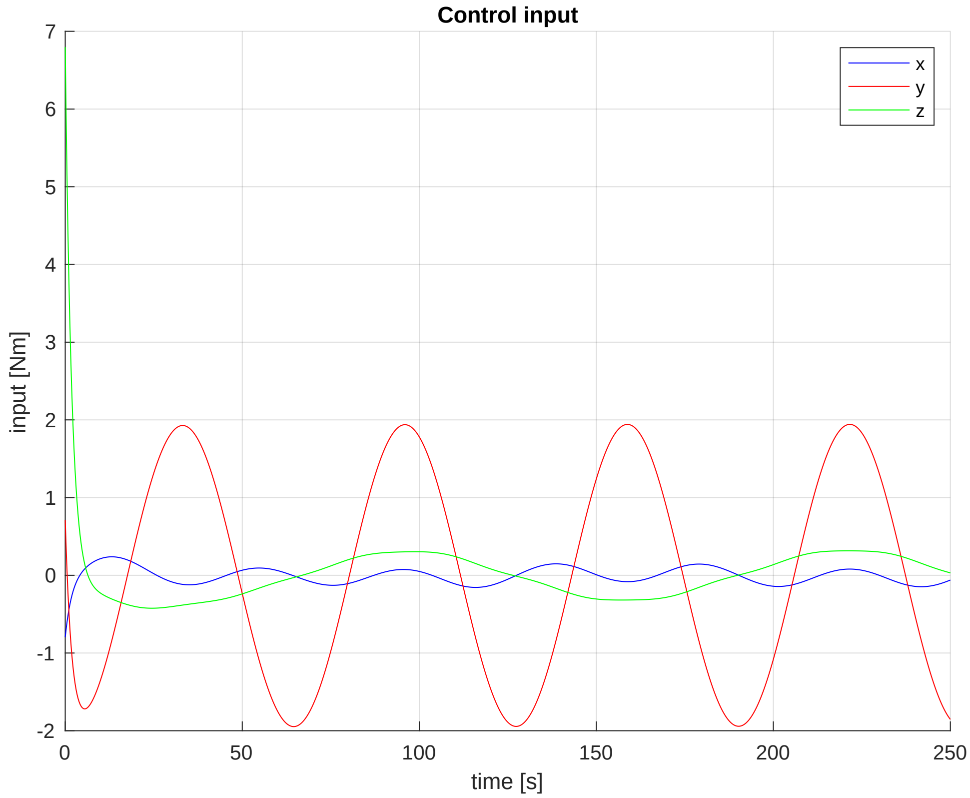

Figure 8: Control Input using control law (13)

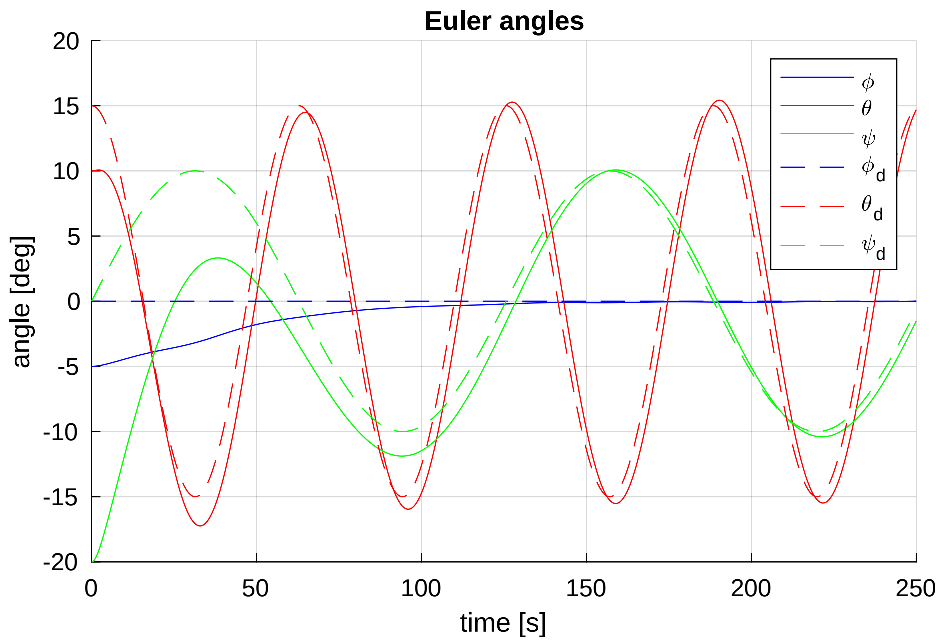

Figure 9: Euler Angles with control law (13)

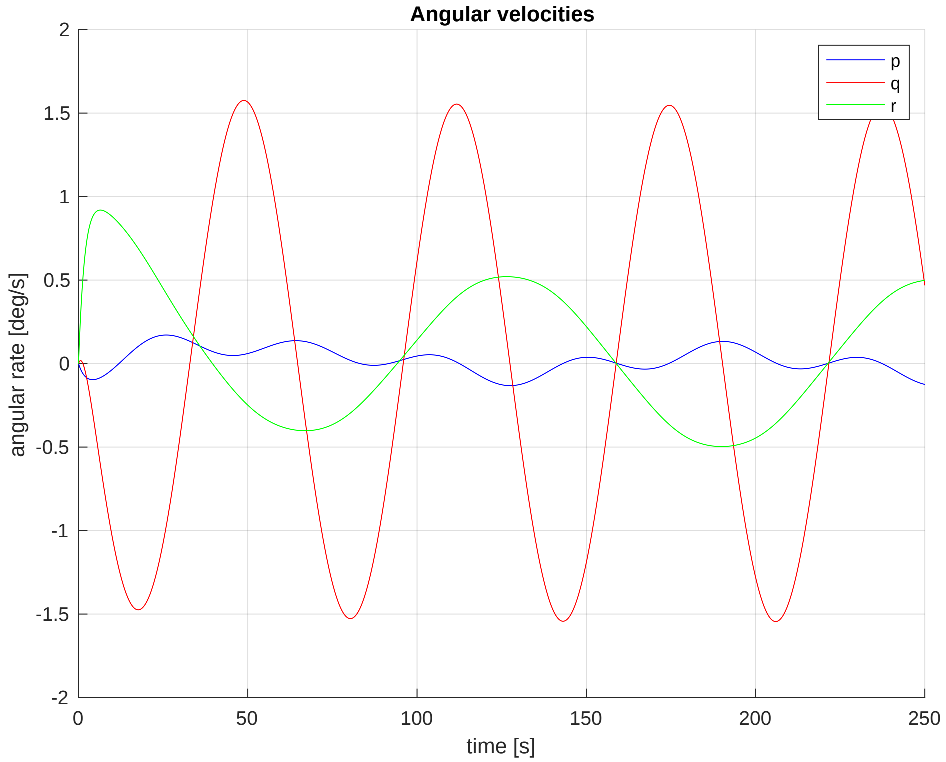

Figure 10: Euler Angular Velocities with (13)

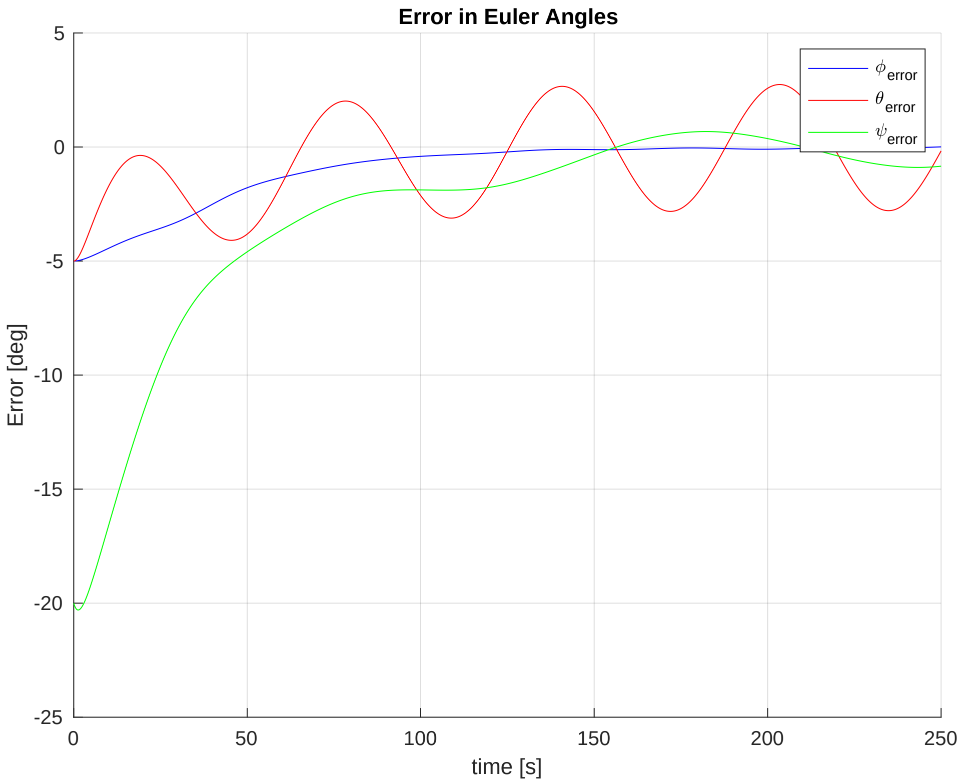

Figure 11: Error euler angles with control law (13)

Problem 1.7

The Lyapunov function can be written as

$$V = \frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\tilde{\boldsymbol{\omega}} + 2k_p(1-\tilde{\eta}) \tag{16}$$

and the derivative as

$$\dot{V} = -\mathbf{K_d}\boldsymbol{\omega}^\top\boldsymbol{\omega} \tag{17}$$

By analyzing the different parts of V, it can be shown that it is positive and radially unbounded. $\frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\tilde{\boldsymbol{\omega}}$ is quadratic and therefore positive. Since $\eta = \sqrt{1-\varepsilon\varepsilon^\top}$, then η is between $0 < \eta < 1$ and $\tilde{\eta}$ $0 < \eta < 1$. This results in $\tilde{\eta} \leq 1$, and the last term also being positive. To check if it is radially undbounded, let $\lim_{\mathbf{x}\to\infty} V(\mathbf{x})$. This will go to ∞, since the quadratic term will go to infinity as ω does, and $\mid \eta \mid \leq 1$.

To prove (17), we derive the expression for $\dot{V}$:

$$\begin{aligned}\dot{V} &= \boldsymbol{\omega}^\top \mathbf{I}_{CG}\dot{\boldsymbol{\omega}} - 2k_p\dot{\eta} \\ &= \boldsymbol{\omega}^\top \mathbf{I}_{CG}\mathbf{I}_{CG}^{-1}(\mathbf{S}(\mathbf{I_{CG}}\boldsymbol{\omega})\boldsymbol{\omega} + \tau) + k_p\tilde{\boldsymbol{\varepsilon}}^\top\tilde{\boldsymbol{\omega}} \\ &= \boldsymbol{\omega}^\top\tau + k_p\boldsymbol{\varepsilon}^\top\boldsymbol{\omega} \\ &= \boldsymbol{\omega}^\top(-\mathbf{K_d}\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\epsilon}}) + k_p\tilde{\boldsymbol{\varepsilon}}^\top\boldsymbol{\omega} \\ &= -\mathbf{K_d}\boldsymbol{\omega}^\top\boldsymbol{\omega}\end{aligned} \tag{18}$$

To prove that the equilibirum point of the closed loop system is asymptotically stable, Lyapunov's direct method was used.

$$\dot{V}(x) = -\begin{bmatrix}\tilde{\mathbf{q}}^\top & \tilde{\boldsymbol{\omega}}^\top\end{bmatrix}\begin{bmatrix}0 & 0 \\ 0 & \mathbf{K_d}I\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{q}} \\ \tilde{\boldsymbol{\omega}}\end{bmatrix} \tag{19}$$

From this it can be observed that $\dot{V}(x)$ is negative semi-definite, resulting in a stable system. But this is not enough to prove asymptotic stability.

To get a definite answer, La Salle's theorem was needed. Let $x^* = \left[\tilde{\mathbf{q}}^* \tilde{\boldsymbol{\omega}}^*\right] = \left[\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^\top \mathbf{0}\right]$. Then the set $\Omega = \left\{\tilde{\boldsymbol{\omega}} = 0, \tilde{\mathbf{q}} \in \mathbf{S^3}\right\}$, is the set of values of $\mathbf{x}$ where $\dot{V}$ is zero.

$$
\begin{aligned}
\dot{\tilde{\boldsymbol{\omega}}} &= \mathbf{I}_{CG}^{-1}(\mathbf{S}(\mathbf{I_{CG}}\boldsymbol{\omega})\tilde{\boldsymbol{\omega}} + \tau) \\
&= \tau \\
&= -\mathbf{K_d}\boldsymbol{\omega} - k_p\tilde{\boldsymbol{\varepsilon}} \\
&= -k_p\tilde{\boldsymbol{\varepsilon}}
\end{aligned} \tag{20}
$$

For the trajectory to stay inside Ω, $\dot{\tilde{\boldsymbol{\omega}}}$ has to be equal to 0. The only solution is $\tilde{\boldsymbol{\varepsilon}} = 0$, giving us the set $M = \left\{\tilde{\boldsymbol{\omega}} = 0, \tilde{\boldsymbol{\varepsilon}} = 0\right\} = \mathbf{x}^*$. Since the largest invariant set in Ω is equal to the equilibrium point, we can conclude that the closed-loop system is asymptotically stable.

Since the state of the system is defined using quaternions, we cannot conclude that the system is GAS. Since the quaternions have double-cover of the SO(3), we can only conclude that the system is locally asymptotically stable.

References

[1] T. Fossen, *Handbook of Marine Craft Hydrodynamics and Motion Control.* John Wiley & Sons, 2011.