SIAM J. COMPUT.
Vol. 16, No. 5, October 1987

© 1987 Society for Industrial and Applied Mathematics
010

ON THE OPTIMAL LAYOUT OF PLANAR GRAPHS WITH FIXED BOUNDARY*

B. BECKER† AND G. HOTZ†

Abstract. The optimal planar layout of planar graphs with respect to the L_1- or L_2-metric leads to NP-hard problems, if one assumes the nodes of the graph to be fixed in the plane (see [FiPa], [Be]).

In this paper we consider the (optimal) layout of graphs with fixed boundary (i.e., graphs, where only the nodes of a given cycle of the graph have fixed positions in the plane). The investigated layouts are straight line embeddings in a continuous part of the plane; the cost of a layout is calculated with help of very general cost functions including the pth power of the usual Euclidean distance metric for $p = 2, 3, \cdots$ (for short, l_p-metric).

For a large class of graphs, which, for example, occur in chip layout problems as the abstract structure of switching circuits, we show the existence and uniqueness of the optimal layout.

The main part of the paper is concerned with planar graphs. We get an interesting characterization of nonplanar layouts of planar graphs, which shows that the optimal layout of a planar graph is planar or at least "quasiplanar." This property makes it possible to decompose the general layout problem into two independent problems:

(i) Find a layout of a circuit that is not necessarily planar, but that has an "allowed crossing behaviour."

(ii) Fix the crossing points and then optimize the layout. Our theorems show that no new contacts (crossing points) will be generated.

In the Appendix we outline some (efficient, polynomial time) methods for the construction of optimal layouts and give as an example the optimal layouts of a recursively defined n-bit adder and multiplier.

Key words. graph with fixed boundary, l_p-metric, (optimal) layout, (quasi) planar, star, convex point

AMS(MOS) subject classifications. 05C10, 57M15, 68C05

1. Introduction. In this paper we develop the concept of "graph with fixed boundary" and present some results on the (optimal) layout of those graphs in the Euclidean plane (with respect to a given cost function).

A graph is a commonly used model to represent the abstract structure of a switching circuit (nodes of the graph equal modules (function blocks or basic cells) of the circuit, edges of the graph equal signal nets interconnecting the modules). If this circuit is to be realized on a (VLSI-) chip, there are some modules, the input- and output-pins, which have to be located on the boundary of the chip area. In our model we therefore assume that a cycle of the graph is given and embedded in the plane which corresponds to the boundary of the chip and whose nodes are identified with the i/o-pins of the circuit. We call this structure "graph with fixed boundary." So a graph with fixed boundary models the abstract structure of a circuit together with topological constraints given by the boundary of the chip and fixed i/o-ports.

For the actual construction of a circuit on a chip it is necessary to place the (remaining) modules of the circuit on the chip area, too, and to route the wires interconnecting the modules according to given design and optimization criteria. These problems are well known in the area of design automation as *placement and routing problems* [Br]. A layout of a graph with fixed boundary in the Euclidean plane within the area prescribed by the fixed boundary may be interpreted as a first placement and

---

* Received by the editors January 2, 1984; accepted for publication (in revised form) December 23, 1986.
† Fachbereich 10, Universität des Saarlandes, D-6600 Saarbrücken, West Germany.

global routing of the corresponding circuit. This is done (and the results of this paper are used) in a CAD-system for the design of integrated circuits being developed at the Universität des Saarlandes. (For detailed information see [HBKM].)

We now give a brief review of other results and methods for placement and routing, in order to relate them to our approach. As pointed out at the beginning, the graph-theoretic model is commonly adopted. Placement and routing then mainly consist of constructing a good layout with respect to a given measure. There are many goals which have to be considered:

- Chip area should be minimized.
- Routing must be possible (according to given design rules).
- Testing has to be possible.
- Heat dissipation levels must be preserved, etc. $\cdots$.

It seems impossible to directly incorporate all these goals into an algorithm, but minimizing a total weighted edge length has turned out to best reflect all the goals and therefore is widely used (see [Br]).

We consider cost functions including the pth power of the usual Euclidean distance metric. Minimizing with respect to the square of the Euclidean length reflects the following idea: the modules are joined together by lines of force, similar to springs or rubber bands, that attract the modules to one another. We then construct an optimal layout (with respect to the l_2-metric), i.e., a layout where the force on each module (not on the boundary) is equal to 0. This is done to produce a balanced distribution of all modules on the chip area. In existing design automation systems placement heuristics are often used which are based on similar ideas (see e.g., "force-directed placement" in [Fi]). We also consider l_p-optimal layouts, which yield an even more balanced distribution and converge to a layout minimizing the longest wire (see [BeOs]).

Today's design automation systems use heuristics (as indicated above) since it has turned out during the last years that many problems in this area are NP-hard (see [SaBh], [Do]). Nevertheless we can construct optimal layouts for a graph with fixed boundary in polynomial time because of the following:

- Layouts are not considered in a discrete part of the plane (=grid) as, for example, in [Sto], but in an area of the plane given by a fixed boundary. The fixed boundary together with the i/o-pins of the circuit function as a "frame spanning the remaining nodes and edges." If the graph is not spanned by the boundary (i.e., the inputs and outputs are not fixed on the boundary) and the nodes are only allowed to be embedded on different grid points inside the boundary, the construction of the optimal layout corresponds to the solution of an NP-complete problem, the "Quadratic Assignment Problem" [SaGo]. On the other hand, the optimal layout for the graph with fixed boundary can be constructed in time less than $O(n^3)$ (n = number of nodes), as is shown in the Appendix.

- The locations of the modules not on the fixed boundary are freely moveable on the chip area. If all locations are fixed in the plane, the problem of finding a planar layout of minimal total weighted edge length with respect to the L_1- or L_2-metric is NP-hard (see [FiPa], [Be]). In our model we show as a main result of this paper that an optimization of a given planar graph does not destroy planarity (i.e., the placement algorithm does not destroy topological properties of the circuit, which have been transferred to the placement and routing phase from the logical topological level of the system (see [BHKM])).

- The resulting optimal layouts have wires running in any direction of the plane. This is not possible in current technologies; however, efforts to generalize the concept of strictly horizontal or vertical wires can be observed (see e.g., cover and preface in

[Bry]). On the other hand, some work has been done to embed the resulting optimal layouts in a grid [Schm].

There is no doubt that a graph with fixed boundary is a very idealized model of a real circuit. In order to construct a realistic physical circuit the optimal layout of the idealized graph has to be transformed into a layout, which realizes the edges and nodes by geometric configurations of given dimensions and thus separates edges and nodes according to given design rules. This can be done with help of an algorithm called "logarithmic pumping," which assigns the necessary configurations to nodes and edges on one side and minimizes the layout area in an iterative process on the other side. For detailed information see [BHKM] and [Schw].

In this paper we restrict ourselves to the study of "graphs with fixed boundary" and their layouts.

We finish the Introduction by outlining the remainder of the paper: First we give the basic definitions and show the existence and uniqueness of the optimal layout of a graph with fixed boundary with respect to a general cost function f including the l_p-metric for $p=2, 3, \cdots$. (In the case $p=1$ we still have the existence but not the uniqueness of the optimal layout.) The main part of the paper is concerned with connected planar graphs with fixed boundary and their layouts. We give a classification of layouts and show that some forms of layouts can never be optimal: Nonplanar layouts of planar graphs, which are not even "quasiplanar," have "convex points," but this contradicts optimality. This result is first proved for triangulated graphs (Lemma 6) and then generalized to planar graphs (Lemmas 7 and 8). Although the lemmas are intuitively clear, their proof is rather lengthy. It requires an exact classification of nonplanarity (Lemmas 2, 3 and 5) and finishes with the calculation of an invariant (Lemma 4). In a last part we show that the optimal layout of triconnected planar graphs is not only quasiplanar but also planar. This is done mainly with help of a technical lemma, the proof of which uses the properties of the cost function f together with analytical tools. In the Appendix we introduce methods to construct an optimal layout and present pictures of the optimal layout of a recursively defined adder and multiplier.

2. Basic definitions. Let $G=(V, E)$ be an undirected simple graph.

A *layout* χ of G in the plane is given by a mapping $\chi: V \to \mathbb{R}^2$: i.e., the edges $e=(v_1, v_2)$ of G are mapped on the straight line segments $\overline{\chi(v_1)\chi(v_2)} =: \chi(e)$. (Note that in this general definition different nodes may be mapped on the same point in the plane.)

$G_{\rho(c)} := (G, c, \rho)$ is called *a graph with fixed boundary*, iff the following hold:

- $G=(V, E)$ is a (undirected, simple) graph with $V = V_{\text{in}} \dot\cup V_c$.
- $c=(V_c, E_c)$ is a nontrivial simple cycle of G.
- $\rho: V_c \to \mathbb{R}^2$ is a layout of c, which maps c onto a convex polygon $\rho(c)$.

$$C_{\rho(c)} := \left\{ x \in \mathbb{R}^2 \,|\, x = \sum_{v \in V_c} \lambda_v \cdot \rho(v) \text{ with } \lambda_v \in [0, 1], \sum_{v \in V_c} \lambda_v = 1 \right\}$$

is a convex set, called the *convex closure* of $\rho(c)$ and denotes $\rho(c)$ together with its interior.

A layout χ of G, denoted by $\chi(G)$, is called $\rho(c)$-respecting, iff $G_{\rho(c)} = (G, c, \rho)$ is a graph with fixed boundary and $\chi|c = \rho$, i.e., the cycle c is mapped on $\rho(c)$ in the layout χ. We give an example of a $\rho(c)$-respecting layout in Fig. 1.

In the following we investigate

$$L(G, c, \rho) := \{\chi \,|\, \chi \text{ is a } \rho(c)\text{-respecting layout of } G\},$$

the *set of* $\rho(c)$*-respecting layouts of* G.

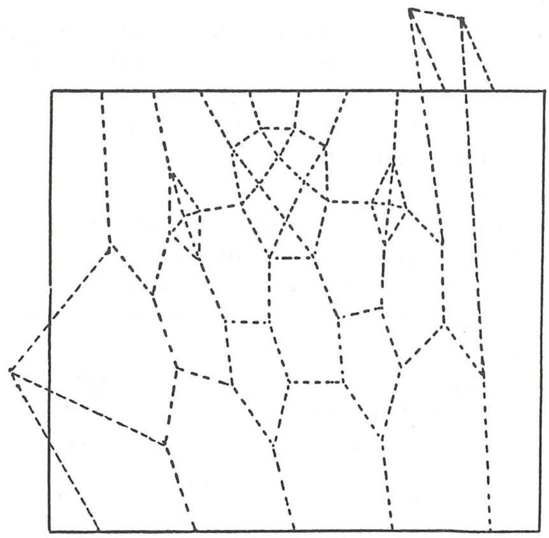


FIG. 1. —— *denotes the fixed boundary*; ---- *denotes the remaining edges.*

Remark. If we have $V_{\text{in}} = \{v_1, v_2, \cdots, v_n\}$ we get the following bijection:

$$\begin{array}{ccl} L(G, c, \rho) & \to & \qquad \mathbb{R}^{2n} \\ \chi & \to & (x_1, y_1, x_2, y_2, \cdots, x_n, y_n) \\ & & \text{with } (x_i, y_i) := \chi(v_i) \\ & & \text{for } i = 1, \cdots, n. \end{array}$$

We therefore identify a $\rho(c)$-respecting layout $\chi(G)$ with a point in $\mathbb{R}^{2n}$ and vice versa.

Next we define some special kind of $\rho(c)$-respecting layouts, where all nodes are placed inside the area prescribed by the boundary. (These layouts "correspond" to layouts of a given circuit on a given chip area.)

A layout $\chi(G)$ is called $\rho(c)$-*bounded*, iff χ is $\rho(c)$-respecting and $\chi(V) \subset C_{\rho(c)}$. We give an example in Fig. 2.

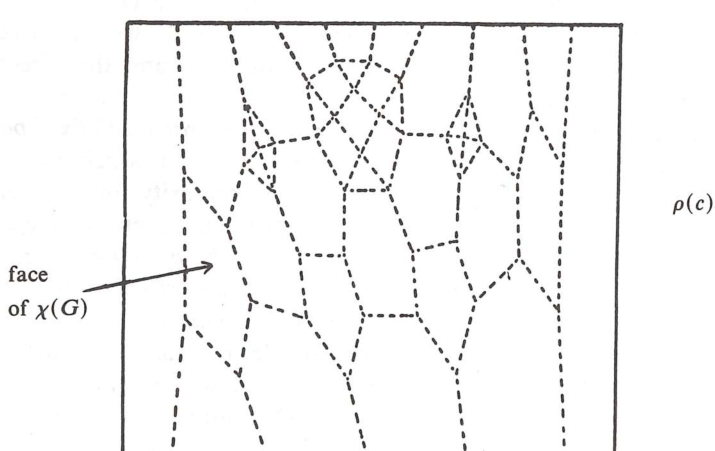


FIG. 2

In the example of Fig. 2 we have a planar embedding inside $\rho(c)$. (Different nodes are mapped on different points, different edges are mapped on disjoint straight line segments (up to common endpoints).)

Henceforth we call a layout $\chi(G)$ *$\rho(c)$-planar,* iff χ is $\rho(c)$-bounded and planar.

A graph G with fixed boundary $\rho(c)$ is called *$\rho(c)$-planar,* iff there exists a layout $\chi(G)$, which is $\rho(c)$-planar.

Remark. There does not exist a $\rho(c)$-planar layout for each nontrivial cycle c of a planar graph. This easily follows from the uniqueness theorem for planar embeddings of triconnected planar graphs [Ho]. Furthermore, one sees from the construction given in [Ho] which cycles c of a planar graph yield planar embeddings and how many "topologically" different layouts exist.

Let $\chi(G)$ be a $\rho(c)$-planar layout of a biconnected graph G. "Removal of $\chi(e)$ for all edges e of G from the plane" splits the plane into connected open regions. Exactly one region ("the outer region") is unbounded. A finite region together with its boundary is called a *face* of $\chi(G)$. Let F be a face, then int (F) denotes the interior of F and ∂F the boundary of F. (See Fig. 2.)

A layout $\chi(G)$ of a biconnected graph G is called $\rho(c)$-planar convex, iff χ is $\rho(c)$-planar and all faces of $\chi(G)$ are convex regions.

We will need these definitions later when we try to find minimal elements of $L(G, c, \rho)$ with respect to certain cost functions. The cost of a layout $\chi(G)$ is defined as follows: Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be nondecreasing and continuous, then $|\chi|_f := \sum_{e\in E} f(|\chi(e)|)$, where $|\chi(e)| :=$ Euclidean length of $\chi(e)$ is the *cost of χ with respect to the cost function f.* In the case $f(x)=x^p\,(p=2,3,\cdots,)$ we call $|\cdot|_f$ l_p-metric. We try to find a layout χ_{opt} with $|\chi_{\text{opt}}|_f=\inf\{|\chi|_f \mid \chi\in L(G,c,\rho)\}$. χ_{opt} is then called *optimal layout* of G with respect to $\rho(c)$ and f.

3. Existence and uniqueness of the optimal layout. We first show the existence of an optimal layout χ_{opt} with respect to a given cost function f and will then see that with some additional assumptions for G and f χ_{opt} is unique.

Let $G'=(V',E')$ be a component of G. If $V'\cap V_c=\emptyset$, then an optimal layout $\chi(G')$ is given by $\chi(v')=(x,y)\in\mathbb{R}^2$, $\forall v'\in V'$. If $V'\supset V_c$, we conclude with the help of the monotony of f and the convexity of $C_{\rho(c)}$.

Remark. Let $\chi(G')$ be a $\rho(c)$-respecting layout which is not $\rho(c)$-bounded. Then there exists a $\rho(c)$-bounded layout $\bar\chi(G')$ with $|\chi(G')|_f\geqq|\bar\chi(G')|_f$.

This means that if there exists an optimal layout of G, we may assume that it is $\rho(c)$-bounded. Together with the compactness of $C_{\rho(c)}$ and the bijection between $L(G,c,\rho)$ and $\mathbb{R}^{2n}$ we get the following theorem.

THEOREM 1 (Existence). *Let $G_{\rho(c)}=(G,c,\rho)$ be a graph with fixed boundary. Then there exists an optimal layout χ_{opt} with respect to $\rho(c)$ and f, which is $\rho(c)$-bounded.*

If we assume connectedness for G and strict convexity for f (as is the case for $f(x)=x^p$ with $p\geqq 2$, but not for $f(x)=x$), we get the uniqueness of χ_{opt}.

THEOREM 2 (Uniqueness). *Let $G_{\rho(c)}=(G,c,\rho)$ be a connected graph with fixed boundary and $f:\mathbb{R}^+\to\mathbb{R}^+$ a continuous, nondecreasing and strictly convex function. Then the optimal layout of G with respect to $\rho(c)$ and f is unique.*

Proof of Theorem 2. For the proof we consider the cost $|\cdot|_f$ and show that $|\cdot|_f$ is strictly convex on $\mathbb{R}^{2n}$. With the help of Theorem 1 we then get the following: The unique minimum of $|\cdot|_f$ defines the optimal $\rho(c)$-bounded layout χ_{opt}.

We now give the exact proof. Let $G=(V,E)$ with $V=V_{\text{in}}\,\dot\cup\, V_c$ and $V_{\text{in}}=\{v_1,\cdots,v_n\}$, $V_c=\{v_{n+1},\cdots,v_m\}$. Then $\rho(c)$ may be identified with $P_c:=(a_{n+1},b_{n+1},\cdots,a_m,b_m)\in\mathbb{R}^{2(m-n)}$ with $(a_i,b_i):=\rho(v_i)$, $\forall i\in\{n+1,\cdots,m\}$. A layout

$\chi \in L(G, c, \rho)$ is given by $P = (x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n}$ and if we define the Euclidean metric by

$$\begin{array}{rccc} h: & \mathbb{R}^4 & \rightarrow & \mathbb{R} \\ & (x_1, y_1, x_2, y_2) & \rightarrow & ((x_1 - x_2)^2 + (y_1 - y_2)^2)^{1/2}, \end{array}$$

the cost function $|\cdot|_f$ is given as the following real-valued function

$$\begin{array}{rcl} |\cdot|_f: \quad \mathbb{R}^{2n} & \rightarrow & \mathbb{R} \\ (x_1, y_1, \cdots, x_n, y_n) & \rightarrow & \displaystyle\sum_{\substack{i,j \in \{1,\cdots,n\} \\ (v_i, v_j) \in E}} f(h(x_i, y_i, x_j, y_j)) \\ & & \displaystyle + \sum_{\substack{i \in \{1,\cdots,n\} \\ j \in \{n+1,\cdots,m\} \\ (v_i,v_j) \in E}} f(h(x_i, y_i, a_j, b_j)) \\ & & \displaystyle + \sum_{\substack{i,j \in \{n+1,\cdots,m\} \\ (v_i,v_j) \in E}} f(h(a_i, b_i, a_j, b_j)). \end{array}$$

A short calculation shows that

(i) h is a convex function, and

(ii) if $((x = (x_1, x_2, x_3, x_4) \neq (x_1, x_2, \bar{x}_3, \bar{x}_4) = \bar{x})$ and $h(x) = h(\bar{x}))$, then h is strictly convex on $\{y \mid y = \lambda x + (1-\lambda)\bar{x}, \lambda \in [0, 1]\}$.

On the other hand, $|\cdot|_f$ is convex as the sum of convex functions $f \cdot h$. In order to show that $|\cdot|_f$ is strictly convex, it suffices to show that one term of the above sum is strictly convex. Because of the connectivity of G we succeed in finding i, j with the following:

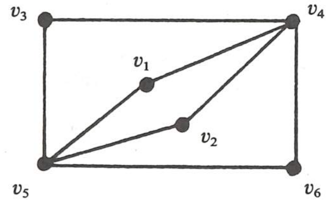
(a) $(x_i, y_i, a_j, b_j) \neq (\bar{x}_i, \bar{y}_i, a_j, b_j)$, or

(b) $(x_i, y_i, x_j, y_j) \neq (\bar{x}_i, \bar{y}_i, \bar{x}_j, \bar{y}_j)$ but $((x_i, y_i) = (\bar{x}_i, \bar{y}_i)) \vee ((x_j, y_j) = (\bar{x}_j, \bar{y}_j))$

for any two layouts $(x_1, y_1, \cdots, x_n, y_n) \neq (\bar{x}_1, \bar{y}_1, \cdots, \bar{x}_n, \bar{y}_n)$. Because of (a), (b) and since f is nondecreasing strictly convex, the corresponding term of $|\cdot|_f$ is strictly convex. So we have: $|\cdot|_f$ is strictly convex. Together with Theorem 1 we get the proof of Theorem 2. □

4. Planarity of the optimal layout. In the following we investigate the optimal layout of a connected $\rho(c)$-planar graph with fixed boundary. We would like to show that an optimization does not destroy the topological properties of the circuit, i.e., the optimal layout of a $\rho(c)$-planar graph is $\rho(c)$-planar, also. But this is not always true, as it is shown in [Gr] for the example in Fig. 3. (We only give the picture.)

We prove the following: Optimal layouts of $\rho(c)$-planar graphs are "quasi-$\rho(c)$-planar." The intuitive meaning of "χ is quasi-$\rho(c)$-planar" is: "If one looks at χ, χ seems to be $\rho(c)$-planar, for example, there is no crossing over of edges." In Fig. 3 the optimal layout χ_{opt} is quasi-$\rho(c)$-planar, but not $\rho(c)$-planar, since $\chi_{\text{opt}}(v_2) = \chi_{\text{opt}}(v_1)$. (See Fig. 4.)



$V_{\text{in}} = \{v_1, v_2\}$
$V_c = \{v_3, v_4, v_5, v_6\}$

FIG. 3

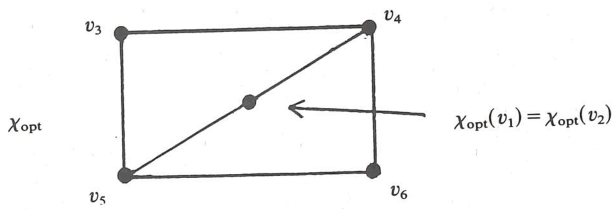


FIG. 4

If we assume that G is triconnected and that the fixed boundary allows a planar layout, we can prove a still better result: the optimal layout is $\rho(c)$-planar.

The subsequent definition of "convex point" is of great importance if one wants to prove the nonoptimality of a given layout.

DEFINITION. A straight line g in $\mathbb{R}^2$ divides the plane into two infinite regions $H_1(g)$, $H_2(g)$ with $H_1(g) \cap H_2(g) = g$. For $v \in V$ the *edge set of* v is defined as $E(v) := \{e \mid e \in E,\ e = (v, w)\}$. The *node set of* v is defined as $V(v) := \{w \mid w \in V,\ e = (v, w) \in E(v)\}$. Let χ be a $\rho(c)$-bounded layout of G. $v \in V_{\text{in}}$ is called a *convex point of* χ, iff the following hold:

(i) $\{\chi(v)\} \subsetneqq \chi(E(v))$;

(ii) There exists a straight line g with $g \cap \chi(E(v)) = \{\chi(v)\}$ and $\chi(E(v)) \subset H_1(g)$.

Remark. χ has a convex point, iff χ looks locally like the example in Fig. 5.

The following lemma proves that an optimal layout can never have convex points.

LEMMA 1. *Let χ be a $\rho(c)$-bounded layout of G with a convex point v, f continuous and increasing. Then χ is not optimal with respect to f.*

Proof. According to the definition of convex point we have the following situation. (See Fig. 6.)

Because of the strict monotony of f there exists an $\varepsilon > 0$, such that a change of $\chi(v)$ vertical to g by ε to $\bar{\chi}(v)$ results in a better layout. (See Fig. 7.)

We have $\sum_{i=1}^n f(g(\chi(v), \chi(v_i))) > \sum_{i=1}^n f(g(\bar{\chi}(v), \chi(v_i)))$ because $g(\chi(v), \chi(v_i)) > g(\bar{\chi}(v), \chi(v_i))$, $\forall i = 1, \cdots, n$. $\square$

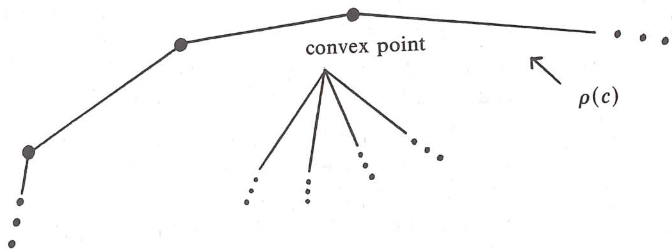


FIG. 5

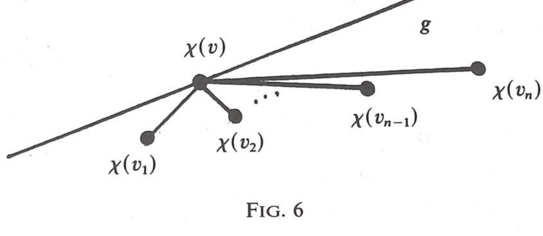


FIG. 6

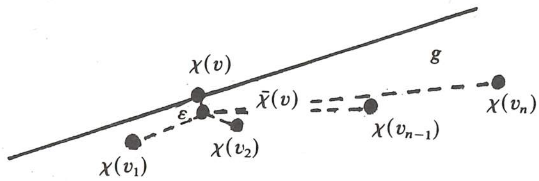


FIG. 7

4.1. Quasiplanarity and optimality. In this part of the paper we show that optimal layouts of $\rho(c)$-planar graphs are quasi-$\rho(c)$-planar. The results are first shown for triangulated graphs, then the generalization to any $\rho(c)$-planar graph is given.

DEFINITION. A graph G with fixed boundary $\rho(c)$ is called $\rho(c)$-triangulated, iff

(i) G is biconnected and $\rho(c)$-planar;

(ii) For any $\rho(c)$-planar layout $\chi_0(G)$ all faces of $\chi_0(G)$ are triangles.

With the help of [Wh] one easily concludes that the faces of a $\rho(c)$-triangulated graph G are uniquely determined. For the following let G be a $\rho(c)$-triangulated graph and let χ_0 be a fixed $\rho(c)$-planar layout of G. The set of triangles of χ_0 is denoted by

$$T(\chi_0) := \{\Delta_{\chi_0}(v_1, v_2, v_3) \mid \Delta_{\chi_0}(v_1, v_2, v_3) \text{ is a face of } \chi_0 \text{ and is "formed" by the points } \chi_0(v_1), \chi_0(v_2), \chi_0(v_3)\}.$$

In order to determine planarity, respectively, nonplanarity, any other layout $\chi(G)$ will be compared with χ_0 locally: for example, we consider the sequence of edges emanating from $\chi_0(v)$ and $\chi(v)$ (called the "stars in $\chi_0(v)$ and $\chi(v)$"). The instrument we use to do this is a mapping $\varphi(\chi_0, \chi)$: $C_{\rho(c)} \to C_{\rho(c)}$. φ maps the triangles of $T(\chi_0)$ onto the corresponding triangles in χ. More precisely, if one considers χ_0 and any point $P \in C_{\rho(c)}$, we have exactly one of the following three possibilities:

(i) $P = \chi_0(v)$ for exactly one $v \in V$.

(ii) There exists exactly one edge $\overline{\chi_0(v_1)\chi_0(v_2)}$ with the following properties: $\overline{\chi_0(v_1)\chi_0(v_2)}$ is an edge of a triangle of $T(\chi_0)$ and P has a unique representation $P = \lambda_1\chi_0(v_1) + \lambda_2\chi_0(v_2)$ with $\lambda_1, \lambda_2 \in (0, 1)$, $\lambda_1 + \lambda_2 = 1$.

(iii) There exists exactly one triangle $\Delta_{\chi_0}(v_1, v_2, v_3)$ of $T(\chi_0)$ and P lies in the interior of $\Delta_{\chi_0}(v_1, v_2, v_3)$. Then P has a unique representation $P = \sum_{i=1}^{3} \lambda_i\chi_0(v_i)$ with $\lambda_i \in (0, 1)$ and $\sum_{i=1}^{3} \lambda_i = 1$.

If χ is any $\rho(c)$-bounded layout of G, $\varphi(:= \varphi(\chi_0, \chi)) : C_{\rho(c)} \to C_{\rho(c)}$ is defined by the following:

$$\varphi(P) = \begin{cases} \chi(\chi_0^{-1}(P)) & P \text{ belongs to case (i),} \\ \lambda_1\varphi(\chi_0(v_1)) + \lambda_2\varphi(\chi_0(v_2)) & P \text{ belongs to case (ii),} \\ \sum_{i=1}^{3} \lambda_i\varphi(\chi_0(v_i)) & P \text{ belongs to case (iii).} \end{cases}$$

We call φ a deformation of χ_0 with respect to χ. The following remark is well known from topology [Re].

Remark. φ is well defined, continuous and surjective.

We illustrate the mapping φ with an example (see Fig. 8), which will give an intuitive idea how to prove the quasiplanarity of the optimal layout.

If we look at the example in Fig. 8, we see that φ changes the "stars" in $\chi_0(v_4)$ and $\chi_0(v_5)$. If we measure the angles of the triangles meeting in $\chi(v_4)$, we have a sum

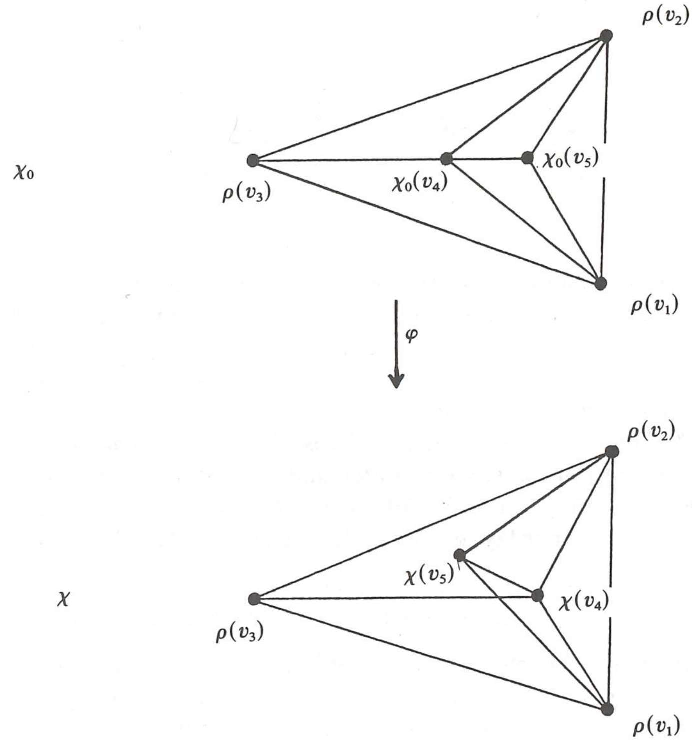


FIG. 8. *χ_0 is a $\rho(c)$-planar layout of a graph G; χ is a non-$\rho(c)$-planar layout of the same graph.*

of more than 2π, if we measure in $\chi(v_5)$, we have less than 2π, whereas for all points of V_{in} in χ_0 we get exactly 2π. This gives the motivation for the following idea: If a layout is not (quasi)-$\rho(c)$-planar, we try to find a point $v \in V_{\text{in}}$, where the star in $\chi_0(v)$ is changed by φ. If there exists such a point, we conclude that there exists a point where the sum of all angles is less than 2π; this point then will be a convex point. ($\chi(v_5)$ in the example of Fig. 8 is a convex point.) The existence of a convex point contradicts optimality according to Lemma 1.

In order to prove this exactly, we first have to give the formal definition of star. For the following let χ be a layout with $|\chi(e)| \neq 0$, $\forall e \in E$, i.e., $\chi(e)$ is not a point for all $e \in E$. For edges of $E(v)$, $v \in V$, we define an equivalence relation "$\sim_{\chi(v)}$":

$e_1 \sim_{\chi(v)} e_2 :\Leftrightarrow \text{card}\,(\chi(e_1) \cap \chi(e_2)) > 1$,
$[e_1]$ then is the equivalence class of e_1 in $\chi(v)$,
i.e., the set of all edges in $E(v)$, which overlap with e_1.

If $e_1, e_2 \in E(v)$ with $\chi(e_1) \cap \chi(e_2) = \{\chi(v)\}$,

$\sphericalangle(\chi(e_1), \chi(e_2)) :=$ the angle in $\chi(v)$ between $\chi(e_1)$ and $\chi(e_2)$
(in clockwise direction from $\chi(e_1)$ to $\chi(e_2)$).

$S_\chi(v) = ([e_1], [e_2], \cdots, [e_n])$ is called *the star in* $\chi(v)$ iff:
(i) $\bigcup_{i=1}^n [e_i] = E(v)$, $(n > 1)$;

(ii) $\forall i, j \in \{1, \cdots, n\}, i \neq j$:

$$(e \in [e_i], e' \in [e_j]) \Rightarrow (\chi(e) \cap \chi(e') = \{\chi(v)\});$$

(iii) For $i = 1, \cdots, n$: $\chi(e_i)$ is the direct left neighbour of $\chi(e_{i+1})$ and $\sphericalangle(\chi(e_i), \chi(e_{i+1})) \leqq \pi$ for $\{e_i, e_{i+1}\} \not\subset E_c$ (we define $e_{n+1} := e_1$!) (i.e., if there exists i with $\sphericalangle(\chi(e_i), \chi(e_{i+1})) > \pi$ (see $S_\chi(v_5)$ in Fig. 8), then we have a convex point and the star is not defined). If $S_\chi(v)$ is defined, it is unique up to cyclic permutation. (Hence we call two stars equal, iff they are equal up to cyclic permutation.)

Let $S_\chi(v) = ([e_1], \cdots, [e_n])$ $(v \in V)$. Then $([e_1], \cdots, [e_n])$ is called a *clockwise enumeration of $E(v)$ in $\chi(v)$*, $([e_n], [e_{n-1}], \cdots, [e_1])$ a *counterclockwise enumeration.* If card $([e_i]) = 1$, $\forall i = 1, \cdots, n$ (i.e., no edges lie upon each other), the star in $\chi(v)$ is called *simple.*

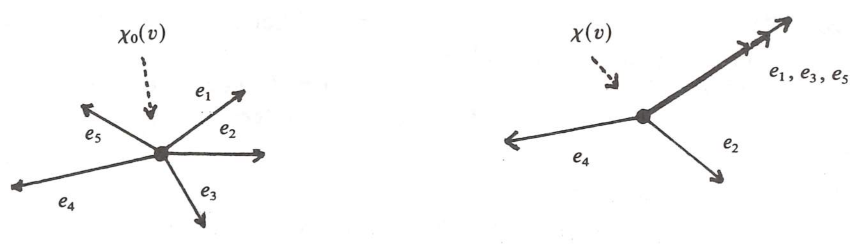
Remark. For χ_0 $S_{\chi_0}(v)$ is defined and simple for all $v \in V$.

We now consider the "image" of $S_{\chi_0}(v)$ for $v \in V$ in χ, which is obtained by collapsing edges adjacent in χ_0 that are in the same equivalence class with respect to "$\sim_{\chi(v)}$". More precisely, let $S_{\chi_0}(v) = (e_1, \cdots, e_n)$. Then (with respect to "$\sim_{\chi(v)}$") there exists $l \in [1, n]$ and $n_1 < n_2 < \cdots < n_l$ $(n_i \in [1, n]$ for $i = 1, \cdots, l)$ with:

$$[e_{n_i}] \neq [e_{n_{i+1}}] \quad \text{for } i = 1, \cdots, l \quad ([e_{n_{l+1}}] := [e_{n_1}]),$$
$$E_{n_{i+1}} := \{e_{n_i+1}, e_{n_i+2}, \cdots, e_{n_{i+1}}\} \subset [e_{n_{i+1}}] \quad \text{for } i = 1, \cdots, l-1,$$
$$E_{n_1} := \{e_1, \cdots, e_{n_1}, e_{n_l+1}, \cdots, e_n\} \subset [e_{n_1}].$$

We call $\chi(S_{\chi_0}(v)) := (E_{n_1}, \cdots, E_{n_l})$ the *image of $S_{\chi_0}(v)$ in χ*. We say $S_{\chi_0}(v)$ *induces* $S_\chi(v)$, iff $\chi(S_{\chi_0}(v))$ is equal to $S_\chi(v)$ (up to cyclic permutation), for short, $\chi(S_{\chi_0}(v)) = S_\chi(v)$. For an example see Fig. 9.

$S_\chi(v)$ is called a *full star,* iff $S_{\chi_0}(v)$ induces $S_\chi(v) := ([e_1], \cdots, [e_n])$ and $\sphericalangle(\chi(e_i), \chi(e_{i+1})) < \pi$ for $[e_i] \cap E_c = \varnothing$ or $[e_{i+1}] \cap E_c = \varnothing$. $S_\chi(v)$ is called a *half star* (along g), iff $S_{\chi_0}(v)$ induces $S_\chi(v)$ and looks like the example in Fig. 10. $S_\chi(v)$ is called an *inverse half star* (along g), iff $\chi(S_{\chi_0}(v))$ is a counterclockwise enumeration of $S_\chi(v)$ and looks like the example in Fig. 10. $S_\chi(v)$ is called a *degenerated star* (along g), iff $S_{\chi_0}(v)$ induces $S_\chi(v)$ and looks like the example in Fig. 11.



(a) $S_{\chi_0}(v) = (\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\})$.

(b) $S_\chi(v) = (\{e_1, e_3, e_5\}, \{e_2\}, \{e_4\})$,
$\chi(S_{\chi_0}(v)) = (\{e_1, e_5\}, \{e_2\}, \{e_3\}, \{e_4\})$.

FIG. 9

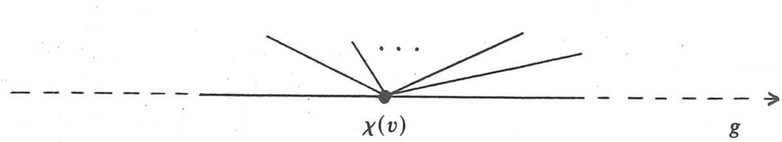


FIG. 10

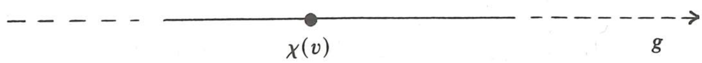


FIG. 11

In the following we give a characterization lemma for planar and quasiplanar layouts.

LEMMA 2. *Let χ be a $\rho(c)$-bounded layout of G. Then we have*:

$$\forall v \in V;\ S_\chi(v) \text{ is defined, simple and induced by } S_{\chi_0}(v)$$
$$\Leftrightarrow$$
$$\chi \text{ is } \rho(c)\text{-planar.}$$

An exact proof of this lemma can be found in [Gr]. Lemma 2 gives the possibility of being able to decide the planarity of χ only by looking at all nodes $v \in V$ and comparing the stars in $\chi(v)$ and $\chi_0(v)$.

We prove an analogous property for the quasiplanarity of a layout χ in Lemma 3. At first we give the formal definition of "quasi-$\rho(c)$-planar."

DEFINITION. Let G be $\rho(c)$-triangulated, $\chi_0(G)\rho(c)$-planar, χ a layout of G (not necessarily $|\chi(e)|>0, \forall e \in E$). $\chi(v)(v \in V)$ is called a *peak*, iff $\exists e \in E(v)$: $|\chi(e)|>0$, $\forall e, e' \in E(v)$ with $|\chi(e)|, |\chi(e')|>0$: $\operatorname{card}(\chi(e) \cap \chi(e'))>1$. χ is called *quasi-$\rho(c)$-planar*, iff

(i) $\forall \Delta_1, \Delta_2 \in T(\chi_0)$: $\varphi(\Delta_1) \cap \operatorname{int}(\varphi(\Delta_2)) = \emptyset$,

(ii) $\forall v \in V$: $\chi(v)$ is not a peak.

Point (i) of the definition corresponds to the intuitive idea of quasiplanar given at the beginning of this chapter: χ is planar up to triangles collapsed to a line or a point. Point (ii) makes sure that there are no "hidden" convex points in regions where triangles are collapsed.

LEMMA 3. *Let χ be a $\rho(c)$-bounded layout of G with $|\chi(e)|>0, \forall e \in E$. Then we have*:

$$\forall v \in V\text{: } S_\chi(v) \text{ is defined and induced by } S_{\chi_0}(v)$$
$$\Rightarrow$$
$$\chi \text{ is quasi-}\rho(c)\text{-planar.}$$

Proof. The proof is a general version of that of Lemma 2 (see [Gr]). We prove 5 facts that will finally give the complete proof.

FACT 1. *There exist no peaks in χ (since $S_\chi(v)$ is defined for all $v \in V$).*

Next we consider neighbourhoods of nodes and edges in χ, χ_0. For a point $v \in V$ we define the following:

The *neighbourhood of v for χ_0*:

$$U_{\chi_0}(v) := \bigcup_{\substack{\Delta_{\chi_0}(v_1, v_2, v_3) \in T(\chi_0), \\ v_i = v \text{ for } i=1 \vee 2 \vee 3}} \Delta_{\chi_0}(v_1, v_2, v_3).$$

The *neighbourhood of v for χ*:

$$U_\chi(v) := \bigcup_{\substack{\Delta_{\chi_0}(v_1, v_2, v_3) \in T(\chi_0), \\ v_i = v \text{ for } i=1 \vee 2 \vee 3}} \Delta_\chi(v_1, v_2, v_3), \quad \text{with } \Delta_\chi(v_1, v_2, v_3) := \varphi(\Delta_{\chi_0}(v_1, v_2, v_3)).$$

In the following we consider an edge $e_0 = (v_0, w_0)$ with $\chi(e_0) \not\subset \rho(c)$ and construct a *generalized neighbourhood* $U_\chi(e_0)$ of e_0 in χ, which consists of several neighbourhoods "surrounding" e_0. Here there are two cases of interest. Assume that e_0 is an edge of a triangle $\Delta \in T(\chi_0)$, which does not degenerate in χ (i.e., $\operatorname{int}(\varphi(\Delta)) \neq \emptyset$):

(i) If $[e_0] = \{e_0\}$ in $\chi(v_0)$, then ($[e_0] = \{e_0\}$ in $\chi(w_0)$ and) we define $U_\chi(e_0) := U_\chi(v_0) \cup U_\chi(w_0)$. (See Fig. 12.)

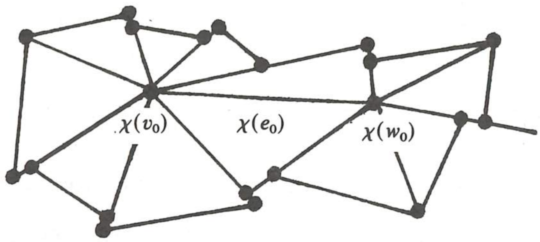


FIG. 12

(ii) If $[e_0] \neq \{e_0\}$, we can show the existence of a cycle in G (resp. $\chi_0(G)$), which collapses (together with its interior) to a straight line segment in the layout χ. This cycle together with its interior and its "surrounding" triangles will give the neighbourhood of e_0 in this case. (For an example, see Fig. 16.)

More precisely we give Fact 2.

FACT 2. *There exists a unique simple cycle* $c_0 = (V_0, E_0)$ *in* G *with the following properties*:

(a) $\{v_0, w_0\} \subset V_0$, $e_0 \in E_0$.

(b) $\chi(c_0)$ *lies on a straight line* g.

(c) *There are exactly two points* $u_1, u_2 \in V_0$ *where* $S_\chi(u_1)$, $S_\chi(u_2)$ *are not degenerated and not half stars along* g (u_1, u_2 *are called endpoints of* c_0 *in* χ), $\forall v \in V_0$, $v \neq u_1, u_2$ *we have*: $S_\chi(v)$ *is a half star* (*along* g).

(d) $\forall v \in V$ "*inside* c_0" (*with respect to* χ_0): $\chi(v)$ *lies on* $\chi(c_0)$ *and has a degenerated star.*

Proof of Fact 2. We start with the construction of the endpoints of c_0. Without loss of generality assume a situation as in Fig. 13. Let g be the straight line defined by e_0. According to the direction of g the relation $P < P'$ is well defined for two points P, P' on g. (For example, $\chi(v_0) < \chi(w_0)$ in Fig. 13.)

Either $S_\chi(v_0)$ is not a half star along g:

– Then v_0 is one of the endpoints of c_0;

or $S_\chi(v_0)$ is a half star along g:

– Then we can continue our considerations with $e_{v_1} := (v_1, v_0)$ instead of e_0, and e_{v_1} is an edge of a triangle $\Delta_{v_1} \in T(\chi_0)$ with int $(\varphi(\Delta_{v_1})) \neq \emptyset$. (See Fig. 14.)

The node $\chi(w_0)$ is treated analogously.

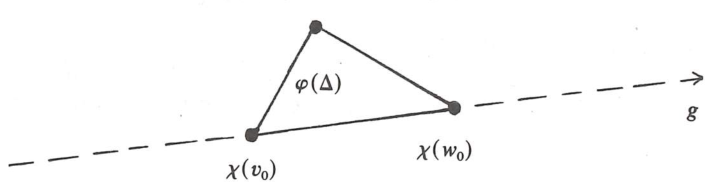


FIG. 13

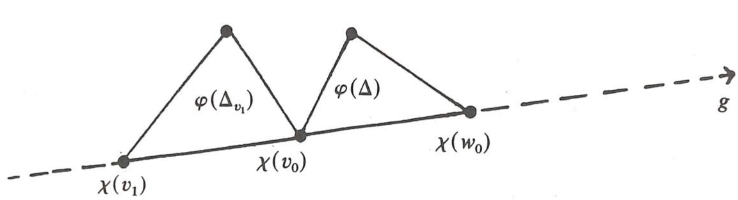


FIG. 14

By induction (see Fig. 15) we finally get a sequence $(v_0, v_1, \cdots, v_n)$ (resp. $(w_0, \cdots, w_m)$) of nodes as follows:

$S_\chi(v_i)$ (resp. $S_\chi(w_j)$) is a half star along g for $i = 1, \cdots, n-1$ (resp. $j = 1, \cdots, m-1$);

$S_\chi(v_n)$ (resp. $S_\chi(w_m)$) is not a half star along g and not degenerated.

If we consider $[e_{v_n}]$ $(e_{v_n} := (v_n, v_{n-1}))$ in $\chi(v_n)$, we find an edge $e_{v_{n+1}} = (v_n, v_{n+1}) \in [e_{v_n}]$ as follows:

$e_{v_n} \neq e_{v_{n+1}}$ and $e_{v_{n+1}}$ is an edge of a triangle $\Delta_{v_{n+1}} \in T(\chi_0)$ with $\operatorname{int}(\varphi(\Delta_{v_{n+1}})) \neq \emptyset$.

Analogously, we get the existence of an edge $e_{w_{m+1}}$ with the corresponding properties.

The edges $e_{v_n}, e_{v_{n-1}} := (v_{n-1}, v_{n-2}), \cdots, e_{v_1}, e_0, e_{w_1} := (w_1, w_0), e_{w_2} := (w_2, w_1), \cdots, e_{w_m} := (w_m, w_{m-1})$ form a first part of the cycle c_0. In a next step we deform the sequence $c := (u_1, \cdots, u_l) := (v_n, v_{n-1}, \cdots, v_0, w_0, \cdots, w_m)$, until we get the second part of the cycle c_0. The following two deformations are applied repeatedly: ($V(c_0)$ ultimately contains the nodes $v_{n-1}, \cdots, v_0, w_0, \cdots, w_{m-1}$ and the nodes inside c_0. At the beginning $V(c_0)$ is empty.)

(i) If there exists $i \in \{2, \cdots, l-1\}$, such that $\chi(u_{i-1}) < \chi(u_i) < \chi(u_{i+1})$ and $\Delta_{\chi_0}(u_{i-1}, u_i, u_{i+1}) \in T(\chi_0)$, then change c to $(u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_l)$ and set $V(c_0) := V(c_0) \cup \{u_i\}$.

(ii) If there exist $i \in \{1, \cdots, l-1\}$ and $u \notin V(c)$, such that $\chi(u_i) < \chi(u) < \chi(u_{i+1})$ and $\Delta_{\chi_0}(u_i, u, u_{i+1}) \in T(\chi_0)$, then change c to $(u_1, \cdots, u_i, u, u_{i+1}, \cdots, u_l)$.

We apply (i) and (ii) as long as possible. Assume that $c = (u'_1, \cdots, u'_k)$. Since the stars in χ are defined and induced by χ_0, the following holds: If there exist $i \in \{1, \cdots, k-1\}$ and $u \notin V(c_0)$, such that $\chi(u)$ lies on g and $\Delta_{\chi_0}(u'_i, u'_{i+1}, u) \in T(\chi_0)$, then there also exists a triangle, such that the application of deformation (i) or (ii) is possible. That is: if neither (i) nor (ii) can be applied, the nodes on $c = (u'_1, \cdots, u'_k)$ are half stars along g for $i = 2, \cdots, k-1$, and $u'_1 = v_n$, $u'_k = w_m$ are no half stars and not degenerated. c defines the second part of the cycle c_0; this finishes the proof of Fact 2.

The construction of c_0 is illustrated in Fig. 16. $U_\chi(e_0)$ is now defined as

$$U_\chi(e_0) := \left(\bigcup_{v \in V_0} U_\chi(v) \right) \cup \left(\bigcup_{v \text{ inside } c_0} U_\chi(v) \right).$$

The following is clear without proof.

FACT 3. *Let* $\Delta \in T(\chi_0)$ *with* $\operatorname{int}(\varphi(\Delta)) \neq \emptyset$. *Then* $\varphi|\Delta$ *is bijective.*

In the next remark we consider curves in $C_{\rho(c)}$ and their inverse images under φ.

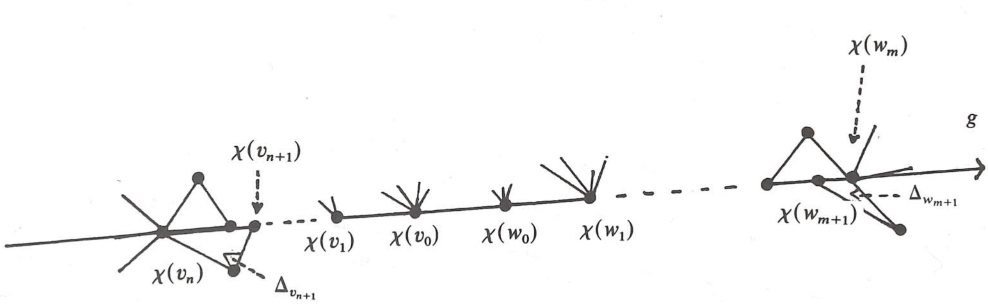


FIG. 15

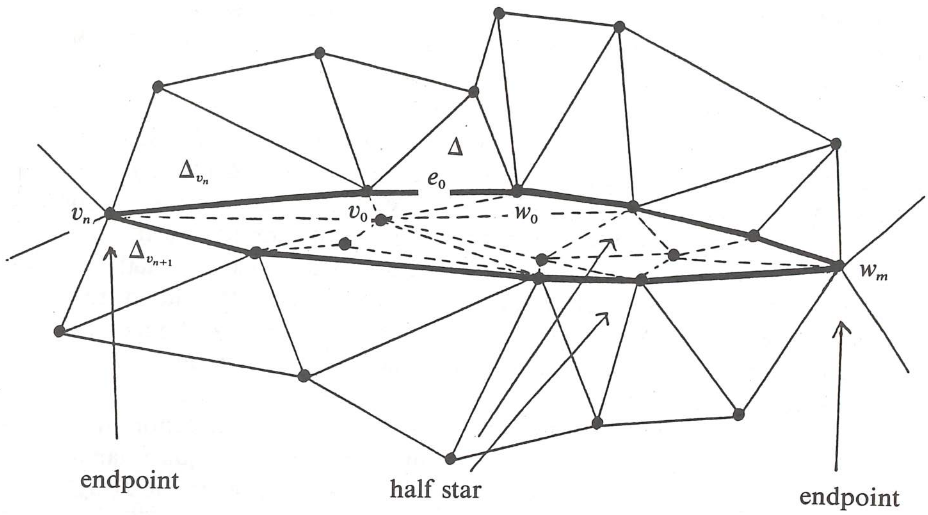


FIG. 16. --- *denotes the triangles, which collapse on g in* χ; —— *denotes the cycle* c_0.

With the help of Facts 2 and 3 we conclude the following:

FACT 4. *Let $\bar{\gamma}$ be a simple curve in the interior of $C_{\rho(c)}$ from $\bar{x}$ to $\bar{y}$ with the following properties*:

(i) $\{\bar{x}, \bar{y}\} \cap \chi(E) = \varnothing$ *and* $\bar{\gamma} \cap \chi(V) = \varnothing$.

(ii) *Let* $\Delta \in T(\chi_0)$ *with* $\operatorname{int}(\varphi(\Delta)) = \varnothing$, *then* $\operatorname{card}(\bar{\gamma} \cap \varphi(\Delta)) \leqq 1$.

Let $x \in \varphi^{-1}(\bar{x})$. Then there exists exactly one simple curve γ with starting point x and $\varphi(\gamma) = \bar{\gamma}$.

Proof of Fact 4. Let $x \in \Delta_1 \in T(\chi_0)$ $(\varphi(x) = \bar{x})$. Then we have a situation as in Fig. 17.

Consider $U_\chi(e_0)$ and use Fact 2 to get the following. There exists exactly one triangle $\Delta_2 \in T(\chi_0)$ with

$$\varphi(\Delta_2) \in U_\chi(e_0), \quad \operatorname{int}(\varphi(\Delta_2)) \neq \varnothing, \quad \varphi(\Delta_2) \cap \varphi(\Delta_1) \cap \bar{\gamma} = \bar{z} \neq \varnothing.$$

γ starts in x and is uniquely determined in Δ_1 (see Fact 3); we call this part of γ γ_1. γ_1 runs from x to a point $z_1(\not\subset \chi_0(V))$ on the boundary of Δ_1 with $\varphi(z_1) = \bar{z}$. There exists exactly one point z_2 on the boundary of Δ_2 with $\varphi(z_2) = \bar{z}$. The only way to continue Δ_1 is to use the uniquely determined simple curve γ_2, which starts in z_1, runs inside the cycle c_0 (which contains collapsing triangles according to Fact 2), ends in z_2 and whose image is $\bar{z}$. In Δ_2 we are then able to continue the curve without any problems. So Fact 4 is proved for the initial case. By induction we finish the proof.

FACT 5. *There exists a point $\bar{y} \in C_{\rho(c)}$ with the following properties*:

(i) $\operatorname{card}(\varphi^{-1}(\bar{y})) = 1$;

(ii) $\bar{y} \cap \chi(E) = \varnothing$.

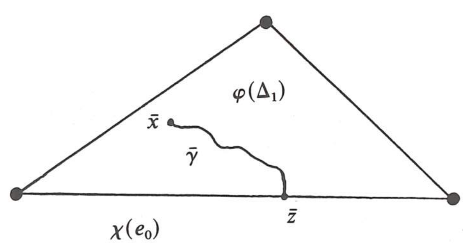


FIG. 17

Proof of Fact 5. Choose $v \in V_c$ with $e_1, e_2 \in E_c \cap E(v)$ and $\sphericalangle(\chi(e_1), \chi(e_2)) \neq \pi$. Then there exists $\varepsilon > 0$ with: $\forall \bar{y} \in C_{\rho(c)}$ with $|\bar{y} - \chi(v)| < \varepsilon$ and $\bar{y} \cap \chi(E) = \emptyset$: card $(\varphi^{-1}(\bar{y})) = 1$.

With Facts 1, 4 and 5 we are able to prove Lemma 3.

Assume the following (in contradiction to the definition of quasiplanar): Let $\bar{x} \in C_{\rho(c)}$. There are $\Delta_1, \Delta_2 \in T(\chi_0)$ with $\bar{x} \in \text{Int}(\varphi(\Delta_i))$, $x_i \in \text{Int}(\Delta_i)$, $\varphi(x_i) = \bar{x}$ $(i = 1, 2)$. (We may assume w.l.o.g., that $\bar{x} \cap \chi(E) = \emptyset$.) Choose a point $\bar{y} \in C_{\rho(c)}$ according to Fact 5 and then a curve $\bar{\gamma}$ from $\bar{x}$ to $\bar{y}$, which fulfills the conditions of Fact 4. (This is possible without any difficulties!) According to Fact 4 there exists exactly one simple curve γ_i with $\varphi(\gamma_i) = \bar{\gamma}$, which starts in x_i $(i = 1, 2)$. On the other hand, there exists exactly one curve γ starting in y with $\varphi(\gamma) = \bar{\gamma}$. This gives $\gamma_1 = \gamma_2$, hence $x_1 = x_2$; that is, the interior of two triangles, which do not degenerate, is disjoint.

This concludes the proof of Lemma 3 together with Fact 1. □

Remark. In Lemma 4 we will implicitly prove the inverse direction of Lemma 3, but this is not necessary for the proof of the nonoptimality of nonquasiplanar layouts.

With the help of the characterization Lemma 3 and an invariant $I(G_{\rho(c)})$ of G, which may be computed by calculating the angles of all triangles meeting in a point $\chi(v)$, we will show that "nonquasiplanarity" implies the existence of convex points.

The *angle* $\sphericalangle(\chi, v)$ *in* v *for* χ is $\sphericalangle(\chi, v) := \sum_{\Delta \in U_\chi(v)} \sphericalangle(\Delta, v)$ with $\sphericalangle(\Delta, v)$ is the angle of Δ in $\chi(v)$, if int $(\Delta) \neq \emptyset$; if int $(\Delta) = \emptyset$, $\sphericalangle(\Delta, v)$ is defined as follows: Let $\chi(e_1), \chi(e_2), \chi(e_3)$ compose Δ

$$\sphericalangle(\Delta, v) = \begin{cases} \pi & \text{if } \exists i, j \in \{1, 2, 3\},\ i \neq j \text{ with } \chi(e_i) \cap \chi(e_j) = \chi(v), \\ 0 & \text{otherwise.} \end{cases}$$

Remark. For $v \in V_{\text{in}}$ we get: $\sphericalangle(\chi_0, v) = 2\pi$.

If we look at the example at the beginning of this section, we see that $\sphericalangle(\chi, v)$ may be greater or less than 2π for points v in V_{in}.

We are now ready to define the *invariant* $I(G_{\rho(c)})$:

$$I(G_{\rho(c)}) := (2n + k - 2)\pi \quad \text{where } k = \text{card}(V_c),\ n = \text{card}(V_{\text{in}}).$$

$I(G_{\rho(c)})$ may be computed in different ways, as we will see in the following claim.

CLAIM 1. *Let* $N = \text{card}(T(\chi_0))$. *Then*

$$\begin{aligned} I(G_{\rho(c)}) &= \sum_{v \in V_{\text{in}}} 2\pi + (k-2)\pi \\ &= \sum_{v \in V_{\text{in}}} \sphericalangle(\chi_0, v) + \sum_{v \in V_c} \sphericalangle(\chi_0, v) \\ &= \sum_{v \in V} \sphericalangle(\chi_0, v) \\ &= N\pi \\ &= \sum_{v \in V} \sphericalangle(\chi, v). \end{aligned}$$

The correctness of Claim 1 is clear, since the number of "triangles" of χ_0 and χ is the same. (Note that for a collapsed triangle exactly one vertex is charged π and the others are charged 0.)

We make the following observations:

CLAIM 2.

(i) $\sphericalangle(\chi_0, v) \leqq \sphericalangle(\chi, v)$ *for* $v \in V_c$.

(ii) *If there exists a* $v_1 \in V$ *with* $\sphericalangle(\chi_0, v_1) < \sphericalangle(\chi, v_1)$, *then there exists a* $v_2 \in V_{in}$ *with* $2\pi = \sphericalangle(\chi_0, v_2) > \sphericalangle(\chi, v_2)$.

(iii) *If* $\sphericalangle(\chi, v) < 2\pi$ *for* $v \in V_{in}$, *then* $\chi(v)$ *is a convex point.*

Proof.

(i) Clear;

(ii) Follows from Claim 1 and (i);

(iii) Follows directly from a consideration of the triangles composing $U_\chi(v)$.

With the help of Claim 2 and Lemma 3 we succeed in finding convex points.

LEMMA 4. *Let* χ *be a* $\rho(c)$*-bounded layout of* G *with* $|\chi(e)| > 0$, $\forall e \in E$. *Then we have*:

$$\chi \text{ is not quasi-}\rho(c)\text{-planar}$$
$$\Rightarrow$$
$$\chi \text{ has a convex point.}$$

Proof. Apply Lemma 3 to get the following: If there exists $v \in V$ with "$S_\chi(v)$ not defined," we have the existence of a convex point in v. (See (iii) in the definition of "star.") So we may assume: there exists $v \in V$ with the following: $S_{\chi_0}(v)$ does not induce $S_\chi(v)$.

Case 1: $v \in V_c$. We conclude $\sphericalangle(\chi_0, v) < \sphericalangle(\chi, v)$ and with Claim 2 we get the existence of a convex point.

Case 2: $v \in V_{in}$.

(i) $\sphericalangle(\chi, v) \neq 2\pi \Rightarrow_{\text{Claim 2}}$ there exists a convex point;

(ii) $\sphericalangle(\chi, v) = 2\pi$.

Without loss of generality assume that in v there is no convex point of χ.

(a) φ "folds" $U_{\chi_0}(v)$ along a line segment of a straight line g. The situation is illustrated in Fig. 18.

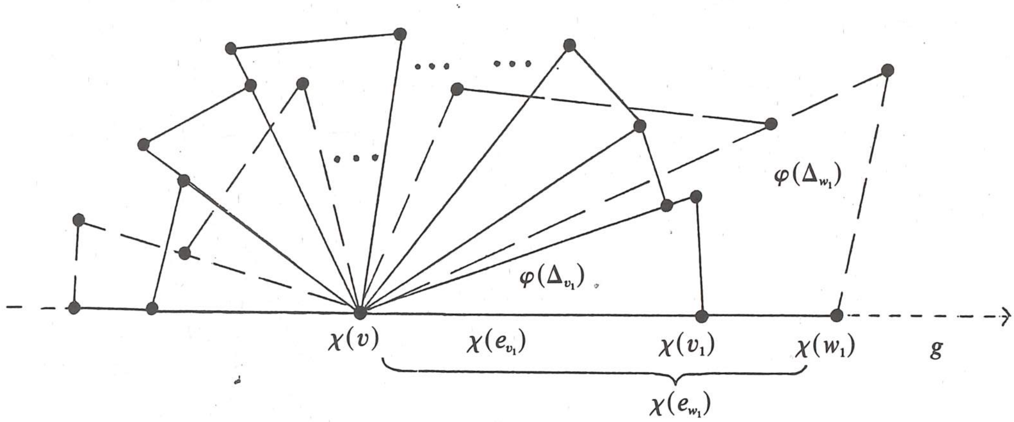


FIG. 18. $\chi(v)$ *is called a folding point along* g.

Consideration of the angle at $v_1(w_1)$ for χ either directly implies the existence of a convex point or we get the following: v_1 is a half star along g and w_1 is an inverse half star along g with respect to χ (or vice versa). In this case we start with the construction of a cycle c_0, analogously to the cycle c_0 in the proof of Fact 2 (for illustration see Fig. 19): at the beginning we have the point $\chi(v)$, the edges $e_{v_1}=(v, v_1)$, $e_{w_1}=(v, w_1)$ and the triangles Δ_{v_1}, Δ_{w_1}, during the construction we find:

– Either a node $v' \in V_{\text{in}}$ with: v' is a convex point or $\sphericalangle(\chi, v') \neq 2\pi$ ($\Rightarrow$ existence of a convex point);

– Or $v' \in V_c$ with: $\sphericalangle(\chi_0, v') < \sphericalangle(\chi, v')$ ($\Rightarrow$ existence of a convex point);

– Or we inductively get sequences $(v, v_1, v_2, \cdots, v_n)$ and $(v, w_1, w_2, \cdots, w_m)$ with: $v, v_n = w_m$ are folding points along g, $v_1, v_2, \cdots, v_{n-1}$ are half stars along g, and $w_1, w_2, \cdots, w_{m-1}$ are inverse half stars along g (with respect to χ). The cycle c_0 is given by $(v, v_1, \cdots, v_n = w_m, w_{m-1}, \cdots, w_2)$ and all nodes inside c_0 have degenerated stars in χ.

Since χ has a convex boundary $\rho(c)$, we inductively find (by construction of new cycles c_0):

– Either a point $v' \in V_{\text{in}}$ with: v' is a convex point or $\sphericalangle(\chi, v') \neq 2\pi$;

– Or a point $v' \in V_c$ with: $\sphericalangle(\chi_0, v') < \sphericalangle(\chi, v')$.

So we finally get the existence of a convex point in Case (a).

(b) There is the following case left: $\sphericalangle(\chi, v) = 2\pi$ and $\chi(S_{\chi_0}(v))$ is a counterclockwise enumeration of the edges in $\chi(v)$. Choose a path of edges from $\chi(v)$ to the boundary $\rho(c)$. Then one finds a node w on this path such that: $\chi(S_{\chi_0}(w)$ is neither a clockwise nor a counterclockwise enumeration of the edges in $\chi(w)$. This gives the existence of a convex point according to Case 1, 2(i), 2(ii)(a). $\square$

With the help of Lemma 4 we get the inverse direction of Lemma 3 and therefore have the following lemma.

LEMMA 5. *Let χ be a $\rho(c)$-bounded layout of G with $|\chi(e)| > 0$, $\forall e \in E$. Then we have*:

$$\chi \text{ is quasi-}\rho(c)\text{-planar,}$$
$$\Leftrightarrow$$
$$\forall v \in V\text{: } S_\chi(v) \text{ is defined and } \chi(S_{\chi_0}(v)) = S_\chi(v).$$

We generalize Lemma 4 to layouts χ, where $|\chi(e)|$ is not necessarily greater than 0, $\forall e \in E$.

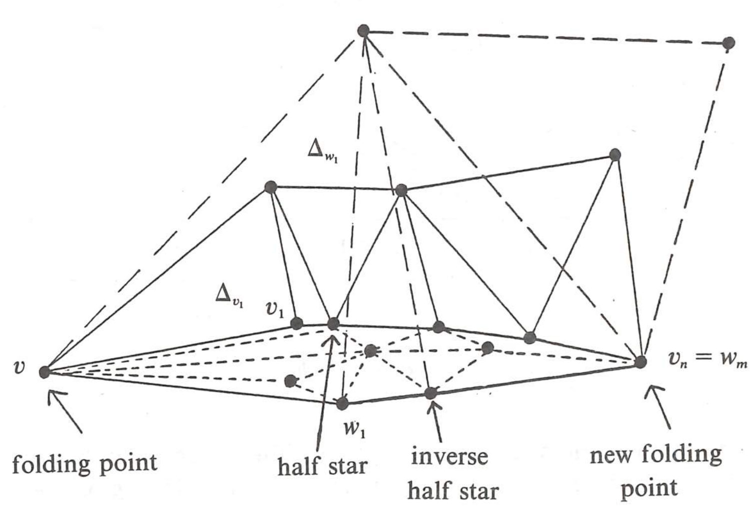


FIG. 19. —— *denotes the triangles collapsed on g*; --- *denotes the "folded" triangles.*

LEMMA 6. *Let χ be a $\rho(c)$-bounded layout of a $\rho(c)$-triangulated graph G. Then we have:*

$$\chi \text{ is not quasi-}\rho(c)\text{-planar,}$$
$$\Rightarrow$$
$$\chi \text{ has a convex point.}$$

Proof. If there exists an edge $e=(v, w)$, which collapses to a point under φ, we consider the $\rho(c)$-triangulated graph, which results from identifying v and w. With induction and Lemma 4 we finish the proof.

It remains to prove our results for general graphs with fixed boundary (not necessarily $\rho(c)$-triangulated). First we have to give the general definition of quasi-planar. Intuitively spoken quasiplanarity of a layout $\chi(G)$ in this case means the following: all connected components of G collapse to points, all biconnected components collapse to straight line segments, the layout of the "remaining" graph is quasiplanar in the sense of our former definition. More formally, let G be any $\rho(c)$-planar graph with fixed boundary, χ a layout of G.

χ has the *a-point property*, iff the following hold:

– Let v be any articulation point of G (then G is split into connected components by v), let $G_v=(V_v, E_v)$ be any of the resulting subgraphs with $c \not\subset G_v$. Then $\chi(w)=\chi(v)$, $\forall w \in V_v$. (See the example in Fig. 20!)

χ has the *a-pair property*, iff the following hold:

– Let (v_1, v_2) be any articulation pair of G (then G is split into connected components by (v_1, v_2)), let $G_{(v_1,v_2)}=(V_{(v_1,v_2)}, E_{(v_1,v_2)})$ be any of the resulting subgraphs with $c \not\subset G_{(v_1,v_2)}$. Then we have the following: If $E_{(v_1,v_2)} \cap E_c=\emptyset$ or $v_1, v_2 \in V_c$ with $\rho(v_1)\rho(v_2) \subset \rho(c)$, then $\chi(w) \in \overline{\chi(v_1)\chi(v_2)}$, $\forall w \in V_{(v_1,v_2)}$. (See the example in Fig. 21.)

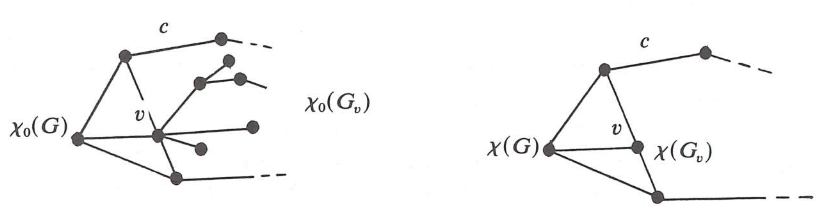


FIG. 20

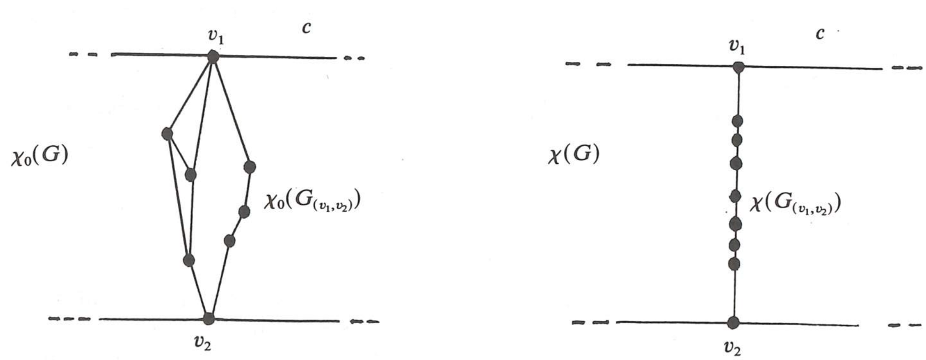


FIG. 21

With the help of these definitions we "construct" the reduced graph G_{red} of G in three steps:

(i) Define G_{red} to be the maximal subgraph of G with:

– $c \subset G_{\text{red}}$,

– $\nexists$ articulation point in G_{red};

(ii) Let (v_1, v_2) be an articulation pair in G_{red}, consider the resulting components $G_{(v_1,v_2)}$ and replace all components with $E_{(v_1,v_2)} \cap E_c = \emptyset$ or $v_1, v_2 \in V_c$ and $\overline{\rho(v_1)\rho(v_2)}$ lies on $\rho(c)$ by an edge $e = (v_1, v_2)$, unless e is already an edge of G_{red};

(iii) Repeat (ii) as long as possible.

The resulting graph is called the *reduced graph* G_{red} *of* G. Any layout χ of G, which fulfills the a-point and a-pair property, may be interpreted as a layout of the reduced graph G_{red}. G_{red} is a $\rho(c)$-planar graph with fixed boundary, and the faces of G_{red} are uniquely determined.

DEFINITION. Let G be a $\rho(c)$-planar graph with fixed boundary. A $\rho(c)$-bounded layout χ of G is *nearly quasi-$\rho(c)$-planar*, iff:

(i) χ has the a-point and a-pair property;

(ii) Let F_1, F_2 be faces of G_{red} in χ, then $F_1 \cap \operatorname{int}(F_2) = \emptyset$;

(iii) $\nexists v \in V$: $\chi(v)$ is a peak ("peak" is defined analogously to the triangulated case).

As a generalization of Lemma 6 we get Lemma 7.

LEMMA 7. *Let χ be a $\rho(c)$-bounded layout of a $\rho(c)$-planar graph G. Then we have*:

$$\chi \text{ is not nearly quasi-}\rho(c)\text{-planar} \;\Rightarrow\; \chi \text{ has a convex point.}$$

Proof.

Case 1. (i), (iii) of the above definition is not fulfilled $\Rightarrow$ we at once get the existence of a convex point.

Case 2. (ii) of the above definition is not fulfilled $\Rightarrow$ let G_T denote any triangulation of G_{red}, let χ_T be the layout of G_T induced by χ, then χ_T is not quasi-$\rho(c)$-planar, which gives the existence of a convex point in this case, also.

Nearly quasi-$\rho(c)$-planar layouts may have faces whose boundaries are not simple or do not define convex sets. These layouts cannot be optimal. Therefore we define:

DEFINITION. Let G be a $\rho(c)$-planar graph with fixed boundary. A $\rho(c)$-bounded layout χ of G is *quasi-$\rho(c)$-planar*, iff

(i) χ is nearly quasi-$\rho(c)$-planar;

(ii) Let F be a face of G_{red} with $\operatorname{int}(F) \neq \emptyset$, then F is a convex set and ∂F is simple. Let F be a face of G_{red} with $\operatorname{int}(F) = \emptyset$, then ∂F consists of two identical straight line segments or a point. (F is "degenerated convex.")

We get a more precise version of Lemma 7.

LEMMA 8. *Let χ be a $\rho(c)$-bounded layout of a $\rho(c)$-planar graph G. Then we have*:

$$\chi \text{ is not quasi-}\rho(c)\text{-planar} \;\Rightarrow\; \chi \text{ has a convex point.}$$

Proof. Since an exact proof requires the same methods as introduced before, we do not go into detail and only indicate the general proceeding: Assume that (ii) in the above definition of "quasiplanar" is not fulfilled. Then we either directly get the existence of a convex point or we can choose a triangulation of G, which after application of Lemmas 4, 6 and 7 gives the existence of a convex point for the triangulated case. This immediately implies the existence of a convex point in the layout χ.

Lemma 8 gives an interesting characterization of nonplanar layouts of planar graphs (with fixed boundary). Either they are quasiplanar, or there exists a convex point.

We summarize our results in Theorem 3.

THEOREM 3. *Let $G_{\rho(c)}$ be a connected $\rho(c)$-planar graph with fixed boundary and $f: \mathbb{R}^+ \to \mathbb{R}^+$ continuous and increasing. Then an optimal layout of G with respect to $\rho(c)$ and f is quasi-$\rho(c)$-planar.*

4.2. Planarity and optimality. We restrict to triconnected planar graphs with fixed boundary and show that the optimal layout is planar convex. We need some preparations:

TECHNICAL LEMMA. *Assume f to be nondecreasing, strictly convex, twice differentiable, $f(0)=f'(0)=0$. Let $N>0$, $y>0$, $x \geqq 0$, $0<\alpha \leqq \pi/2$, for $\varepsilon>0$ define $x' := (x^2+\varepsilon^2)^{1/2}$, $y' := (y^2+\varepsilon^2-2y\varepsilon \cos \alpha)^{1/2}$. Assume a situation as in Fig. 22. Then there exists $\bar{\varepsilon}>0$, such that*:

$$\forall 0<\varepsilon \leqq \bar{\varepsilon}:\ N(f(x')-f(x))<f(y)-f(y').$$

Proof.

Case 1. $x>0$. If we choose ε small enough, we get with the help of the properties of f and the Mean Value Theorem

$$N(f(x')-f(x))<Nf'(2x)(x'-x), \qquad f'(y/2)(y-y')<f(y)-f(y').$$

Therefore it suffices to show that

$$\exists \bar{\varepsilon}>0: \forall \varepsilon \leqq \bar{\varepsilon}: M(x'-x)<y-y' \quad \text{with } M := \frac{Nf'(2x)}{f'(y/2)}>0.$$

An exact calculation shows that there exists a polynomial P with

$$\begin{aligned} &M(x'-x)<y-y' \\ &\Leftrightarrow Mx+y>M(x^2+\varepsilon^2)^{1/2}+(y^2+\varepsilon^2-2y\varepsilon \cos \alpha)^{1/2} \\ &\Leftrightarrow Mx+y>\varepsilon \cdot P(\varepsilon). \end{aligned}$$

This finishes the proof of Case 1.

Case 2. $x=0$. We have to show that $Nf(\varepsilon)<f(y)-f(y')$. As in Case 1 we get, for small enough ε,

$$Nf(\varepsilon)<Nf'(\varepsilon)\varepsilon, \qquad f'(y/2)(y-y')<f(y)-f(y').$$

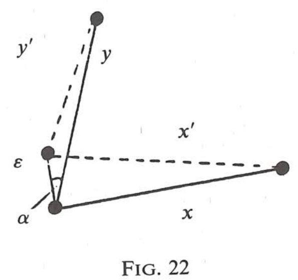


FIG. 22

Therefore it suffices to show that

$$\exists \bar{\varepsilon} > 0 \colon \forall \varepsilon \leqq \bar{\varepsilon} \colon Mf'(\varepsilon)\varepsilon < y - y' \quad \text{with } M := \frac{N}{f'(y/2)} > 0.$$

We have the following:

$$\begin{aligned}
&Mf'(\varepsilon)\varepsilon < y - y' \\
&\Leftrightarrow y - Mf'(\varepsilon)\varepsilon > (y^2 + \varepsilon^2 - 2y\varepsilon \cos\alpha)^{1/2} \\
&\Leftrightarrow y^2 - 2Myf'(\varepsilon)\varepsilon + M^2(f'(\varepsilon))^2\varepsilon^2 > y^2 + \varepsilon^2 - 2y\varepsilon\cos\alpha \\
&\Leftrightarrow 2y\cos\alpha > \varepsilon(1 - M^2(f'(\varepsilon))^2) + 2Myf'(\varepsilon).
\end{aligned}$$

Because of $\lim_{\varepsilon \to 0} f'(\varepsilon) = 0$, the proof is finished. □

The technical lemma gives us a method to improve layouts. In order to be able to apply this lemma, we assume from now on, that f fulfills the following conditions as above: f is nondecreasing, strictly convex, twice differentiable and $f(0) = f'(0) = 0$.

LEMMA 9. *Let χ be $\rho(c)$-bounded layout of G, which is quasi-$\rho(c)$-planar but not $\rho(c)$-planar convex. Then χ is not optimal.*

Proof. We first consider two special layout situations, which will later help us to prove that a layout χ with degenerated convex faces cannot be optimal.

(i) Assume a situation as in Fig. 23. F is a face collapsed to a line and g is the straight line defined by F. There exists a point v on F with:

$$(*) \qquad \chi(E(v)) \cap (H_1(g)\backslash g) \neq \varnothing, \qquad \chi(E(v)) \cap (H_2(g)\backslash g) = \varnothing.$$

Now we use the technical lemma to show that in this situation the layout is not optimal. Let $\{v_1, \cdots, v_n\} \subset V(v)$ be the vertices with $\chi(v_i) \in (H_1(g)\backslash g)$. We know that $n \geqq 1$ because of (*). Let $\{w_1, \cdots, w_m\} \subset V(v)$ be the vertices with $\chi(w_i) \in g$. We know that $m \geqq 2$ because of v on F.

We choose $\varepsilon > 0$ and change the placement of $\chi(v)$ by ε to a direction vertical to g. We call the new point in $H_1(g)$ $\bar{\chi}(v)$. (See Fig. 24.) Then we have

$$\begin{aligned}
&\sum_{i=1}^{m} (f(g(\chi(w_i), \bar{\chi}(v))) - f(g(\chi(w_i), \chi(v)))) \\
&\qquad < m \cdot \max_{i=1,\cdots,m} \{(f(g(\chi(w_i), \bar{\chi}(v))) - f(g(\chi(w_i), \chi(v)))\} \\
&\qquad < f(g(\chi(v_1), \chi(v))) - f(g(\chi(v_1), \bar{\chi}(v))) \\
&\qquad\qquad\qquad \text{(technical lemma with } \varepsilon \text{ sufficiently small)} \\
&\qquad < \sum_{i=1}^{n} (f(g(\chi(v_i), \chi(v))) - f(g(\chi(v_i), \bar{\chi}(v)))).
\end{aligned}$$

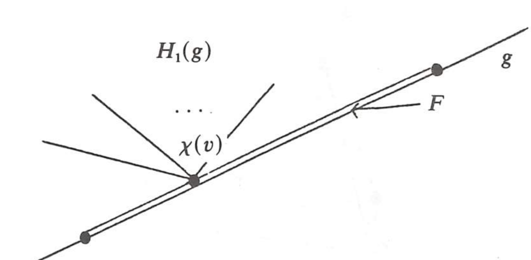


FIG. 23

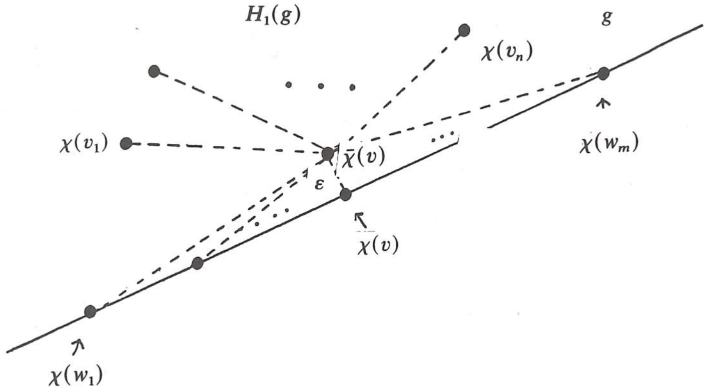


FIG. 24

This means that there exists an $\varepsilon > 0$ such that

$$\sum_{w \in V(v)} f(g(\chi(w), \bar{\chi}(v))) < \sum_{w \in V(v)} f(g(\chi(w), \chi(v))).$$

Therefore situation (i) can never occur in an optimal layout.

(ii) Assume the following situation:

$$\chi(v_1) = \chi(v_2) \quad \text{for } v_1 \in V(v_2)$$

and there does not exist a straight line g such that $\chi(E(v_1)) \subset g$ or $\chi(E(v_2)) \subset g$ (see the example in Fig. 25). The "part of the circle in $\chi(v_1)$ from $(\chi(w_1), \chi(v_1))$ in clockwise orientation to $(\chi(w_2), \chi(v_1))$" is called *sector* $S(w_1, v_1, w_2)$. Since χ is quasi-$\rho(c)$-planar, there exist points $\chi(w_1)$, $\chi(w_2)$, $\chi(w_3)$, $\chi(w_4)$, such that

(a) $w_1, w_2 \in V(v_1)$, $w_3, w_4 \in V(v_2)$,

(b) $\chi(E(v_1)) \subset S(w_1, v_1, w_2)$, $\chi(E(v_2)) \subset S(w_3, v_2, w_4)$,

(c) $\operatorname{int}(S(w_1, v_1, w_2)) \cap \operatorname{int}(S(w_3, v_2, w_4)) = \emptyset$.

With χ quasi-$\rho(c)$-planar it follows that the angle $\sphericalangle(w_1, v_1, w_2) \leqq \pi$, or $\sphericalangle(w_3, v_2, w_4) \leqq \pi$. Assume that $\sphericalangle(w_1, v_1, w_2) \leqq \pi$. With the help of the technical lemma we succeed in showing (analogously to (i)) that χ is not optimal. (See Fig. 26.)

It now remains to show the following for a quasi-$\rho(c)$-planar layout χ, which is not $\rho(c)$-planar convex: There exists a face F respectively, points v_1, v_2, such that (i)

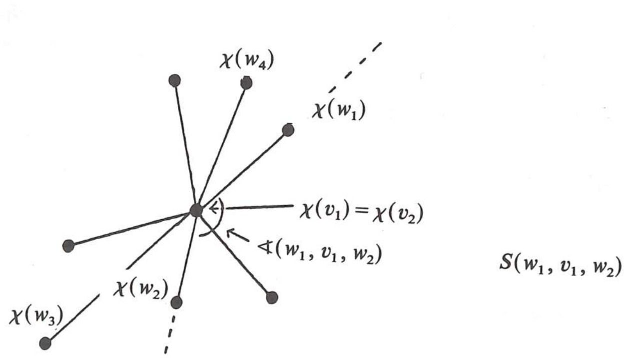


FIG. 25

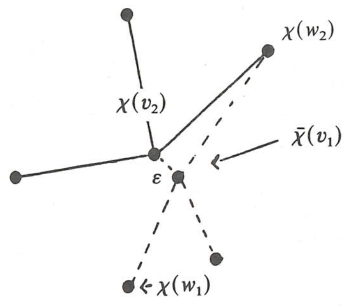


FIG. 26

or (ii) is applicable. Assume all faces of χ which are degenerated to a line do not allow application of (i). We have to show that there exist points which allow application of (ii). Let F be a face of χ collapsed to a line g. If the application of (ii) at one of the "endpoints" of F is possible, the proof is finished. Otherwise, we get the existence of a set S of faces (incl. F) degenerated to g and nested into each other. One of the endpoints of this set then allows the application of (ii). (See Fig. 27!) More exactly, S is defined as follows:

(i) $F \in S$;

(ii) All faces in S are degenerated to the line g;

(iii) $\bigcup_{F' \in S} F' := g_S$ is a (connected) straight line segment of g;

(iv) Exactly the endpoints P_1, P_2 of g_S correspond to nodes v_i $(i=1,2)$ of G with $\chi(E(v_i)) \not\subset g$ $(i=1,2)$.

From the quasiplanarity of χ we know that the endpoints of g_S do not correspond to peaks. We conclude that P_1 or P_2 must be the image of at least two vertices of G. Otherwise we would have a contradiction to the triconnectivity of G. So we conclude that (ii) is applicable. Therefore, we may assume that there are no faces of χ collapsed to a line. If there rest vertices v_1, v_2 with $\chi(v_1)=\chi(v_2)$, then situation (ii) is applicable. So the proof of Lemma 9 is finished. □

If we combine Theorem 3 and Lemma 9 we get Theorem 4.

THEOREM 4. *Let $G_{\rho(c)}$ be a triconnected $\rho(c)$-planar graph with fixed boundary and let $f: \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing, strictly convex, twice differentiable function with $f(0)=f'(0)=0$. Then the optimal layout of G with respect to $\rho(c)$ and f is $\rho(c)$-planar convex.*

COROLLARY. *A careful reading of the proof of Lemma 9 gives the following: The triconnectivity of G is not necessary in Theorem 4; $G=G_{\text{red}}$ is sufficient.*

5. Concluding remark. (i) If one considers functions f_e, $\forall e \in E$ instead of one function f and defines the cost of a layout χ as $|\chi| := \sum_{e \in E} f_e(|\chi(e)|)$, all the results remain valid. (Especially layouts of "weighted graphs with fixed boundary" may be considered.)

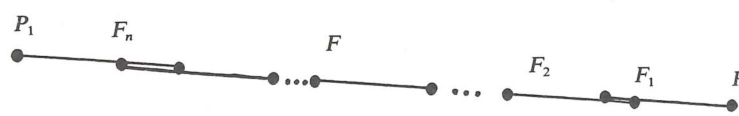


FIG. 27

(ii) If one considers "the limit" of the optimal layouts of a graph with respect to the l_p-metric for $p=2, 3, \cdots$, one gets a layout, which minimizes the maximal length of an edge of the graph and balances the length of the remaining ones. With the help of Theorems 3 and 4 we are able to prove the quasiplanarity of this layout, also (see [BeOs]).

Appendix. Apart from the existence and properties of optimal layouts their construction, too, is a point of interest. We studied approximation algorithms for the construction of the optimal layout with respect to cost functions $f(x)=x^p$. The cases $f(x)=x^2$ and $f(x)=x^p$ $(p>2)$ have to be distinguished.

In the case $p=2$ the solution of the optimization problem is equivalent to the solution of two (sparse) linear equation systems. This at once gives a polynomial time algorithm for the computation of the exact solution. On the other hand, Jacobi or Gauss–Seidel relaxation can be applied and converge to the optimal layout. (Iteration steps may yield nonplanar intermediate layouts. Therefore as termination criterion a planarity test for the approximation is needed. Such a test can be done in linear time by use of Lemma 2. For details see [GrHo].)

The complexity of the approximation algorithms depends on the spectral radius of the relevant matrix. There exists a class of graphs (see [StBu, p. 257]), for which the optimal layout can be computed in time $O(n^2)$ ($n=$number of the nodes), if the precision of the calculation is given. Experimental results yield linear time for the considered graphs, but we still did not succeed in the exact computation of the spectral radius for these cases. In the case $p>2$ methods of steepest descent were studied. Details for both cases ($p=2$ and $p>2$) together with the corresponding program listing can be found in [Gr]. With the help of multigrid methods it is possible to approximate the solution of the optimization problem for $p=2$ in linear time [St]. Whether this can be extended to $p>2$ is not known at this time. (For further information about multigrid methods see [HaTr].)

We finish with two examples: Consider the "net" in Fig. 28, which recursively defines an n-bit adder. We apply the given recursion for $n/2$ down to 2 (Ad_2 corresponds to a full adder) and thus finally we get a graph corresponding to the switching circuit

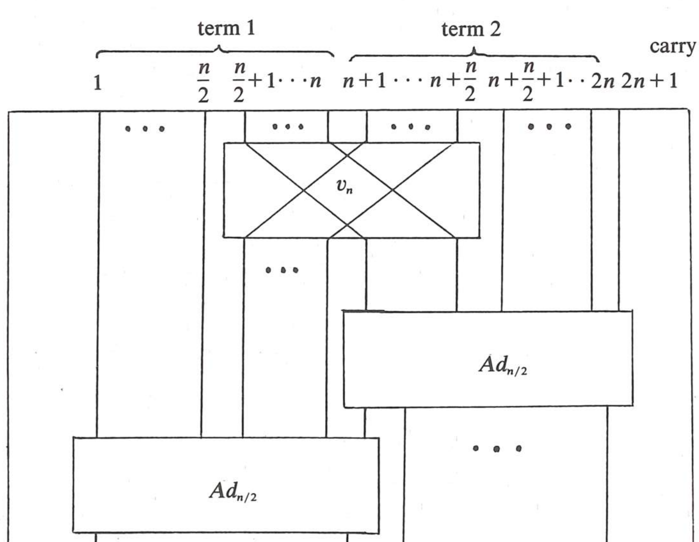


FIG. 28

of an n-bit adder. If the recursion is given, the above process is done automatically in the logical-topological level of the design system CADIC (for details see [BHKM]). The resulting graph is input for the optimization algorithms.

A first very small example is given by the optimal layouts of a 4-bit adder defined by the net of Fig. 28. (See Fig. 29.)

The approximation algorithms were tested for many (more complex) circuits (for example: carry-look-ahead adder, conditional-sum-adder, multiplier, memories, combinational logic, ...).

We finish by giving the l_2-optimal layout of a fast multiplier presented in [LuVu]. The multiplier can be described with the help of four recursive equations in a manner

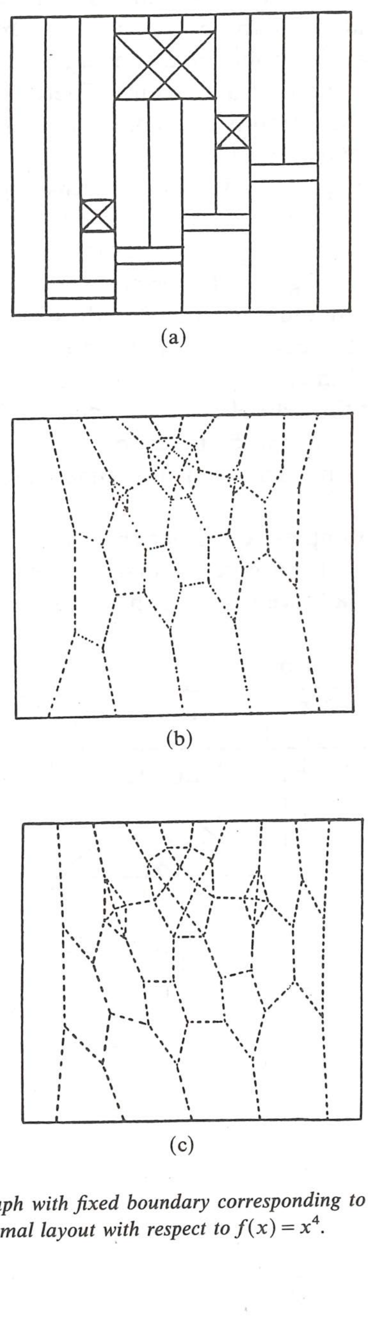


FIG. 29. (a) *Layout of a graph with fixed boundary corresponding to a 4-bit adder*; (b) *Optimal layout with respect to* $f(x)=x^2$; (c) *Optimal layout with respect to* $f(x)=x^4$.

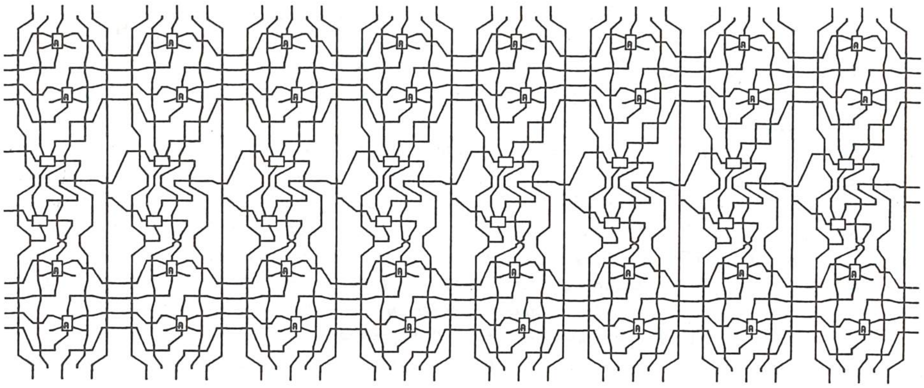


FIG. 30. *Expansion of the recursively defined network for the fast multiplier.*

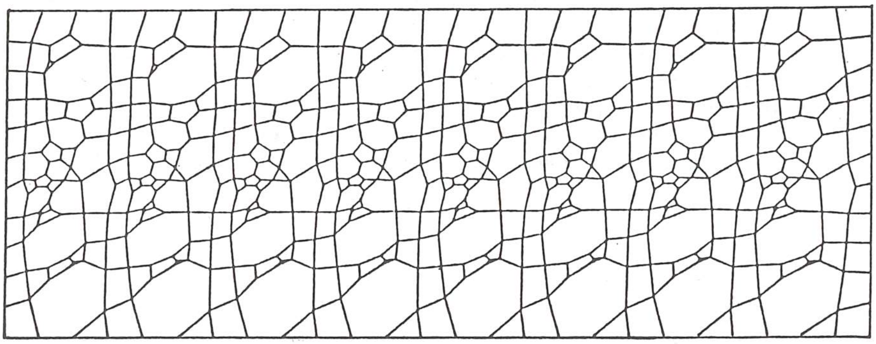


FIG. 31. *Optimal layout with respect to* $f(x)=x^2$.

analogous to the above adder (see [BHKM]). Here we start directly with the graph with fixed boundary constructed out of the equations (Figs. 30 and 31).

Acknowledgments. The authors thank U. Groh for many helpful comments and a careful reading of the manuscript.

REFERENCES

[Be] B. BECKER, *Über die kreuzungsfreie, rechtwinklige Einbettung von gewichteten Graphen in die Ebene*, Ph.D. dissertation, Universität des Saarlandes, Saarbrücken, 1982.

[BeOs] B. BECKER AND H. G. OSTHOF, *Layouts with wires of balanced length*, Proc. Symposium on Theoretical Aspects of Computer Science, Springer-Verlag, Berlin, 1985, pp. 21–31.

[BHKM] B. BECKER, G. HOTZ, R. KOLLA AND P. MOLITOR, *Ein CAD-System zum Entwurf integrierter Schaltungen*, Technical Report, 16/1984, SFB 124 B1, Universität des Saarlandes, Saarbrücken.

[Br] M. A. BREUER, *Design Automation of Digital Systems, Vol.* 1: *Theory and Techniques*, Prentice-Hall, Englewood Cliffs, NJ, 1972.

[Bry] R. BRYANT, ed., *Third Caltech Conference on* VLSI, Springer-Verlag, Berlin, 1983.

[Do] W. E. DONATH, *Complexity theory and design automation*, Proc. 17th Design Autom. Conf., New York, IEEE Computer Soc. Press, 1980, pp. 412–419.

[Fi] C. J. FISK, D. L. CASKEY AND L. L. WEST, *Automated circuit card etching layout*, Proc. IEEE, 55, 1967, pp. 1971–1982.

[FiPa] M. J. FISCHER AND M. S. PATTERSON, *Optimal tree layout* (*preliminary version*), Proc. 12th Annual ACM Symposium on the Theory of Computing, 1980, pp. 177–189.

[Gr] U. GROH, *Optimale Einbettungen von Graphen mit festem Rand*, Diplomarbeit, Universität des Saarlandes, Saarbrücken, 1983.

[GrHo] U. GROH AND G. HOTZ, *Ein Planaritätstest für planar-konvexe Grapheneinbettungen mit linearer Komplexität*, Beiträge zur Algebra und Geometrie, 18 (1984), pp. 191–200.

[HaTr] W. HACKBUSCH AND U. TROTTENBERG, eds., *Multigrid methods. Proceedings of the conference held at Köln-Porz*, November 23–27, 1981, Lecture Notes in Mathematics 960, Springer-Verlag, Berlin, 1981.

[Ho] G. HOTZ, *Einbettung von Streckenkomplexen in die Ebene*, Math. Annalen, 167 (1966), pp. 214–223.

[HBKM] G. HOTZ, B. BECKER, R. KOLLA AND P. MOLITOR, *Ein logisch-topologischer Kalkül zur Konstruktion von integrierten Schaltkreisen*, Informatik Forschung und Entwicklung, 1 (1986), pp. 38–47.

[LuVu] W. K. LUK AND J. VUILLEMIN, *Recursive Implementation of Optimal Time* VLSI *Integer Multipliers*, Proc. IFIP Congress 83, Amsterdam, North-Holland, 1983, pp. 155–168.

[Re] K. REIDEMEISTER, *Einführung in die kombinatorische Topologie*, Vieweg, Braunschweig, 1932.

[SaBh] S. SAHNI AND A. BHATT, *The complexity of design automation*, Proc. 19th Design Autom. Conf., New York, IEEE Computer Soc. Press, 1980, pp. 402–411.

[SaGo] S. SAHNI AND T. GONZALEZ, *P-complete approximation problems*, J. Assoc. Comput. Mach., 23 (1976), pp. 555–565.

[Schm] F. J. SCHMITT, *Rechtwinklige Verdrahtung optimaler Layouts*, Diplomarbeit, Universität des Saarlandes, Saarbrücken, 1985.

[StBu] J. STOER AND R. BULIRSCH, *Einführung in die Numerische Mathematik* II, Springer-Verlag, Berlin, 1978.

[Schw] K. SCHWORM, *Layouts unter Berücksichtigung von Leitungs-breiten*, Diplomarbeit, Universität des Saarlandes, Saarbrücken, 1985.

[Sto] J. A. STORER, *The node cost measure for embedding graphs on the planar grid*, Proc. 12th Annual Symposium on the Theory of Computing, 1980, pp. 201–210.

[St] K. STÜBEN, *Algebraic Multigrid* (AMG): *Experiences and Comparisons*, Arbeitspapiere der GMD Bonn Nr. 23, März, 1983.

[Wh] H. WHITNEY, *A set of topological invariants for graphs*, Amer. J. Math., 55 (1933), pp. 231–235.