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Foldingfree Coverings and Graph Embeddings

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Abstract

In this paper we outline some useful properties and relations between two function classes called foldingfree functions and foldingfree coverings. The notions *foldingfree function* and *covering* are generalizations of *topological function* and *covering*. It is shown that the two classes are equivalent under certain connectivity assumptions.

This result can be applied, e.g., to optimization problems for chip layouts. (Thereby the l_p-norm is used as a cost function to optimize or balance the edge lengths of the graphs modelling the chip's circuit structure.)

1 Introduction

In many areas of computer science, e.g. VLSI-design, networks and operations research, there are interesting problems which can be summarized in the following task:

> *Find an embedding of a graph in a given space which is suitable according to some (optimizing) criteria.*

We concentrate our interest on VLSI-design and outline some embedding problems in this area. If an integrated circuit is modelled with the help of a graph, gates or basic cells are considered as the nodes and the interconnecting signal nets as the edges of the graph. The realization of such a circuit now requires a "planarization" of the corresponding graph, which from the beginning has not to be planar. But if one makes some topological assumptions for the layout of a graph, as it is usual, e.g., in the wellknown algorithms for homotopic routing, then it is possible to consider the problem of the placement of modules or of routing as a topological problem of planar graphs. This leads to a model which was first described in 1965 in [Ho] and which is used in the design system CADIC ([BHKM]) as underlying structure.

In addition layouts of circuits shall fulfill certain conditions, e.g. neither extremely long signal nets (signal delay) nor very short ones (design rules) are allowed. One possibility to fulfil such requirements is to optimize or balance the edge lengths according to certain cost functions. Those functions are often based on so-called force models, which have been developed from several authors independently (s. e.g. [BeHo], [Bl], [FCW], [QuBr]). We only mention one possible cost function, the l_p-norm. Using this cost function the sum of the p-th power of the edge lengths is to be minimized (s. e.g. [Gr]). One borderline case of the l_p-norm is the l_∞-norm (s. z.B. [BeOs]). To illustrate the force model we choose the l_2-norm: Imagine springs instead of the edges between the nodes of the graph. On the outside we then fix the nodes of a "boundary" and permit all other nodes "inside" to move freely. This system moves into a balanced layout, the *optimal* layout according to this cost function.

One question that remains is whether the optimal layout still is planar if the starting layout was. In [BeHo] it is proved that this is true for a very general class of optimizations, following [BeOs] this is not always the case for optimizations according to the l_∞-norm. Since optimization always has to be done by approximation algorithms where some of the calculated layouts can be nonplanar, the planarity theorem in [BeHo] is not sufficient. It is rather necessary to test planarity before stopping the approximation. An efficient planarity test in the case of convex elementary regions is given in [GrHo]. The planarity theorem itself and this test are based on an application of the monodromy lemma and on a generalization of this lemma resp..

Whereas in [GrHo] the monodromy lemma could be applied almost directly, for the goals of [BeHo] it was necessary to generalize the definition of the covering space. This has been done there for piecewise linear functions which is a bit unsatisfactory from a pure mathematical point of view. In this paper the generalization of the covering space is now done for *foldingfree coverings*, where the notion *foldingfree* is a generalization of *piecewise linear*. It is shown that under certain circumstances - which are fulfilled, e.g., for the planarity theorem - foldingfree coverings are even foldingfree. The application of this result to the restricted case of piecewise linear functions leads to a proof of the planarity theorem which can be arranged more clearly. Moreover, our results on foldingfree functions can be used to imply a generalization of the planarity theorem to graph embeddings on the sphere surface.

The paper is structured as follows:

In section 2 the notions *foldingfree* and *foldingfree covering* are defined. All in all a function is foldingfree if the inverse image sets of simply connected balls are simply connected. A function p is a foldingfree covering if there is a simply connected ball U about each image point, s.t. the restriction of p on any connected component V of $p^{-1}(U)$ is a foldingfree function.

In section 3 it is shown that the inverse image set of a point is connected in the case of foldingfree functions. For foldingfree coverings one can prove that the number of connected components of the inverse image set of all points is the same.

Section 4 considers inverse image sets of simply connected sets and shows how to find a deformation of a path with the help of a given deformation of its image path.

In section 5 the preceeding results are used to show that for foldingfree coverings $p\colon \tilde{X} \to X$ the restriction of p on any connected component of $\tilde{X}$ leads to a foldingfree covering if certain compactness assumptions are fulfilled.

2 Definitions and Simple Examples

We now consider functions between n-dimensional manifolds in $\mathbf{R}^m$ (with or without boundary). We analyse relations between local and global properties of these functions and we follow, as far as possible, the proof of the monodromy lemma as given in [Co].

Roughly speaking, this lemma holds for functions with the following property: every restriction of the function on any - sufficiently small - subset of the definition domain is topological. This property therefore is called *locally topological.*

The contents of the lemma is now the following: given a path in the image domain and a point in the definition domain, which is mapped onto the starting point of the path, then there is exactly one inverse image path starting from the given point.

With the help of this lemma one can prove the following implication: If there is a locally topological function where one given point has only one inverse image point, and if the image domain is arcwise connected, then all points have exactly one inverse image point. This can be seen if one assumes that there is another point with more than one inverse image point. Then there exists a path between these two points. Using the lemma one can find unique inverse image paths starting from each of the inverse image points of the second point. But these two paths end in the only inverse image point of the first point. Looking at the path in the other direction one thus finds two inverse image paths starting from the same point, contradicting the lemma. Under these conditions a locally topological function is shown to be globally topological, too.

We are now interested in assertaining similar relations between a local and a global property under weaker assumptions. If a function is topological, then any image point has exactly one inverse image point. This condition shall be weakened considering domains, i.e. balls about image points and requiring that the inverse image set of each domain is simply connected. We start with some definitions and notions concerning the spaces considered.

The spaces we deal with are n-dimensional manifolds $X \subseteq \mathbf{R}^m$, called *real manifolds*. As neighborhoods of a point u are considered so-called *(ε-)balls about u* in X, $U_\varepsilon(u) := X \cap \{x \in \mathbf{R}^m \mid |x-u| < \varepsilon\}$. Such an ε-ball is called *simple* if it is simply connected. An ε-ball about an inner point u is called *complete*, if it is homeomorphic to the unit sphere U^n, about a boundary point v it is said to be *complete* if it is homeomorphic to the hemisphere H^n, where v corresponds to the center of this hemisphere.

We now consider the following type of functions between real manifolds:

Definition 1 (foldingfree) *A function $f: \tilde{X} \to X$ is called foldingfree, iff the two following conditions hold:*

1. *f is continuous and surjective.*
2. *Every simple ball U has a simply connected inverse image set $f^{-1}(U)$.*

It is easy to show that every topological function is foldingfree. In this sense the notion *foldingfree* is a generalization of the notion *topological*. Another example, given without further explanation, is the following:

Example 1 *The function* $f: \mathbf{R} \to \mathbf{R}$ *where*

$$f(x) := \begin{cases} n & \text{if } x \in [2n-1, 2n] \\ x - n & \text{if } x \in (2n, 2n+1) \end{cases}$$

(see figure 1) is foldingfree.

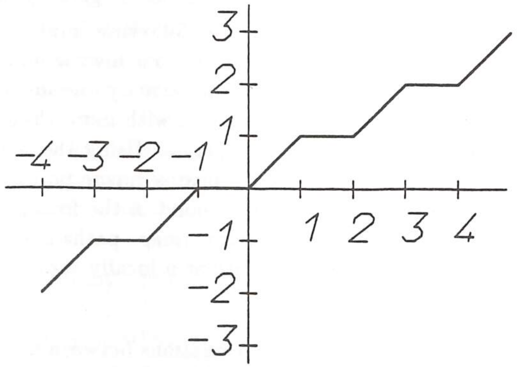

Figure 1

The formal definition of the "local" property is now:

Definition 2 (foldingfree covering) *A function* $p: \tilde{X} \to X$ *is called a foldingfree covering, iff for each point* $x \in X$ *there exists a simple ball* U*, such that the following two conditions hold:*

1. *Every connected component* V *of* $p^{-1}(U)$ *is open and simply connected.*

2. $p \mid V: V \to U$ *is foldingfree.*

We call the simple ball U *about* x *elementary neighborhood (of* x *and all points inside* U*) and the connected components* V *the corresponding elementary components.*

For condition (2) all simple balls inside an elementary neighborhood are again elementary neighborhoods. With the help of this one can easily derive that foldingfree coverings are continuous and surjective, too.

Of course topological functions are foldingfree coverings as well. Further examples are:

Example 2 1. *The function* f *in Example 1 is a foldingfree covering.*

2. *The function* $p: \{0,1\} \times \mathbf{R} \to \mathbf{R}$ *with* $(\xi, x) \mapsto f(x)$*, where* $f: \mathbf{R} \to \mathbf{R}$ *is defined as in Example 1, is a foldingfree covering.*

In the following we investigate relations between the local and the global property of functions between compact, n-dimensional real manifolds $\tilde{X}$ and X. In addition we assume the image space X to be connected. (Because X is compact we then yield that X is arcwise connected.)

We examine inverse images of points, paths and connected sets and achieve, finally, the announced theorems.

3 Inverse Images of Points

According to the definition of foldingfree functions, the inverse image sets of simple balls - of each size - are simply connected. In Lemma 1 we will show that in the limit case of a point the inverse image remains at least connected.

In the following we use the notion *inverse class* for a connected component of the inverse image set of a point.

Lemma 1 *Let* $f: \tilde{X} \to X$ *be a foldingfree function and* x *any point in* X*.*
Then $f^{-1}(x)$ *is connected, i.e. each image point has exactly one inverse class.*

Proof: Assume $f^{-1}(x)$ is not connected, i.e. there exist two in $f^{-1}(x)$ closed subsets K_1 and K_2 of $f^{-1}(x)$, s.t. $K_1 \cup K_2 = f^{-1}(x)$, $K_1 \cap K_2 = \emptyset$. Then there exist open sets M_1, M_2, $M_i \supset K_i$ with $\overline{M_1} \cap \overline{M_2} = \emptyset$. It is clear that $f^{-1}(x) \subset M_1 \cup M_2$ holds (s. Figure 2). We consider now a sequence U_n of simple balls about x with $\bigcap_{n \in \mathbf{N}} U_n = \{x\}$ and $U_n \supset U_{n+1}$. Then we know, because $x \in U_n$, that $f^{-1}(U_n) \cap M_i \neq \emptyset$, $(i = 1,2)$ $\forall\, n \in \mathbf{N}$. Moreover $f^{-1}(U_n)$ is (simply) connected, since f is foldingfree, and thus: $V_n := f^{-1}(U_n) \setminus (\overline{M_1} \cup \overline{M_2}) \neq \emptyset \;\; \forall\, n \in \mathbf{N}$, and V_n is open, because we substract a closed set from an open one.
With $V_n \subset \tilde{X}$ we know that V_n is bounded. Since $U_n \supset U_{n+1}$ also $f^{-1}(U_n) \supset f^{-1}(U_{n+1})$ holds and thus $V_n \supseteq V_{n+1}$.

We now distinguish between the two following cases:

1. case: $\bigcap_{n \in \mathbf{N}} V_n \neq \emptyset$

Let be $\tilde{x} \in \bigcap_{n \in \mathbf{N}} V_n$. Because $f^{-1}(x) = \bigcap_{n \in \mathbf{N}} f^{-1}(U_n)$ we yield $\bigcap_{n \in \mathbf{N}} V_n \subset f^{-1}(x)$, i.e. $f(\tilde{x}) = x$. But then there exists an $i \in \{1,2\}$ s.t. $\tilde{x} \in K_i$ in contradiction to the definition of V_n.

2. case: $\bigcap_{n \in \mathbf{N}} V_n = \emptyset$

One can easily derive that there exists an $N \in \mathbf{N}$ and a point $\tilde{x}$ with $\tilde{x} \in \partial V_n \ \forall\, n > N$. Then this point $\tilde{x}$ is accumulation point of a sequence $(\tilde{x}_n)$ with $\tilde{x}_n \in V_n$. Since f is continuous, $f(\tilde{x})$ has to be an accumulation point of $(f(\tilde{x}_n))$. On the other side is $f(\tilde{x}_n) \in U_n$ and thus this sequence converges towards x. So again $f(\tilde{x}) = x$ holds, i.e. $\tilde{x} \in K_i$ for an $i \in \{1, 2\}$, in contradiction to the definition of V_n. ∎

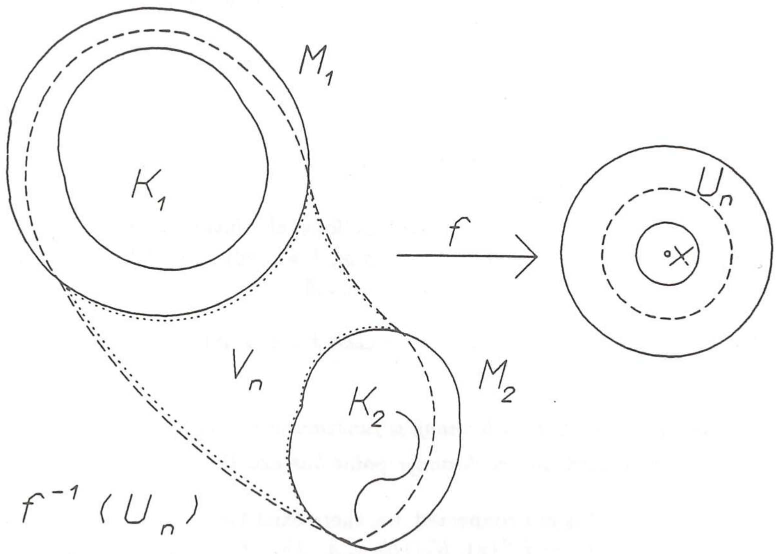

Figure 2

For the investigation of inverse image sets in foldingfree coverings we use the following definition:

Definition 3 (characteristic) *Let $g\colon \tilde{X} \to X$ be a function. The characteristic of a point $x \in X$ with respect to g ($char_g(x)$) is given by the number of its inverse classes, i.e. the number of connected components of $g^{-1}(x)$.*
If all points in X have the same characteristic according to g, then we call this the characteristic of the whole function. Note that the characteristic of a function is not always defined.

According to Lemma 1 foldingfree functions have characteristic 1. As to foldingfree coverings we just know the following: Every point has finite characteristic, since we can cover the compact space $\tilde{X}$ with a finite number of elementary components, inside of these every point has exactly one inverse class. We show that the characteristic of foldingfree coverings is defined too. This is a first step towards the proof, that *foldingfree function* and *folding covering with characteristic 1* are equivalent notions.

Lemma 2 *Let $p\colon \tilde{X} \to X$ be a foldingfree covering.*
Then all points have the same characteristic according to p, i.e. the characteristic of function is defined.

Proof: First we define the set

$$X_c := \{x \in X \mid char_p(x) = c\}$$

which is a subset of X. X is connected, i.e. the only subsets of X, which are open and clo in X at the same time, are X itself and the empty set. Now we show, that X_c is open closed in X, thus $X_c = \emptyset$ or $X_c = X$. Then we yield for any $x \in X$ that $X_{char_p(x)} =$ i.e. all points in X have the same characteristic.

First we remark that for every point $x \in X$ and any elementary neighborhood $U(x)$ of t point holds: $char_p(z) = char_p(z') \ \ \forall\, z, z' \in U(x)$. This can easily be seen from the f that $p \mid V$ is foldingfree, if V is any elementary component corresponding to $U(x)$ and t $char_{p|V} = 1$. So the characteristic of all points in $U(x)$ is equal to the number of element components corresponding to $U(x)$.

X_c is open:

Consider a point $x \in X_c$ and an elementary neighborhood $U(x)$ of x. Because of the ab remark we know, that $U(x) \subset X_c$, i.e. X_c is open.

X_c is closed:

Consider a point $x \in \partial X_c$ and an elementary neighborhood $U(x)$ of x. If $x \in \partial X_c$, then th exists $x' \in X_c \cap U(x)$. From the above remark we know, that then $char_p(x) = char_p(x') =$ So $x \in X_c$, i.e. X_c is closed.

4 Inverse Images of Simply Connected Sets

Before stating relations between *foldingfree functions* and *coverings* we need furt information about inverse images of simply connected sets. To do this we define the noti *suitable ball* which is defined for both types of mappings. With the help of this term we c prove the following lemmata for foldingfree functions and coverings as well.

Definition 4 (suitable ball) *Let be $g\colon \tilde{X} \to X$ a foldingfree function or covering. simple ball $U \subseteq X$ is called suitable, if the following holds:*

1. *Any connected component V of the inverse image of U is open and simply connec with $g(V) = U$.*

2. *Any point in such a simple ball U has exactly one inverse class in V.*

In the following we call a connected component of the inverse image of a suitable ball an inverse component.

It follows from the definition, that in the case of foldingfree coverings the elementary neighborhoods are suitable balls, for foldingfree functions any simple ball is suitable.

From now on we consider a foldingfree function or covering $g\colon \tilde{X} \to X$.

Lemma 3 *Let be U_1 and U_2 suitable balls in X, V_1 and V_2 corresponding inverse components with $V_1 \cap V_2 \neq \emptyset$. Then it holds:*

$$\begin{aligned} g(V_1 \cap V_2) &= U_1 \cap U_2 \\ g(V_1 \setminus V_2) &= U_1 \setminus U_2 \\ g(V_2 \setminus V_1) &= U_2 \setminus U_1 \end{aligned}$$

Proof: It is clear that $g(V_1 \cap V_2) \subseteq U_1 \cap U_2$. Assume now that $(U_1 \cap U_2) \setminus g(V_1 \cap V_2) \neq \emptyset$.

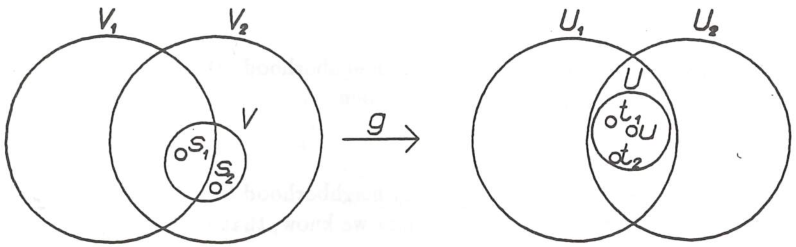

Figure 3

Then there exists a $u \in \partial g(V_1 \cap V_2) \cap (U_1 \cap U_2)$. Let be U a suitable ball about u, s.t. $U \subset U_1 \cap U_2$. For $u \in \partial g(V_1 \cap V_2)$ there exists $t_1 \in U \cap g(V_1 \cap V_2)$ and thus there is $s_1 \in V_1 \cap V_2$ with $g(s_1) = t_1$. Let V now be an inverse component corresponding to U with $s_1 \in V$. Because of $u \in \partial g(V_1 \cap V_2)$, there also exists $t_2 \in U \setminus g(V_1 \cap V_2)$, and a corresponding $s_2 \in V \setminus (V_1 \cap V_2)$. W.l.o.g. assume $s_2 \notin V_1$. We know that V is simply connected, i.e. especially arcwise connected, $V \cap V_1 \neq \emptyset$, $V \setminus V_1 \neq \emptyset$ and $g(V) = U \subset g(V_1) = U_1$, although V_1 is a connected component of $g^{-1}(U_1)$. Contradiction! So we yield $g(V_1 \cap V_2) = U_1 \cap U_2$.
With the help of this we know now that all points in $U_1 \cap U_2$ don't have any inverse images in $V_1 \setminus V_2$ resp. $V_2 \setminus V_1$, because such an inverse image must be connected with the one in $V_1 \cap V_2$ and thus belongs to V_1 and V_2. This completes the proof. ∎

Lemma 4 *Let be U_1 and U_2 suitable balls in X, V_1 and V_2 corresponding inverse component with nonempty intersection. If there exists a path in $U_1 \cap U_2$ between two points $t_1, t_2 \in U_1 \cap U_2$ then there also exists a path in $V_1 \cap V_2$ between any two inverse images of t_1 and t_2.*

Proof: Because of lemma 3 there are two points $s_1, s_2 \in V_1 \cap V_2$ with $g(s_i) = t_i$, $(i = 1, 2)$ As assumed there is a path from t_1 to t_2 in $U_1 \cap U_2$. Consider a finite cover of this path with suitable balls $W_1, \ldots, W_k$ inside $U_1 \cap U_2$. Now we find inside V_1 unique corresponding inverse components of the W_i, called $Q_1, \ldots, Q_k$.

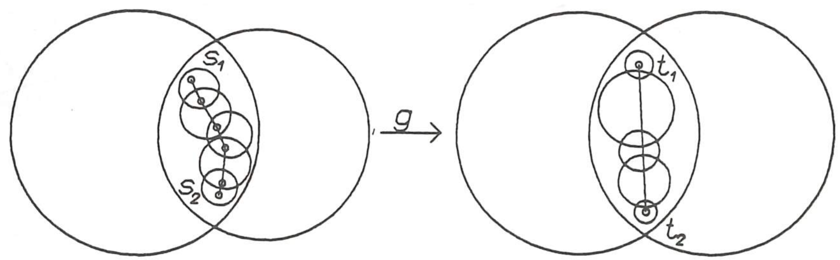

Figure 4

Apply lemma 3 again, to see that these inverse components are inside $V_1 \cap V_2$, and since $g \mid V_1$ has characteristic 1, we know $Q_i \cap Q_{i+1} \neq \emptyset$, $i = 1, \ldots, k-1$. For the same reason i $s_1 \in Q_1$ and $s_2 \in Q_k$ resp. . Now we can find points $q_i \in Q_i \cap Q_{i+1}$ $(i = 1, \ldots, k-1)$ inside the intersection of two successive inverse components. Since any inverse component is arcwise connected, there is a path from s_1 to s_2 passing the points q_i inside the Q_i $(i = 1, \ldots, k)$ i.e. in $V_1 \cap V_2$. ∎

Consider now $g\colon \tilde{X} \to X$ a foldingfree function or covering, $\alpha\colon [0,1] \to \tilde{X}$ a path, s.t. the image path $g\alpha\colon [0,1] \to X$ is cyclic. Let $\phi\colon [0,1] \times [0,1] \to X$ be a deformation of $g\alpha$ to a point y. What we are going to do now is the following: We overlay the image set o the deformation ϕ with a finite lattice, s.t. each mesh of the lattice lies inside a suitable ball. Using the two lemmata mentioned above we then find a corresponding lattice with meshes inside the corresponding inverse components. This allows us to perform an analogous deformation with α.

First observe that ϕ is a continuous function on a compact set, i.e. ϕ is uniformly continuous and $Im\phi$ is compact. With the help of the compactness of $Im\phi$ it is easy to show that there exists an $\varepsilon > 0$, s.t. for all $x \in Im\phi$ the simple ε-ball about x is suitable. With the help o ϕ uniformly continuous we find a $\delta := \delta(\varepsilon)$, s.t. for all $x := \phi(t,s) \in Im\phi$ the image of the square $R := [t-\delta, t+\delta] \times [s-\delta, s+\delta]$ is contained in the suitable ball $U_\varepsilon(x)$ (s. figure 5)

$$\phi(R) \subset U_\varepsilon(x)$$

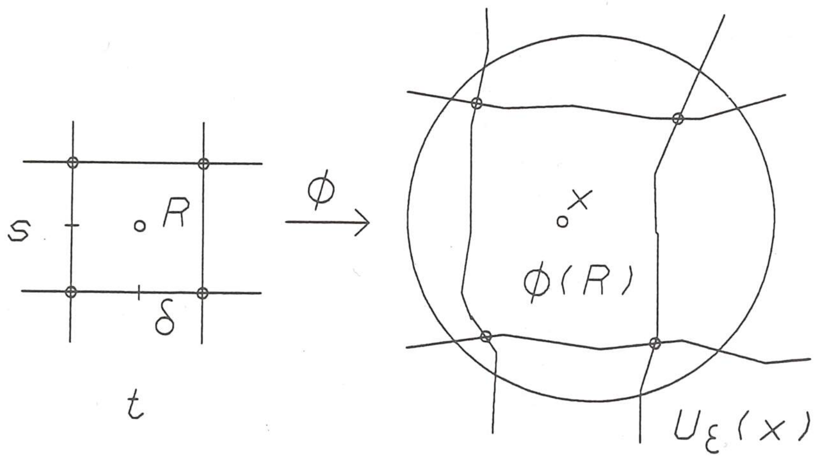

Figure 5

The images $\phi(R)$ of such squares are to become the meshes of the mentioned lattice. To achieve this we choose δ (w.l.o.g.) such that - in addition to the above property - the intervall $[0,1]$ is partitioned into $k \in \mathbf{N}$ intervals of size 2δ. I.e. $[0,1] \times [0,1]$ is divided in squares

$$R_{n,m} := [2n\delta, 2(n+1)\delta] \times [2m\delta, 2(m+1)\delta]$$

with $n, m \in \{0, \ldots, k-1\}$. Applying ϕ to these squares we yield *meshes* $\phi(R_{n,m}) \subset U_\varepsilon(x_{n,m})$, where $x_{n,m} := \phi((2n+1)\delta, (2m+1)\delta)$ is the "center" of the mesh. It is clear that the set of all $U_\varepsilon(x_{n,m})$ is a finite cover of $Im\phi$. Starting with $R_{o,o}$ we are now able to construct successively a lattice in $\tilde{X}$ for the deformation of α. First we search the elementary components corresponding to the neighborhoods which build the cover of the lattice in X. For that purpose we define two sets C and F, C contains at the beginning all elementary neighborhoods which build the cover of the image lattice, i.e. $C := \{U_\varepsilon(x_{n,m}) | n, m \in \{0, \ldots, k-1\}\}$; F will contain those elementary neighborhoods, whose components are already found, i.e. we start with F to be the empty set. Consider now $\phi(R_{o,o})$. There is a unique corresponding inverse component which contains $\alpha(0)$. Thus we take $\phi(R_{o,o})$ out of C and put it into F.

Inductively we can find an elementary neighborhood W_1 in C, which has a nonempty intersection with a W_2 in F. Consider any point in the intersection of these two neighborhoods. Such a point has a unique inverse class in the inverse component Q_2 corresponding to the elementary neighborhood W_2 in F. Now there is exactly one inverse component Q_1 containing this inverse class which corresponds to the neighborhood W_1 in C. Aided by the above lemmata we know that the component Q_1 has "correct" intersections with all already constructed components of the inverse image cover. More precisely, exactly the intersections of the inverse components are mapped onto the intersections of the neighborhoods of the image cover, each point in an intersection of two or more neighborhoods has an inverse class in the intersection of the corresponding components of these neighborhoods. So we can take W_1 out of C and put it into F.

We continue until C is empty. We call the inverse components $V_{n,m}$. As mentioned above we find inverse classes of all "lattice points" $(2n\delta, 2m\delta)$, $(n, m \in \{0, \ldots, k-1\})$ because of Lemma 3 inside the intersections of the "correct" $V_{n,m}$'s. With Lemma 4 there are paths inside the intersections as in $Im\phi$, building meshes each lying totally inside an inverse component $V_{n,m}$. Thus the analogous lattice can be constructed.

Because each mesh of the inverse lattice is homotopic to zero, one easily sees, that any deformation which works "meshwise" in X can be imitated using the analogous lattice in $\tilde{X}$. In particular, one easily can see that a cyclic path α is homotopic to zero, too.

We summarize the above observations in the following Lemma:

Lemma 5 *Let be $g: \tilde{X} \to X$ a foldingfree function or covering, $\alpha: [0,1] \to \tilde{X}$ a path, s.t. the image path $g\alpha: [0,1] \to X$ is cyclic. Let $\phi: [0,1] \times [0,1] \to X$ be a deformation of $g\alpha$ to a point y. Then we can find an "analogous" deformation of α, as described above.*

5 Characterizing Theorems

After these preparations we are now able to formulate a first relation between *foldingfree functions* and *coverings*:

Theorem 1 *Let $\tilde{X}$ and X be compact, real n-dimensional manifolds (with or without boundary), in addition let X be connected.*
Then the following statements concerning a function $g: \tilde{X} \to X$ are equivalent:

- *g is foldingfree*
- *g is a foldingfree covering with characteristic 1.*

Proof:
'$\Longrightarrow$'

Let be g a foldingfree function. With Lemma 1 we know that all points have characteristic 1. It remains to show that g is a foldingfree covering.

For that purpose consider any point $x \in X$ and a simple ball U about it. Because g is foldingfree, the inverse image set $g^{-1}(U)$ is simply connected, i.e. there is exactly one inverse component $V = g^{-1}(U)$ which is open and simply connected.

We still have to prove that the restriction $g \mid V$ is a foldingfree function.

It is clear that $g(V) = U$ because g is surjective and $V = g^{-1}(U)$. Since $U_\delta(z) \cap U \subseteq U$, we have $g^{-1}(U_\delta(z) \cap U) \subseteq g^{-1}(U) = V$, and thus $(g \mid V)^{-1}(U_\delta(z) \cap U) = g^{-1}(U_\delta(z) \cap U)$. As to the simple connectivity of $g^{-1}(U_\delta(z) \cap U)$ consider any cyclic path α inside this set. Then $g\alpha$ is cyclic too, and both paths fulfil the conditions of Lemma 5. So α is homotopic to zero.

This yields that g is a foldingfree covering with characteristic 1.

'$\Longleftarrow$'

Let now g be a foldingfree covering with characteristic 1. We have already mentioned that foldingfree coverings are continuous and surjective functions. Consider any simple ball U about any point in X. Then every cyclic path α in $g^{-1}(U)$ is homotopic to zero, because we can imitate the deformation made with its image path $g\alpha$ as described in Lemma 5. Thus $g^{-1}(U)$ is simply connected.
We yield that the function g is foldingfree. ■

We consider now the more general case of foldingfree coverings with characteristic n greater than 1. Because of Lemma 2 we know that for a foldingfree covering all points have the same characteristic which means that the characteristic of such a function is defined. What we now can prove is that the domain $\tilde{X}$ is partitioned into n *leaves*, i.e. connected components (which are indeed simply connected), s.t. the restriction of the function on any leaf is foldingfree, if the domain X is simply connected.

Theorem 2 *Let $\tilde{X}$ and X be compact, real n-dimensional manifolds (with or without boundary), X simply connected and $p\colon \tilde{X} \to X$ a foldingfree covering of characteristic n.*
Then $\tilde{X}$ is the disjoint union of n connected components $\tilde{X}_i$ $(i = 1, \ldots, n)$ with $p \mid \tilde{X}_i$ is a foldingfree function.

Proof: Consider any connected component $\tilde{X}_o$ of $\tilde{X}$. Because of Theorem 1 it suffices to show that $p \mid \tilde{X}_o$ has characteristic 1. We will do this in two steps. First we prove that $p(\tilde{X}_o) = X$, i.e. each image point has at least one inverse image point in $\tilde{X}_o$. Afterwards we show that each point has at most one inverse class in $\tilde{X}_o$.

$char_{p|\tilde{X}_o} \geq 1$:

We prove that $p(\tilde{X}_o)$ is equal to X. Because X is connected, it suffices to show that $p(\tilde{X}_o)$ is at the same time open and closed in X.

As to open, assume we have an $x \in p(\tilde{X}_o)$. Then there is an inverse image point $\tilde{x} \in \tilde{X}_o$ of x. Consider now an elementary neighborhood U about x and a corresponding elementary component V to U, which contains $\tilde{x}$. Because V is connected, $\tilde{X}_o$ connected component and $\tilde{x} \in V \cap \tilde{X}_o$, it follows that $V \subset \tilde{X}_o$. But then $p(V) = U \subset p(\tilde{X}_o)$, i.e. $p(\tilde{X}_o)$ is open.

As to closed, assume we have an $x \in \partial p(\tilde{X}_o)$ and again an elementary neighborhood U of x. Then there is a point $x_1 \in U \cap p(\tilde{X}_o)$. Since $x_1 \in p(\tilde{X}_o)$, we find an inverse image point $\tilde{x}_1 \in \tilde{X}_o$ of x_1. Let be V the elementary component corresponding to U which contains this point $\tilde{x}_1$. Again we know that $V \subset \tilde{X}_o$ and thus $p(V) = U \subset p(\tilde{X}_o)$, which means in particular that $x \in p(\tilde{X}_o)$, i.e. $p(\tilde{X}_o)$ is closed.

$char_{p|\tilde{X}_o} \leq 1$:

First remark that $\tilde{X}_o$ is arcwise connected: because it is connected and compact we can find a finite cover of $\tilde{X}_o$ with arcwise connected elementary components. Assume now that there are two points z_1, z_2 in $\tilde{X}_o$ with $p(z_1) = p(z_2)$, s.t. z_1 and z_2 belong to two different inverse classes of $p(z_i)$, $i = 1, 2$. Consider a path α from z_1 to z_2 in $\tilde{X}_o$. Then $p\alpha$ is cyclic. Because X is simply connected we can find a deformation of $p\alpha$ to a point y and two corresponding lattices in X resp. $\tilde{X}_o$. Perform now a meshwise deformation on the two lattices in the following way: First we have the actual paths α and $p\alpha$ rotating around the collection of all meshes. We will now inductively deform the actual paths such that one mesh after the other is transferred to the outside until only $\phi(R_{o,o})$ and its analogon in $\tilde{X}_o$ resp. remain inside the actual path. Notice that this means that the actual paths both always start and end in the same points. The image path then is part of $U_\varepsilon(\phi(x_{o,o}))$. In $\tilde{X}_o$ the actual path leads from z_1 to z_2, thus must be part of two different elementary components since two different inverse classes cannot lie in the same component. However, these components are components of the same elementary neighborhood and have nonempty intersection, which also is impossible!

So the characteristic of $p \mid \tilde{X}_o$ equals 1 and the theorem is proved. ■

We now resume our results in the following theorem:

Theorem 3 *Let be $\tilde{X}$ and X compact, real n-dimensional manifolds (with or without boundary), X be simply connected.*

1. *If $\exists\, x \in X$ with $char_g(x) \geq 1$, then*
   g is a foldingfree covering iff g is composed of $char_g(x)$ foldingfree functions.
2. *If $\exists\, x \in X$ with $char_g(x) = 1$, then this means:*
   g is a foldingfree covering iff g is foldingfree.

Proof: The proof follows directly with Lemma 2, Theorem 1 and Theorem 2. ■

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