The Decidability of the Knot Problem

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Abstract

AFLs are normalforms for knot representations based on projections of the knot on a plane H, which can be decomposed intwo simple curves L, A as introduced by K. F. Gauss. We decompose A in a sequence $\alpha := (x_1, x_2, ..., x_n)$ of segments, which can be considered as projections of arcs of an arcade with arcs in the upper half space H^+ and arcs in the lower half space H^-. We may assume that the x_i are alternating projejctions from H^+ and H^-. α plays the role of a ccordinate system to describe L by a word over an alphabet, which represents, in which direction L crosses the arc projections. The pairs (α, L) are the AFLs we use as normalforms for the knot projections. An AFL is reduced, when it cannot be simplified by applications of Reidemeister-moves of type R1 and R2. To each AFL there exists up to isomorhpy only one reduced normalform.

The transformations of knot projections into AFLs are sequences of moves of loops, which are segmnts of L, along simple prefix segments of the actual string L onto the actual arcade. For this moves exist many strategies. We use only two of them ν^1 and ν^2, which are uniguely determined by its behavior relativ to any pair (s_i, t_j) of arcs the move of the loop passes through respectiv over. After each such move we reduce the patially given AFL by R1- and R2-moves.

The discussion of these moves is guided by the following idea: Let K be a knotprojection and K' generated by the application of a Reidemeister-move on K. If $\nu := (\nu_1, \nu_2, ...\nu_p)$ is a sequence of moves of the strategies ν^1, ν^2 transformig K into a reduced AFL (α, L), then there exists a sequence $\nu' := (\nu'_1, \nu'_2, ..., \nu_{p'})$ transforming K' into a AFL (α', L') isomorphic to (α, L).

This idea needs an extension of the strategies to work. In some cases after the move of type ν^i of the loop and the reduction we have to do a redraw of the loop out of the arcade and to apply a move of type $\nu^j \neq \nu^i$ on this configuration. On this base we are able to realize the idea.

This construction defines a nondeterministic algorithm to decide the equivalence problem of knots. The number of arcs which may be generated by these constructions is bounded by 2^p, p the number of crossing points of K. So we are able to decide the equivalence of two knot projections K, K' in time $2^{O(max\{p,p'\})}$.

We use the description of the AFLs by words of a formal language, but this description is not essential for the construction. A translation of the constructions into a formal system would make it unreadable.

1 Introduction

1.1 The Idea and Definitions

For an introduction in knot theory see [4],[3]. The author has discussed in [5] and [6] a class of knot representations which had been suggested to him by Kurt Reidemeister [1]. These representations are based on a remark of K.F. Gauss [2] that each knot has projections on the plane which can be decomposed in two strings without intersecting itself. Figure 1 shows as an example the projection of a trefoil knot and a decomposition of the projection in two simple segments defined by the points A, B. Reidemeister handeled both strings in an unsymmetrical manner. One of the strings called faden L remains in the plan, the other one he moves into the $\mathbf{R^3}$ forming an arcade α with arcs alternating on the upper- and the lower side of the plane. The theory becomes simpler if we use projections on the $2-sphere$, the surface S of a ball and build the arcade on S.

We may assume that the projection of the arcade forms a straight line on S. The projections of the upper arcs we will represent by red lines the projections of the arcs under the plane by blue lines. We assume the knot to be oriented and the orientation transferred to the projection. The projections of the sequence of arcs are numbered and oriented corresponding to the orientation of the knot. The i-th arc of the arcade gets the name s_i if it is the projection of an upper arc and the name t_i in the other case. We may assume that the sequence of the projections of the arcs of the arcade alternates in its color. An arcade therefore can be considered as an alternating finite sequence α

$$s_1, t_2, s_3, t_4, ... \quad \text{or} \quad t_1, s_2, t_3, s_4, ...$$

ending with a red arcade s_n or a blue arcade t_n. The pairs (α, L) are the arcade-string-configurations (Arkaden-Faden-Lagen) introduced as knot

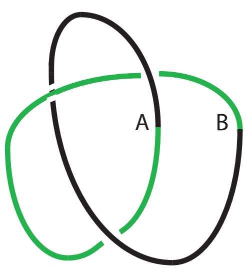


Figure 1

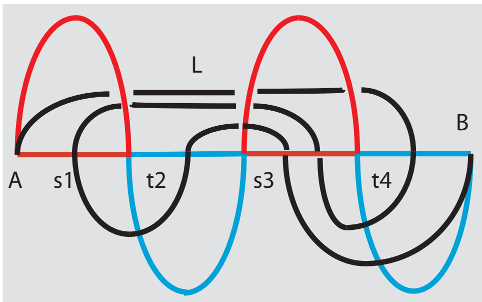


Figure 2

representations by Reidemeister [5]. Figure 2 shows such a representation of the knot 4_1 [3] p.363. For shortness we call the arcade-string-configurations AFL. We assign to each AFL (α, L) a signature

$$\sigma_\alpha(L) := x_{i_1}^{\epsilon_1} * x_{i_2}^{\epsilon_2} * \cdots * x_{i_n}^{\epsilon_n} \quad \text{for} \quad x_{i_l} \in \{s, t\} \quad \text{and} \quad \epsilon_i \in \{+1, -1\}$$

which is defined as follows: Let

$$P_1, P_2, ..., P_n$$

the sequence of the crossing-points of L with the arcade α in the order they appear on L relative to the orientation of the knot. The alphabet element x_{i_l} belongs to the point P_l. If P_l is crossing-point of L with a red arc of α, then we define $x = s$ else $x = t$ and we define $i_l := k$ if the arc is the k-th element in the enumeration of the arcs of α. We define $\epsilon_l := 1$ iff L crosses x_k from left to right else $\epsilon_l := -1$ relative to the knot orientation. For our example Figure 2 we get relative to both of the possible orientations of the knot

$$\sigma_\alpha(L) = t_1 * s_2^{-1} * t_3$$

The exponents of the variables are independent from the orientation of the knot. Changing the orientation results in the reflection of the signature and the numbering of the arcs. Turning the knot round the arcade as axis does not change the signature up to the trivial isomorphism, which exchanges s and t variables.

In [5],[6] it has been shown that reductions of the signature $\sigma(L)$ by applying sequences of substitutions of the following rules (1),...,(4) generate words, which not in each case represent AFLs but always projections of the same knot. But it is possible to apply the reduction rules in such orders that each reduction step corresponds to an AFL, which can be constructed from the original AFL by applying a sequence of the Reidemeister moves of type R1 and R2. Here we use that the arcade is build on S.

$$x_{i_l}^{\epsilon} * x_{i_l}^{-\epsilon} \longrightarrow 1 \quad \text{for} \quad x \in \{s, t\}, \tag{1}$$

$$x_{i_1}^{r} \longrightarrow 1 \quad \text{for} \quad i_1 = n, \quad x_{i_n}^{r} \longrightarrow 1 \quad \text{for} \quad i_n = 1, \tag{2}$$

$$x_j \longrightarrow x_{j-2} \quad \text{for} \quad i > 0, j > i+1, \quad \text{and} \quad x_{i+1} \quad \text{not in} \quad \sigma(F), \tag{3}$$

$$i \longrightarrow i-1 \quad \text{for} \quad i > 1, \quad x_1 \quad \text{not in} \quad \sigma(F) \tag{4}$$

We use this rules to assign to each knot $\mathbf{K}$ a formal languages $\mathrm{L_K}$ over an infinite alphabet X.

$$S := \{s_i^\epsilon : i \in \mathrm{N}, \epsilon \in \{-1, 1\}\}, \quad T := \{t_i^\epsilon : i \in \mathrm{N}, \epsilon \in \{-1, 1\}\}, \quad X := S \cup T.$$

The languages have the special structure

$$\mathrm{L} \subset (S \cdot T)^* \cup (T \cdot S)^* \cup T \cdot (S \cdot T)^* \cup (S \cdot T)^* \cdot S.$$

The operation $\cdot$ is the product, which concatenates the sequences of the free monoid X^*. To each knot $\mathbf{K}$ belongs an infinite set of knot projections K and to each projection *normalization* to AFLs (α_K, F_K), which we describe uniquely by their signature $\sigma_\alpha(F_K)$. We choose special normalization algorithms and define for each oriented knot projection $\mathcal{K}$ of the not orientd knot $\mathbf{K}$ relative to the chosen class of normalizations the language

$$\mathrm{L}_{\mathbf{K}} := \{\sigma_\alpha(L) : (\alpha, L) \quad \text{is a normalization of a projection} \quad \mathcal{K} \quad \text{of} \quad \mathbf{K}\}$$

Two knots $\mathbf{K}, \mathbf{K'}$ are equal iff $\mathrm{L}_{\mathbf{K}} = \mathrm{L}_{\mathbf{K'}}$ holds. This means that the word problem of these languages is equivalent to the knot problem.
We define two special types ν^1 and ν^2 of moves of points of $\mathcal{K}$ on the arcades. The *normalization* ν are sequences of such moves. We prove that to each Reidemeister-move $\mathcal{K} \longrightarrow \mathcal{K}'$ and normalization ν of $\mathcal{K}$ there exists a normalization ν' of $\mathcal{K}'$ such that the resulting AFLs can be transformed into each other by Reidemeister-moves of type R1 and R2. This means that the corresponding signatures may be reduced on the same short-word. The reductions are achieved by R1- and 2-moves corresponding to the rules (1),...,(4). The proof needs the assumption that the Reidemeister move transforming $\mathcal{K}$ into $\mathcal{K}'$ does not concern parts of the arcade.

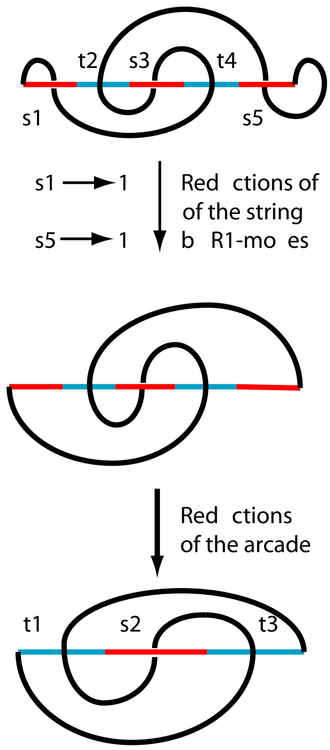


Figure 3

1.2 Examples

Example 1 - see Figure 3 - shows the reduction of an AFL representing a trefoil. We see the elementary relation between the Reidemeister moves $R1$ and $R2$ and the reduction rules applied on the signatures

$$\sigma_1 = s_5^{-1} \cdot t_2 \cdot s_3^{-1} \cdot t_4 \cdot s_1^{-3}, \quad \sigma_2 = t_2 \cdot s_3^{-1} \cdot t_4, \quad \sigma_3 = t_1 \cdot s_2^{-1} \cdot t_3$$

Example 2 - Figure 4 - shows the reduction of an AFL representing a circle. The Reidemeister moves of type $R1$ and $R2$ in the graphics correspond to the following reduction steps of the related signatures.

$$\begin{aligned}
\sigma_1 &= s_1^{-1} \cdot t_2 \cdot s_3^{-1} \cdot s_3 \cdot t_4^{-1} \\
&\downarrow \quad s_3 \cdot s_3^{-1} \longrightarrow 1 \\
\sigma_2 &= s_1^{-1} \cdot t_2 \cdot t_4^{-1} \\
&\downarrow \quad s_3 \longrightarrow t_2, t_4 \longrightarrow t_2 \\
\sigma_3 &= s_1^{-1} \cdot t_2 \cdot t_2^{-1} \\
&\downarrow \quad t_2 \cdot t_2^{-1} \longrightarrow 1 \\
\sigma_4 &= s_1^{-1} \\
&\downarrow \quad t_2 \longrightarrow s_1, s_1 \longrightarrow 1 \\
\sigma_5 &= 1
\end{aligned}$$

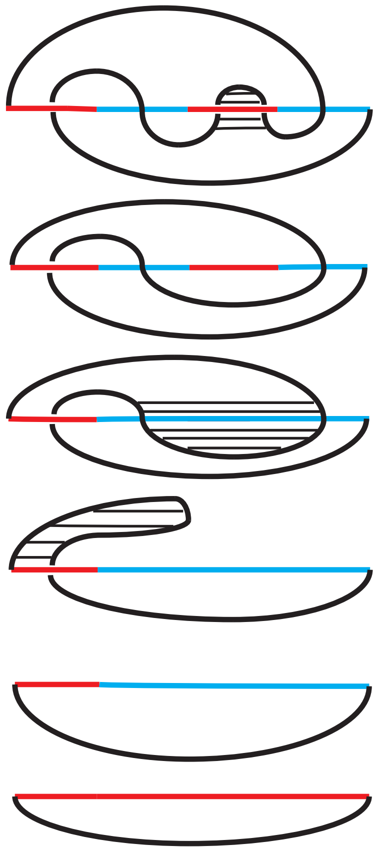


Fig re 4

2 Reductions

2.1 The Reduction of the Signatures

It is well known from the theory of free groups that the application of the reductions (1)

$$x_i \cdot x_i^{-1} \longrightarrow 1 \quad \text{and} \quad x_i^{-1} \cdot x_i \longrightarrow 1$$

is commutative. In other words the result of reducing a word w relative to this rules completely to a word $\bar{w}$ is independent from the order the rules will be applied. This remains true if we add the reduction rules (2): If the first of the rules (2) can be applied to w, then w has the form

$$w = x_1^{\epsilon} \cdot \tilde{w} \longrightarrow \tilde{w}$$

If a production of (1) can be applied on w then it holds

$$w = w_1 \cdot x^{\eta} \cdot x^{-\eta} \cdot w_2$$

If $w_1 \neq 1$ then we can write $w_1 = x_1^{\epsilon} \cdot w_1'$ and have as results for the two possible reductions

$$w = x_1^{\epsilon} \cdot w_1' \cdot x_i^{\eta} \cdot x_i^{-\eta} \cdot w_2 \longrightarrow x_1^{\epsilon} \cdot w_1' \cdot w_2 \longrightarrow w_1' \cdot w_2$$

$$w = x_1^{\epsilon} \cdot w_1' \cdot x_i^{\eta} \cdot x_i^{-\eta} \cdot w_2 \longrightarrow w_1' \cdot x_i^{\eta} \cdot x_i^{-\eta} \cdot w_2 \longrightarrow w_1' \cdot w_2$$

If $w_1 = 1$ then the correspond reductions have the form

$$w = x_1^{\epsilon} \cdot x_1^{-\epsilon} \cdot w_2 \longrightarrow w_2$$

$$w = x_1^{\epsilon} \cdot x_1^{-\epsilon} \cdot w_2 \longrightarrow x_1^{-\epsilon} \cdot w_2 \longrightarrow w_2$$

We see that the resulting short words under reductions of type {(1),(2)} are independent from the order we apply the reduction rules.
We now discuss the the complete reduction system {(1),(2),(3),(4)}. The rules (3) and (4) identify some variables and in connection with this produce a shift in the variable names. It is clear that each production, which could be applied before an application of a rule from (3),(4) can be applied afterwards too in some cases with shifted variables. But after an application of rules from (3),(4) there may be more reductions applicable on base of the variable identifications. On base of our statement about the unique short-wort under the application of the rules from (1),(2) we see that we get an uniquely determined short-word, if we before each application of a rule from (3),(4) reduce the word relative to (1),(2) completely and reduce the result of the last application of a rule from (3),(4) relative to (1),(2) completely. From this it follows together with the observation that each reduction applicable before a variable identification can be applied after this application with a shifted rule it follows that for each word w there exists one and only on short-word. So we have

Lemma 1 *For each word $w \in \mathbf{L_K}$ there exists exactly one short-word. The short-word $\tilde{w}$ of w can be constructed in linear time relative to the length $|w|$ of w.*

2.2 The Reduction of the AFL

Let (α, L) be an AFL and $w := \sigma_\alpha(L)$ the signature of L relative to the arcade α. It is obvious that each Reidemeister move R2 of the string L, which removes two crossing-points with the arc α_i of the arcade α, corresponds to a reduction of the signature of the type

$$w = w_1 \cdot x_i^\epsilon \cdot x_i^{-\epsilon} \cdot w_2 \longrightarrow w_1 \cdot w_2$$

But not to each such decomposition of the signature corresponds a Reidemeister move R2, which removes the related crossing-points of the AFL. There exist AFLs with the signature w that has a decomposition

$$w = x_1^\eta \cdot w_1 \cdot x_i \cdot x_1^{-1} \cdot w_2 \cdot x_i \cdot x_i^{-1} \cdot w_3 \cdot x_1 \cdot x_1^{-1} \cdot w_4,$$

which corresponds to a situation as described by Figure 5.

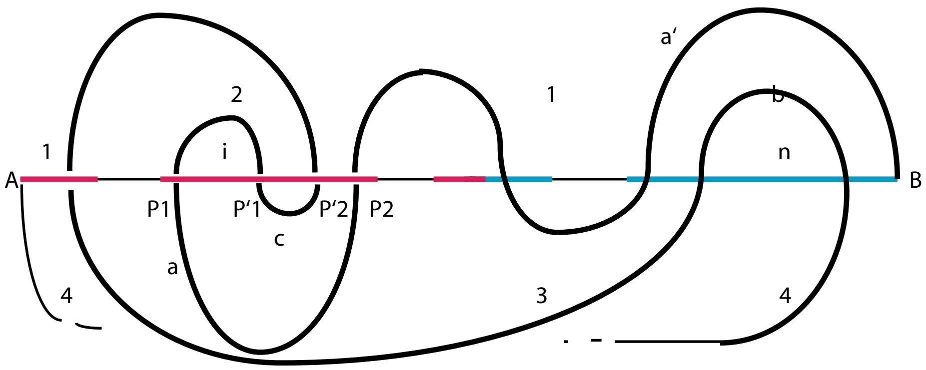


Figure 5

Because each AFL has only a finite number of crossing-points to each pair $x_i^\epsilon \cdot x_i^{-\epsilon}$, which appears in the signature, there exists a segment a of L, that crosses the arc α_i in two Points P_1, P_2 which are neighbors on L but not necessarily on α_i. The same difficulty concerns the segment that corresponds to the prefix x_1^η of the signature. But because the string L and the arcade α are simple curves there exist in these cases other segments b of the string L with pairs P'_1, P'_2 of crossing-points, which are neighbors on L and situated on the arcade between P_1, P_2. By iterating this argument we find in each case a segment c of L with crossing-points, which are neighbors on α_i and on c. This means that there exists a sequence of reductions of the signature which corresponds to a sequence of Reidemeister moves of type R1 and R2, which removes the corresponding crossing-points. From this it follows [6] the

Theorem 1 *To each AFL (α, L) with the signature $w := \sigma_\alpha(L)$ there exists a AFL (α', L') and the signature $w' := \sigma_{\alpha'}(L')$ such that it holds: $w \longrightarrow w'$,*

w' is reduced and (α, L) can be transformed by a sequence of Reidemeister moves of type R1 and R2 and arc reductions applied on (α, L) into $(\alpha'.L')$.

2.3 Tapes Round Middle Lines

We assume an arcade α being given on the sphere S beginning in point A and ending in point B. C and D are points on S different from A, B. M is a simple oriented curve from C to D. We define the signature $\sigma_\alpha(M)$ as we did it for the strings of the AFLs. c and d are line segments, c bisected by C and d by D (Fig. 6.1) M_1 and M_2 are oriented simple curves such that the configuration describes a tape with sides M_1 and M_2 and M as middle line. We may consider the tape generated by a move of c along M into d.

Observation 1: If none of the endpoints of the arcs of α lies on the tape and c, d do'nt intersect α then it holds $\sigma(M_1) = \sigma(M) = \sigma(M_2)$.

Observation 2: Each Reidemeister move R2, which can be applied to reduce the signature $\sigma_\alpha(M)$ can be applied to the tape if it is thin enough.

Observation 3: The example (Fig. 6.2) proves that in general it is not true that it follows from

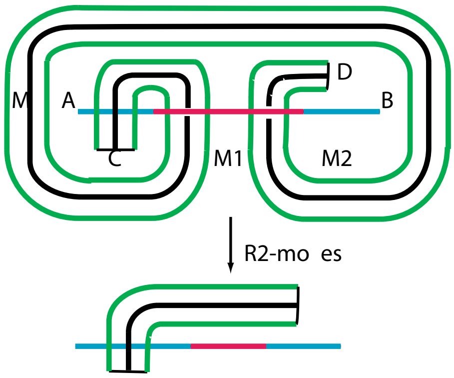


Figure 6.1

$$\sigma_\alpha(M) \to 1$$

that there exists a sequence of moves R2 to move M with fixed A and B into a line without any intersection with α. But if M is a segment of the string L of an AFL (α, L), then there exists a neighborhood M' of M on L such that this is true for M'. (Fig. 6.3) This follows from Theorem 1.

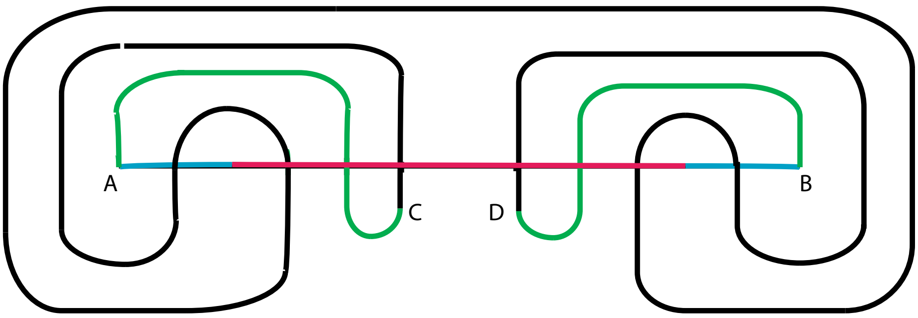


Figure 6.2

We discuss now tapes along middle lines M ending in the starting point A of the arcade as described in (Fig 6.3). The starting point of M is C. We assume the tape being generated by R3 moves, which move a crossing point E generated by a R1 move on M. Let M' be the curve generated from M by applying first the R1 move and then the sequence of R3 moves shifting the loop along M as indicated in (Fig. 6.3). We get for the signatures

$$\sigma_\alpha(M') = \sigma_\alpha(M) \cdot x_1^\epsilon \cdot \sigma_\alpha(M)^{-1} \cdot \sigma_\alpha(M) \to \sigma_\alpha(M) \cdot x_1^\epsilon \to \sigma_\alpha(M).$$

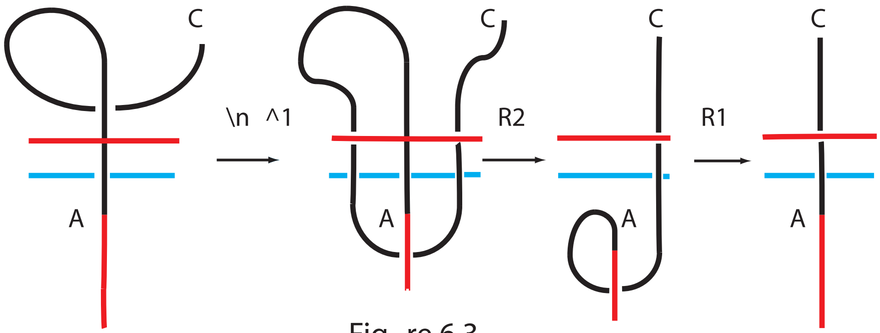


Fig re 6.3

We see in Fig.6.3 that M' can be reduced by R2 moves and one R1 move in point A to the original segment M.
We are now interested in the signature of the border line M^* of a tape with M' as middle line as described in Fig. 6.4. The green line describes this border line. We assume that the green line under crosses the segment M. It is important to note that shifting the loop along M we always pass the arcs in the same color as M does.
The colored space shows how the green line can be reduced by R2 moves to a curve, with the same signature as the line would have after a corresponding move along M.

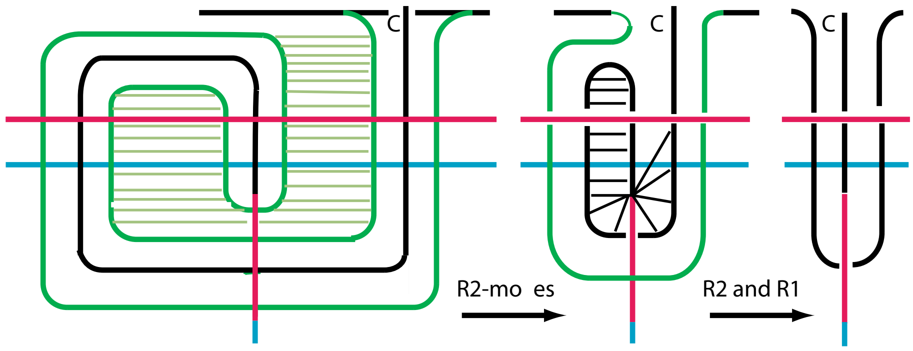


Figure 6.4

$$\begin{aligned}\sigma_\alpha(M^*) &= (\sigma_\alpha(M) \cdot x_3^\epsilon \cdot \sigma_\alpha(M)^{-1}) \\ &\cdot \quad (\sigma_\alpha(M) \cdot x_1^\epsilon \cdot \sigma_\alpha(M)^{-1}) \cdot (\sigma_\alpha(M) \cdot x_1^{-\epsilon} \cdot \sigma_\alpha(M)^{-1}) \\ &\to \quad \sigma_\alpha(M) \cdot x_3^\epsilon \cdot \sigma_\alpha(M)^{-1} \to \sigma_{\alpha'}(M) \cdot x_2^\epsilon \cdot \sigma_{\alpha'}(M)^{-1}\end{aligned}$$

The last step in this reduction is based on the reduction of α by removing the arc belonging to x_1. By reducing M' to M we are able to reduce α' to the original arcade α by removing the arc x_1. So after the reduction we get the same signatures for M and the green tape along M.
From example Fig. 6.3 we see that there exist configurations, which make it necessary that the move of the loop along the middle line passes all arcs with the same color as the middle line does. We will speak in this connection about long moves of type ν^1 .
But we can not restrict these moves to this strategy, because there are examples, as see in Fig. 6.5,

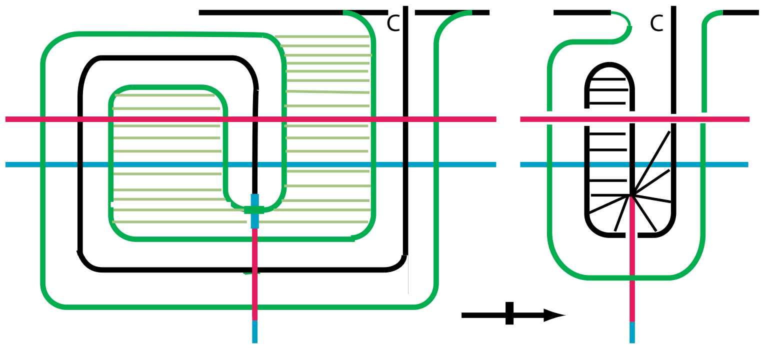


Figure 6.5
that make it necessary to *move the loop in a maximal distance from the middle line* such that the color always is the same as in the final step moving over A. Figure 6.6 describes an example, which is a variant of Fig. 6.4.

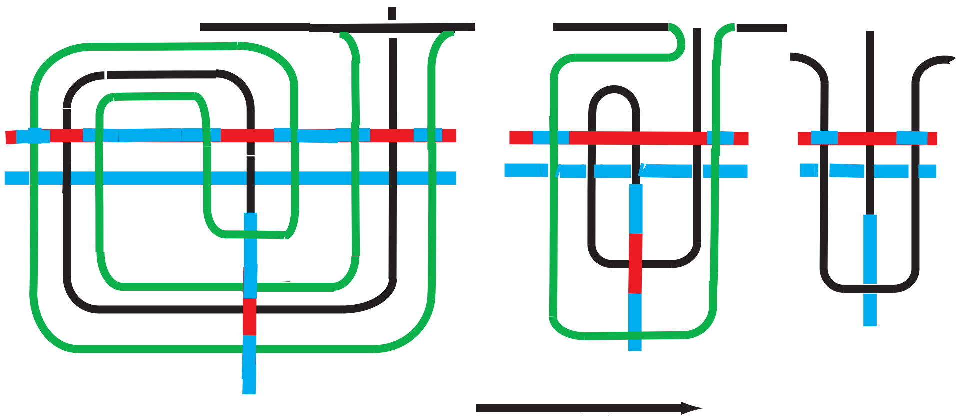


Figure 6.6
The figures differ in the way the green curve passes the arcs of the arcade. Indeed we have not the same arcade, but a transformed version. In a small

neighborhood of each crossing point of the green line we color the arcade blue if it was red there. This means that we transform the arcade α by introducing new blue and red arcs such that the green loop the arcade always passes over blue arcs. We see that in this configuration the reduction to the original configuration is possible too (Fig. 6.6).
We generalize this observation. Let α be an arcade and $M := (C, A)$ a simple string. Let N be a simple loop, which is the border of a tape Π generated by moving a circle with its middle point on M from C to A. We assume the radius of the circle so small that Π crosses the arcade exactly in the same arcs as M it does and that the tape does not overlap itself. Then we are able to assign to each crossing point P of M with the arcade two crossing points Q and Q' of N such that there does'nt exist another crossing point of N and the arcade between Q and Q'. Now we transform the arcade α as follows: Let γ the color of the arc x_i, then we successively substitute each arc x_i with color of x_i unequal γ and crossing point P by 5 new arcs, two with color γ and Q respective Q' as center point and no other crossing points in it and the three arcs of the original color covering the rest of the original arc x_i. If this process has been applied onto the neighborhood of each crossing point P with color unequal γ we have constructed the arcade α'.

Lemma 2 *Let be N_1 and N_2 the two segments of N corresponding to M. We assume that there is given a sequence of R2-moves applied on M and reductions applied on α transforming M into the simple string $M^* := (C, A)$ and reducing α to an arcade α^* such that $\sigma_{\alpha^*}(M^*)$ is a reduced word. Then it follows that there exists a sequence of R2-moves of M, N_1 and N_2 in a sequence of simple strings and a sequence of arcade reductions of α' into a tape with M^* as middle line and N_1^*, N_2* as border and an arcade α^+ such that*

$$\sigma_{\alpha^+}(M^*), \quad \sigma_{\alpha^+}(N_1^*), \quad \sigma_{\alpha^+}(N_2^*)$$

are reduced words.

Proof: From the assumption it follows, that there exists a sequence of R2-moves of M, which transforms (α, M) into a reduced system (α^*, M^*) keeping C fixed. We may restrict our discussion to one R2-move of the tape. To a R2-move of M belong two crossing points P_1, P_2 of an arc x_i with a segment of M. The two segments f on M and g on x_i between this points on α and M are simple strings which together form the border of two open subsets of the surface S of the ball. One of these two sets does'nt contain any parts of α and M. We call this region F (Fig.6.7). There exist two segments f_1, f_2 parallel to f on N_1 and N_2. We may assume that f_1 lies in F with end points

Q_1 and Q_2 on g. There are two possibilities: N_1 over crosses g in Q_1 and Q_2 or it under crosses in both points. We may restrict to the case that the behavior of N_1 in these points differ from that of M. This means that the colors of the arcs of α' and α differ in this case. We assume the color of f in α to be red and the color of the arcs around Q_1 and Q_2 are blue.

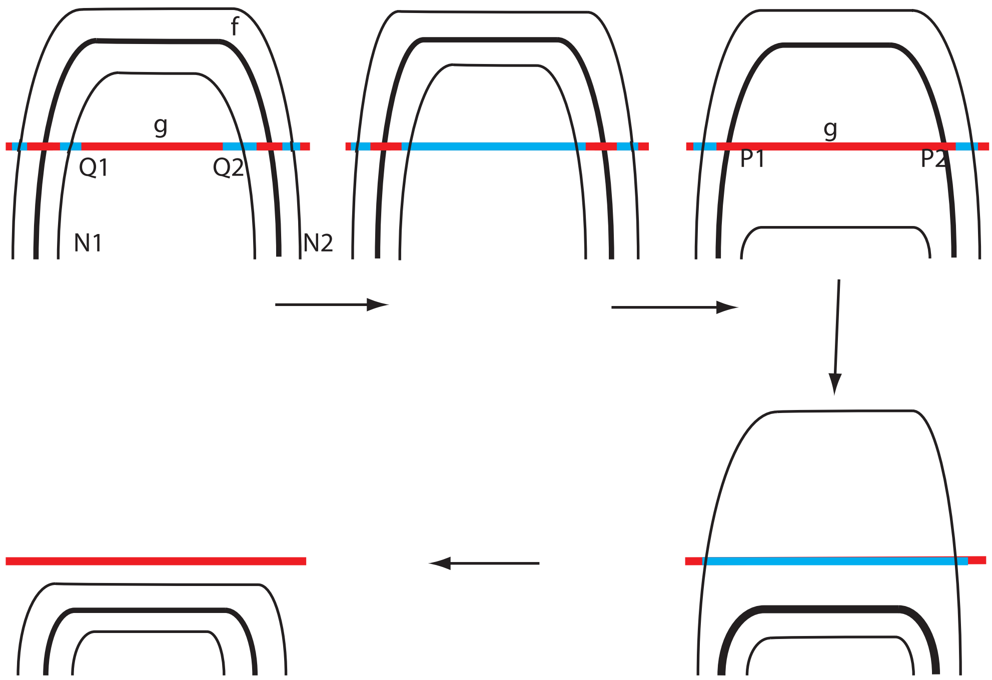


Figure 6.7

Because there are no crossing points on g between Q_1 and Q_2 we my reduce the red arc of α' between the two blue arcs on f. Then we by a R2-move may reduce N_1 such that f_1 is moved out of F. Now we may reduce the blue arc between Q_1 and Q_2 and after this may apply a R2-move to M to reduce f. Now we reduce again the arcade by substituting the red arcade by a blue one and are able to reduce N_2 by moving f_2 over the blue arc. Finally we now we color the blue arc red and may apply the next induction step. From this the lemma follows.

Corollary 1 *Let $M = M_1 \cdot M_2$ a decomposition of M into two segments M_1 and M_2 and $N_i = N_{i,1} \cdot N_{i,2}$ the corresponding decompositions of N_1 and N_2. If there exists a R2-reduction of M, which reduces M_1 to a segment without crossing points with the arcade, then this holds for the corresponding reductions of N_i and its decompositions too.*

2.4 Reduced Signatures of the Trefoil Knot

Figure 7 represents three AFLs of the trefoil knot as easily to be seen. This means that we are able to transform the first representation by Reidemeister moves into into the second one. We find before the moves (5)

$$\sigma_\alpha(L) = s_1 t_2^{-1} s_3 \quad (5)$$

$$\sigma_{\alpha'}(L') = t_2 s_1 t_2^{-1} t_2^{-1} s_3 t_2 \quad (6)$$

and after the transformation for the new AFL (α', L') the signature (6). We see that both signatures are reduced and that they are different. But if we choose an other segment of the second knot projection for building up the AFL (α^*, L^*) as shown in the last example of figure 7 we get for the signature

$$\begin{aligned}\sigma_{\alpha^*}(L^*) &= s_1 t_2^{-1} s_3 s_1^{-1} s_3^{-1} s_1 s_1^{-1} s_3 \\ &\rightarrow s_1 t_2^{-1} s_3 s_1^{-1} \rightarrow s_1 t_2^{-1} s_3\end{aligned}$$

the original reduced signature. The green segment indicates the move, which has to be done in the normalization process.

The idea behind this paper is to prove that this special case can be generalized in the sense that the normalization process to construct an AFL from a knot projection can be defined such that in each knot projection there exists a segment to build up the arcade such that the reduced signatures of both AFLs, the signature of the original one and the signature of the AFL generated by Reidemeister moves, are equal.

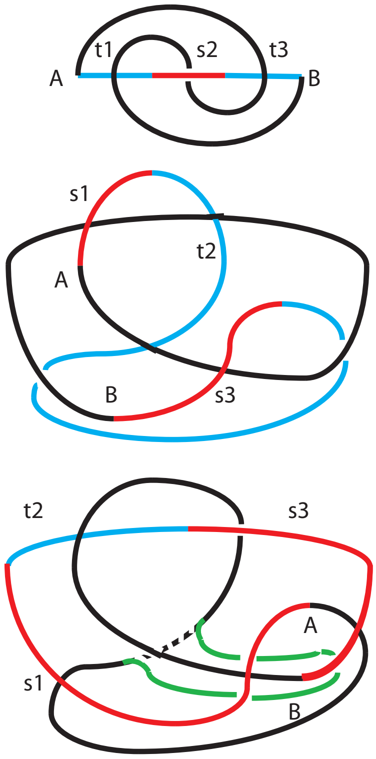


Figure 7

3 Invariance Properties of the Signatures

In the subsection 2.4 we presented examples of different AFLs (α, L) and (α', L') representing the same knot but having signatures $\sigma_\alpha(L)$ and $\sigma_{\alpha'}(L')$, which reduces to different words w respective w'. In this section we will discuss invariance properties of the reduced signatures.

3.1 Reidemeister Moves on the Surface of the Ball

It is obvious that there exists a sequence of Reidemeister moves to transform the projections of two knots on the surface S of the ball in each other iff the two knots are equivalent. But a stronger property holds on S:
Let K and K' to projections of the same knot and a a segment of K without any crossing points of K. Then there exists a sequence $seq := (\mu_1, \mu_2, ..., \mu_{k-1})$ of Reidemeister moves moving K into a projection K^* isomorphic to K', such that none of the moves of seq touches a. This means that none of the moves of seq moves any parts of a, and it will not move over segments of K or its successors K_i during the transformation process under or over a. We call such move sequences a-**normal**. We define

$$K_1 := K \quad \text{and} \quad K_{i+1} := \mu_i(K_i) \quad \text{for} \quad i = 1, ..., k-1.$$

If K and K' are projections of the same knot, then there exists such a sequence with $K^* = K_k$ isomorphic to K'. We use the segment a to construct on base of K_i AFL (α_i, L_i) representing the knot. Then we will prove that for each of our transformations of K_i into an AFL there exists such a transformation of K_{i+1} into an AFL such that the two reduced AFLs are isomorphic. The difficulty is the fact that we need the transformation applied on K_i to construct the corresponding transformation for K_{i+1}. So we have to guess the right one or to construct all transformations, which have to be considered. We considere two different strategies ν^1, ν^2 extended by a redrawing feature. Locally it is not to see which one we should apply. Therefore in the case that a concrete move sequence is not available, we have to use all the possible strategies. The construction of a AFL for given (K, a) we call **a-normalization**. Applying the possible normalization on each pair (K, a), a an allowed segment of K we get sets of reduced words $W(K, a)$ for (K, a). Two knot projections K, K' are equivalent iff it holds

$$(\bigcup_a W(K, a)) \bigcap (\bigcup_{a'} W(K', a') \neq \emptyset \tag{7}$$

The size of the set of allowed segments a of K is bounded by $3 * p - 4$, if p is the number of crossing points of K. The number of the possible normalization is exponential in the number of crossing points of K. But each case can be done effectively. The reduced forms of the signatures are computable and are uniquely determined. This means that the equivalence problem for knots is decidable.

3.2 The Normalization

Let K be an oriented knot projection and a a segment without any crossing points on K and A as start and B as end point. Let be $P_1, P_2, ..., P_n$ the set of crossing points of K in the order the points appear the first time if we walk from A to B against the orientation of the knot. We define the process of normalization by induction.

Definition 1 *If the set of crossing points is empty nothing is to do. In the other case there exists a simple segment* $seg_1 := (P_1, A)$ *oriented as* K *and a segment* c_1 *crossing* seg_1 *in* P_1 *which does not cross the knot in any other point. We move* c_1 *in a loop with* seg_1 *and a prefix of* a *as middle line such that the final loop never touches* K *beside in the new crossing point* P_1' *on* a*. If* c_1 *is under crossing in* P_1 *then we color* a *red in the other case blue. The loop together with* c_1 *defines the tape* $tape_1$*.*
Assume the points $P_1, ..., P_i$*,* $i < n$ *beeing moved onto* a *and it resulted in the knot projection* K_{i+1}*, an arcade* α_{i+1} *on* a *and in the simple segment* seg_{i+1} *from* P_{i+1} *to* A *on* K_{i+1}*. We define the move* $i+1$ *as follows: We choose a small segment* c_{i+1} *crossing* seg_{i+1} *in* P_{i+1} *and not touching* K_{i+1} *any where else. We move* c_{i+1} *along* seg_{i+1} *as middle line such that the loop does not touch* K_{i+1} *besides of crossings of arcs of* α_{i+1} *and end the move after having reached an inner point* P'_{i+1} *of the first arc of* α_{i+1}*. If* c_{i+1} *was under-grossing and the first arc has the color red, then we don't modify the first arcade. In the other case we construct a new red and a blue arc from the old red arc. This we do such that* P_{i+1} *lies on the blue arc and the old crossing points of the red arc keep their color. If* c_{i+1} *is over crossing then we proceed as before only changing the role of blue and red.*
The construction is not yet uniquily defined. We have to define how the loop behaves crossing arcs. We use two different strategies ν^1 *and* ν^2*:*
Definition *of* ν^1*: We demand that the loop* $loop^1_{i+1}$ *is always riding on the segment and never jumping over an arc or moving over a point, in which arcs of different color met. We define as tape* $tape_{i+1}$ *the region with the loop and* c_{i+1} *as border and* seg_{i+1} *as middle line. This means for the signature* $\sigma_{\alpha_{i+1}}(loop^1_{i+1}) = w \cdot x_1^{\epsilon} \cdot w^{-\epsilon}$ *with* $w := \sigma_{\alpha_i}(seg_i)$ *if the color of the first arc does not change under the move. In the over case we get* w *by a variable shift from* $\sigma_{\alpha_i}(loop_i)$*.*
Definition *of* ν^2*: The projection of the loop* $loop^2_{i+1}$ *generated by* ν^2 *is the same as of the loop* $loop^1_{i+1}$ *generated by* ν^1*. The difference between these moves is as follows: The color of the variables in* $\sigma_{\alpha_{i+1}}(loop^2_{i+1})$ *is for all the same. This means that we have to construct new arcs in the case that* $loop^1$ *crosses arcs of a color different from* x_1*.*
Definition *for* ν*:* $seg_{i+1} := seg_i \cdot loop_{i+1}$*. The sequence* $\nu := (\nu_1^{\eta_1}, \nu_2^{\eta_2}, ..., \nu_n^{\eta_n})$*,*

n number of crossing points of the knot projection K we define as the normalization process belonging to (K, a).

The definition describes a set of sequences of normalization moves. In each such sequence the $tape_{i+1}$ is packed in $tape_i$. because the loops of index i may be used as parts of the middle line of moves with index $j > i$ the normalization process may generate configurations with an exponentially growing complexity.
The normlisation ν generates up to isomorphisms uniquely an AFL

$$(\alpha, L) := \nu(K, L)$$

from a given knot projection K and a simple crossing point free segment a of K. In the next sections we will prove, that for a-normal Reidemeister moves μ of (K, a) and an a-normalization ν of K there always exists an a-normalization ν' such that the following invariance diagram holds

$$\begin{array}{ccccc}
 & & \mu & & \\
(K, a) & & \longrightarrow & & (K', a) \\
\nu \downarrow & & & & \downarrow \nu' \\
(\alpha, L) & & & & (\alpha', L') \\
\sigma \downarrow & & & & \downarrow \sigma \\
\sigma_\alpha(L) & \rightarrow & w & \leftarrow & \sigma_{\alpha'}(L') \\
 & \varrho & & \varrho &
\end{array}$$

$w := \varrho(w')$ describes the result of the total reduction of w'.

3.3 The R1 and R2 Invariance

In section 2.3 we discussed the following situation in a normalization process: A simple segment seg with the endpoints (C,A) constructed by a partial normalization has been changed by a R1-move into the segment seg', with a crossing point (Fig. 6.3). The following normalization step transforms this segment into a simple segment seg^*. This segment will become a prefix of L' of the AFL, which the normalization generates from the transformed knot projection. We have seen that this segment may be reduced by R2 and R1 moves relative to the arcs of the current arcade into seg. We proved that the tapes generated by the following normalization step may be reduced by R2-moves in each other too (Fig. 6.4). From this it follows the same result for the following normalization steps as far as it concerns the loop of the part of the tape with seg^+. The later normalization steps may use loops of tapes as middle lines, which were deformed in consequence of our R1 move. We noticed that the loop of the normalization step just after the construction of seg^* can be reduced by R2-moves to their original middle line. Observation 1 in section 1.5 inductively applied states that the the string L' of the AFL (α', L') can be reduced by R2-moves and one R1-move to L. Essential is that the move along the segment onto the arcade never jumps over an arc in the over crossing case an never jumps under an arc in the under crossing case. New arcs by this process are only generated if the first arc is red and the moved loop is over crossing or if the first arc is blue an the loop is under crossing. It follows

Lemma 3 *Let K be a knot projection and a a segment of K free of crossing points. If α is an arcade generated by an a-normalization of K then the reduced signature $\sigma_\alpha(L)$ remains invariant under a-normal R1 moves of K.*

We now come to the discussion of the invariance of signatures under the R2-move. We have to consider configurations as described by the left parts of the two diagrams of Fig. 8. We assume that the knot projection K is partially normalized an that the actual state of this process is represented in the abstract diagrams of Fig.8. A subsegment the segment b' of K has been moved by the R2-move represented by the green tape $tape_1$ under crossing the segment b, which is free of crossing points and generated by the partial normalization. The following normalization step μ can be described by two sequences ν^1 respective ν^2 of R3-moves along b. This moves may be considered as a sequence of R3-moves of $tape_1$. The result is the tape $tape_2$ with green colored border in the right parts of Fig. 8. $tape_2$ under and over crosses the arcs, in the figure 8 represented symbolically by

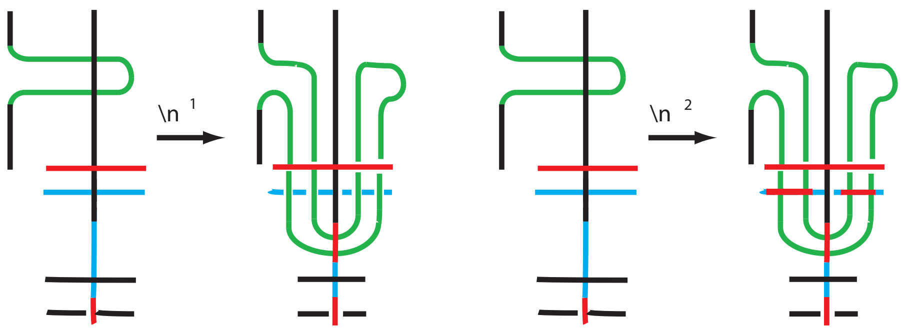


Figure 8

parts of a red and a blue arc, in the same manner as b does. Because the arcade in the diagram is beginning with a blue arc and $tape_1$ is under crossing we have to substitute the first arc t_1 of the arcade by two arcs s_1, t_2 and to shift the indices of the following arcs by 1. Tape $tape_2$ is now under crossing the arc s_1. The segments, which crossed t_1 before this normalization step are now crossing t_2. We see that the border loop of $tape_2$ can be reduced by a sequence of R2 moves relative to the arcs, such that we get the configuration before that application of μ. From subsection 1.5 it follows that this holds too for all the tapes with this middle line, which by the normalization process after the considered configuration will be generated. Inductively this follows for all segments, which are new in consequence of μ. So it follows

Lemma 4 *Let a be a segment of the knot K without any crossing points and (α, L) the arcade generated by an a-normalization ν of K. If K' has been generated by a a-normal R2-move and (α', L') by an suitable chosen extension ν' of ν from K' then the reduced signatures of both $AFLs$ are equal.*

We summarize: Let K and K' projections of a knot, which can be transformed in each other by a sequence of R1- and R2-moves and a a simple crossing point free segment of K, then there exists a sequence of a-normal R1- and R2-moves, which transform K into a knot projection K^* isomorphic to K'. The sets of the a-normalization of K and K^* construct sets of AFLs, which define sets of reduced signatures $W(K, a)$ and $W(K^*, a)$. The relation $W(K, a) \bigcap W(K^*, a) \neq \emptyset$ is an invariant under R1- and R2-moves. Because K' and K are isomorph it follows that we are able to find a simple crossing free segment a' of K' such that the relation $W(K', a') = W(K^*, a)$ holds. If we define

$$W(K) := \bigcup_a W(K, a) \tag{8}$$

then we see that $W(K) \cap W(K') \neq \emptyset$ holds if K and K' are $(R1, R2)$-equivalent.

3.4 The R3 Invariance

We first notice that we have not to discuss the R3-moves in the general form because we are able to reduce it to a special move, which we call R'3-move.

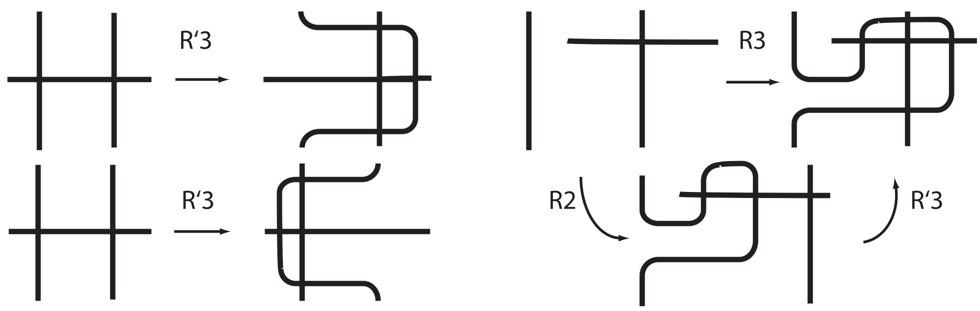


Fig re 9

It is explained by the upper left part of Figure 9. The right part of this figure shows the equivalence of the R3-move and the R'3-move modulo a R2-move. The lower left part of Figure 9 states the equivalence of applying the R'3-move to the left or the right vertical line. To prove the invariance of the signature under R'3-moves we have to discuss the four possible start configurations for the R'3-moves as explained by Fig. 10. This figure describes four configurations (a),(b),(c) and (d) of segments u_3, u_2, u_1, in the order of the orientation of the knot and the first arc of the arcade α. A is the endpoint of u_1 and the start point of α.

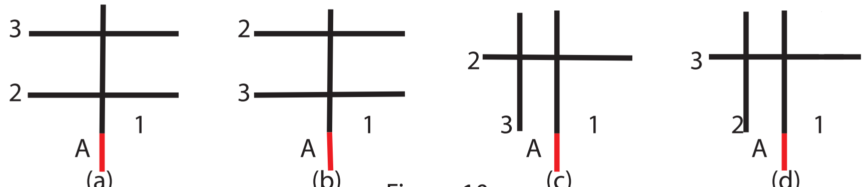


Fig re 10

We first state that the configurations of the cases (a) and (b) relative to the R'3-move are equivalent. Fig. 11.a.

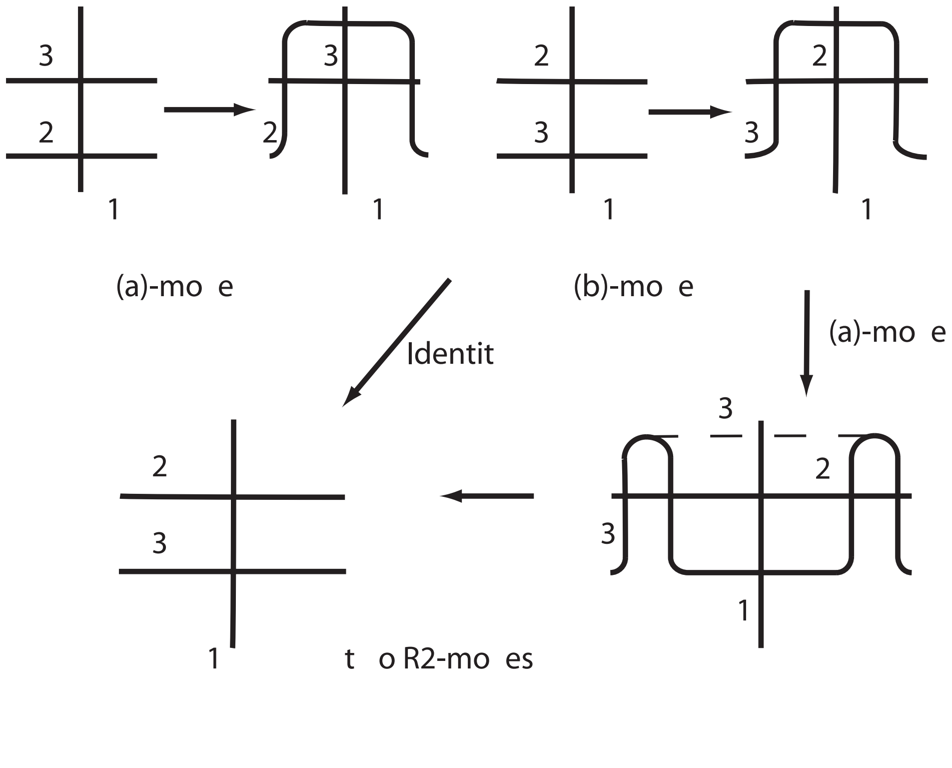


Figure 11.a

We have to compair the results of normalization applied before the R'3-move with the result after it. As first step in this direction we prove the equivalence of (a) and (b). A (a)-move means a R'3-move concerning the confiuration (a) and the meaning of (b)-,(c)- and (d)-move is analogue Fig. 11.a and 11.b. The diagram in figure 11a proves, that the (a)- and (b)-moves are equivalent modulo R2-moves. This proof can be applied to prove the equivalence of the cases of (c) and (d) with the configurations we get by reflection of it on a vertical line. We now prove that the cases (c) and (d) are equivalent to (a) modulo R2-moves. We do this explicitly only for the case (c) because the proof for (d) can be based analogously on the case (b), which is equivalent to (a) modulo R2-moves.

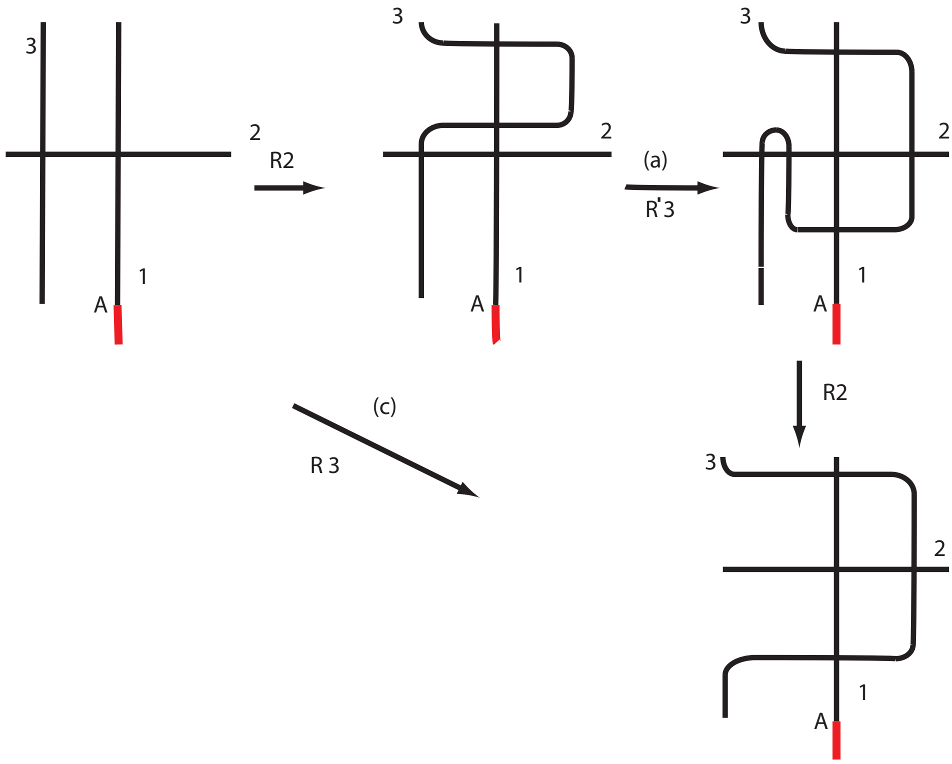


Figure 11.b

The proof is completely represented by the diagrams of Fig. 11.b. In the next step we consider the refinement of these cases by considering the over and under crossing of the segments.

3.5 The Invariance under R'3-moves

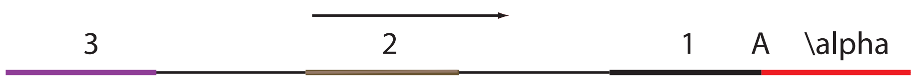
In the discussion of case (a) we have to consider some sub cases concerning over and under crossing of the different segments. We assume a knot projection K to be given and a normalization ν of K, which transforms K into a AFL (α, L). We assume the knot projection K' to be generated by a R'3-move applied on K and discuss the influence of this move on the relation between the AFLs (α, L) and the AFLs (α', L') generated by possible variants of the normalization ν and ν'.

3.5.1 Redrawing Loops

We will see that there are configurations of knot projections K and normalization ν of K onto an AFL (α, L) such that we are not able to choose a normalization ν' of K' generated by a R'3-move applied on K such that the generated AFL (α', L') is equivalent to (α, L). In these cases we use a third procedure to construct an AFL (α', L') such that it is equivalent to (α, L). This procedure we call **redrawing.**
This proedure is defined as follows: After each normalization step concerning the move of a segment of a crossing point in a loop on the arcade we reduce the partially constructed arcade completely by applications of R1- and R2-moves. After this process we allow the redrawing of the reduced loop such tat it not more crosses arcs of the arcade. Now we move the loop again on the arcade but using the complementary strategy of the normalization step before. We will prove that we are able to construct by applying normalization steps of the types ν^1, ν^2 and redrawing on K' such that the resulting AFL (α', L') is equivalent to (α, L). For shortness we call this extended procedure again **normalization**.

3.5.2 The Invariance in Case (a)

We represent u_1 by the *black* segment, u_2 by the color *malt* and u_3 by the color *plum*.



The Figure 12 represents the state of the normalization ν after having moved all the segments crossing u_1 between u_2 and A. The move of u_2 and u_3 to the arcade is done by the normalization steps ν_2 respective ν_1. It is represented by the Diagram in the left part of Fig. 12. The upper left diagram represents the R'3 move. We discuss three sub cases:

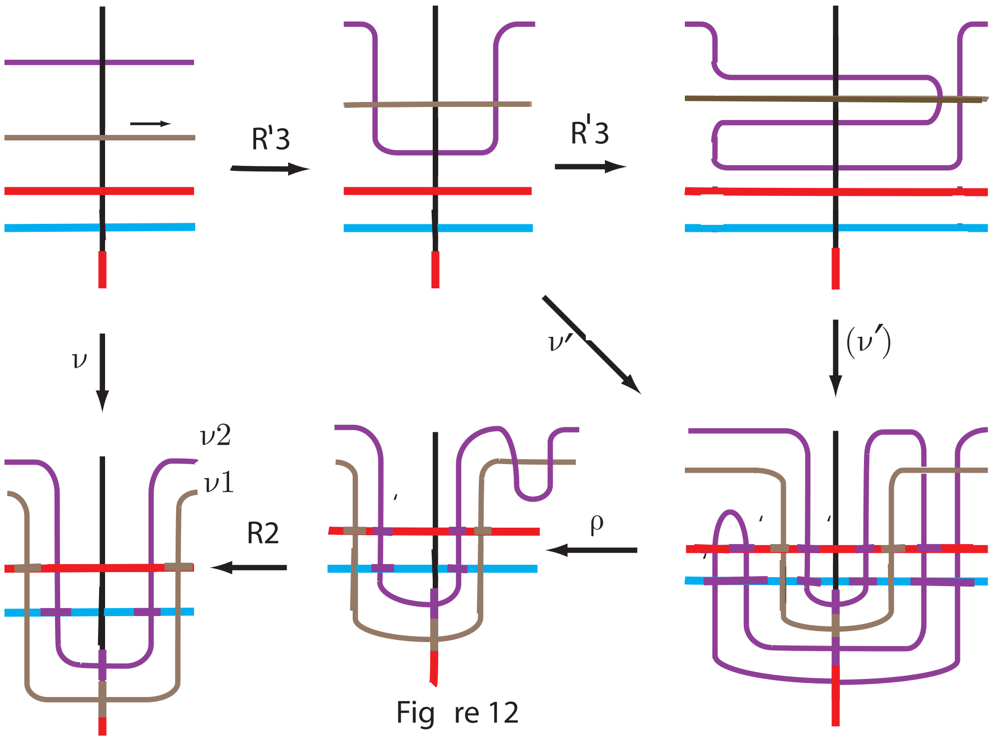


Fig re 12

(1) : malt > plum > black. We here assume the *malt* segment u_2 over crosses the moved *plum* segment u_3. The following R'3-move together with (ν) indicates the idea how to choose the normalization ν' on the *plum* loop. The variables (x, y) in Fig. 12 have to be complemented by the number of the diagrams, in which they appear, as indices. So we have to discuss the normalization by considering the possible colors *blue*, *red*, which the variables $x_1, y_1, x_2, y_2, z_3, x_3, y_3$ may have as values depending from the type of the moves ν_1, ν_2. We try to organize the normalization ν' such that it can be reduced to the original normalization ν. From the assumption that u_2 and u_3 over cross u_1 it follows that the loops of the moved segments u_2, u_3 cross the same blue arcs as u_1 it does. But the variables x_1, y_1 may be *blue* or *red* depending from the normalization strategy we have chosen for u_2 and u_3. But we are not free to choose the colors for the variables x_3 and y_3 independently, because we assume *malt* over crosses *plum*. From $x_3 = red$ it follows $y_3 = red$. We are free in choosing the color of z_3. We choose $color(z_3) = color(y_3)$ to be able to apply the R2-reductions ϱ of diagram (3) to (2). This means that we choose $x_2 := x_3, y_2 := y_3$. But the R2-move from (2) to (1) is possible if and only if $x_1 = x_3, y_1 = y_3$ holds. So it follows

Lemma 5 *Under the assumption that u_2 and u_3 over cross u_1 and R'3 moves u_3 between u_1 and u_2 it follows: To each normalization ν of K onto the AFL*

(α, L) *there exists a a normalization* ν' *of* K*' onto a* AFL (α', L') *such that* (α', L') *can be transformed by a sequence of R2-moves into* (α, L)*, if the relation*

$$color(x_1) = red \longrightarrow color(y_1) = red$$

holds.

Now we have to discuss the **case**

$$color(x_1) = red \quad \text{and} \quad color(y_1) = blue,$$

which is excluded by the assumption of the lemma.

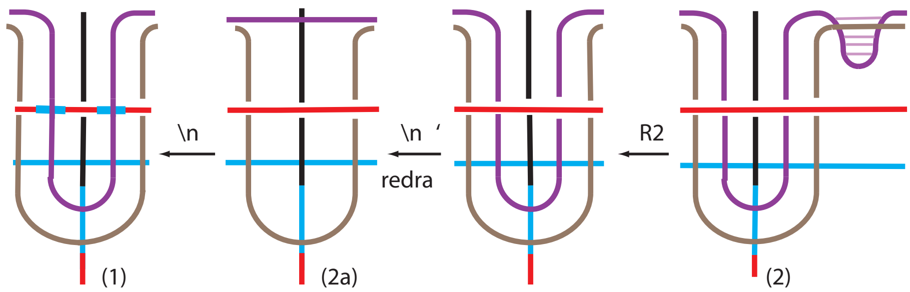


Figure 13

We choose in this case

$$color(y_3) := color(z) := color(x_3) := color(x_1)$$
$$\text{and} \quad color(x_2) := color(y_2) := color(x_1).$$

Under this assumption all relations represented in Fig. 12 remain true beside of the relation R2 between the configurations (2) and (1). We insert therefore an additional configuration (2a) between (1) and (2) (Fig. 13). We redraw the *plum* loop from the arcade (2) and apply on this configuration the normalization ν as before the first R'3-move in figure 12. The result of this operation is then isomorphic to the configuration (1) of figure 12.

(2) : plum > malt > black. We now relate in the discussion to Fig. 14.

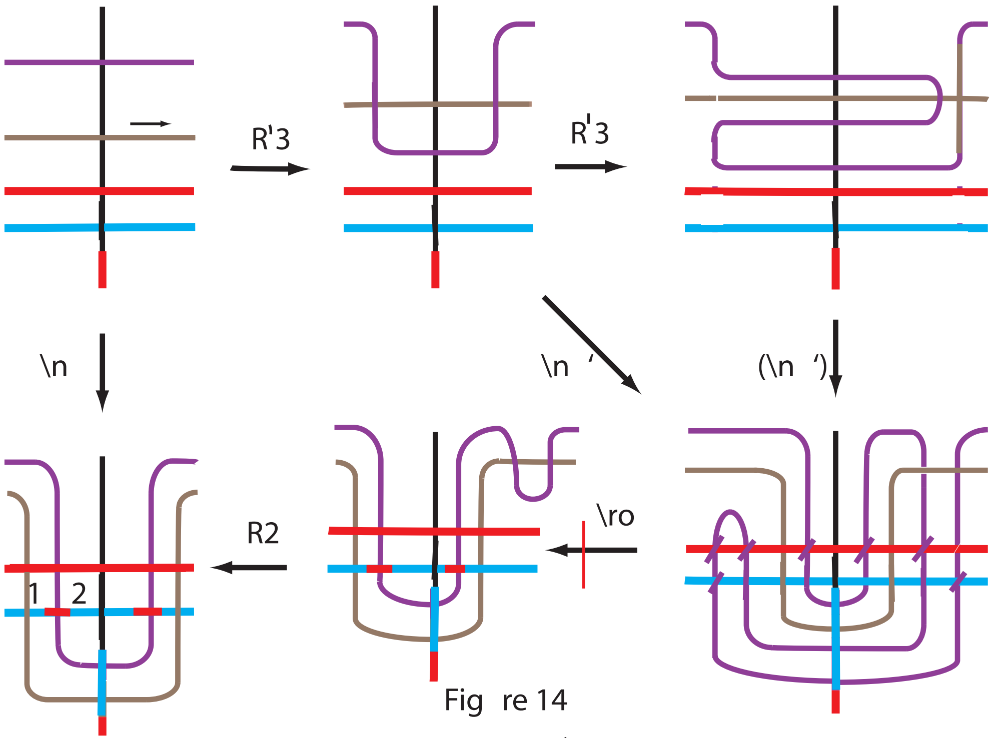


Fig re 14

If the types of the normalization ν_1 and ν_2 are equal and if we choose the types of the normalization after he R'3-move of the same type, then the diagrams commute. Fig. 14 represents the case that the type of ν_1 is ν^2 and the type of ν_2 is ν^1. In this case we have to choose for the four normalization concerning the two crossing points of *malt* and *plum* and the crossing points of *mult*, *plum* with u_1 of the second diagram in the upper line of Fig. 14 the type ν^1 to be able to reduce the effect of the R'3-move on the normalization. But the result is not the same as after the application of ν onto the original configuration. Therefore we redraw the *plum* loop and after this we apply the normalization of type ν^2, which produces the same result as ν did.
In the case type of ν_2 equal ν^2 we are free to choose the types of the three moves of the *plum* crossing points as we like it to get the desired result without needing a redraw of a loop. Our result is symmetrical to the result of the previous case. Relating to the variables introduced in Fig. 14 we have proven the

Lemma 6 *Under the assumption that u_2 and u_3 over crosses u_1 and R'3 moves u_3 over u_2 it follows: To each normalization ν of K onto the AFL (α, L) there exists a normalization ν' of K' onto (α', L') such that (α', L') can be transformed by a sequence of R2 moves into (α, L), if the relation*

$$color(x_1) = blue \longrightarrow color(x_2) = blue$$

holds. In the case color(x_1) = *blue and color*(x_2) = *red we need to achieve this result a reduction after each normalization step and a redraw f the loop and after this the normalization of type* ν^2 *of this loop.*

(3) : malt > black > plum The following discussion of this case is based on Fig. 15. The names of the color variables introduced in this figure are to be completed by the number of the diagrams in which the variables appear as index. The color of the crossing points which are colored with blue or red is determined by the assumption of this case. The color of the other crossing points is determined uniquely by the color of the crossing points in the position symmetrical relative to the black axes.

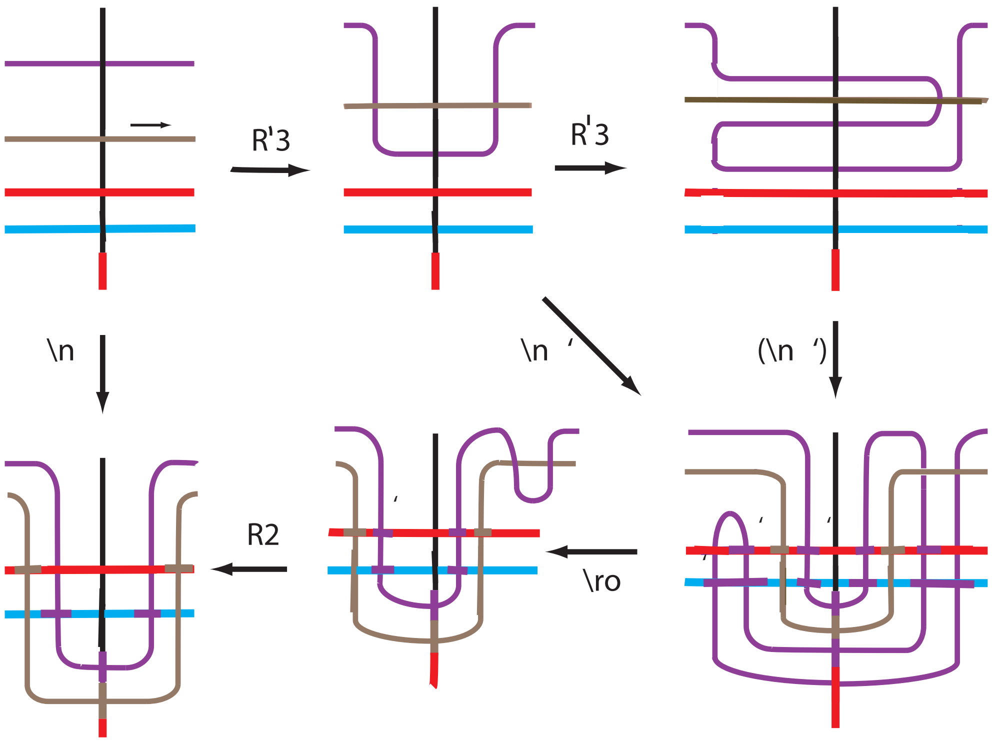


Figure 15

The values for the variables x_1, y_1 are uniquely determined by the strategies used in the normalization steps generating the *malt* and *plum* loop. So we have to consider four cases and to show that we can choose the normalization strategies after having applied the R'3 move such that the diagrams of figure 15 commute. We always set

$$color(x_2) := color(x_3) := color(x_1)$$

and

$$color(x'_2) := color(x'_3), \quad color(y_2) := color(y_3)$$

The value of y'_3 we are free to choose and we set $y'_3 := y_3$. The value of y_3 can be chosen blue or red because *plum* is under crossing *malt* and the normalization leading segment is the malt one and it over-crosses the blue arcs. $color(x'_3)$ depends from $color(x_3)$.
Case $color(x_3) = red$: In this case we have to choose $color(x'_3) := red$. In this case it follows that we can choose

$$\begin{aligned} color(x'_3) &:= color(x_3) = color(x_1) \\ \text{and} \quad color(y_3) &= color(y_1) \end{aligned}$$

This means that Diagram (3) of Fig. 15 can be reduced by R2-moves to (2) of Fig. 15 and (2) to (1) by a R2 move. We see in this case there exists a normalization of K' to a reduced AFL (α', L'), which is isomorphic to the reduced AFL generated by the given normalization ν of K.
$color(x_1) = blue$: In this case we are free to choose the colors for x'_3 and y_3 as we like. We set

$$color(x'_3) := red \quad \text{and} \quad color(y_3) := color(y_1)$$

and we see that all the diagrams commute.

Lemma 7 *Under the assumption that u_3 under crosses u_1 and u_2 over crosses u_1 there exists for each normalization ν of K and each R'3 move of K into a knot projection K' a normalization ν' of K' such that the reduced generated AFLs are isomorphic.*

Sub Cases: The three remaining subcases are symmetrical to the three discussed sub cases. By reflection of the knot projections K and K' and of the related AFLs on the projection plane the cases will be translated as represented in the following pattern

$$\begin{aligned} malt > plum > black &\longleftrightarrow black > plum > malt, \\ plum > malt > black &\longleftrightarrow black > malt > plum, \\ malt > black > plum &\longleftrightarrow plum > black > malt. \end{aligned}$$

It follows that the discussions of the the first free subcases can be formally translated in the discussions of the three cases on the right side of the pattern.
We summerize the results of this subsection:

Lemma 8 *Under the assumption, that the segments u_3, u_2, u_1 of the knot are configured as in the cases (a) and (b) of Fig. 10 the following holds: If the knot projection K has been moved into the knot projection K' by a R'3-move of u_3 over or between or under the cross point of u_2 and u_1, then there exists to each normalization ν of K onto an AFL (α, L) a normalization ν' of K' extended by redrawing of loops onto an AFL (α', L'), such that the two reduced AFLs are isomorphic on the surface S of the ball.*

4 The Complexity of the Equivalence Problem of Knots

We have seen that projections of equivalent knots on the surface S of a ball can be moved into isomorphic projections by sequences of Reidemeister moves, which do not touch a given crossing free segment a and a certain neighborhood of it at all. We defined sequences of moves of crossing points along crossing-free segments onto the arcade build on a. Part of this moves are substitutions of arcs of the arcade by two or more new arcs.
We used two different strategies for this long moves: One strategy ν^1 consists in moving the loop always in contact with segment under over arcs as the segment does. We spoke in this connection of *riding on the line*. The second strategy ν^2 moves the loop not between arc and line but jumps over the arc, which makes it in most cases necessary to build new arcs. This strategy we called *distance move*. Both strategies reduce the number of crossing points lieing not on the arcade strict monotonously. If the knot projection has n crossing points then there exist 2^n many different sequences of normalization of the knot projection onto a AFL. Each move of a Crossing point onto the arcade generates a loop twice as long as the segment it has to move along. Therefore the number of crossing points of the loops moved to the arcade may crow exponentially but not faster. But this may over estimate the crowing of the arcade, because a loop shifted to the arcade in most cases will not pass all the over loops shifted before it. This means that the size of the AFLs we construct by this procedure from a knot with n crossing points is exponentially bounded. So in the worst case we have to expect expontial many AFLs of expontial size. This bound holds too if we bring in account the extension of the normalization by the redrawing strategy. The size of the arcades may be reduced considerably by reducing the patially generated arcade after each normalisation step.
For knot projections $\mathcal{K}$ and $\mathcal{K}'$, which can be transformed by a Reidemeister move in each other we proved the following result: There exists to each

normalisation ν of $\mathcal{K}$ a normalisation ν' of $\mathcal{K}'$ such that the two reduced AFLs generated are isomorphic. This property is transitv. So it follows the

Theorem 2 *The equivalence problem of two knot projections $\mathcal{K}_1$ and $\mathcal{K}_2$ with n_1 respective n_2 crossing points is decidable in time $O(2^{2*n})$ with $n := max\{n_1, n_2\}$.*

Using the definition (8) of $W(\mathcal{K})$ we define for knots $\mathbf{K}$

$$W(\mathbf{K}) := \bigcap_{\mathcal{K} \text{ a projection of } \mathbf{K}} W(\mathcal{K}) \tag{9}$$

From Lemma 8 it follows for all knots $\mathbf{K}$

$$W(\mathbf{K}) \neq \emptyset \quad \text{and} \quad W(\mathbf{K}) \quad \text{is finite}$$

5 Concluding Remarks

1. The product $\sigma_\alpha(L) * \sigma_{\alpha'}(L')$ of the signatures of two AFLs is equal to a signature $\sigma_{\tilde{\alpha}}(\tilde{L})$ of an AFL $(\tilde{\alpha}, \tilde{L})$ representing the Schubert product of the knots represented by (α, L) and (α', L'). The reductions (2) then translate into a drilling of the AFL $(\tilde{\alpha}, \tilde{L})$.
2. From the invariance of (9) under Reidemeister moves ist follows that homomorphisms

$$\phi : F(X) \longrightarrow G$$

from the free Group generated by X on groups G may be used to define knot invariants. Every knot invariant should be definable on this base.
3. Probably we overestimated the complexity of the algorithm by far, because knot projections with a number of crossing points much larger as that of a projection with a minimal number of crossing points will be reduced during the normalisation considrably and it seems not very likely that segments to be moved by the normalisations are following in a order that the move of the succesoor segment has to follow the loop of a predecessor frequently. So one could guess, that the problem to compute a small hull of $W(\mathbf{K})$ is only seldom exponentially large relative to the minimal number of crossing points of a projection of $\mathbf{K}$.
If we are only interested in weaker invariants as $\phi(W(\mathbf{K}))$ then during the normalisation stronger reductions may be available, which will decrease the complexity. If G is the free commutative group generated by X than this is the case.

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