

### Problem 1

biased coin lands on heads with probability  $p$   
it's flipped 10 times  
6 heads is the result  
find the conditional probability that the first 3 outcomes are H,T,T

let  $X$  = the number of heads flipped out of 10 trials

$P(\text{first three flips are H,T,T} \mid X = 6 \text{ for 10 flips}) = ?$

$X$  is then binomial with  $(n,p) = (10,p)$

$$P(X=6) = \binom{10}{6} p^6 (1-p)^4$$

The probability that the first three flips is H,T,T is  $p(1-p)^2$

$$P(\text{H,T,T} \mid X = 6 \text{ for 10 flips}) = \frac{P(\text{HTT} \cap X=6)}{P(X=6)}$$

$P(\text{HTT} \cap X = 6)$  is the number of ways to get  $X=6$  when our first 3 outcomes are HTT and we do 10 tosses total

so we have things like H,T,T,H,H,H,H,T,T and H,T,T,H,H,H,H,T,H,T

this is like deciding the first 3 rolls and then out of 7 choosing 5 to be success

$$\text{so } P(\text{HTT} \cap X = 6) = p(1-p)^2 \binom{7}{5} p^5 (1-p)^2 = \binom{7}{5} p^6 (1-p)^4$$

$$\text{Then } P(\text{H,T,T} \mid X = 6 \text{ for 10 flips}) = \frac{P(\text{HTT} \cap X=6)}{P(X=6)} = \frac{\binom{7}{5} p^6 (1-p)^4}{\binom{10}{6} p^6 (1-p)^4} = \frac{\binom{7}{5}}{\binom{10}{6}} = \frac{\frac{7!}{5!2!}}{\frac{10!}{6!4!}} = \frac{1}{10}$$

### Problem 2

let  $X$  = the number of boxes needed to get a prize

let  $X_1$  = the number of boxes needed to get the first unique coupon

let  $X_2$  = the number of boxes needed to get the second unique coupon

...

up to letting  $X_5$  = the number of boxes needed to get the fifth and final unique coupon

then

$$E[X] = E[X_1] + E[X_2] + E[X_3] + E[X_4] + E[X_5]$$

$E[X_1] = 1$  since you'll always need exactly 1 box to get the first unique coupon

$X_2$  is a geometric distribution since we want the first success with probability  $\frac{4}{5}$  since there are still 4 unique to us coupons we can obtain out of 5 total unique coupons  
the expectation value for a geometric distribution is  $\frac{1}{p}$

$$\text{hence } E[X_2] = \frac{5}{4}$$

$X_3$  is also geometric but now  $p = \frac{3}{5}$  and so  $E[X_3] = \frac{5}{3}$

it seems clear enough from here that  $E[X_4] = \frac{5}{2}$  and  $E[X_5] = 5$

then we have

$$E[X] = E[X_1] + E[X_2] + E[X_3] + E[X_4] + E[X_5] = 1 + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + 5 = \frac{137}{12} \approx 12$$

We rounded up to 12 since we can't buy a fraction of a box

### Problem 3

let  $X$  = the number of rainy days in a given year

we're given that  $P[X \geq 1] = 1/2$

we know that  $P[X=0] + P[X \geq 1] = 1$

$$\Rightarrow P[X=0] = \frac{1}{2}$$

We can model  $X$  as a poisson distribution and set up an equation from  $P[X=0]$  to calculate  $\lambda$

$$\frac{1}{2} = P[X=0] = \frac{\lambda^0}{0!} e^{-\lambda} = e^{-\lambda}$$

taking the ln of both sides

$$\begin{aligned} \ln\left(\frac{1}{2}\right) &= -\lambda \ln(e) \\ \ln\left(\frac{1}{2}\right) &= \ln(1) - \ln(2) = -\ln(2) \\ \ln(e) &= 1 \end{aligned}$$

so we have

$$\begin{aligned} -\ln(2) &= -\lambda \\ \Rightarrow \lambda &= \ln(2) \end{aligned}$$

let  $Y$  = the number of rainy days in 265 days

since poisson is over a period of time and we have changed the time interval, our  $\lambda$  changes as well

let  $\lambda_y$  be the  $\lambda$  for the  $Y$  distribution

$$\text{then } \lambda_y = \frac{250}{365} \lambda = \frac{50}{73} \ln(2)$$

$$\text{we want } P[Y=3] = \frac{(\lambda_y)^3}{3!} e^{-\lambda_y} = \frac{\left(\frac{50}{73} \ln(2)\right)^3}{3!} e^{-\frac{50}{73} \ln(2)}$$

this can be simplified to

$$= \frac{\frac{50^3}{73^3} * (\ln(2))^3}{6} * \frac{1}{2^{\frac{50}{73}}}$$

from here we can approximate  $\left(\frac{50}{73}\right) \approx \left(\frac{5}{7}\right) \ln(2) \approx \frac{7}{10}$  to get

$$\frac{1}{6} * \left(\frac{5}{7}\right)^3 * \left(\frac{7}{10}\right)^3 * \left(\frac{1}{2^{\frac{5}{7}}}\right) = \frac{5^3 * 7^3}{6 * 7^3 * 10^3 * 2^{\frac{5}{7}}} = \frac{1}{48 * 2^{\frac{5}{7}}} \approx 0.0127$$

so the  $P[Y=3] \approx 0.0127$  or about a  $\frac{1}{100}$  chance of it raining 3 times in 250 days

This seems reasonable since there's a 50% chance it doesn't rain for a year.

The question said approximate so I figured the approximations here were fine. In case only the approximation of  $X$  being a Poisson distribution is fine, here is the complete calculation

$$P[Y=3] = \frac{\left(\frac{50}{73} \ln(2)\right)^3}{3!} e^{-\frac{50}{73} \ln(2)} = 0.01109$$