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Data601
HW8

Problem 1

$$F_Y(y) = P[2X^2 \le y] = P\left[X \le \sqrt{\frac{y}{2}}\right]$$

$$\Rightarrow F_Y(y) = \{ \begin{matrix} \sqrt{\frac{y}{2}},\ for\ y \in [0,2] \\ 0,\ otherwise \end{matrix}$$

to find $f_Y(y)$ we can take the derivative with respect to y of $F_Y(y) = \sqrt{\frac{y}{2}}$

$$\Rightarrow f_Y(y) = \frac{\partial}{\partial y}\left(\frac{y}{2}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{y}{2}\right)^{-\frac{1}{2}} * \left(\frac{1}{2}\right) = \frac{1}{4} * \left(\frac{2}{y}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{4\sqrt{y}} = \frac{1}{2\sqrt{2y}}$$

$$\Rightarrow f_Y(y) = \{ \begin{matrix} \frac{1}{2\sqrt{2y}},\ for\ y \in (0,2] \\ 0,\ otherwise \end{matrix}$$

Problem 2

a)

to find k we can use the formula $\int_{-\infty}^{\infty} f_X(x)dx = 1$
which gives us

$$\int_1^\infty \frac{k}{x^3}dx = k\int_1^\infty x^{-3}dx = k[\ \frac{x^{-2}}{-2}\Big|_1^\infty = k[\ -\frac{1}{2x^2}\Big|_1^\infty = k\left[-(0) - \left(-\frac{1}{2}\right)\right] = \frac{k}{2} = 1$$

$$\Rightarrow k = 2$$

b)

we can use the formula for expectation to find that

$$\mathrm{E}[X^\beta] = \int_1^\infty x^\beta * \frac{2}{x^3}\ dx = 2\int_1^\infty x^{\beta-3}\ dx = 2[\ \frac{x^{\beta-2}}{\beta-2}\Big|_1^\infty$$

from here we can see that when $\beta > 2$ we'll end up with x^∞ which wont converge

the denominator will be 0 when $\beta = 2$

we're left with the scenario that $\beta < 2$ and this will converge

Problem 3

a. $f_Y(y) = P[Y = \lfloor X \rfloor]$
$\Rightarrow P[Y = 0] = P[X < 1] = F_X(1) = 1 - e^{-1}$
N.B. we don't need to worry about Y being less than 0 since we're looking at a probability space

$P[Y = 1] = P[1 \leq X < 2] = F_X(2) - F_X(1) = (1 - e^{-2}) - (1 - e^{-1}) = e^{-1} - e^{-2}$

$P[Y = 2] = P[2 \leq X < 3] = F_X(3) - F_X(2) = (1 - e^{-3}) - (1 - e^{-2}) = e^{-2} - e^{-3}$

It's a pattern and we can deduce that

$$P[Y = y] = e^{-y} - e^{-y-1}$$

hence

$$f_Y(y) = \{\begin{matrix} e^{-y} - e^{-y-1},\ y \geq 0 \\ 0,\ otherwise \end{matrix}$$

b) Notice from the density function of Y that we no longer have a continous random variable in Y. There is no $P[1.1 \leq Y \leq 1.9]$ for example. We need to use the discrete formula for the expectation value.
We need to use
E[Y] = $\sum_0^n y \ * P[Y = y]$
I'm struggling to find the right words to describe n. n$\in \mathbb{N}$ but is also the $\lim_{n\to\infty} \sum_0^n y \ * P[Y = y]$

$$E[Y] = \sum_0^{n\to\infty} y \ * (e^{-y} - e^{-y-1})$$

$$= [0 \ * (1 - e^{-1})] + [1 \ * (e^{-1} - e^{-2})] + [2 \ * (e^{-2} - e^{-3})] + \cdots$$

$$= e^{-1} - e^{-2} + 2e^{-2} - 2e^{-3} + 3e^{-3} + \cdots$$

$$= e^{-1} + e^{-2} + e^{-3} + e^{-4} + \cdots$$

$= \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} + \cdots$

using the infinite geometric series formula
The general formula for the sum of an infinite geometric series with first term a and common ratio r (where r < 1)
$\mathrm{S} = \frac{\mathrm{a}}{1-\mathrm{r}}$

$\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} + \cdots = \frac{e^{-1}}{1-e^{-1}} = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}$

Problem 4

a)

we want to solve $P[X \geq 250{,}010] = ?$

We'll use the normal approximation to the binomial.
We have
$\mathrm{n} = 1{,}000{,}000$
$\mathrm{p} = \frac{1}{4}$

$$E[X] = np = 250{,}000$$

$$\mathrm{var[X]} = np(1-p) = 250{,}000\left(\frac{3}{4}\right) = 187{,}500$$

$$\sigma = \sqrt{var[X]} = 250\sqrt{3}$$

$$P\left[\frac{X - E[X]}{\sqrt{var[X]}} \geq \frac{250{,}010 - 250{,}000}{250\sqrt{3}} = \frac{2}{5\sqrt{3}}\right] = \frac{1}{\sqrt{2\pi}}\int_{\frac{2}{5\sqrt{3}}}^{\infty} e^{-\frac{x^2}{2}}dx$$

plugging into wolfram alpha we get $P[X \geq 250{,}010] \approx 0.4087$

b)

we want to find n such that $P\left[\frac{X - E[X]}{\sqrt{var[X]}} \geq \frac{n - 250{,}000}{250\sqrt{3}}\right] \approx 0.90$

$$\mathrm{t}\ P\left[\frac{X - E[X]}{\sqrt{var[X]}} \geq \frac{n - 250{,}000}{250\sqrt{3}}\right] = \frac{1}{\sqrt{2\pi}}\int_{\frac{n-2\quad ,000}{250\sqrt{3}}}^{\infty} e^{-\frac{x^2}{2}}dx \approx 0.90$$

plugging into wolfram alpha we get $n \approx 249{,}445$

when I plug back into the formula and have wolfram alpha solve the intergral we get 0.9000029
This indicates that it's a pretty accurate result

c)

we want $P[X \geq 0.24n] \approx 0.90$

$$P[X \geq 0.24n] = P\left[\frac{X - E[X]}{\sqrt{var[X]}} \geq \frac{0.24n - n\left(\frac{1}{4}\right)}{\sqrt{n\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)}} = \frac{\frac{24n}{100} - \frac{25n}{100}}{\sqrt{\frac{3n}{16}}} = \frac{-\frac{n}{100}}{\frac{\sqrt{3n}}{4}} = \frac{-n}{25\sqrt{3n}} = -\frac{\sqrt{n}}{25\sqrt{3}}\right]$$

$$\approx \frac{1}{\sqrt{2\pi}}\int_{-\frac{\sqrt{n}}{25\sqrt{3}}}^{\infty} e^{-\frac{x^2}{2}}dx = 0.90$$

Plugging into wolfram alpha and solving for n we get $n \approx 3079.45$

rounding up, we'll say $n = 3080$

if we plug back into our equation we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\frac{2\sqrt{2310}}{75}}^{\infty} e^{-\frac{x^2}{2}} dx$$

plugging into wolfram alpha we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\frac{2\sqrt{2310}}{75}}^{\infty} e^{-\frac{x^2}{2}} dx \approx 0.900002$$

so our answer is verified to be very close.