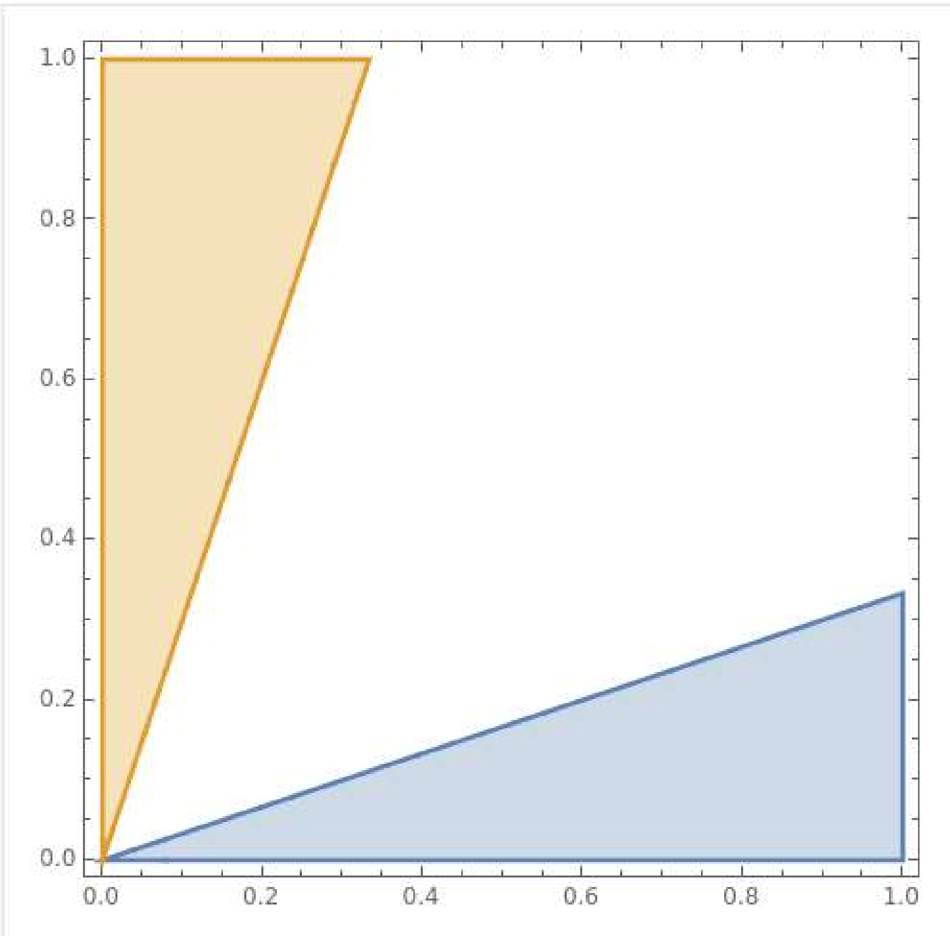
Bradley Scott
Data601
HW9

Problem 1

X~ uniform on [0,1]
Y~ uniform on [0,1]

if X > Y we want P[X $\geq$ 3Y] = P[X-3Y $\geq$ 0]
if Y > X we want P[Y $\geq$ 3X] = P[Y - 3X $\geq$ 0]

It's likely easiest to see the areas when we graph these



Graph is from wolframalpha

Let C = the areas covered by the shaded region

Since X and Y are uniform we know that $P[\, X \in C,\, Y \in C] = area(C)$

The orange triangle has area $\frac{1}{2}(1)\left(\frac{1}{3}\right) = \frac{1}{6}$

Similarly, the blue triangle has area $\frac{1}{2}(1)\left(\frac{1}{3}\right) = \frac{1}{6}$

We know it's at 1/3 since that is when the other random variable is at it's maximum, 1, for still meeting the inequalities P[X-3Y $\geq$ 0] and P[Y - 3X $\geq$ 0].

So the probability that the larger of the two random variable is 3 times as big as the other is $2 * \left(\frac{1}{6}\right) = \frac{1}{3}$

Problem 2

part a)

We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx\,dy = 1$

The problem states $p_{X,Y}(x,y)$ which we normally reserve for the density function of a discrete random variable but since this is on an interval of (0,1) I'm thinking it was meant to be $f_{X,Y}(x,y)$ for the density function of a continuous random variable.

so we can plug into that equation to get

$$\int_0^1\int_0^1 cxy(1-x)dxdy = 1$$

$$\Rightarrow c\int_0^1\int_0^1 xy - x^2y\,dx\,dy = 1$$

$$\Rightarrow c\int_0^1 \left.\frac{x^2y}{2} - \frac{x^3y}{3}\right|_0^1 dy = 1$$

$$\Rightarrow c\int_0^1 \frac{y}{2} - \frac{y}{3}dy = 1$$

$$\Rightarrow c\left(\left.\frac{y^2}{4} - \frac{y^2}{6}\right|_0^1\right) = 1$$

$$\Rightarrow c\left(\frac{1}{4} - \frac{1}{6}\right) = 1$$

$$\Rightarrow c\left(\frac{3-2}{12}\right) = 1$$

$$\Rightarrow c = 12$$

part b)

X and Y are independent if

$$f_{X,Y}(x,y) = f_X(x) * f_Y(y)\ \forall\ x,y$$

We can find $f_X(x)$ and $f_Y(y)$ by integrating $f_{X,Y}(x,y)$ over the opposite random variable for the density function we are seeking

$$f_X(x) = 12\int_0^1 xy - x^2y\,dy = 12\left(\frac{xy^2}{2} - \frac{x^2y^2}{2}\Big|_0^1\right) =$$

$$= 12\left(\frac{x}{2} - \frac{x^2}{2}\right) = 6x(1-x)$$

$$f_Y(y) = 12\int_0^1 xy - x^2y\,dx = 12\left(\frac{x^2y}{2} - \frac{x^3y}{3}\Big|_0^1\right)$$

$$= 12\left(\frac{y}{2} - \frac{y}{3}\right) = 12\left(\frac{3y-2y}{6}\right) = 2y$$

$$f_{X,Y}(x,y) = 12xy(1-x)$$

$$f_X(x) * f_Y(y) = 6x(1-x) * 2y = 12xy(1-x)$$

Hence, $f_{X,Y}(x,y) = f_X(x) * f_Y(y)\ \forall\ x,y$

and so X and Y are independent

part c)

$$\text{E[Y]} = \int_{-\infty}^{\infty} y\ * f_Y(y)dy = \int_0^1 y\ * 2y\,dy = \frac{2y^3}{3}\Big|_0^1 = \frac{2}{3}$$

part d)

$$\text{E[X]} = \int_0^1 x\ * [6x(1-x)]dx = 6\int_0^1 x^2 - x^3dx = 6[\ \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$$

$$\text{E}[X^2] = \int_0^1 x^2\ * [6x(1-x)]dx = 6\int_0^1 x^3 - x^4dx = 6[\ \frac{x^4}{4} - \frac{x^5}{5}\Big|_0^1 = 6\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{3}{10}$$

$$\text{var[X]} = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20}$$

part e)

$$f_{X+Y}(a) = P[X = a,\ Y = x - a] = \int_{-\infty}^{\infty} f_X(a) f_Y(x-a)da$$

$$= \int_0^1 6a(1-a) * 2(x-a)da = 12\int_0^1 xa\ - a^2 - xa^2 + a^3 da$$

$$= 12[\ \frac{xa^2}{2} - \frac{a^3}{3} - \frac{xa^3}{3} + \frac{a^4}{4}\Big|_0^1 = 12\left[\frac{x}{2} - \frac{1}{3} - \frac{x}{3} + \frac{1}{4}\right] = 12\left(\frac{6x - 4 - 4x + 3}{12}\right) = 2x - 1$$

Since this is across the sample space of X+Y and both X and Y go from 0 to 1, this is from 0 to 2

$$f_{X+Y}(a) = \{ \begin{matrix} 2x\ - 1\ when\ 0 < X + Y < 2 \\ 0,\ otherwise \end{matrix}$$

Is this ok as is? Or should I be doing something like X + Y = Z? but then I get confused on how I can just say 2z-1 for the density function

Problem 3

let b_1 denote battery 1 and b_2 denote battery 2

Then the probability that b_1 fails is given by

$$f_{b_1}(x) = \{ \begin{matrix} e^{-x},\ x \geq 0 \\ 0,\ otherwise \end{matrix}$$

and the probability that b_2 fails is given by

$$f_{b_2}(x) = \{ \begin{matrix} 2e^{-2x}\ ,\ x \geq 0 \\ 0,\ otherwise \end{matrix}$$

We want $P[Y = b_1 \mid X > 2]$ where X is the time a battery lasted and Y is which battery was selected

$$P[Y = b_1 \mid X > 2] = \frac{P[Y=b_1 \cap X>2]}{P[X>2]} = \frac{P[Y=b_1]*P[X>2|Y=b_1]}{P[Y=b_1]*P[X>2|Y=b_1] + P[Y=b_2]*P[X>2|Y=b_2]}$$

$P[Y = b_1] = \ P[Y = b_2] = \frac{1}{2}$ since we're randomly selecting between battery 1 and battery 2

$P[X > 2|Y = b_1] = 1\ - P[X \leq 2\ |Y = b_1] =$ *1−∫_0^2▒〖e^(−x) dx〗=*
$1 - [\ - e^{-x}|_0^2 = 1 - (-e^{-2} + e^0)$ *=1+e^(−2)−1=e^(−2)*

$P[X > 2|Y = b_2] = 1\ - P[X \leq 2\ |Y = b_2] =$ *1−2∫_0^2▒〖e^(−2x) dx〗=*

$$1\ - 2[\ -\frac{e^{-2x}}{2}\Big|_0^2 = 1\ - (-e^{-4} + 1) = e^{-4}$$

Plugging in we get

$$\frac{P[Y=b_1]*P[X>2|Y=b_1]}{P[Y=b_1]*P[X>2|Y=b_1]+\ P[Y=b_2]*P[X>2|Y=b_2]}=\frac{\frac{e^{-2}}{2}}{\frac{e^{-2}}{2}+\frac{e^{-4}}{2}}=\frac{e^{-2}}{e^{-2}+e^{-4}}=$$

$$\frac{\frac{1}{e^2}}{\frac{1}{e^2}+\frac{1}{e^4}}=\frac{1}{1+\frac{1}{e^2}}=\frac{e^2}{e^2+1}\approx 0.88$$

So if a randomly selected battery has lasted over 2 months, there's approximately a 88% chance that it is the first battery.