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Data601
HW11

Problem 1

We have

$X = weekly\ allowance\ in\ dollars,\ X \geq 0$

$E[X] = 6$

$var[X] = 6$

We want

$P[X \leq 12] \leq c$

(a)

Using Markov inequality

$$P[X \geq t] \leq \frac{E[X]}{t}$$

$$\Rightarrow P[X \geq 12] \leq \frac{6}{12} = \frac{1}{2}$$

(b)

Using Chebyshev inequality

$$P[X - E[X] \geq t] \leq \frac{var[X]}{t^2}$$

$$\Rightarrow P[X \geq 12] = P[X - 6 \geq 6] \leq \frac{6}{6^2} = \frac{1}{6}$$

(c)

Using one sided Chebyshev inequality

$$P[X - E[X] \geq t] \leq \frac{var[X]}{var[X] + t^2}$$

$$\Rightarrow P[X \geq 12] = P[X - 6 \geq 6] \leq \frac{6}{6 + 6^2} = \frac{1}{7}$$

Problem 2

We have

$X_1,\ X_2,\ X_3,\ and\ X_4$ are random variables

$E[X_i] = 0\ \forall\ i \in \{1,2,3,4\}$

$var[X_i] = 1\ \forall\ i \in \{1,2,3,4\}$

$$var[X_i] = E\left[X_i^2\right] - (E[X_i])^2 = E\left[X_i^2\right] - 0\ (by\ our\ given\ mean\ being\ 0)$$

$\Rightarrow E[X_i^2] = 1 \; \forall \, i \in \{1,2,3,4\}$

Since we have pairwise independence we have
$\text{cov}[X_i, X_j] = 0 \; \forall \, i, j \in \{1,2,3,4\} \; and \; i \neq j$

$cov[X_i,\, X_j] = E[X_i X_j] - E[X_i]E[X_j] = 0$

$E[X_i] = 0 \; \forall \, i \in \{1,2,3,4\}$

$\Rightarrow E[X_i X_j] = 0 \; \forall \; i, j \in \{1,2,3,4\} \; and \; i \neq j$

(a)

$corr[(X_1 + X_2), (X_2 + X_3)] = \; ?$

$$corr[(X_1 + X_2), (X_2 + X_3)] = \frac{cov[(X_1 + X_2), (X_2 + X_3)]}{\sqrt{var[(X_1 + X_2)]} * \sqrt{var[(X_2 + X_3)]}}$$

$var[(X_1 + X_2)] = E[(X_1 + X_2)^2] - (E[X_1 + X_2])^2$

$= E[X_1^2 + 2X_1X_2 + X_2^2] - (E[X_1]^2 + 2E[X_1]E[X_2] + E[X_2]^2)$

$= E[X_1^2] \; - E[X_1]^2 + E[X_2^2] - E[X_2]^2 + 2E[X_1]E[X_2] - 2E[X_1]E[X_2]$

$= var[X_1] + var[X_2] = 2$

similarly, $var[(X_2 + X_3)] = \; 2$

$cov[(X_1 + X_2), (X_2 + X_3)] = E[(X_1 + X_2 - E[X_1 + X_2]) * (X_2 + X_3 - E[X_2 + X_3])]$

$E[X_1 + X_2] = E[X_1] + E[X_2] = 0 + 0 = 0$

similarly, $E[X_2 + X_3] = 0$

$= \; E[(X_1 + X_2)(X_2 + X_3)] = E[X_1X_2 + X_1X_3 + X_2^2 + X_2X_3]$

$= E[X_1X_2] + E[X_1X_3] + E[X_2^2] + E[X_2X_3] = 0 + 0 + E[X_2^2] + 0 \; (see\text{" }we\text{ have" } - by\ pairwise\ independence)$

$= E[X_2^2] = 1 \; (see \text{ "}we\text{ have" } - mean\ being\ 0\ and\ variance\ being\ 1)$

$$corr[(X_1 + X_2), (X_2 + X_3)] = \frac{cov[(X_1 + X_2), (X_2 + X_3)]}{\sqrt{var[(X_1 + X_2)]} * \sqrt{var[(X_2 + X_3)]}} = \frac{1}{\sqrt{2} * \sqrt{2}} = \frac{1}{2}$$

(b)

$$corr[(X_1 + X_2),(X_3 + X_4)] = \ ?$$

$$corr[(X_1 + X_2),(X_3 + X_4)] = \frac{cov[(X_1 + X_2),(X_2 + X_3)]}{\sqrt{var[(X_1 + X_2)]} * \sqrt{var[(X_2 + X_3)]}}$$

We can tell from part (a) that the covariance is going to be 0 but here's me showing my work.

$$cov[(X_1 + X_2),(X_3 + X_4)] = E[(X_1 + X_2 - E[X_1 + X_2]) * (X_3 + X_4 - E[X_3 + X_4])]$$

$$E[X_1 + X_2] = E[X_1] + E[X_2] = 0 + 0 = 0$$

similarly, $E[X_3 + X_4] = 0$

$= E[(X_1 + X_2)(X_3 + X_4)] = E[X_1X_3 + X_1X_4 + X_2X_3 + X_2X_4]$

$= E[X_1X_3] + E[X_1X_4] + E[X_2\ X_3] + E[X_2X_4] = 0 + 0 + 0 + 0$ $(see\ \text{"}we\ have\text{"} - by\ pairwise\ independence)$

$= 0$

$$corr[(X_1 + X_2),(X_3 + X_4)] = \frac{cov[(X_1 + X_2),(X_2 + X_3)]}{\sqrt{var[(X_1 + X_2)]} * \sqrt{var[(X_2 + X_3)]}} = 0$$

Problem 3

(a)

$Z \sim N(0,1)$

Want $E[Z^4]$

$$E[Z^4] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^4 e^{-\frac{z^2}{2}} dz$$

let $u = z$ and $dv = z^3 e^{-\frac{z^2}{2}} dz$

then $du = dz$ and $v = (z^2 + 2)(-e^{-\frac{z^2}{2}})$

N.B. I used wolfram alpha to do the integration to get v and confirmed it by hand but it's a lot and I don't think the integration work is paramount to the result here

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^4 e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{(2\pi)}}[-ze^{-\frac{z^2}{2}}(z^2+2)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}(z^2+2)dz]$$

$-ze^{-\frac{z^2}{2}}(z^2+2)\Big|_{-\infty}^{\infty} = -z^3e^{-\frac{z^2}{2}} - 2ze^{-\frac{z^2}{2}}\Big|_{-\infty}^{\infty}$ both of these are odd functions so this goes to 0

$\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}(z^2+2)dz = \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz + 2\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$

$$\int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz = E[Z^2]$$

$$2\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = 2 * (1)\ because\ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz\ is\ the\ proability\ space = 2$$

$\int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz + 2\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = E[Z^2] + 2(1)$

we know that since $var[Z] = 1$ and $E[Z] = 0$ we know $E[Z^2] = 1$

$E[Z^2] + 2(1) = 1 + 2 = 3$

$E[Z^4] = 3$

(b)

$X \sim N(\alpha,\ \sigma^2)$

$E[X^4] = ?$

$$E[X^4] = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

This isn't an integral that we can easily solve.
I know we haven't talked about moment generating functions in class but they're in the book and I'm assuming that is the method we are supposed to solve this with.

The moment generation function $M(t) = E[e^{tX}] = e^{\alpha t + \frac{\sigma^2 t^2}{2}}$ for normal random variables with $N(a,\ \sigma^2)$

$E[X^4] = M^{(4)}(0)$ (this is to say the 4th derivative of the moment generating function at 0

$$M^{(4)}(t) = \frac{\partial^4}{\partial^4 t}\left(e^{\alpha t\ \frac{\sigma^2 t^2}{2}}\right) = [6\sigma^2(\alpha + \sigma^2 t)^2 + (\alpha + \sigma^2 t)^4 + 2\sigma^4]e^{\alpha t + \frac{\sigma^2 t^2}{2}}$$

$\Rightarrow M^{(4)}(0) = 6\sigma^2\alpha^2 + \alpha^4 + 3\sigma^4$

$$E[X^4] = 6\sigma^2\alpha^2 + \alpha^4 + 3\sigma^4$$