Bradley Scott
Data601
HW13

Problem 1

$E[X_1] = \frac{1}{2}\int_{-1}^{1} x(1+\theta x)dx = \frac{1}{2}\left[\ \frac{x^2}{2} + \frac{x^3\theta}{3}\right|_{-1}^{1} = \frac{1}{2}\left[\left(\frac{1}{2}+\frac{\theta}{3}\right) - \left(\frac{1}{2}-\frac{\theta}{3}\right)\right] = \frac{\theta}{3}$

$$E[k\overline{X_n}] = l\, E[\overline{X_n}] = kE\left[\frac{X_1 + \cdots + X_n}{n}\right] = k * \frac{nE[X_1]}{n} = \frac{k\theta}{3}$$

Want $E[k\overline{X_n}] = \theta$ for an unbiased estimator

so setting $\frac{k\theta}{3} = \theta$ we get k = 3

Problem 2

part 1

$E[X] = \int_{\theta}^{\infty} x * \left(\lambda e^{-\lambda(x-\theta)}\right)dx = \theta + \frac{1}{\lambda}$ (used wolfram alpha and verified)

$E[X^2] = \int_{\theta}^{\infty} x^2 * \left(\lambda e^{-\lambda(x-\theta)}\right)dx = \theta^2 + \frac{2\theta}{\lambda} + \frac{2}{\lambda^2}$ (used wolfram alpha and verified)

$$E[X] = \frac{x_1 + \cdots + x_k}{k} = \bar{X} = \theta + \frac{1}{\lambda}\ (equation\ 1)$$

$$E[X^2] = \frac{x_1^2 + \cdots + x_k^2}{k} = S = \theta^2 + \frac{2\theta}{\lambda} + \frac{2}{\lambda^2}\ (equation\ 2)$$

from equation 1

$$\theta = \bar{X} - \frac{1}{\lambda}$$

putting this into equation 2

$$S = \left(\bar{X} - \frac{1}{\lambda}\right)^2 + \frac{2\left(\bar{X} - \frac{1}{\lambda}\right)}{\lambda} + \frac{2}{\lambda^2}$$

$$\Rightarrow S = \bar{X}^2 - \frac{2\bar{X}}{\lambda} + \frac{1}{\lambda^2} + \frac{2\bar{X}}{\lambda} - \frac{2}{\lambda^2} + \frac{2}{\lambda^2}$$

$$\Rightarrow S = \bar{X}^2 + \frac{1}{\lambda^2}$$

$$\Rightarrow \lambda = \sqrt{\frac{1}{S - \bar{X}^2}} \; Note\ that\ because\ \lambda > 0\ we\ don't\ take\ the\ negative\ root$$

plugging this into equation 1 we get

$$\theta = \bar{X} - \frac{1}{\lambda} = \bar{X} - \frac{1}{\sqrt{\frac{1}{S - \bar{X}^2}}} = \bar{X} - \sqrt{S - \bar{X}^2}$$

So we have

$$\lambda = \sqrt{\frac{1}{S - \bar{X}^2}}$$

and

$$\theta = \bar{X} - \sqrt{S - \bar{X}^2}$$

part 2

$$x_1 = 3,\ x_2 = 0.5,\ x_3 = 2.5,\ x_4 = 2,\ x_5 = 5$$
$$x_6 = 3.5,\ x_7 = 10,\ x_8 = 9,\ x_9 = 18,\ x_{10} = 1.5$$

$$\bar{X} = \frac{3 + 0.5 + 2.5 + 2 + 5 + 3.5 + 10 + 9 + 18 + 1.5}{10} = 5.5$$

$$S = \frac{3^2 + 0.5^2 + 2.5^2 + 2^2 + 5^2 + 3.5^2 + 10^2 + 9^2 + 18^2 + 1.5^2}{10} = 56.4$$

$$\lambda = \sqrt{\frac{1}{S - \bar{X}^2}} = \sqrt{\frac{1}{56.4 - 5.5^2}} \approx 0.195$$

$$\theta = \bar{X} - \sqrt{S - \bar{X}^2} = 5.5 - \sqrt{56.4 - 5.5^2} \approx 0.386$$

Problem 3

part 1

$$L = f(x_1, \ \dots, \ x_n; \theta, \ \lambda) = \lambda e^{-\lambda(x_1 - \theta)} * \dots * \lambda e^{-\lambda(x_n - \theta)} = \lambda^n e^{-\lambda(\sum_{i=1}^{n} x_i - n\theta)}$$

$$\ln(L) = \ln\left(\lambda^n e^{-(\sum_{i=1}^{n} x_i - n\theta)}\right) = nln(\lambda) - \lambda\left(\sum_{i=1}^{n} x_i - n\theta\right)$$

$$\frac{\partial \ln(L)}{\partial \theta} = \lambda n = 0$$

This isn't helpful because we know $\lambda >$ 0 and if n is 0 then we are not estimating anything

Instead we need to think about how we can maximize ln(L) with respect to θ
We know that $x_i > \theta$ but when θ approaches the smallest x_i is when we would maximize ln(L) with respect to θ so

$$\theta = \min(x_1, \ \ldots, \ x_n)$$

$$\frac{\partial \ln(L)}{\partial \lambda} = \frac{n}{\lambda} - \left(\sum_{i=1}^{n} x_i - n\theta\right) = 0$$

$$\Rightarrow \lambda = \frac{n}{\left(\sum_{i=1}^{n} x_i - n\theta\right)}$$

part 2

$$x_1 = 3, \ x_2 = 0.5, \ x_3 = 2.5, \ x_4 = 2, \ x_5 = 5$$
$$x_6 = 3.5, \ x_7 = 10, \ x_8 = 9, \ x_9 = 18, \ x_{10} = 1.5$$

n = 10

$$\theta = 0.5$$

$$\lambda = \frac{10}{\big(3 - 10(0.5)\big) + \big(0.5 - 10(0.5)\big) + \big(2.5 - 10(0.5)\big) + \cdots + \big(1.5 - 10(0.5)\big)}$$

$$= \frac{10}{(-2) + (-4.5) + (-2.5) + (-3) + (0) + (-1.5) + (5) + (4) + (13) + (-3.5)}$$
$$= \frac{10}{50} = \frac{1}{5}$$