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Data601
HW10

Problem 1

a.

$$cov[2X - Y,\ X + 3Y] = E[(2X - Y)(X + 3Y)] - E[2X - Y]E[X + 3Y]$$

$$= E[2X^2 + 6XY - XY\ - 3Y^2] - (2E[X] - E[Y])(E[X] + 3E[Y])$$

$$=2E[X^2] + 6E[X]E[Y] - E[X]E[Y] - 3E[Y^2] - (2E[X]^2 + 6E[X]E[Y] - E[Y]E[X]) - 3E[Y]^2)$$

$$= 2(E[X^2] - E[X]^2) - 3(E[Y^2] - E[Y]^2) + 5(E[XY] - \ E[X]E[Y])$$

Notice this is the same as

$$= 2var[X] - 3var[Y] + 5cov[X.Y] = 2(3) - 3(4) + 5(1) = -1$$

b.

$$\rho(2X - Y, X + 3Y) = \frac{cov[2X - Y,\ X + 3Y]}{\sqrt{var[2X - Y]} * \sqrt{var[X + 3Y]}}$$

we know that $cov[2X - Y,\ X + 3Y] = -1$

$$var[2X - Y] = E[(2X - Y)^2] - (E[2X - Y])^2$$

In order to make this go faster we can just solve for the general case and then use the formula

$$var[aX + bY] = E[(aX + bY)^2] - (E[aX + bY])^2$$

$$= E[\ a^2X^2 + 2abXY + b^2Y^2] - (aE[X] + bE[Y])^2$$

$$=a^2E[X^2] + 2abE[XY] + b^2E[Y^2] - (a^2E[X]^2 + 2abE[X]E[Y] + b^2E[Y]^2)$$

$$= a^2(E[X^2] - E[X]^2] + b^2(E[Y^2] - E[Y]^2) + 2ab(E[XY] - E[X]E[Y])$$

$$= a^2var[X] + b^2var[Y] + 2abcov[X, Y]$$

This implies that

$$var[2X - Y] = 4var[X] + (-1)^2var[Y] + 2(2)(-1)cov[X, Y] = 4(3) + 4\ - 4(1) = 12$$

and that

$$var[X + 3Y] = (1)^2var[X] + (3)^2var[Y] + 2(1)(3)cov[X, Y] = 1(3) + 9(4) + 6(1) = 45$$

then

$$\rho(2X - Y, X + 3Y) = \frac{cov[2X - Y,\ X + 3Y]}{\sqrt{var[2X - Y]} * \sqrt{var[X + 3Y]}} = -\frac{1}{\sqrt{12} * \sqrt{45}} = -\frac{1}{6\sqrt{15}}$$

Problem 2

We know this is a discrete random variable since we only have 2 elements in the sample space

hence we can let X(1) = x_1 and X(2) = x_2

similarly

Y(1) = y_1 and Y(2) = y_2

Then E[X] = $x_1 * P[X = x_1] + x_2 * P[X = x_2]$

similarly, E[Y] = $y_1 * P[Y = y_1] + y_2 * P[Y = y_2]$

and E[XY] = $\sum_{i,j\in\{1,2\}} x_i y_j P[X = x_i,\ Y = y_j] =$
$x_1y_1P[X = x_1,\ Y = y_1] + x_1y_2P[X = x_1,\ Y = y_2] + x_2y_1P[X = x_2,\ Y = y_1] + x_2y_2P[X = x_2,\ Y = y_2]$

we know that cov(X,Y) = 0

$$\Rightarrow E[XY] - E[X]E[Y] = 0$$

$$\Rightarrow x_1y_1P[X = x_1,\ Y = y_1] + x_1y_2P[X = x_1,\ Y = y_2] + x_2y_1P[X = x_2,\ Y = y_1] + x_2y_2P[X = x_2,\ Y = y_2] = (x_1 * P[X = x_1] + x_2 * P[X = x_2])(y_1 * P[Y = y_1] + y_2 * P[Y = y_2])$$

looking at RHS (right hand side)

$$(x_1 * P[X = x_1] + x_2 * P[X = x_2])(y_1 * P[Y = y_1] + y_2 * P[Y = y_2])$$

$$= x_1y_1P[X = x_1]P[Y = y_1] + x_1y_2P[X = x_1]P[y = y_2] + x_2y_1P[X = x_2]P[y = y_1] + x_2y_2P[X = x_2]P[Y = y_2]$$

so we have

$$x_1y_1P[X = x_1,\ Y = y_1] + x_1y_2P[X = x_1,\ Y = y_2] + x_2y_1P[X = x_2,\ Y = y_1] + x_2y_2P[X = x_2,\ Y = y_2] = x_1y_1P[X = x_1]P[Y = y_1] + x_1y_2P[X = x_1]P[y = y_2] + x_2y_1P[X = x_2]P[y = y_1] + x_2y_2P[X = x_2]P[Y = y_2]$$

$\Rightarrow \sum_{i,j\in\{0,1\}} x_i y_j P[X = x_i,\ Y = y_j] = \sum_{i,j\in\{0,1\}} x_i y_j P[X = x_i]P[Y = y_j]$

$\Rightarrow P[X = x_i,\ Y = y_j] = P[X = x_i]P[Y = y_j]$ which means X and Y are independent

Problem 3

$$f_{X+Y}(x) = \int_0^x f_X(t)f_Y(x-t)dt = \int_0^x e^{-t}e^{-(x-t)}dt = e^{-x}\int_0^x dt = xe^{-x}$$

$$\Rightarrow f_{X+Y}(x) = \begin{cases} xe^{-x}, \ x \geq 0 \\ 0 \ otherwise \end{cases}$$