

Problem 1

(a)

$$\int_0^2 cx \, dx = 1 \Rightarrow \frac{cx^2}{2} \Big|_0^2 = 1 \Rightarrow 2c = 1$$

$$\Rightarrow c = \frac{1}{2}$$

(b)

$$f_x(x) = \frac{x}{2}$$

$$F_x(x) = \int_0^x f_x(x) \, dx = \int_0^x \frac{x}{2} \, dx = \frac{x^2}{4} \Big|_0^x = \frac{x^2}{4}$$

or more formally

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x^2}{4}, & 0 < x < 2 \\ 1, & x \geq 2 \end{cases}$$

(c)

$$P[X > 1] = 1 - P[X \leq 1] = 1 - \frac{x^2}{4} \Big|_0^1 = \frac{3}{4}$$

(d)

$$E[X] = \int_0^2 x * \frac{x}{2} \, dx = \frac{1}{2} * \frac{x^3}{3} \Big|_0^2 = \frac{8}{6} = \frac{4}{3}$$

(e)

$$\text{var}[X] = E[X^2] - (E[X])^2$$

$$= \int_0^2 x^2 * \frac{x}{2} \, dx - \left(\frac{4}{3}\right)^2$$

$$= \frac{1}{2} * \frac{x^4}{4} \Big|_0^2 - \frac{16}{9} = 2 - \frac{16}{9} = \frac{2}{9}$$

(f)

$Y = e^x$ and we want to find $f_Y(y)$

we need to use the $f_Y(y) = f_X(g^{-1}(y)) * \frac{1}{|g'(g^{-1}(y))|}$ formula

finding $g^{-1}(y)$

$$y = e^x$$

$$x = e^y \Rightarrow \ln(x) = y \Rightarrow g^{-1}(y) = \ln(y)$$

finding $f_X(g^{-1}(y))$

$$f_X(x) = \frac{x}{2} \text{ and } g^{-1}(y) = \ln(y)$$

$$\Rightarrow f_X(g^{-1}(y)) = \frac{\ln(y)}{2}$$

finding $g'(y)$

$$g(y) = e^y \Rightarrow g'(y) = e^y$$

finding $g'(g^{-1}(y))$

$$g'(y) = e^y \text{ and } g^{-1}(y) = \ln(y)$$

$$\Rightarrow g'(g^{-1}(y)) = e^{\ln(y)} = y$$

finding $\frac{1}{|g'(g^{-1}(y))|}$

$$\frac{1}{|g'(g^{-1}(y))|} = \frac{1}{y}$$

Finally we have

$$f_Y(y) = f_X(g^{-1}(y)) * \frac{1}{|g'(g^{-1}(y))|} = \frac{\ln(y)}{2} * \frac{1}{y} = \frac{\ln(y)}{2y}$$

and on the intervals from $e^0 = 1$ and e^2

so more formally we have

$$f_Y(y) = \begin{cases} 0, & y \leq 1 \\ \frac{\ln(y)}{2y}, & y \in (1, e^2) \\ 0, & y \geq e^2 \end{cases}$$

(g)

$$e^X = Y \Rightarrow E[e^X] = E[Y]$$

$$E[Y] = \int y * f_Y(y) dy = \int_1^{e^2} y * \left(\frac{\ln(y)}{2y} \right) dy = \frac{1}{2} \int_1^{e^2} \ln(y) dy$$

we need to use integration by parts to solve this

using the formula $\int u dv = uv - \int v du$

let $u = \ln(y)$ and $dv = dy$

then

$$du = \frac{1}{y} dy \text{ and } v = y$$

so we have

$$\begin{aligned} & \frac{1}{2} [y \ln(Y)|_1^{e^2} - \int_1^{e^2} y * \frac{1}{y} dy] \\ &= \frac{1}{2} [e^2(2 * \ln(e)) - 1(\ln 1) - y|_1^{e^2}] \\ &= \frac{1}{2} [2e^2 - 0 - (e^2 - 1)] = \frac{1}{2} [e^2 + 1] \approx 4.19 \end{aligned}$$

Problem 2

let Y = the daily demand

X = the number of liters bought

$(2 - \frac{x}{400})$ = cost per liter

$y > x$ is when the demand is higher than what we bought so we sell everything we have

$y \leq x$ is when the demand is less than what we bought so we only sell up to y

$$\text{Then profit} = \begin{cases} 3x - x(2 - \frac{x}{400}), & \text{when } y > x \\ 3y - x(2 - \frac{x}{400}), & \text{when } y \leq x \end{cases}$$

so profit is a function of both x and y

we can calculate the expected value of the profit by breaking it into two integrals and using our formula for expected value = $y * P[y]$. In this case $P[y] = 1/100$ for all y

$$E[\text{profit}] = \int_0^x [3y - x(2 - \frac{x}{400})] * \frac{1}{100} dy + \int_x^{100} [3x - x(2 - \frac{x}{400})] * \frac{1}{100} dy$$

solving will give us the expected profit in terms of x

$$\begin{aligned} E[\text{profit}] &= \frac{1}{100} * [\frac{3y^2}{2} - 2xy + \frac{x^2y}{400}]_0^x + 3xy - 2xy + \frac{x^2y}{400} \Big|_x^{100} = \\ &= \frac{1}{100} [\frac{3x^2}{2} - 2x^2 + \frac{x^3}{400} + 300x - 200x + \frac{x^2}{4} - 3x^2 + 2x^2 - \frac{x^3}{400}] = \\ &= \frac{1}{100} \left[-\frac{x^2}{2} + 100x + \frac{x^2}{4} - x^2 \right] = \frac{x}{100} \left[100 - \frac{5x}{4} \right] = x - \frac{x^2}{80} \end{aligned}$$

we could graph this equation and find the maximum or we could test the inflection points by taking the derivative and setting it equal to 0 and solving for x. We would then test that value vs the extremes to find the maximum.

$$\frac{d}{dx} \left(x - \frac{x^2}{80} \right) = 1 - \frac{x}{40} = 0 \Rightarrow x = 40$$

so then we need to test $x = 0, 40$ and 100 and see which gives the highest value in the equation $x - \frac{x^2}{80}$.

0 gives us 0 .

100 gives us -25

40 gives us 20 as the expected profit. So $x = 40$ (buying 40 liters of milk) maximizes the profit.

Problem 3

We know that $f_X(x)$ must define a probability space such that $\int_{-\infty}^{\infty} f_X(x) = 1$ so we have

$$\int_0^1 (a + bx^2) dx = 1$$

$$\Rightarrow ax + \frac{bx^3}{3} \Big|_0^1 = 1$$

$$\Rightarrow a + \frac{b}{3} = 1$$

$$\Rightarrow 3a + b = 3$$

We also know that $E[X] = \frac{5}{8}$ and can use the formula $E[X] = \int_{-\infty}^{\infty} x * f_X(x) dx$ so we have

$$\int_0^1 x * (a + bx^2) dx = \frac{5}{8}$$

$$\Rightarrow \int_0^1 ax + bx^3 dx = \frac{5}{8}$$

$$\Rightarrow \frac{ax^2}{2} + \frac{bx^4}{4} \Big|_0^1 = \frac{5}{8}$$

$$\Rightarrow \frac{a}{2} + \frac{b}{4} = \frac{5}{8}$$

$$\Rightarrow 4a + 2b = 5$$

So now we have two equations and two unknowns

a	b	num
3	1	3
4	2	5

multiplying the first row by -2 and adding we get

$$(4-6)a = (5-6)$$

$$-2a = -1$$

$$a = \frac{1}{2}$$

plugging $a = \frac{1}{2}$ into the second row we get

$$4\left(\frac{1}{2}\right) + 2b = 5$$

$$\Rightarrow 2 + 2b = 5$$

$$\Rightarrow 2b = 3$$

$$\Rightarrow b = \frac{3}{2}$$