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HW5
DATA601

Problem 1

We have X(n) = $\frac{1}{2}(n-3)(n-5)$

Plugging in for each n$\in S = \{1,2,3,4,5,6\}$

$X(1) = \frac{1}{2}(1-3)(1-5) = 4$

$X(2) = \frac{1}{2}(2-3)(2-5) = \frac{3}{2}$

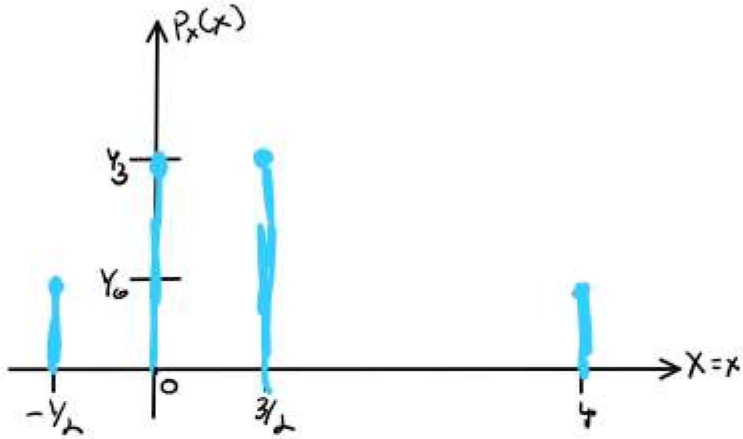
$X(3) = \frac{1}{2}(3-3)(3-5) = 0$

$X(4) = \frac{1}{2}(4-3)(4-5) = -\frac{1}{2}$

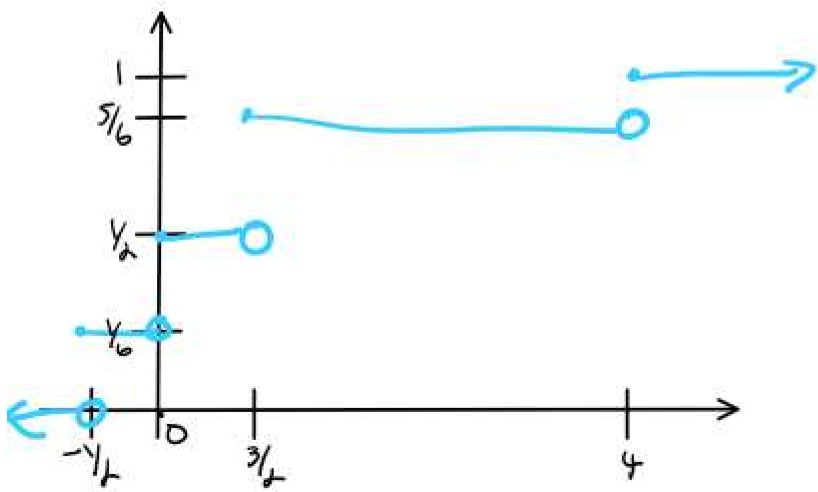
$X(5) = \frac{1}{2}(2-3)(2-5) = 0$

$X(6) = \frac{1}{2}(2-3)(2-5) = \frac{3}{2}$

Now we can make the probability mass function



We can also make the cumulative distribution function



$\mathrm{E}[\mathrm{X}] = \frac{1}{6}\left[4 + \frac{3}{2} + 0 + \left(-\frac{1}{2}\right) + 0 + \frac{3}{2}\right) = \frac{13}{12}$

$\mathrm{var}[\mathrm{X}] = E[X^2] - (E[X])^2$

$$(E[X]^2) = \left(\frac{13}{12}\right)^2 = \frac{13^2}{12^2}$$

$$E[X^2] = \frac{1}{6}\left[4^2 + \left(\frac{3}{2}\right)^2 + 0 + \left(-\frac{1}{2}\right)^2 + 0 + \left(\frac{3}{2}\right)^2\right] = \frac{83}{24}$$

$$var[X] = \frac{83}{24} - \frac{169}{144} = \frac{329}{144}$$

Problem 2

fair three sided die part

$X = \left(\frac{n}{3}\right)^2$
$P(\{n\}) = \frac{1}{3} \ \forall\, n \in \{1,2,3\}$

$E[X] = \frac{1}{3}\left[\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{3}\right)^2\right] = \frac{14}{27}$

fair six sided die part

$X = \left(\frac{n}{6}\right)^2$

$P(\{n\}) = \frac{1}{6}$

$E[X] = \frac{1}{6}\left[\left(\frac{1}{6}\right)^2 + \left(\frac{2}{6}\right)^2 + \left(\frac{3}{6}\right)^2 + \left(\frac{4}{6}\right)^2 + \left(\frac{5}{6}\right)^2 + \left(\frac{6}{6}\right)^2\right] = \frac{91}{216}$

fair sided 1000 sided die

$X = \left(\frac{n}{1000}\right)^2$

$P(\{n\}) = \frac{1}{1000} \ \forall\, n \in \{1,2,3,\dots,1000\}$

$E[X] = \frac{1}{1000} \ * \ \sum_{i=1}^{1000}\left(\frac{i}{1000}\right)^2 = \frac{1}{1000^3} \ * \ \sum_{i=1}^{1000} i^2$

the summation is a squared partial sum which has the following formula

$$\sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$$

so then

$E[X] = \frac{1}{1000^3} \ * \frac{1000(1000+1)[(2*1000)+1)}{6} = \frac{1}{1000^3} * \frac{1000(1001)(2001)}{6} \approx \frac{1}{1000^3} \ * \frac{2*1000^3}{6} = \frac{1}{3}$

Problem 3

Part a

P({H}) = p

P({T}) = (1-p)

Games goes until two heads or two tails happens.

There is 4 ways this can happen. HH, TT, HTH, THT.

Their probabilities are as follows

P({H,H}) = p^2

P({T,T}) = $(1-p)^2$

P({H,T,H}) = $p(1-p)p = p^2(1-p)$

P({T,H,T}) = $(1-p)p(1-p) = p(1-p)^2$

P({T,H,H}) = $p^2(1-p)$

P({H,T,T}) = $p(1-p)^2$

let X = the number of rolls when the game is done

then X = 2 for P({H,H}) and P({T,T})

and X = 3 for P({H,T,H}), P({T,H,T}), P({T,H,H}), and P({H,T,T})

so for E[X] we have

$$
\begin{aligned}
\text{E[X]} &= 2\,[p^2 + (1-p)^2] + 3\,[2p^2(1-p) + 2p(1-p)^2] \\
&= 2p^2 + 2(1-p)^2 + 6p^2(1-p) + 6p(1-p)^2 \\
&= 2p^2 + 2(p^2 - 2p + 1) + 6p^2 - 6p^3 + 6p(p^2 - 2p + 1) \\
&= 2p^2 + 2p^2 - 4p + 2 + 6p^2 - 6p^3 + 6p^3 - 12p^2 + 6p \\
&= -2(p^2 - p - 1)
\end{aligned}
$$

Part b

let X = the number of coin tosses when the game ends

X=2 happens when we have {H,H} or {T,T}
P({H,H}) = p^2

P({T,T}) $= (1-p)^2$

so P(X=2) $= p^2 + (1-p)^2 = p^2 + 1 - 2p + p^2 = 2p^2 - 2p + 1$

X=3 happens when we have {T,H,H} or {H,T,T}
P({T,H,H}) $= p^2(1-p)$
P({H,T,T}) $= p(1-p)^2$

so P(X=3) $= p^2(1-p) + \ p(1-p)^2 = p^2 - p^3 + p(1 - 2p + p^2) = p^2 - p^3 + p - 2p^2 + p^3 = p - p^2 = p(1-p)$

X=4 happens when we have {H,T,H,H} or {T,H,T,T}
P({H,T,H,H}) $= p^3(1-p)$
P({T,H,T,T}) $= p(1-p)^3$

so P(X=4) $= p^3(1-p) + \ p(1-p)^3 = -2p^4 + 4p^3 - 3p^2 + p$

note that this is equal to $(p - p^2)(2p^2 - 2p + 1)$

so we have P(X=4) = P(X=3) * P(X=2)

X=5 happens when we have {T,H,T,H,H} and {H,T,H,T,T}
P({T,H,T,H,H}) $= p^3(1-p)^2$
P({H,T,H,T,T}) $= p^2(1-p)^3$

So P(X=5) $= p^3(1-p)^2 + \ p^2(1-p)^3 = p^4 - 2p^3 + p^2 = p^2(p^2 - 2p + 1) = p^2(1-p)^2 = [P(X=3)]^2$

X=6 happens when we have {H,T,H,T,H,H} and {T,H,T,H,T,T}
P({H,T,H,T,H,H}) $= p^4(1-p)^2$
P({T,H,T,H,T,T} $= p^2(1-p)^4$

so P(X=6) $= p^4(1-p)^2 + \ p^2(1-p)^4 = p^2(1-p)^2(2p^2 - 2p + 1)$

note that this is the same as saying P(X=6) = P(X=5)*P(X=2)

so summarizing we have
P(X=2) $= 2p^2 - 2p + 1$
P(X=3) $= p(1-p)$
P(X=4) = P(X=3) * P(X=2)
P(X=5) $= [P(X=3)]^2$
P(X=6) = P(X=5)*P(X=2) $= [P(X=3)]^2 \ * P(X=2)$

it seems like we have two emerging patterns based on if x is even or odd

based on this I would predict that P(X=7) $= [P(X=3)]^3$ and P(X=8) = P(X=7) * P(X=2) $= [P(X=3)]^3 \ * P(X=2)$

X=7 happens when we have {T,H,T,H,T,H,H} and {H,T,H,T,H,T,T}
P({T,H,T,H,T,H,H}) = $p^4(1-p)^3$
P({H,T,H,T,H,T,T}) = $p^3(1-p)^4$

so P(X=7) = $p^4(1-p)^3 + p^3(1-p)^4 = p^3(1-p)^3 = [P(X=3)]^3$

x=8 happens when we have {H,T,H,T,H,T,H,H} and {T,H,T,H,T,H,T,T}
P({H,T,H,T,H,T,H,H}) = $p^5(1-p)^3$
P({T,H,T,H,T,H,T,T}) = $p^3(1-p)^5$

so P(X=8) = $p^5(1-p)^3 + p^3(1-p)^5 = p^3(1-p)^3(2p^2-2p+1) = P(X=7) * P(X=2) = [P(X=3)]^3 * P(X=2)$

So now we can derive a formula for when X is even or X is odd

X = 2y has P(2y) = $[p(1-p)]^{y-1} * (2p^2-2p+1)$

converting for x = 2y ⇒ $y = \frac{x}{2}$

when x is even P(X=x) = $[p(1-p)]^{\frac{x}{2}-1} * (2p^2-2p+1)$

X=2y+1 has P(2y+1) = $[p(1-p)]^{y}$

converting for x = 2y+1 ⇒ $y = \frac{x-1}{2}$

when x is odd P(X=x) = $[p(1-p)]^{\frac{x-1}{2}}$

to put them into summations we need to make x = 2x for evens and x = 2x+1 for odds so we get

$$E[X] = (2p^2-2p+1)\sum_{i=1}^{\infty}(2i)[p(1-p)]^{i-1} + \sum_{i=1}^{\infty}(2i+1)[p(1-p)]^{i}$$

wolfram alpha gives the following answer

$$\sum_{i=1}^{\infty}(2i)[p(1-p)]^{i-1} = -\frac{2(p-1)}{(1-p)(1-p+p^2)^2}$$

and

$$\sum_{i=1}^{\infty}(2i+1)[p(1-p)]^i = \frac{-p^4+2p^3-4p^2+3p}{(p^2-p+1)^2}$$

combining all the terms we have

E[X] = $(2p^2-2p+1) \;*\; -\frac{2(p-1)}{(1-p)(1-p+p^2)^2} + \frac{-p^4+2p^3-4p^2+3p}{(p^2-p+1)^2}$

This we can simplify

$$\frac{2p^2+2(1-p)^2-p^4+2p^3-4p^2+3p}{(1-p+p^2)^2} = \frac{-p^4+2p^3-2p^2+3p+2(p^2-2p+1)}{(p^2-p+1)^2} =$$

$$\frac{-p^4+2p^3-p+2}{(p^2-p+1)^2} = \frac{-(p^4-2p^3+p-2)}{(p^2-p+1)^2} = \frac{-(p^3(p-2)+(p-2))}{(p^2-p+1)^2} =$$

$$\frac{-(p-2)(p^3+1)}{(p^2-p+1)^2}$$

from here we can express (p^3+1) as $p^3+1^3 = (p+1)(p^2-p+1^2)$

so we end up with

$$E[X] = \frac{(2-p)(p+1)}{p^2-p+1} = \frac{2p+2-p^2-p}{p(p-1)+1} = \frac{2+p(1-p)}{p(p-1)+1}$$

Problem 4

Let X = # of children seated next together

lets find P(X=3)

we have 3! ways to choose the children all sitting next to each other and 5! ways to choose the adults together

we have a total of (8-1)! =7! different arrangments total

(reduced by 1 because $c_1,\ c_2, c_3,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5$ around the round table is the same as $c_2, c_3,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5, c_1$)

so then P(X=3) = $\frac{3!*5!}{7!}$

lets find P(X=2)

we can start by choosing the 5 non children spots and arranging them such that we can only have a group of 2 children. This will be 5! ways.

There is $\binom{3}{2} = 3$ ways we can arrange the grouped children

There is 4 gaps of 2 that we can make around the table

we have 2 ways to arrange each grouping of children

e.g. $c_1,\ c_2$ and $c_2,\ c_1$

Finally, we have the same denominator as above with 7!

so then P(X=2) $= \frac{5!*3*4*2}{7!} = \frac{4}{7}$

When X=2 we get 1 dollar, and when X=3 we get two dollars

let Y = the amount of money we get

$E[Y] = 1 * \left(\frac{4}{7}\right) + 2 \ * \left(\frac{1}{7}\right) = \frac{6}{7} \approx 0.86$