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Data601
HW7

Problem 1

(a)

$\int_0^2 cx\,dx = 1 \Rightarrow \frac{cx^2}{2}\Big|_0^2 = 1 \Rightarrow 2c = 1$

$$\Rightarrow c = \frac{1}{2}$$

(b)

$f_x(x) = \frac{x}{2}$

$F_X(x) = \int_0^x f_X(x)dx = \int_0^x \frac{x}{2}dx = \frac{x^2}{4}\Big|_0^x = \frac{x^2}{4}$

or more formally

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ \frac{x^2}{4}, & 0 < x < 2 \\ 1, & x \ge 2 \end{cases}$$

(c)

$$P[X > 1] = 1 - P[X \le 1] = 1 - \frac{x^2}{4}\Big|_0^1 = \frac{3}{4}$$

(d)

$E[X] = \int_0^2 x * \frac{x}{2}\,dx = \frac{1}{2} * \frac{x^3}{3}\Big|_0^2 = \frac{8}{6} = \frac{4}{3}$

(e)

$$var[X] = E[X^2] - (E[X])^2$$

$= \int_0^2 x^2 * \frac{x}{2}\,dx - \left(\frac{4}{3}\right)^2$

$= \frac{1}{2} * \frac{x^4}{4}\Big|_0^2 - \frac{16}{9} = 2 - \frac{16}{9} = \frac{2}{9}$

(f)

$Y = e^x$ and we want to find $f_Y(y)$

we need to use the $f_Y(y) = f_X\left(g^{-1}(y)\right) * \frac{1}{|g'(g^{-1}(y))|}$ formula

finding $g^{-1}(y)$

$$y = e^x$$

$x = e^y \Rightarrow \ln(x) = y \Rightarrow g^{-1}(y) = \ln(y)$

finding $f_X\left(g^{-1}(y)\right)$

$f_X(x) = \frac{x}{2}$ and $g^{-1}(y) = \ln(y)$

$$\Rightarrow f_X\left(g^{-1}(y)\right) = \frac{\ln(y)}{2}$$

finding $g'(y)$

$$g(y) = e^y \Rightarrow g'(y) = e^y$$

finding $g'(g^{-1}(y))$

$g'(y) = e^y$ and $g^{-1}(y) = \ln(y)$

$$\Rightarrow g'\left(g^{-1}(y)\right) = e^{\ln(y)} = y$$

finding $\frac{1}{|g'(g^{-1}(y))|}$

$$\frac{1}{|g'\left(g^{-1}(y)\right)|} = \frac{1}{y}$$

Finally we have

$$f_Y(y) = f_X\left(g^{-1}(y)\right) * \frac{1}{|g'(g^{-1}(y))|} = \frac{\ln(y)}{2} * \frac{1}{y} = \frac{\ln(y)}{2y}$$

and on the intervals from $e^0 = 1$ and e^2

so more formally we have

$$f_Y(y) = \{\begin{matrix} 0,\ y \le 1 \\ \frac{\ln(y)}{2y},\ y \in (1, e^2) \\ 0,\ y \ge e^2 \end{matrix}$$

(g)

$E[e^x] = \int_0^2 e^x * \frac{x}{2}\, dx =$ *1/2 ∫_0^2▒〖〖xe〗^x dx〗*

this will require integration by parts

let u = x, dv = $e^x dx$

then $du = dx$ and $v = e^x$

using the formula $\int udv = uv - \int vdu$

we get

$$\frac{1}{2}\left[xe^x\Big|_0^2 - \int_0^2 e^x dx\right] = \frac{1}{2}\left(2e^x - (e^2-1)\right) = \frac{1}{2}(2e^x - e^x + 1) = \frac{1}{2}(e^x+1)$$

Problem 2

let Y = the daily demand

X = the number of liters bought

$\left(2 - \frac{x}{400}\right)$ = cost per liter

y > x is when the demand is higher than what we bought so we sell everything we have

y ≤ x is when the demand is less than what we bought so we only sell up to y

Then profit = $\begin{cases} 3x - x\left(2-\frac{x}{400}\right), \; when\; y > x \\ 3y - x\left(2-\frac{x}{400}\right), \; when\; y \le x \end{cases}$

so profit is a function of both x and y

we can calculate the expected value of the profit by breaking it into two integrals and using our formula for expected value = y*P[y]. In this case P[y] = 1/100 for all y

E[profit] = $\int_0^x \left[3y - x\left(2-\frac{x}{400}\right)\right] * \frac{1}{100} dy + \int_x^{100} \left[3x - x\left(2-\frac{x}{400}\right)\right] * \frac{1}{100} dy$

solving will give us the expected profit in terms of x

E[profit] = $\frac{1}{100} * \left[\frac{3y^2}{2} - 2xy + \frac{x^2y}{400}\Big|_0^x + 3xy - 2xy + \frac{x^2y}{400}\Big|_x^{100}\right] =$

$\frac{1}{100}\left[\frac{3x^2}{2} - 2x^2 + \frac{x^3}{400} + 300x - 200x + \frac{x^2}{4} - 3x^2 + 2x^2 - \frac{x^3}{400}\right] =$

$$\frac{1}{100}\left[-\frac{x^2}{2} + 100x + \frac{x^2}{4} - x^2\right] = \frac{x}{100}\left[100 - \frac{5x}{4}\right] = x - \frac{x^2}{80}$$

we could graph this equation and find the maximum or we could test the inflection points by taking the derivative and setting it equal to 0 and solving for x. We would then test that value vs the extremes to find the maximum.

$$\frac{d}{dx}\left(x - \frac{x^2}{80}\right) = 1 - \frac{x}{40} = 0 \Rightarrow x = 40$$

so then we need to test x = 0, 40 and 100 and see which gives the highest value in the equation $x - \frac{x^2}{80}$.

0 gives us 0.

100 gives us - 125

40 gives us 20 as the expected profit. So x = 40 (buying 40 liters of milk) maximizes the profit.

Problem 3

We know that $f_X(x)$ must define a probability space such that $\int_{-\infty}^{\infty} f_X(x) = 1$ so we have

$$\int_0^1 (a + bx^2)dx = 1$$

$$\Rightarrow ax + \frac{bx^3}{3}\bigg|_0^1 = 1$$

$$\Rightarrow a + \frac{b}{3} = 1$$
$$\Rightarrow 3a + b = 3$$

We also know that E[X] $= \frac{5}{8}$ and can use the formula E[X] $= \int_{-\infty}^{\infty} x * f_X(x)dx$ so we have

$$\int_0^1 x \ * (a + bx^2)dx = \frac{5}{8}$$

$$\Rightarrow \int_0^1 ax + bx^3 \ dx = \frac{5}{8}$$

$$\Rightarrow \frac{ax^2}{2} + \frac{bx^4}{4}\bigg|_0^1 = \frac{5}{8}$$

$$\Rightarrow \frac{a}{2} + \frac{b}{4} = \frac{5}{8}$$

$$\Rightarrow 4a + 2b = 5$$

So now we have two equations and two unknowns

a	b	num
3	1	3
4	2	5

multiplying the first row by -2 and adding we get
(4-6)a = (5-6)
-2a = -1
a $= \frac{1}{2}$

plugging a $= \frac{1}{2}$ into the second row we get

$$4\left(\frac{1}{2}\right) + 2b = 5$$
$$\Rightarrow 2 + 2b = 5$$
$$\Rightarrow 2b = 3$$
$$\Rightarrow b = \frac{3}{2}$$