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Data 601
HW 5

Problem 1

We have X(n) = $\frac{1}{2}(n-3)(n-5)$

Plugging in for each n$\in S = \{1,2,3,4,5,6\}$

$\mathrm{X}(1) = \frac{1}{2}(1-3)(1-5) = 4$

$\mathrm{X}(2) = \frac{1}{2}(2-3)(2-5) = \frac{3}{2}$

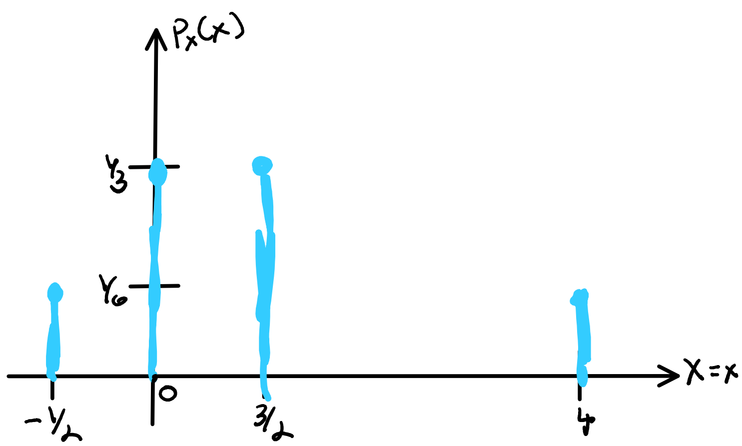
$\mathrm{X}(3) = \frac{1}{2}(3-3)(3-5) = 0$

$\mathrm{X}(4) = \frac{1}{2}(4-3)(4-5) = -\frac{1}{2}$

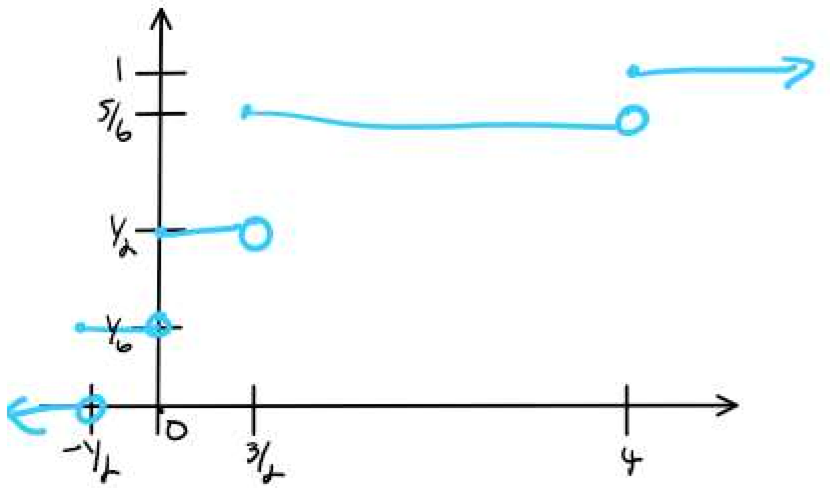
$\mathrm{X}(5) = \frac{1}{2}(2-3)(2-5) = 0$

$\mathrm{X}(6) = \frac{1}{2}(2-3)(2-5) = \frac{3}{2}$

Now we can make the probability mass function



We can also make the cumulative distribution function



$\mathrm{E[X]} = \frac{1}{6}\left[4 + \frac{3}{2} + 0 + \left(-\frac{1}{2}\right) + 0 + \frac{3}{2}\right) = \frac{13}{12}$

$\mathrm{var[X]} = E[X^2] - (E[X])^2$

$$(E[X]^2) = \left(\frac{13}{12}\right)^2 = \frac{13^2}{12^2}$$

$$E[X^2] = \frac{1}{6}\left[4^2 + \left(\frac{3}{2}\right)^2 + 0 + \left(-\frac{1}{2}\right)^2 + 0 + \left(\frac{3}{2}\right)^2\right] = \frac{83}{24}$$

$$var[X] = \frac{83}{24} - \frac{169}{144} = \frac{329}{144}$$

Problem 2

fair three sided die part

$\mathrm{X} = \left(\frac{n}{3}\right)^2$
$\mathrm{P(\{n\})} = \frac{1}{3}\ \forall\, n \in \{1,2,3\}$

$\mathrm{E[X]} = \frac{1}{3}\left[\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{3}\right)^2\right] = \frac{14}{27}$

fair six sided die part

$\mathrm{X} = \left(\frac{n}{6}\right)^2$

P({n}) $= \frac{1}{6}$

E[X] $= \frac{1}{6}\left[\left(\frac{1}{6}\right)^2 + \left(\frac{2}{6}\right)^2 + \left(\frac{3}{6}\right)^2 + \left(\frac{4}{6}\right)^2 + \left(\frac{5}{6}\right)^2 + \left(\frac{6}{6}\right)^2\right] = \frac{91}{216}$

fair sided 1000 sided die

X $= \left(\frac{n}{1000}\right)^2$

P({n}) $= \frac{1}{1000}\ \forall\, n \in \{1,2,3,\dots,1000\}$

E[X] $= \frac{1}{1000} \ * \ \sum_{i=1}^{1000}\left(\frac{i}{1000}\right)^2 = \frac{1}{1000^3} \ * \ \sum_{i=1}^{1000} i^2$

the summation is a squared partial sum which has the following formula

$$\sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$$

so then

E[X] $= \frac{1}{1000^3} \ * \frac{1000(1000+1)[(2*1000)+1)}{6} = \frac{1}{1000^3} * \frac{1000(1001)(2001)}{6} \approx \frac{1}{1000^3} \ * \frac{2*1000^3}{6} = \frac{1}{3}$

Problem 3

Part a

P({H}) = p

P({T}) = (1-p)

Games goes until two heads or two tails happens.

There is 4 ways this can happen. HH, TT, HTH, THT.

Their probabilities are as follows

P({H,H}) $= p^2$

P({T,T}) $= (1-p)^2$

P({H,T,H}) $= p(1-p)p = p^2(1-p)]$

P({T,H,T}) $= (1-p)p(1-p) = p(1-p)^2$

let X = the number of rolls when the game is done

then X = 2 for P({H,H}) and P({T,T})

and X = 3 for P({H,T,H}) and P({T,H,T})

so for E[X] we have

$$\mathrm{E[X]} = 2\,[p^2 + (1-p)^2] + 3\,[p^2(1-p) + p(1-p)^2]$$

$$= 2p^2 + 2(1-p)^2 + 3p^2(1-p) + 3p(1-p)^2$$

$$= 2p^2 + 2(p^2 - 2p + 1) + 3p^2 - 3p^3 + 3p(p^2 - 2p + 1)$$

$$= 2p^2 + 2p^2 - 4p + 2 + 3p^2 - 3p^3 + 3p^3 - 6p^2 + 3p$$

$$= p^2 - p + 2$$

Part b

let X = the number of coin tosses still needed to get two heads or two tails

let X_H = X when the first roll is heads

let X_T = X when the first roll is tails

then $E[X_H] = 1 + p + (1-p)E[X]$

the 1 is for the first roll

the p is for when we roll another heads and the game is over

the $(1-p)E[X]$ is for when we roll a tails

then $\mathrm{E}[X_T] = 1 + (1-p) + pE[X]$

using similar logic as above

Then

$\mathrm{E[X]} = \mathrm{P(\{H\})}*\mathrm{E}[X_H] + \mathrm{P(\{t\})}*\mathrm{E}[X_T]$

$\mathrm{E[X]} = p \; * (1 + p + (1-p)E[X]) + (1-p) * (1 + (1-p) + pE[X])$

simplifying we get

$\mathrm{E[X]} = p + p^2 + \; p(1-p)E[X] + (1-p) + (1-p)^2 + p(1-p)E[X]$

$$E[X](1-2p(1-p)) = p + p^2 + 1 - p + p^2 - 2p + 1 = 2p^2 - 2p + 2$$

E[X] $= \frac{2p^2-2p+2}{2p^2-2p+1}$

We can check our answer by putting p=0 or p=1 and we should get E[X] = 2 (which we do)
We can also check our answer by putting p=.5 and we should get E[X] = 3 (which we do)

Problem 4

Let X = # of children seated next together

lets find P(X=3)

we have 3! ways to choose the children all sitting next to each other and 5! ways to choose the adults together
we have a total of (8-1)! =7! different arrangments total
(reduced by 1 because $c_1,\ c_2, c_3,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5$ around the round table is the same as $c_2, c_3,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5, c_1$)

so then P(X=3) = $\frac{3!*5!}{7!}$

lets find P(X=2)

we can start by choosing the 5 non children spots and arranging them such that we can only have a group of 2 children. This will be 5! ways.
There is $\binom{3}{2} = 3$ ways we can arrange the grouped children
There is 4 gaps of 2 that we can make around the table
we have 2 ways to arrange each grouping of children
e.g. $c_1,\ c_2$ and $c_2,\ c_1$
Finally, we have the same denominator as above with 7!

so then P(X=2) $= \frac{5!*3*4*2}{7!} = \frac{4}{7}$

When X=2 we get 1 dollar, and when X=3 we get two dollars

let Y = the amount of money we get

E[Y] $= 1 * \left(\frac{4}{7}\right) + 2 * \left(\frac{1}{7}\right) = \frac{6}{7} \approx 0.86$