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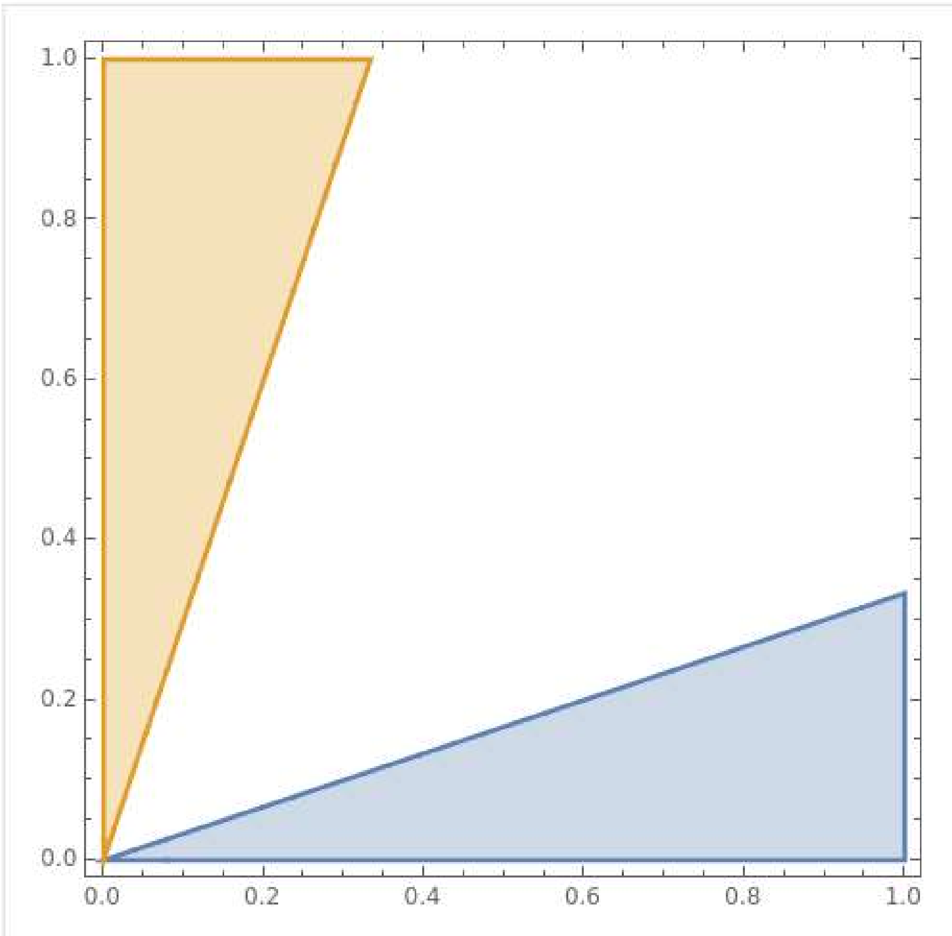
Data601
HW9

Problem 1

X~ uniform on [0,1]
Y~ uniform on [0,1]

if X > Y we want P[X $\geq$ 3Y] = P[X-3Y $\geq$ 0]
if Y > X we want P[Y $\geq$ 3X] = P[Y - 3X $\geq$ 0]

It's likely easiest to see the areas when we graph these



Graph is from wolframalpha

Let C = the areas covered by the shaded region

Since X and Y are uniform we know that $P[\ X \in C,\ Y \in C] = area(C)$

The orange triangle has area $\frac{1}{2}(1)\left(\frac{1}{3}\right) = \frac{1}{6}$

Similarly, the blue triangle has area $\frac{1}{2}(1)\left(\frac{1}{3}\right) = \frac{1}{6}$

We know it's at 1/3 since that is when the other random variable is at it's maximum, 1, for still meeting the inequalities P[X-3Y $\geq$ 0] and P[Y - 3X $\geq$ 0].

So the probability that the larger of the two random variable is 3 times as big as the other is $2 * \left(\frac{1}{6}\right) = \frac{1}{3}$

Problem 2

part a)

We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx\, dy = 1$

so we can plug into that equation to get

$$\int_0^1 \int_0^1 cxy(1-x)dxdy = 1$$

$$\Rightarrow c\int_0^1 \int_0^1 xy - x^2y\, dx\, dy = 1$$

$$\Rightarrow c\int_0^1 \frac{x^2y}{2} - \frac{x^3y}{3}\bigg|_0^1 dy = 1$$

$$\Rightarrow c\int_0^1 \frac{y}{2} - \frac{y}{3} dy = 1$$

$$\Rightarrow c\left(\frac{y^2}{4} - \frac{y^2}{6}\bigg|_0^1\right) = 1$$

$$\Rightarrow c\left(\frac{1}{4} - \frac{1}{6}\right) = 1$$

$$\Rightarrow c\left(\frac{3-2}{12}\right) = 1$$

$$\Rightarrow c = 12$$

part b)

X and Y are independent if

$$f_{X,Y}(x,y) = f_X(x) * f_Y(y)\ \forall\, x, y$$

We can find $f_X(x)$ and $f_Y(y)$ by integrating $f_{X,Y}(x,y)$ over the opposite random variable for the density function we are seeking

$$f_X(x) = 12\int_0^1 xy - x^2y\,dy = 12\left(\frac{xy^2}{2} - \frac{x^2y^2}{2}\bigg|_0^1\right) =$$

$$= 12\left(\frac{x}{2} - \frac{x^2}{2}\right) = 6x(1-x)$$

$$f_Y(y) = 12\int_0^1 xy - x^2y\,dx = 12\left(\frac{x^2y}{2} - \frac{x^3y}{3}\bigg|_0^1\right)$$

$$= 12\left(\frac{y}{2} - \frac{y}{3}\right) = 12\left(\frac{3y-2y}{6}\right) = 2y$$

$$f_{X,Y}(x,y) = 12xy(1-x)$$

$$f_X(x) * f_Y(y) = 6x(1-x) * 2y = 12xy(1-x)$$

Hence, $f_{X,Y}(x,y) = f_X(x) * f_Y(y)\ \forall\ x,y$

and so X and Y are independent

part c)

$\mathrm{E}[\mathrm{Y}] = \int_{-\infty}^{\infty} y\ * f_Y(y)dy = \ \int_0^1 y\ * 2y\ dy = \frac{2y^3}{3}\Big|_0^1 = \frac{2}{3}$

part d)

$\mathrm{E}[\mathrm{X}] = \int_0^1 x\ * [6x(1-x)]dx = 6\int_0^1 x^2 - x^3 dx = 6[\ \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$

$\mathrm{E}[X^2] = \int_0^1 x^2\ * [6x(1-x)]dx = 6\int_0^1 x^3 - x^4 dx = 6[\ \frac{x^4}{4} - \frac{x^5}{5}\Big|_0^1 = 6\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{3}{10}$

$\mathrm{var}[\mathrm{X}] = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20}$

part e)

let X + Y = Z

We have to break up this problem into two sections

1. $0 \le z \le 1$
2. $1 < z \le 2$

For part 1 we have

$F_{X+Y}(Z \leq 1) = \int_0^z \int_0^{z-x} 12xy(1-x)dy\, dx = \frac{z^4}{2} - \frac{z^5}{5}$ (I solved by hand for this one but also plugged into wolfram alpha)

$$\frac{\partial}{\partial z}\left(\frac{z^4}{2} - \frac{z^5}{5}\right) = f_{X+Y}(Z) = -z^3(z-2)$$

For part 2 we have

$$F_{X+Y}(Z \leq 1) = \int_{z-1}^{1} \int_{z-x}^{1} 12xy(1-x)dy\, dx = \frac{1}{10}(z-2)^3(2z^2+7z-2)$$

(I plugged into wolfram alpha. I did attempt solving by hand but it got really ugly, really fast)

$$\frac{\partial}{\partial z}\left(\frac{1}{10}(z-2)^3(2z^2+7z-2)\right) = f_{X+Y}(Z) = (z-2)^2(z^2+2z-2)$$

So our final answer would be

$$f_Z(z) = \begin{cases} (z-2)^2(z^2+2z-2), & 1 < z \leq 2 \\ -z^3(z-2), & 0 \leq z \leq 1 \\ 0, & otherwise \end{cases}$$

as a way to check our answer

$\int_0^1 -z^3(z-2)dz = 0.3$ and $\int_1^2 (z-2)^2(z^2+2z-2)\ dz = 0.7$

which adds to 1.0 so the density function serves as a probability space.

Problem 3

let b_1 denote battery 1 and b_2 denote battery 2

Then the probability that b_1 fails is given by

$$f_{b_1}(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

and the probability that b_2 fails is given by

$$f_{b_2}(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

We want $P[Y = b_1 \mid X > 2]$ where X is the time a battery lasted and Y is which battery was selected

$$P[Y = b_1 \mid X > 2] = \frac{P[Y=b_1 \cap X>2]}{P[X>2]} = \frac{P[Y=b_1]*P[X>2|Y=b_1]}{P[Y=b_1]*P[X>2|Y=b_1]+ P[Y=b_2]*P[X>2|Y=b_2]}$$

$P[Y = b_1] = P[Y = b_2] = \frac{1}{2}$ since we're randomly selecting between battery 1 and battery 2

$$P[X > 2|Y = b_1] = 1 - P[X \le 2 \,|Y = b_1] = 1-\int_0^2 e^{-x}\,dx =$$
$$1 - [\,-e^{-x}|_0^2 = 1 - (-e^{-2} + e^{0}) = 1+e^{-2}-1 = e^{-2}$$

$$P[X > 2|Y = b_2] = 1 - P[X \le 2 \,|Y = b_2] = 1-2\int_0^2 e^{-2x}\,dx =$$
$$1 - 2[-\frac{e^{-2x}}{2}\Big|_0^2 = 1 - (-e^{-4} + 1) = e^{-4}$$

Plugging in we get

$$\frac{P[Y = b_1] * P[X > 2|Y = b_1]}{P[Y = b_1] * P[X > 2|Y = b_1] + P[Y = b_2] * P[X > 2|Y = b_2]} = \frac{\frac{e^{-2}}{2}}{\frac{e^{-2}}{2} + \frac{e^{-4}}{2}} = \frac{e^{-2}}{e^{-2} + e^{-4}} =$$

$$\frac{\frac{1}{e^2}}{\frac{1}{e^2} + \frac{1}{e^4}} = \frac{1}{1 + \frac{1}{e^2}} = \frac{e^2}{e^2 + 1} \approx 0.88$$

So if a randomly selected battery has lasted over 2 months, there's approximately a 88% chance that it is the first battery.