MSML610: Advanced Machine Learning

9.1: Reasoning Over Time

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References: - AIMA 14: Probabilistic reasoning over time - https://github.com/rlabbe/Kalman-and-Bayesian-Filters-in-Python

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- ***Reasoning Over Time***
  - Definitions
  - Defining Temporal Inference Tasks
  - Solving Temporal Inference Tasks

- Reasoning Over Time
  - ***Definitions***
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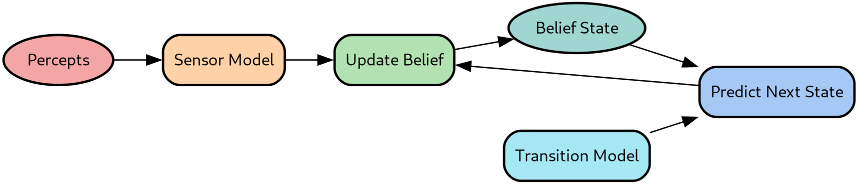
Static vs Dynamic Probabilistic Reasoning

- **Static probabilistic reasoning**
  - Random variables have *fixed* values
  - E.g., repairing a car:
    - Broken parts remain broken during diagnosis
    - Observed evidence stays fixed
- **Dynamic probabilistic reasoning**
  - Random variables *change over time*
    - E.g., tracking a plane's location, economic activity
  - E.g., treating a diabetic patient
    - Assess patient state, decide insulin dose
    - Evidence: insulin doses, food intake, blood sugar (change over time)
    - Time dependency (e.g., metabolic activity, time of day)

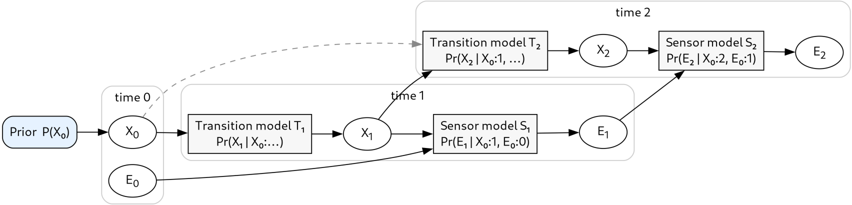
Agents in Partially Observable Environments



- Agents in *partially observable environments* track current state using:
  1. **Belief state**
     - Store possible world states
     - Use probability theory to quantify belief
     - Posterior distribution of current state given all evidence
  2. **Belief state + Transition model**
     - Predict world evolution in next step
  3. **Sensor model + Percepts**
     - Update belief state
- *Idea*: handle time by making each quantity a function of time
  - $\underline{\boldsymbol{X}}_{a:b}$ represents variables in $[a, b]$

Agent: Model Components

1. State of the world: $\underline{\boldsymbol{X}}_t$
   - Not directly observable
2. Prior probability of the state at time 0: $\underline{\boldsymbol{X}}_0$
3. Evidence variables: $\underline{\boldsymbol{E}}_t$
   - Observable
4. Transition model: $\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{0:t-1})$
   - Models world evolution
   - Specifies probability distribution of state $\underline{\boldsymbol{X}}_t$ given previous values
5. Sensor model: $\Pr(\underline{\boldsymbol{E}}_t | \underline{\boldsymbol{X}}_{0:t}, \underline{\boldsymbol{E}}_{0:t-1})$
   - Models generation of evidence variables $\underline{\boldsymbol{E}}_t$



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Discrete vs Continuous Time Models

- **Discrete time models**
  - View world as time slices ("snapshots")
    - Equal time intervals, equispaced samples
    - Label times $t = 0, 1, 2, \ldots$
  - Each slice contains random variables:
    - Hidden RVs (e.g., $\underline{\boldsymbol{X}}_t$)
    - Observable RVs (e.g., $\underline{\boldsymbol{E}}_t$)
- **Continuous time models**
  - Model uncertainty over continuous time with stochastic differential equations (SDEs)
  - Discrete time models approximate SDEs

Markov Property

- **In general**, current state $\underline{\boldsymbol{X}}_t$ depends on a growing number of past states:

$$\Pr(\underline{\boldsymbol{X}}_t | history) \triangleq \Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{0:t-1}) = \Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_0, \underline{\boldsymbol{X}}_1, \ldots, \underline{\boldsymbol{X}}_{t-1})$$

  - Of course, there can't be dependency from the future $\underline{\boldsymbol{X}}_{t+k}$ with $k > 0$
- **Markov property**: current state depends (conditionally) only on a finite fixed number of k previous states:

$$\begin{aligned}\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{0:t-1}) &= \Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_0, \underline{\boldsymbol{X}}_1, \ldots, \underline{\boldsymbol{X}}_{t-k-1}, \underline{\boldsymbol{X}}_{t-k}, \ldots, \underline{\boldsymbol{X}}_{t-1}) \\ &= \Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{t-k:t-1})\end{aligned}$$

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Markov Process

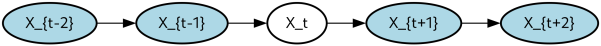
- **Markov processes** (aka Markov chains) have the Markov property

$$\Pr(\underline{\boldsymbol{X}}_t | \mathit{history}) = \Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{t-k:t-1}) \ \forall k, t$$

- **First-order Markov process**: current state $\underline{\boldsymbol{X}}_t$ depends only on previous state $\underline{\boldsymbol{X}}_{t-1}$:

$$\Pr(\underline{\boldsymbol{X}}_t | \mathit{history}) = \Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{t-1}) \ \forall k, t$$

  - Next state depends only on previous state, not full history
  - *Intuition*: system "forgets" everything except immediate last state
  - Bayesian network for a first-order Markov process:



  - E.g., probability of rain today depends only on yesterday, $\Pr(R_t | R_{t-1}) \ \forall t$

- **Second-order Markov process**: current state $\underline{\boldsymbol{X}}_t$ depends only on $\underline{\boldsymbol{X}}_{t-1}$ and $\underline{\boldsymbol{X}}_{t-2}$

Time-Homogeneous Process

- Even with the Markov assumption, there is an infinite number of probability distributions, one for each t

$$\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{t-1:t-k})$$

- **Time-homogeneous** (aka stationarity): probability remains constant by translation over t

$$\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{0:t-1}) = \Pr(\underline{\boldsymbol{X}}_{t-k} | \underline{\boldsymbol{X}}_{0:t-k-1}) \; \forall k, t$$

  - *Intuition*: even if process evolves, governing laws remain unchanged
  - E.g., in the real-world, most physical laws are constant

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First-Order Time-Homogeneous Process

- **First-order Markov property**

$$\Pr(\underline{\boldsymbol{X}}_t | \mathit{history}) = \Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{t-1})$$

- **Time-homogeneous**

$$\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{0:t-1}) = \Pr(\underline{\boldsymbol{X}}_{t-k} | \underline{\boldsymbol{X}}_{0:t-k-1}) \ \forall k, t$$

- **First-order time-homogeneous**: putting both properties together, one conditional probability table suffices:

$$\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{t-1}) = \Pr(\underline{\boldsymbol{X}}_{t-k} | \underline{\boldsymbol{X}}_{t-k-1}) \ \forall k, t$$

  - E.g., rain probability for today depends only on yesterday and is constant: $\Pr(R_t | R_{t-1}) = f(R_{t-1}) \ \forall t$

Sensor Model

- **In general**, evidence variables $\underline{\boldsymbol{E}}_t$ depend on:
  - Previous state of the world $\underline{\boldsymbol{X}}_{0:t}$
  - Previous sensor values $\underline{\boldsymbol{E}}_{0:t-1}$

$$\Pr(\underline{\boldsymbol{E}}_t | \underline{\boldsymbol{X}}_{0:t}, \underline{\boldsymbol{E}}_{0:t-1})$$

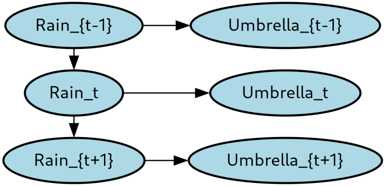
- **Sensor Markov property**
  - Assume sensor value $\underline{\boldsymbol{E}}_t$ depends only on current state $\underline{\boldsymbol{X}}_t$, not on previous sensor values

$$\Pr(\underline{\boldsymbol{E}}_t | \underline{\boldsymbol{X}}_{0:t}, \underline{\boldsymbol{E}}_{0:t-1}) = \Pr(\underline{\boldsymbol{E}}_t | \underline{\boldsymbol{X}}_t)$$

  - In a Bayesian network, even if $\underline{\boldsymbol{X}}_t$ and $\underline{\boldsymbol{E}}_t$ are contemporaneous, the arrow goes from $\underline{\boldsymbol{X}}_t \to \underline{\boldsymbol{E}}_t$ since the world causes the sensor to take on particular values

Sensor Model: Rain Example

- Consider the Bayesian network for the **rain model**
- The world causes the sensor to take specific values $\underline{\boldsymbol{X}}_t \rightarrow \underline{\boldsymbol{E}}_t$
  - E.g., $Rain_t \rightarrow Umbrella_t$, since rain "causes" the umbrella to appear
  - Inference goes the other direction: *"see the umbrella, guess if it's raining"*
- The **transition model** is $\Pr(Rain_t | Rain_{t-1})$
  - $\Pr(R_t | R_{t-1} = T) = 0.7$
  - $\Pr(R_t | R_{t-1} = F) = 0.4$
  - The sum doesn't have to be 1 since it's a conditional probability
- The **sensor model** is $\Pr(Umbrella_t | Rain_t)$
  - $\Pr(U_t | R_t = T) = 0.9$ (people forget the umbrella)
  - $\Pr(U_t | R_t = F) = 0.2$ (people are paranoid)



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Prior Probability

- Complete system specification needs **prior probability of state variables** at initial time, $\Pr(\underline{\boldsymbol{X}}_0)$
  - Represents initial belief about system state before observations
  - Crucial for initializing state estimation process
- E.g.,
  - $\underline{\boldsymbol{X}}_0$ represents position and velocity of a moving object
  - $\Pr(\underline{\boldsymbol{X}}_0)$ could be a Gaussian distribution centered around an initial guess of object's position and velocity with uncertainty

First-Order Markov Process: Joint Distribution

- **Goal**: model a sequence of states $\underline{\boldsymbol{X}}_0, \underline{\boldsymbol{X}}_1, \ldots, \underline{\boldsymbol{X}}_t$ and observations $\underline{\boldsymbol{E}}_1, \ldots, \underline{\boldsymbol{E}}_t$ over time, i.e., $\Pr(\underline{\boldsymbol{X}}_{0:t}, \underline{\boldsymbol{E}}_{1:t})$
- **Express the joint distribution** of n random variables using the chain rule:

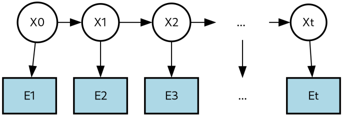
$$\Pr(\underline{\boldsymbol{X}}_1, \ldots, \underline{\boldsymbol{X}}_n) = \prod_{i=1}^{n} \Pr(\underline{\boldsymbol{X}}_i | \underline{\boldsymbol{X}}_{0:i-1})$$

- Like for Bayesian networks **factorize joint distribution** according to graph dependencies:

$$\Pr(\underline{\boldsymbol{X}}_1, \ldots, \underline{\boldsymbol{X}}_n) = \prod_{i=1}^{n} \Pr(\underline{\boldsymbol{X}}_i | \text{parents}(\underline{\boldsymbol{X}}_i))$$

- **First-order Markov assumption**:

$$\Pr(\underline{\boldsymbol{X}}_i | \underline{\boldsymbol{X}}_{0:i-1}) = \Pr(\underline{\boldsymbol{X}}_i | \underline{\boldsymbol{X}}_{i-1})$$



- **First-order Markov sensor model**:

$$\Pr(\underline{\boldsymbol{E}}_i | \underline{\boldsymbol{X}}_{0:i}, \underline{\boldsymbol{E}}_{1:i-1}) = \Pr(\underline{\boldsymbol{E}}_i | \underline{\boldsymbol{X}}_i)$$

First-Order Markov Process: Joint Distribution

- Putting everything together, the joint distribution probability for a time-homogeneous first-order Markov process:

$$\Pr(\underline{\boldsymbol{X}}_{0:t}, \underline{\boldsymbol{E}}_{1:t}) = \Pr(\underline{\boldsymbol{X}}_0) \prod_{i=1}^{t} \Pr(\underline{\boldsymbol{X}}_i | \underline{\boldsymbol{X}}_{i-1}) \Pr(\underline{\boldsymbol{E}}_i | \underline{\boldsymbol{X}}_i)$$

$$= \text{prior} \times \prod_{i} \text{transition model} \times \text{sensor model}$$

- **Remarks**:
  - The state evolves probabilistically from the previous state (transition model)
  - This structure reduces complexity and enables tractable inference
  - A Bayesian network can represent a temporal model by modeling time with indices t, i.e., "unrolling the model"
- **Problem**: infinite t, even assuming the Markov property

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Improving Approximation of Real-World Systems

- Is first-order Markov process a **reasonable approximation of reality**?
  - Particle following random walk is well represented by Markov process
  - In umbrella example, rain depends only on previous day
- **How to improve the approximation**
  1. *Increase order of Markov process model*
     - Model "rarely rains more than two days in a row" with second-order Markov model $\Pr(Rain_t | Rain_{t-1}, Rain_{t-2})$
  2. *Increase number of state variables*
     - Add $Season_t$ to incorporate historical records
     - Transition model becomes more complicated
  3. *Increase number of sensor variables*
     - Add $Location_t, Temperature_t, Humidity_t, Pressure_t$
     - Simplifies modeling of state

- Reasoning Over Time
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Inference Tasks in Temporal Models

- There are **several tasks** in temporal inference

Task	**Description**	**Estimate**
Filtering	Estimate *current* state given past / current obs	$\Pr(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{E}}_{1:t})$
Prediction	Estimate *future* state given past / current obs	$\Pr(\underline{\boldsymbol{X}}_{t+k} \mid \underline{\boldsymbol{E}}_{1:t})$ for $k > 0$
Smoothing	Estimate *past* state given past, current, and *future* obs	$\Pr(\underline{\boldsymbol{X}}_k \mid \underline{\boldsymbol{E}}_{1:T})$ for $T < k$
Most likely explanation	Find most probable sequence of states given the evidence	$\text{argmax}_{\underline{\boldsymbol{x}}_{1:T}} \Pr(\underline{\boldsymbol{X}}_{1:t} \mid \underline{\boldsymbol{E}}_{1:t})$
Learning	Learn model parameters or structure from data	θ of a model

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Task 1: Filtering

- **Filtering** (aka "state estimation") computes the posterior distribution of the *current state* given *all evidence to date*:

$$\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{E}}_{1:t} = \underline{\boldsymbol{e}}_{1:t})$$

  - Estimate probability of rain today, given all umbrella observations so far $\Pr(Rain_t | Umbrella_{1:t})$
- **Application**:
  - Filtering needed by a rational agent to *track current state*
  - Agent believes current state $\Pr(\underline{\boldsymbol{X}}_{t-1})$ at time $t-1$
  - New evidence $\underline{\boldsymbol{e}}_t$ arrives for time t
  - Agent updates belief about current state $\Pr(\underline{\boldsymbol{X}}_t)$ at time t
- "Filtering" refers to filtering out noise in a signal by estimating system parameters

Task 2: Prediction

- **Prediction** involves predicting the posterior distribution over a *future state*, given *all evidence to date*:

$$\Pr(\underline{\boldsymbol{X}}_{t+k} | \underline{\boldsymbol{E}}_{1:t} = \underline{\boldsymbol{e}}_{1:t}) \text{ with } k > 0$$

  - E.g., compute the probability of rain three days from now:

$$\Pr(Rain_{t+3} | Umbrella_{1:t})$$

- **Application**
  - Prediction helps rational agents evaluate actions based on expected outcomes

Task 3: Smoothing

- **Smoothing** compute posterior distribution over a *past state* given *all past, present, and future evidence*:

$$\Pr(\underline{\boldsymbol{X}}_k | \underline{\boldsymbol{e}}_{1:t}) \text{ with } 0 \leq k < t$$

  - **Note**: you have information about the "future" of the evidence, but not the state
  - Smoothing provides a better state estimate by incorporating more future evidence
  - E.g., compute the probability it rained last Wednesday, given all observations up to today
- The term "smoothing" refers to the state estimate being smoother than filtering

Task 4: Most-Likely Explanation

- **Most-likely explanation** finds the sequence of states $\underline{\boldsymbol{X}}_{1:t}$ most likely to have generated observations $\underline{\boldsymbol{E}}_{1:t}$:

$$\operatorname{argmax}_{\underline{\boldsymbol{X}}_{1:t}} \Pr(\underline{\boldsymbol{X}}_{1:t} | \underline{\boldsymbol{E}}_{1:t})$$

  - E.g.,
    - Umbrella appeared on 3 days, not on the fourth
    - Most likely explanation: rained for 3 days, then stopped

- **Applications**

  - Speech recognition: most likely sequence of words given sounds
  - Digital processing: reconstruct bit strings over a noisy channel

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Task 5: Learning

- **Learning** involves estimating the transition model $\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{X}}_{0:t-1})$ and the sensor model $\Pr(\underline{\boldsymbol{E}}_i | \underline{\boldsymbol{X}}_i)$ from observations
- Learning benefits from smoothing rather than filtering for better state estimates
  - Smoothing uses all data to estimate states, leading to more accurate models
  - E.g., in weather prediction, smoothing uses past, present, and future data to better estimate current weather state

- Reasoning Over Time
  - Definitions
  - Defining Temporal Inference Tasks
  - ***Solving Temporal Inference Tasks***

Solving Task 1: Filtering

- **Filtering** computes the posterior distribution of the *current state* given *all evidence to date*, i.e., $\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{E}}_{1:t} = \underline{\boldsymbol{e}}_{1:t})$
- A practical filtering algorithm updates the current state estimate $\underline{\boldsymbol{X}}_{t+1}$ using the previous state $\underline{\boldsymbol{X}}_t$ and the new evidence $\underline{\boldsymbol{e}}_{t+1}$
  - Instead of recomputing each state by going over the entire history of the percepts
  - Aka "recursive state estimation"

$$\Pr(\underline{\boldsymbol{X}}_{t+1} | \underline{\boldsymbol{e}}_{1:t+1}) = f(\Pr(\underline{\boldsymbol{X}}_t | \underline{\boldsymbol{e}}_{1:t}), \underline{\boldsymbol{e}}_{t+1})$$
$$NextState = f(PreviousState, \underline{\boldsymbol{e}}_{t+1})$$

- **Why?**
  - Time and space requirements for updating must be constant for a (finite) agent to keep track of current state indefinitively
- **Is it possible?**
  - What is the formula $f(...)$?

Recursive Filtering: Update Formula

- Compute the state at time $t+1$ with all the evidence up to that time
- Assume that state and evidence are scalar and not vector

$$
\begin{aligned}
&\Pr(X_{t+1}|e_{1:t+1}) \\
&= \Pr(X_{t+1}|e_{1:t}, e_{t+1}) && \text{Divide up the evidence} \\
&= \alpha \Pr(e_{t+1}|X_{t+1}, e_{1:t}) \Pr(X_{t+1}|e_{1:t}) && \text{Bayes rule given} \\
&= \alpha \Pr(e_{t+1}|X_{t+1}) \Pr(X_{t+1}|e_{1:t}) && \text{Markov sensor assumption} \\
&= \alpha \Pr(e_{t+1}|X_{t+1}) \sum_{x_t} \Pr(X_{t+1}|x_t, e_{1:t}) \Pr(x_t|e_{1:t}) && \text{Condition on current state} \\
&= \alpha \Pr(e_{t+1}|X_{t+1}) \sum_{x_t} \Pr(X_{t+1}|x_t) \Pr(x_t|e_{1:t}) && \text{Markov assumption}
\end{aligned}
$$

- It has the expected form:

$$\Pr(X_{t+1}|e_{1:t+1}) = f(\Pr(X_t|e_{1:t}), e_{t+1})$$

Recursive Filtering: Update Formula

- The update formula for the state is:

$$\Pr(X_{t+1}|e_{1:t+1}) = \alpha \Pr(e_{t+1}|X_{t+1}) \sum_{x_t} \Pr(X_{t+1}|x_t) \Pr(x_t|e_{1:t})$$

- The next state is “Sensor model x Transition model x Recursive state”
  - Sensor model: $\Pr(e_{t+1}|X_{t+1})$
  - Transition model: $\Pr(X_{t+1}|x_t)$
  - Recursive term: $\Pr(x_t|e_{1:t})$

Recursive Filtering: Intuition

- Recursive state estimation updates the state belief as new evidence arrives

$$\Pr(X_{t+1}|e_{1:t+1}) = \alpha \Pr(e_{t+1}|X_{t+1}) \sum_{x_t} \Pr(X_{t+1}|x_t) \Pr(x_t|e_{1:t})$$

  in **two steps**

  1. **Prediction step**: Use the transition model to predict the next state based on the current belief

$$\Pr(X_{t+1}|e_{1:t}) = \sum_{x_t} \Pr(X_{t+1}|x_t) \Pr(x_t|e_{1:t})$$

     - Intuition: Project the current belief forward using the model of system evolution

  2. **Update step**: Incorporate the new observation to refine the prediction

$$\Pr(X_{t+1}|e_{1:t+1}) = \alpha \Pr(e_{t+1}|X_{t+1}) \Pr(X_{t+1}|e_{1:t})$$

     - Intuition: Correct the prediction using the likelihood of the new evidence

- Maintain $\Pr(X_t|e_{1:t})$, the probability of the current state given all past evidence
  - E.g., in a weather model, if it was likely to rain today and rain usually continues, the prediction leans toward rain tomorrow
  - Seeing an umbrella supports this and updates the belief accordingly

Forward update

- We achieved:

$$\Pr(\underline{\boldsymbol{X}}_{t+1}|\underline{\boldsymbol{e}}_{1:t+1}) = \alpha \Pr(\underline{\boldsymbol{e}}_{t+1}|\underline{\boldsymbol{X}}_{t+1}) \sum_{x_t} \Pr(\underline{\boldsymbol{X}}_{t+1}|\underline{\boldsymbol{x}}_t) \Pr(\underline{\boldsymbol{x}}_t|\underline{\boldsymbol{e}}_{1:t})$$
$$= f(\Pr(\underline{\boldsymbol{X}}_t|\underline{\boldsymbol{e}}_{1:t}), \underline{\boldsymbol{e}}_{t+1})$$

- The filtered estimate $\underline{\boldsymbol{f}}_{1:t} = \Pr(\underline{\boldsymbol{X}}_t|\underline{\boldsymbol{e}}_{1:t})$ is propagated forward and updated by each transition and new observation

$$\underline{\boldsymbol{f}}_{1:t+1} = Forward(\underline{\boldsymbol{f}}_{1:t}, \underline{\boldsymbol{e}}_{t+1})$$

  starting with the initial condition $\underline{\boldsymbol{f}}_{1:0} = \Pr(\underline{\boldsymbol{X}}_0)$
  - This is called “forward update”
- This process allows efficient online inference without storing the full history
  - Time and space requirements for updating is constant
  - A (finite) agent can keep track of current state indefinitively

Solving Task 2: Prediction

- Prediction is equivalent to filtering without updating the state with new evidence, since there is no evidence
  - Only the transition model is needed, not the sensor model
- The rule predicting state $\underline{\boldsymbol{X}}_{t+k+1}$ given state $\underline{\boldsymbol{X}}_{t+k}$ and evidence $\underline{\boldsymbol{E}}_{1:t}$ is:

$$\Pr(\underline{\boldsymbol{X}}_{t+k+1} | \underline{\boldsymbol{e}}_{1:t}) = \sum_{\underline{\boldsymbol{x}}_{t+k}} \Pr(\underline{\boldsymbol{X}}_{t+k+1} | \underline{\boldsymbol{x}}_{t+k}) \Pr(\underline{\boldsymbol{x}}_{t+k} | \underline{\boldsymbol{e}}_{1:t})$$

- This equation can be used recursively to advance over time
  - Predicting even a few steps ahead generally incurs large uncertainty

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Solving Task 3: Smoothing

- You want to calculate the probability distribution over the hidden state at time k, given all evidence up to time t (in the future!)

$$\Pr(X_k | e_{1:t}) \text{ where } 0 \leq k < t$$

  - Filtering gives $\Pr(X_k | e_{1:k})$ using past and present evidence
  - Smoothing refines the estimate of past states using later evidence
- **Example**
  - You're tracking whether it was raining yesterday
  - You had some evidence up to yesterday (e.g., a cloudy sky)
  - Today you see puddles on the ground
  - That new observation supports the idea that yesterday was raining

Task 3: Smoothing: Update Formula

- Using the same math as for filtering and the two key assumptions of Markov process and Markov sensor
- **Forward Pass (aka filtering):**
  - Move forward through time, using the filtering algorithm to compute:
  $$f_{1:k} = \Pr(X_k | e_{1:k})$$
  - This gives you a "best guess" of the state at time k, based only on evidence up to k
- **Backward Pass (aka smoothing):**
  - Move backward through time from time t, computing:
  $$b_{k+1:t} = \Pr(e_{k+1:t} | X_k)$$
  - This captures how likely the future evidence is, given a particular value of X_k
- **Combine them:**
  - Multiply forward and backward messages to get:
  $$\Pr(X_k | e_{1:t}) \propto f_{1:k} \times b_{k+1:t}$$

Task 4: Most Likely Explanation: Intuition 1/2

- You are tracking the weather (sunny or rainy) based on whether someone carries an umbrella
  - You can't see *Weather* directly (hidden state), but you observe umbrellas (which is a noisy observation)
  - You have 5 observations $Umbrella = [T, T, F, T, T]$
- **Question**: what is the most likely sequence of *Weather* states that explains the *Umbrella* observations?
  - You know something about:
    - the transition model (i.e., "it tends to rain several days in a row")
    - the sensor model (i.e., "people often forget the umbrella")
- Mathematically:

$$\text{argmax}_{x_{1:t}} \Pr(x_{1:t} | e_{1:t}) = \text{argmax}_{Weather_{1:t}} \Pr(Weather_{1:t} | Umbrella_{1:t})$$

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Task 4: Most Likely Explanation: Intuition 2/2

- **Naive approach**: Use smoothing to choose the most likely state at each time step
  - Cons
    - Might lead to an implausible overall path
    - Suboptimal since the question addresses joint probability and we are not using all the information (only one step at the time!)
- **Viterbi algorithm**:
  - Constructs a path through a state-time graph with states as nodes and transitions as edges
  - Finds the most likely entire path through the hidden states
- **Key difference:**
  - In speech recognition, find the most likely word sequence behind a noisy audio signal
    - Smoothing: Best guess per time step (may miss non-English words or suboptimal sequence)
    - Viterbi: Best overall path (maximizes joint probability of the entire sequence)

Viterbi Algorithm: Intuition

- **Goal**: Find the most likely sequence of hidden states given observations

1. Initialization
   - At $t = 1$, estimate probability of starting in each state using initial state distribution and observation likelihood
2. Recursion via dynamic programming
   - For each $t > 1$, for each state x_t:
   - Compute maximum probability path to x_t from any previous state
   - Use:
     - $\Pr(x_t | x_{t-1})$: transition model
     - $\Pr(e_t | x_t)$: sensor model
     - Best path probability to x_{t-1} from prior step
   - Store probability and corresponding back-pointer to x_{t-1}
3. Termination and backtrace
   - At final time $t = T$, identify state with highest final probability
   - Trace back through stored pointers to reconstruct optimal path

Viterbi Algorithm: Example 1/2

- You observe a friend carrying an umbrella over 3 days
  - $Umbrella = [Yes, Yes, No]$
- You want to infer the most likely sequence of hidden *Weather* states
  - States: $S = \{\text{Sunny}, \text{Rainy}\}$ (weather)
  - Observations: $O = \{\text{Yes}, \text{No}\}$ (umbrella)
  - Initial Probabilities:

$$\Pr(\text{Sunny}) = 0.6, \quad \Pr(\text{Rainy}) = 0.4$$

  - Transition Probabilities:

$$\begin{aligned} \Pr(\text{Sunny} \to \text{Sunny}) &= 0.7, \quad \Pr(\text{Sunny} \to \text{Rainy}) = 0.3 \\ \Pr(\text{Rainy} \to \text{Sunny}) &= 0.4, \quad \Pr(\text{Rainy} \to \text{Rainy}) = 0.6 \end{aligned}$$

  - Observation (Emission) Probabilities:

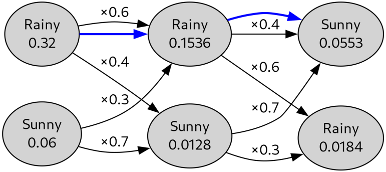
$$\begin{aligned} \Pr(\text{Yes}|\text{Sunny}) &= 0.1, \quad \Pr(\text{No}|\text{Sunny}) = 0.9 \\ \Pr(\text{Yes}|\text{Rainy}) &= 0.8, \quad \Pr(\text{No}|\text{Rainy}) = 0.2 \end{aligned}$$

Viterbi Algorithm: Example 2/2

- Viterbi table

Day	State	Probability	Backpointer
1	Sunny	$0.6 \times 0.1 = \mathbf{0.06}$	—
	Rainy	$0.4 \times 0.8 = \mathbf{0.32}$	—
2	Sunny	$\max(0.06 \times 0.7,\ 0.32 \times 0.4) \times 0.1 = \mathbf{0.0128}$	Rainy
	Rainy	$\max(0.06 \times 0.3,\ 0.32 \times 0.6) \times 0.8 = \mathbf{0.1536}$	Rainy
3	Sunny	$\max(0.0128 \times 0.7,\ 0.1536 \times 0.4) \times 0.9 = \mathbf{0.0553}$	Rainy
	Rainy	$\max(0.0128 \times 0.3,\ 0.1536 \times 0.6) \times 0.2 = \mathbf{0.0184}$	Rainy

- Final most probable state
  - Sunny (Day 3)
- Find the most likely sequence
  - Rainy $\rightarrow$ Rainy $\rightarrow$ Sunny



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