Lecture Notes on

Group Theory for Optimization and Machine Learning

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1 Basics of Group Theory

Definition. Group: A non-empty set G with a binary operation (denoted by $\cdot$) is a group if the following three properties are satisfied:

(Associativity): For all $a, b, c \in G$,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

(Identity): There exists $e \in G$ s.t.

$$a \cdot e = e \cdot a = a$$

(Inverse): For any $a \in G$, there exists $b \in G$ s.t.

$$a \cdot b = b \cdot a = e$$

and b is denoted by a^{-1}.

Definition. Abelian Group: A commutative group under an operation with the property that for any elements $a, b \in G$, $a \cdot b = b \cdot a$. Examples are $\mathbb{R}^n$ with $+$ and $\mathbb{Z}^n$ with $+$.

Definition. Let G be a group. A subset H in G is a **subgroup** if H is also a group under the binary operation of G, denoted by $H < G$.

Definition. The **direct product** $G_1 \times G_2$ of groups G_1 and G_2 is $\{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$ with operation

$$(g_1, g_2) \cdot (h_1, h_2) = (g_1 \cdot h_1, g_2 \cdot h_2)$$

2 Machine Learning