3X03 Assignment 3

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1 Problem 5

Both the strategies produce interpolating polynomials that are within the error tolerance. For Chebyshev, the interpolation points, or nodes, are chosen based on $x_i = \frac{a+b}{2} + \frac{b-a}{2}\cos(\frac{(2i+1)\pi}{2n})\ \forall i = 0 \ldots n-1$ on interval $[a, b]$. On the other hand, the other method evenly distributes points along the interval based on $x_i = a + ih\ \forall i = 0 \ldots n$ where $h \triangleq \frac{b-a}{n}$. The Chebyshev method requires fewer points to satisfy the bound, and more of the nodes are slight more concentrated towards the edges of the interval, as the spacing follows a cosine-based distribution.

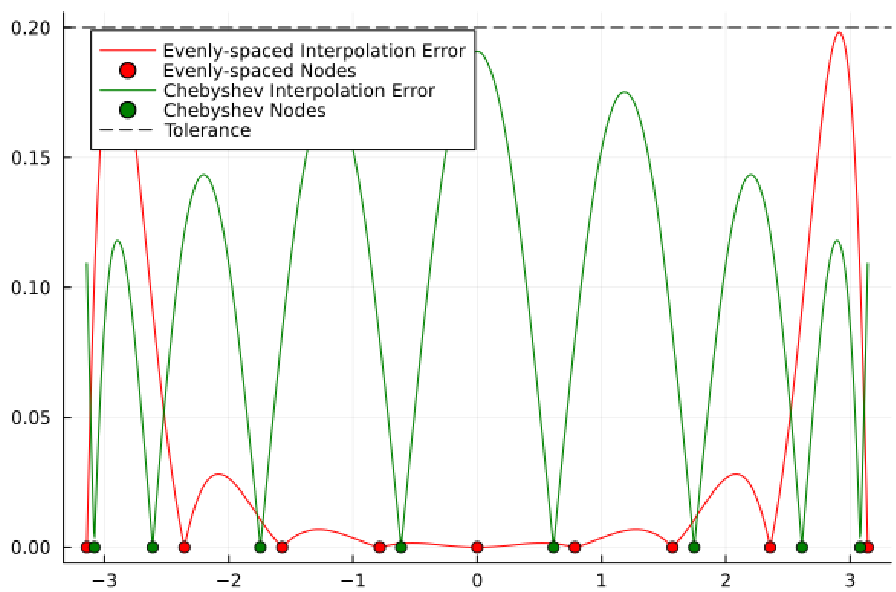

Figure 1: Plot of interpolating polynomial using Chebyshev nodes and even-spacing

2 Problem 7

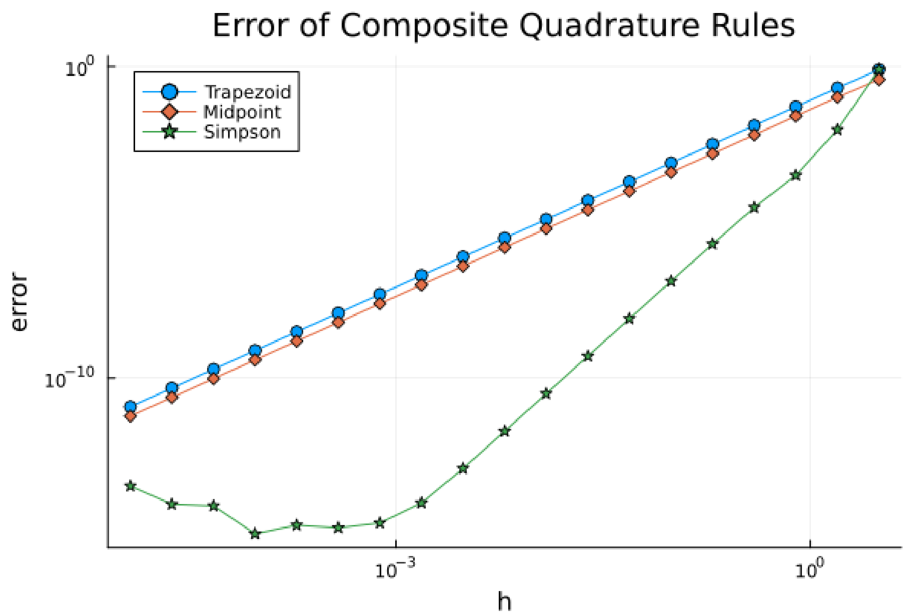

Figure 2: Plot of Error using various quadrature rules.

2.1 Part A

The absolute error using the composite trapezoidal rule is given by:

$$\text{error} = \frac{f''(\eta)(b-a)h^2}{12}$$

$$\begin{aligned}\log(\text{error}) &= \log h^2 + \log(f''(\eta)(b-a)) - \log 12 \\ &= 2\log h + \log(f''(\eta)(b-a)) - \log 12\end{aligned}$$

which implies the slope is 2. The absolute error using composite midpoint rule is given by:

$$\text{error} = \frac{f''(\mu)(b-a)h^2}{24}$$

$$\begin{aligned}\implies \log(\text{error}) &= \log h^2 + \log(f''(\mu)(b-a)) - \log 24 \\ &= 2\log h + \log(f''(\mu)(b-a)) - \log 24\end{aligned}$$

which implies the slope is 2. Lastly, the absolute error using the composite simpson rule is given by

$$\text{error} = \frac{f''(\mu)(b-a)h^4}{180}$$

$$\implies \log(\text{error}) = \log h^4 + \log(f''(\mu)(b-a)) - \log 180$$
$$= 4\log h + \log(f''(\mu)(b-a)) - \log 180$$

which implies the slope is 4.

2.2 Part B

The reason for the offset between the two parallel lines is the distinct intercepts of both methods when plotted on a log-log graph. For the composite trapezoidal rule, this is $\log(f''(\eta)(b-a)) - \log 12$, where $\eta \in [a,b]$. While for the composite midpoint rule, this is $\log(f''(\mu)(b-a)) - \log 24$, $\mu \in [a,b]$. Since the values of $h \leq 1$ are used to produce the graph, it is reasonable to assume in each sub interval used in either the composite trapezoidal or midpoint rule is small enough that $f''(\mu_i) \approx f''(\eta_i) \implies f''(\mu) \approx f''(\eta)$. Thus,

$$E_{trap} = \log(f''(\eta)(b-a)) - \log 12 \geq \log(f''(\eta)(b-a)) - \log 24$$
$$\approx \log(f''(\mu)(b-a)) - \log 24 = E_{mid}$$

which accounts for the offset.

2.3 Part C

As the step-size of the composite Simpson rule approaches zero, the error approaches machine precision. Due to the limited precision of Float64, below this precision, this will cause truncation errors, resulting in the abnormal behavior for $h < 10^{-3}$. Conversely, when $h > 10^{-3}$, the error behaves as predicted in theory (with slope 4), as evidenced by linear segment which larger slope than both the trapezoidal and midpoint rules.

3 Problem 9

3.1 Part A

The integral computed by composite Simpson's rule differs from the integral computed using adaptive Simpson's rule, as it applies a uniform subdivision of the interval regardless of the complexity of the function. This naive approach might overestimate or underestimate the function in regions where it varies significantly. Adaptive Simpson's rule subdivides the interval dynamically and refines the subdivisions until an error tolerance is met. Adaptive Simpson's rule is also more accurate, because it adapts to the local behavior of the function and reduces inefficiency by improving approximations in regions requiring high precision. In contrast, composite Simpson's rule is more inefficient, as a lot of computing power is wasted over regions of the function which do not require a lot of precision.

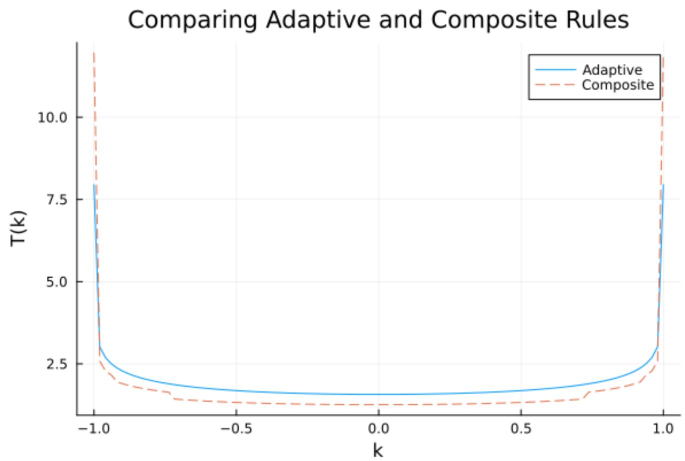

Figure 3: Plot comparing adaptive and composite Simpson's rule.

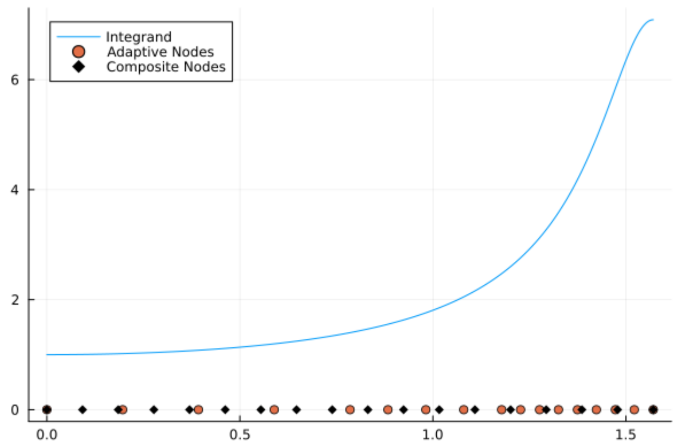

Figure 4: Plot comparing distribution of nodes with adaptive and composite Simpson's rule.

3.2 Part B

As seen in Figure 4, the distribution of composite nodes are uniform, while the composite nodes are more heavily concentrated on [1.0, 1.5]. This substantiates the claims made in part a), since $\theta \to \pi/2 \implies 1 - k^2 \sin^2\theta \to 0$, meaning that a higher number of sub divisions will be required near $\theta \approx \pi/2$ to reach the tolerance since f won't behave smoothly. As such, the adaptive Simpson's rule will select more nodes in this range, which is what is seen in Figure 4. Furthermore, the distribution of composite nodes is equally reasonable, since the composite Simpson's rule is expect to evenly sub-divide the interval.