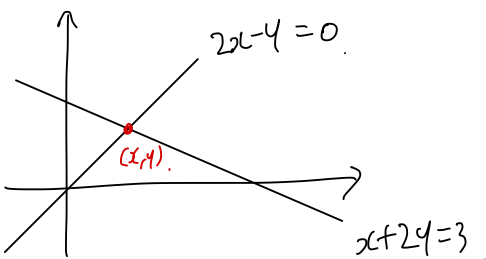
$$\begin{cases} 2x - y = 0 \\ x + 2y = 3 \end{cases} \Longrightarrow \begin{array}{c|cc} & \text{col 1} & \text{col 2} \\ \hline \text{row 1} & 2 & -1 \\ \text{row 2} & 1 & 2 \end{array} \qquad \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

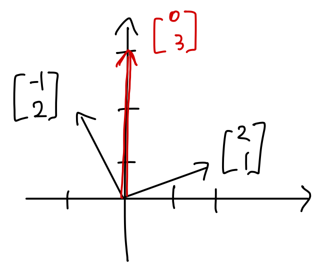
→ row picture.

→ column picture.

1) row picture.



2) Column picture.



$$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

: linear combination of columns.

Matrix multiplication.

$$AB = C$$

$$\begin{bmatrix} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \\ \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} \\ & \boxed{c_{ij}} & \\ \\ \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$= \sum_{n=1}^{k} a_{in} \cdot b_{nj}.$$

3가지 관점.

1). $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$.

$AB = C$

C의 column = A의 column들의 linear combination.

2). $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = 1\begin{bmatrix} 5 & 6 \end{bmatrix} + 2\begin{bmatrix} 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \end{bmatrix}$.

C의 row = B의 row들의 linear combination.

$AB = C$.

3). $\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}_{3\times 1}\underbrace{\begin{bmatrix} 1 & 6 \end{bmatrix}}_{1\times 2} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\begin{bmatrix} 0 & 0 \end{bmatrix}$.

$\rightarrow 3\times 2$. $\rightarrow 3\times 2$.

AB = Sum of { columns of (A) × rows of (B) }.

Transpose.

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}.$$

rows → columns.
columns → rows.

$$(A^T)_{ij} = A_{ji}.$$

Symmetric Matrix

$$\Rightarrow A^T = A.$$

$$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}.$$

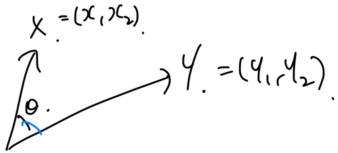
$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}.$$

$$(A \cdot B)^T = \boxed{B^T \cdot A^T}$$

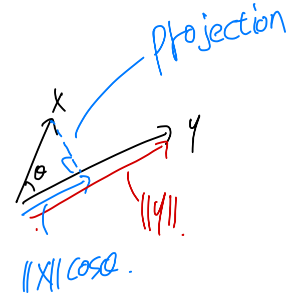
$R^T R$ is always symmetric!

$$(R^T R)^T = R^T \underbrace{\boxed{(R^T)^T}}_{R}.$$

Inner Product



$$\begin{aligned} x \cdot y &= \|x\| \cdot \|y\| \cdot \cos\theta \\ &= x_1 y_1 + x_2 y_2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= x^T y \end{aligned}$$



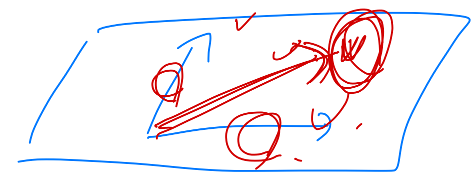
i) $x \perp y \Rightarrow x^T y = 0$

ii) $x^T x = \|x\|^2$

Vector Space (8 Rules).

$w + v$, cv are in the space.

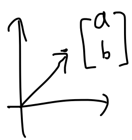
$\Rightarrow$ $\boxed{\text{all combinations } cw + dv \text{ are in the space.}}$



basis.

임의의 벡터 V는 basis의 선형 결합으로 표현 가능

coordinate.



$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

basis.

basis : $\begin{cases} \text{independent.} \\ \text{span the space.} \end{cases}$

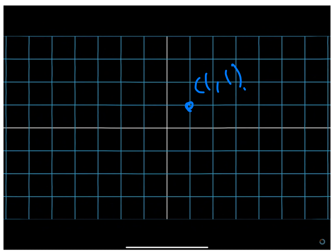
Def. number of basis
= dimension of the space.

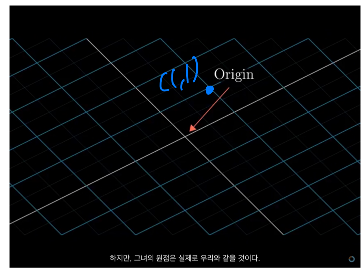
ex) in $\mathbb{R}^3$.

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Another basis.

$$\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\5\\7\end{bmatrix}.$$

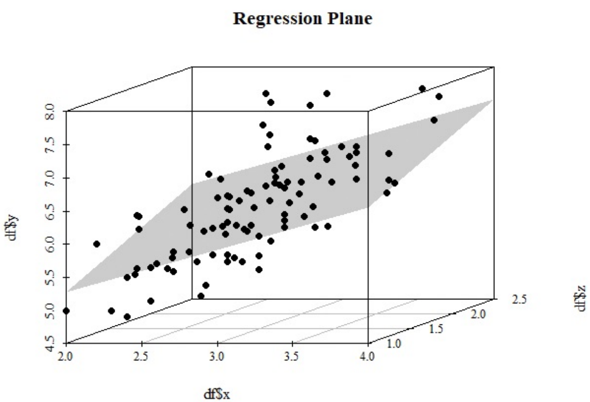


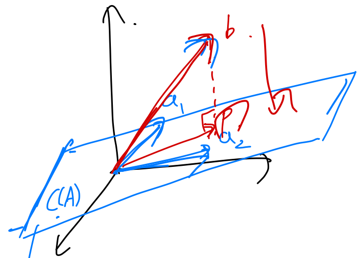


$(1, 2, 3) = (\; ? \;).$

$$\begin{bmatrix}1&2&3\\1&2&5\\2&5&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}. \overset{\text{change of basis.}}{\Longrightarrow} \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1&2&3\\1&2&5\\2&5&7\end{bmatrix}^{-1}\begin{bmatrix}1\\2\\3\end{bmatrix}.$$

Least Square.





Plane of a_1, a_2.

= Column space of $A = [a_1 \ \ a_2]$.

Why project?

Because $Ax=b$ may have no solution.

So, change b the closest vector in C(A).

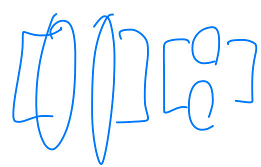
$\Rightarrow$ solve $A\hat{x} = \underset{\text{proj of } b \text{ on to } C(A)}{p}$ instead.

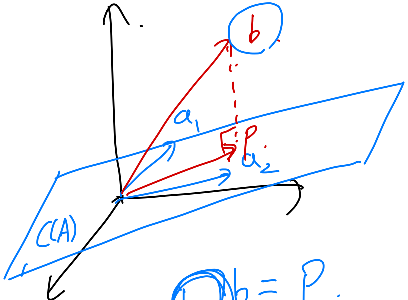
$Ax=b$.

$$\boxed{A\hat{x}=P}$$

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

$$Ax=b.$$





$b = p$.

$$\boxed{p = A\hat{x}, \quad \text{Find } \hat{x}.}$$

idea: $b - A\hat{x}$ is perp. to plane.

$$\Rightarrow a_1^T(b - A\hat{x}) = 0, \quad a_2^T(b - A\hat{x}) = 0.$$

$$\begin{bmatrix} \text{---} \; a_1^T \; \text{---} \\ \text{---} \; a_2^T \; \text{---} \end{bmatrix} \begin{bmatrix} b - A\hat{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow A^T(b - A\hat{x}) = 0.$$

$$\boxed{A^TA\hat{x} = A^Tb} \rightarrow \text{Normal Equation.}$$

$$\boxed{Ax = b}$$

$Ax = b$

$A^TA\hat{x} = A^Tb$

↳ invertible if A has indep. columns.

$\hat{x} = (A^TA)^{-1}A^Tb$.

$\therefore \; P = A(A^TA)^{-1}A^T$

projection Matrix.

$P_C = \boxed{}$

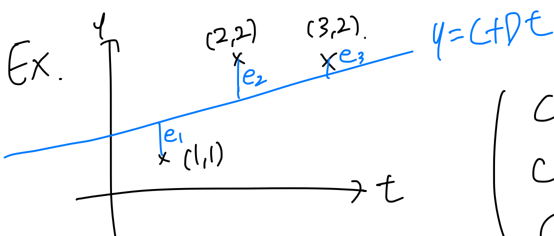
$$P^2 = A\cancel{(A^TA)^{-1}A^T} \cdot \cancel{A}(A^TA)^{-1}A^T$$

$(P^2 = P)$.

$= A(A^TA)^{-1}A^T$.

$= P$.

Ex.

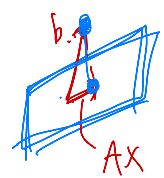


$$\begin{cases} C + D = 1 \\ C + 2D = 2 \\ C + 3D = 3 \end{cases}$$

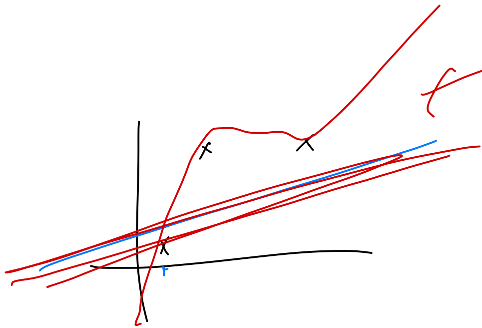
$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}. \Longrightarrow A^TAx = A^Tb.$$

minimize $\|Ax - b\|^2 = \|e\|^2$

$= e_1^2 + e_2^2 + e_3^2$.

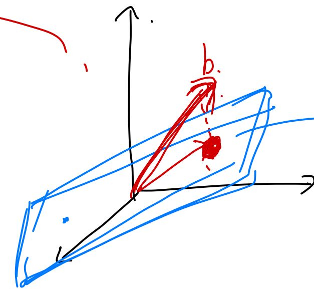


cf.



$\Longleftrightarrow$

same meaning.



→ function space.

내적 X.

Eigen Values & Eigen Vectors.

$$x \xrightarrow{A} Ax$$

like a function.

Ax

I'm specially interested Ax comes out parallel to x.

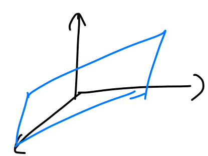
$\Rightarrow$ **eigen vector.**

$$\boxed{Ax = \lambda x}$$

e.vector.

e.value.

Ex. Projection matrix.

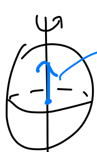


what are x's and λ's for projection matrix?

$\Rightarrow$ Any x in the plane: $Px = x$, $\lambda = 1$.

Any $x \perp$ plane: $Px = 0$, $\lambda = 0$.

Ex. 3D Rotation.



How to find E. values, E. vectors?

$$Ax = \lambda x \Rightarrow \underbrace{(A - \lambda I)}_{\text{must be singular}} x = 0.$$

$\therefore \det(A - \lambda I) = 0.$ (n차 방정식)

Find λ first.

Finding x is just elimination. (finding nullspace).

Diagonalization.

Not all matrices are diagonalizable.

$\boxed{\text{If } A \text{ has } n\text{'s eigen vectors. } (= \text{full rank}).}$

$$A\cancel{x} = \lambda \cancel{x}. \Rightarrow \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}.$$

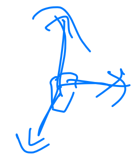
$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

$$\Rightarrow AS = S\Lambda.$$

$$\Rightarrow A = S\Lambda S^{-1}.$$

If A is symmetric

$\Rightarrow$ e.vectors are perpendicular.



$\Rightarrow$ S is orthogonal. $(A^T = A^{-1})$ $\qquad Q^TQ = I$.

$\Rightarrow$ $A = S\Lambda S^T$. $\qquad \boxed{Q^T = Q^{-1}}$

Spectral Theorem.

Every real symmetric A can be diagonalized by an orthogonal matrix Q

$$A = Q\Lambda Q^T = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}.$$

$$P = A(A^TA)^{-1}A^T$$

$$x(x^Tx)^{-1}x^T$$

$$\cancel{1(x)^{-1}}$$

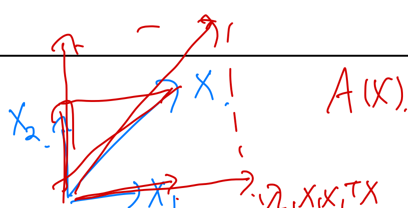
$$= \boxed{xx^T}$$

$$= \lambda_1 \boxed{x_1x_1^T} + \lambda_2 x_2x_2^T + \cdots + \lambda_n x_n x_n^T.$$

$n \times n$ projection matrix.

$$Ax = (\lambda_1 x_1 x_1^T)x + (\lambda_2 x_2 x_2^T)x + \cdots + (\lambda_n x_n x_n^T)x.$$

A becomes a combination of one-dimensional projections.



Means ...

$$Ax = Q\Lambda Q^T x$$

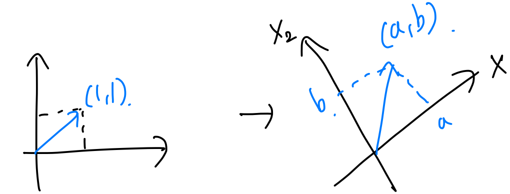
change basis to eigen vectors.

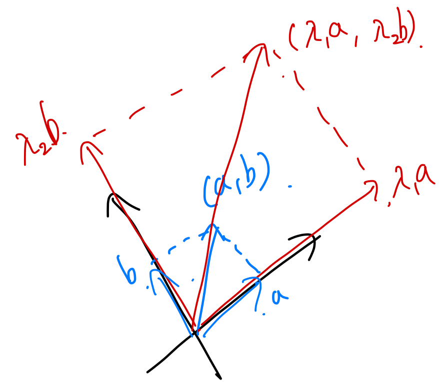
$$= Q\Lambda(Q^T x)$$

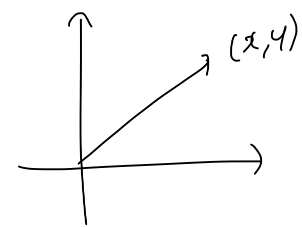
scalar multiplication.

$$= Q(\Lambda Q^T x)$$

change basis to original.







Singular Value Decomposition (SVD).

A : not symmetric. $\Rightarrow$ not diagonalizable.

but. A^TA, AA^T are always symmetric!

$$\Rightarrow \begin{cases} A^TA = V\Lambda V^T. \\ AA^T = U\Lambda U^T. \end{cases}$$

V : eigen vectors of A^TA.

U : eigen vectors of AA^T.

Λ : eigen values of A^TA or AA^T.

$(= \sigma^2)$.

$$\Lambda = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix}.$$

i). $(A^TA)v_j = \lambda_j v_j \Longrightarrow v_j^T(A^TA)v_j = \lambda_j v_j^T v_j$.

$$(Av_j)^T(Av_j) = \lambda_j$$

$$\|Av_j\|^2 = \lambda_j$$

σ $\qquad \sigma_j^2$. $\qquad Av_j = \sigma_j^2$.

$\therefore \Lambda = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \\ & & & \sigma_n^2 \end{bmatrix}$

ii). $(A^TA)v_j = \sigma_j^2 v_j$.

$$AA^T(Av_j) = \sigma_j^2(Av_j)$$

Av_j is eigen vector of AA^T?

So, the unit eigen vector is

$$\frac{Av_j}{\sigma_j} = u_j.$$

$Av_1 = \sigma_1 u_1$.

$Av_2 = \sigma_2 u_2$. $\Rightarrow [A][v_1 \cdots v_r]$

$\vdots$

$Av_r = \sigma_r u_r$.

In other words,

$$AV = U\Sigma.$$

$$A \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}.$$

$$AV = U\Sigma.$$

$$\rightarrow \underbrace{A = U\Sigma V^T}_{\text{SVD}}.$$

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T)$$

$$= V\Sigma^T(U^TU)\Sigma V^T$$

$$= V\Sigma^T\Sigma V^T$$

$$= V\Lambda V^T.$$

$$AA^T = \cdots = U\Lambda U^T.$$

Eckart-Young Theorem.

$$A = U\Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T. \quad (\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_r).$$

$$A_k = U_k \Sigma_k V_k^T = \sigma_1 u_1 v_1^T + \cdots + \sigma_k u_k v_k^T.$$

If B has rank k, then $\|A - B\| \geq \underline{\|A - A_k\|}$.

(SVD is the best approximation)

= closest rank k matrix.

$\Rightarrow$ "$\boxed{\sigma_1 u_1 v_1^T}$ is <u>the most principal part of</u> A".

$\sigma_2 u_2 v_2^T$.

$\sigma_3 u_3 v_3^T$.