# ACYLINDRICALLY HYPERBOLIC GROUPS AND COUNTING PROBLEMS

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ABSTRACT. We show that Morse elements are generic in acylindrically hyperbolic groups. As an application, we observe that fully irreducible outer automorphisms are generic in the outer automorphism group of a finite-rank free group.

**Keywords.** Acylindrical action, Morse, weak proper discontinuity, counting problem, genericity

**MSC classes:** 20F67, 30F60, 57K20, 57M60, 60G50

## 1. INTRODUCTION

In non-positively curved manifolds and groups, certain geodesics or group elements exhibit hyperbolicity. A quasi-geodesic $\gamma$ is said to be *Morse* if every quasi-geodesic of uniform quality connecting points on $\gamma$ lies in a common neighborhood of $\gamma$. A group element $g$ is called a *Morse element* if its orbit $\{g^i\}_{i \in \mathbb{Z}}$ is an unbounded Morse quasi-geodesic in the group.

In globally hyperbolic spaces such as CAT($-1$) spaces and Gromov hyperbolic spaces, every geodesic is Morse (of uniform quality). This corresponds to the fact that every infinite-order element in a word hyperbolic group is loxodromic and is Morse. Furthermore, "most" elements in a word hyperbolic group are Morse. To formulate this, given a group $G$ and its generating set $S$, let $B_S(n)$ be the collection of group elements whose $S$-word length is at most $n$. We can ask if the proportion of Morse elements in $B_S(n)$ tends to 1 as $n$ tends to infinity. This is indeed the case when $G$ is an infinite word hyperbolic group [Dan], [GTT18], [Yan20].

Morse elements are found in many other groups with flat parts. One classic example is the mapping class group Mod($\Sigma$) of a finite-type hyperbolic surface $\Sigma$, whose Morse elements are precisely pseudo-Anosov mapping classes. In [Cho24a], the author proved that the asymptotic density of pseudo-Anosovs in the mapping class group is 1.

In this paper we focus on *acylindrically hyperbolic groups*, which are vast generalizations of hyperbolic groups that include Mod($\Sigma$), Out($F_N$), rank-1 CAT(0) groups and many 3-manifold groups, as well as many Coxeter groups , one-relator groups and free-by-cyclic groups ([BF02], [BF09], [CF10], [BF10], [MO15], [Sis18]). Despite its generality, acylindrically hyperbolic groups have shown to share many dynamical properties with word hyperbolic groups. We refer the readers to the first few pioneering papers in this direction: [BF02], [Ham08], [DGO17], [Sis18].

We add one more property to this list: genericity of Morse elements. Our main theorem is:

**Theorem A.** *Let $G$ be an acylindrically hyperbolic group. Then for every finite generating set $S$ of $G$, we have*

$$\lim_{n \to +\infty} \frac{\#\{g \in B_S(n) : g \text{ is Morse}\}}{\#B_S(n)} = 1.$$

This generalizes W. Yang's result on groups with strongly contracting element [Yan20]. This can be also compared with A. Sisto's theorem that simple random walks on acylindrically hyperbolic groups favor Morse elements ([Sis18, Theorem 1.6]). In fact, non-elementary random walks on any Gromov hyperbolic space favor loxodromics [CM15], [MT18], but one cannot hope such a result for counting problems (see the following subsections).

In view of the equivalent definitions of acylindrically hyperbolic groups in [Osi16] (especially in relation to [BF02]), Theorem A is a restatement of the following more explicit theorem.

**Theorem 1.1.** *Let $G$ be a group generated by a finite set $S \subseteq G$. Suppose that $G$ acts on a Gromov hyperbolic space $X$ and that there exists $g \in G$ that is a loxodromic isometry of $X$ with the WPD property (cf. Definition 2.5). Then for any $M > 0$, we have*

$$\lim_{n \to +\infty} \frac{\#\big\{g \in B_S(n) : g \text{ is a WPD loxodromic element with } \tau_X(g) > M\big\}}{\#B_S(n)} = 1.$$

Indeed, if $g \in G$ serves as a WPD loxodromic on a Gromov hyperbolic space, then $g$ is a Morse element in $G$ ([Sis16, Theorem 1], [Osi16, Theorem 1.4]).

We note a theorem by B. Wiest [Wie17] that was applied to the mapping class group by M. Cumplido and B. Wiest [CW18]: for any finitely generated group $G$ having a non-elementary action on a Gromov hyperbolic space, the density of loxodromics is bounded away from 0. Hence, the main point of Theorem A and 1.1 is that the density has limit 1. Such a claim does not hold for general non-elementary actions.

Two important examples of acylindrically hyperbolic groups beyond hierarchically hyperbolic groups (HHGs) are $\mathrm{Out}(F_N)$ and $\mathrm{Aut}(F_N)$, the outer automorphism group and the automorphism group of the free group of rank $N \geq 3$. Theorem A tells us that most elements are Morse in large word metric balls in these groups.

We can say more by focusing on a specific $\mathrm{Out}(F_N)$-action, namely, the one on the free factor complex $\mathcal{FF}_N$ studied by M. Bestvina and M. Feighn [BF14]. Bestvina and Feighn proved that:

(1) $\mathcal{FF}_N$ is Gromov hyperbolic,
(2) the elements of $\mathrm{Out}(F_N)$ that are loxodromic isometries of $\mathcal{FF}_N$ are precisely the fully irreducible outer automorphisms, and
(3) every fully irreducible outer automorphism has the WPD property.

Using this action and building upon the stability result in [KMPT22], we prove:

**Theorem 1.2.** *Let $G = \mathrm{Out}(F_N)$ be the outer automorphism group of the free group of rank $N$ for some $N \geq 2$. Then for any finite generating set $S$ of $G$, we have*

$$\lim_{n \to +\infty} \frac{\#\big\{g \in B_S(n) : g \text{ is an ageometric triangular fully irreducible element}\big\}}{\#B_S(n)} = 1.$$

This is a counting version of Kapovich–Maher–Pfaff–Taylor's result that random walks on $\mathrm{Out}(F_N)$ favor ageometric triangular fully irreducibles [KMPT22, Theorem A]. There are also versions of random walk theory on $\mathrm{Aut}(F_N)$ and $\mathrm{Out}(F_N)$ using "non-backtracking" paths ([KKS07], [KP15]). In particular, I. Kapovich and C. Pfaff proved that non-backtracking random walks favor geometric triangular fully irreducibles as well.

We record a cute application to the mapping class group $\mathrm{Mod}(\Sigma)$. It seems hard to apply the method of [Cho24a] to general non-elementary subgroups of $\mathrm{Mod}(\Sigma)$, as there is no distance formula for them. However, since they still act on the curve complex $\mathcal{C}(\Sigma)$ with a WPD loxodromic element, we observe that:

**Corollary 1.3.** *Let $G \leq \mathrm{Mod}(\Sigma)$ be a non-elementary subgroup of the mapping class group and let $S$ be a finite generating set of $G$. Then for any $M > 0$, we have*

$$\lim_{n \to +\infty} \frac{\#\big\{g \in B_S(n) : g \text{ is pseudo-Anosov with stretch factor} \geq M\big\}}{\#B_S(n)} = 1.$$

In particular, pseudo-Anosovs are generic in the Torelli group. This generalizes the result of I. Gekhtman, S. Taylor, and G. Tiozzo regarding word hyperbolic groups acting on a Gromov hyperbolic space [GTT18, Theorem 1.12].
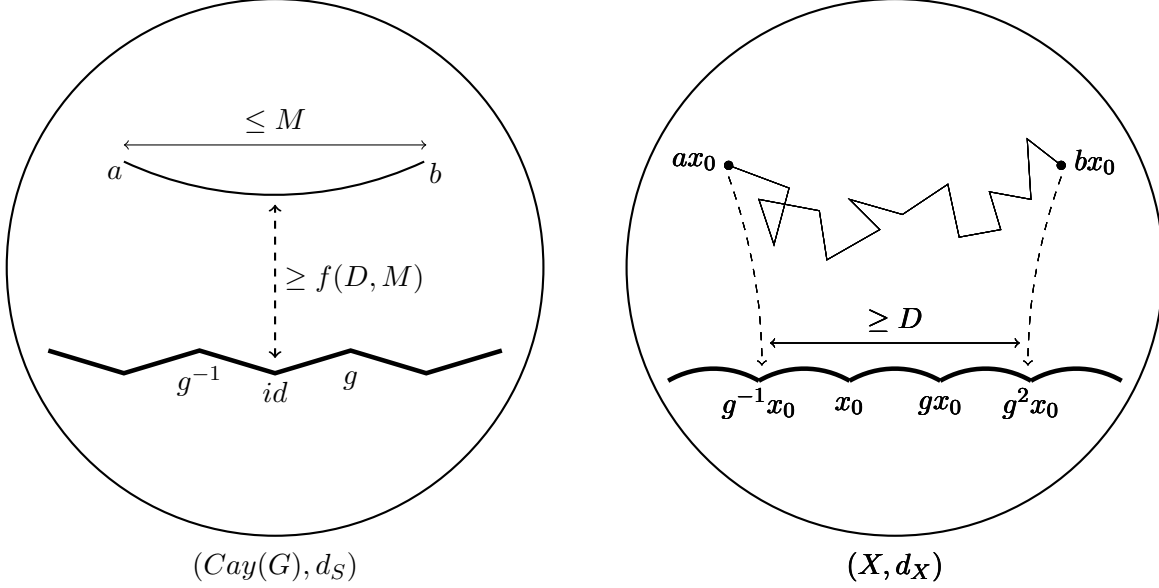
FIGURE 1. Schematics for $f(D, M)$ in Subsection 1.1

## 1.1. Comparison with other groups.

To better illustrate Theorem 1.1, let us compare four groups that act on a Gromov hyperbolic space: the free group $F_2$ of rank 2, the mapping class group $\mathrm{Mod}(\Sigma)$, the outer automorphism group $\mathrm{Out}(F_N)$ of the free group of rank $n \geq 2$, and the direct product $F_2 \times F_3$ of two free groups. All of these act on some Gromov hyperbolic space.

Let $G$ be a group acting on a hyperbolic space $X$ and let $S$ be a finite generating set of $G$. Given a group element $g \in G$ and a basepoint $x_0 \in X$, let us define a function $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ as follows. For any $M$-short word metric geodesic $[a, b] \subseteq G$, if $a$ and $b$ are $D$-apart along $\{g^i x_0\}_{i \in \mathbb{Z}}$, then $[a, b]$ must pass through an $f(D, M)$-neighborhood of $\{g^i\}_{i \in \mathbb{Z}}$ in $G$.

First, $F_2$ has proper action on its own Cayley graph $Cay(F_2)$. This implies that any coarse stabilizer of $v \in F_2$ is finite. Furthermore, each $g \in F_2 \setminus \{id\}$ has the so-called *strong contracting property*: if a geodesic $[a, b] \subseteq F_2$ makes nontrivial progress along $\{g^i\}_{i \in \mathbb{Z}}$, then $[a, b]$ passes through a bounded neighborhood of $\{g^i\}_{i \in \mathbb{Z}}$. In other words, $f(D, M)$ is constant in $M$ for large enough $D$.

Second, $\mathrm{Mod}(\Sigma)$ acts on the ambient curve complex $\mathcal{C}(\Sigma)$ and tuples of subsurface curve complexes $\mathcal{C}(U), U \subsetneq \Sigma$. Fixing a simple closed curve $x_0 \in \mathcal{C}(\Sigma)$, each $g \in \mathrm{Mod}(\Sigma)$ gives rise to shadows $d_U(x_0, gx_0)$ on various $\{\mathcal{C}(U) : U \subseteq \Sigma\}$, using which the word metric on $\mathrm{Mod}(\Sigma)$ can be (coarsely) estimated via distance formula [MM00]. One consequence of the distance formula and is the *weakly contracting property* of pseudo-Anosov orbits [Beh06], [DR09]. Explicitly, for each pseudo-Anosov mapping class $g$, there exists $\epsilon > 0$ such that if an $M$-short geodesic $[a, b] \subseteq \mathrm{Mod}(\Sigma)$ makes progress $D$ along $\{g^i\}_{i \in \mathbb{Z}}$, then $[a, b]$ passes through a $f(D, M) := (M \cdot e^{-\epsilon D})$-neighborhood of $\{g^i\}_{i \in \mathbb{Z}}$.

There is no direct analogue of the distance formula for $\mathrm{Out}(F_N)$. As a result, we do not know whether fully irreducible outer automorphisms (which are analogues of pseudo-Anosovs) have the weakly contracting property on the Cayley graph of $\mathrm{Out}(F_N)$. However, every fully irreducible outer automorphism $g$ has the WPD property (for various hyperbolic actions, cf. [BF10], [Man14], [BF14]), i.e., the joint coarse stabilizer of $g^i$ and $g^j$ is finite when $|i - j|$ is large. This leads to an implicit contracting property, i.e., for every $D$ and $M$ the value of $f(D, M)$ is finite.

Finally, consider a trivial projection of $F_2 \times F_3$ onto the first factor $F_2$. This gives rise to a natural action of $F_2 \times F_3$ on $Cay(F_2)$. This action has not only a large point stabilizer, but also a large global stabilizer. Namely, $\{id\} \times F_3$ acts trivially on $Cay(F_2)$. In addition, there is no contraction along loxodromics on $F_2$, i.e., $f(D, M) = +\infty$. In general, if $X \times Y$ is a product space,

3

| | $\delta$-hyperbolic space | $f(D,M)$ for a fixed $D$ | Density of non-loxodromics |
|---|---|---|---|
| $F_2$ | $Cay(F_2)$ | constant in $M$ | $\lesssim \lambda^{-n}$ for some $\lambda > 1$ |
| $\mathrm{Mod}(\Sigma)$ | $\mathcal{C}(\Sigma)$ | linear in $M$ | $\lesssim n^{-k} \ (\forall k)$ |
| $\mathrm{Out}(F_N)$ | $\mathcal{FF}_N$ | finite | tends to 0 |
| $F_2 \times F_3$ | $Cay(F_2)$ | $+\infty$ | can be bounded away from 0 |

FIGURE 2. Properties of the four actions and the density estimates

a $D$-long geodesic $\gamma$ can have $D$-large projection onto $X \times \{id\}$, regardless of the distance of $\gamma$ from $X \times \{id\}$. Figure 2 summarizes the discussion so far.

The more information we have about the growth of $f(D,M)$, the better asymptotics of the density of non-loxodromics we can prove. In $F_2$, the proportion of non-loxodromic elements in $B_S(n)$ decays exponentially fast in $n$. This is proved by W. Yang [Yan20] in groups with strongly contracting elements, including relatively hyperbolic groups and small cancellation groups.

For the mapping class group, the function $f(D,M)$ grows at most linearly in $M$. Using this property, it is shown in [Cho24a] that the density of non-pseudo-Anosovs in $B_S(n)$ decays faster than $n^{-k}$ for any $k > 0$. Similar growth behaviour of $f(D,M)$ is observed in HHGs with Morse elements, because loxodromics on the top curve space have the weakly contracting property. Rank-1 CAT(0) groups also fall into this category, as the strongly contracting property of a rank-1 element on the CAT(0) space implies its weak contracting property in the group.

Without control of $f(D,M)$, loxodromics can either be generic or non-generic depending on the generating set $S$. Indeed, there exist two finite generating sets $S$ and $S'$ of $F_2 \times F_3$, such that loxodromics (for the action on $Cay(F_2)$) are generic in $S$ but not in $S'$. We refer readers to [GTT18, Example 1]. This simple example also tells us that genericity of Morse elements of a group may not be preserved through a quasi-isometry.

This paper deals with $\mathrm{Out}(F_N)$ and others of its ilk. There is no *a priori* control on the growth of $f(D,M)$ for acylindrically hyperbolic groups. Our main point is that, nonetheless, the finiteness of $f(D,M)$ is sufficient to conclude the genericity of loxodromics.

## 1.2. **Another side of the story: random walks.**
There are two popular models to sample a random element in a group $G$. One is the counting method as in Theorem A. Namely, we consider a large word metric ball and choose an element with respect to the uniform measure. The other one is the random walk model: we put a probability measure $\mu$ on a generating set $S$ of $G$ (e.g., the uniform measure when $S$ is finite) and investigate its $n$-fold convolution $\mu^{*n}$.

For example, given a $G$-action on a Gromov hyperbolic space $X$, one can ask if $\mathbb{P}_{\mu^{*n}}(g \text{ is loxodromic})$ converges to 1 as $n$ tends to infinity. This is closely related to a description of a typical sample path drawn on $X$, called *ray approximation* or *geodesic tracking*, that was pursued for word hyperbolic groups by V. Kaimanovich [Kai94]; see [Kai00] also. It was J. Maher's observation that neither the properness of $X$ nor the properness of the action is necessary. As a result, Maher proved in [Mah11] that $\mathbb{P}_{\mu^{*n}}(g \text{ is pseudo-Anosov})$ converges to 1 in the mapping class group (I. Rivin independently proved this result using different method in [Riv08]).

Maher's observation was later generalized by D. Calegari and J. Maher [CM15], and once again by J. Maher and G. Tiozzo in [MT18]: they proved that $\mathbb{P}_{\mu^{*n}}(g \text{ is loxodromic})$ converges to 1 as long as the $G$-action on $X$ is non-elementary (i.e., $S$ generates two independent loxodromics). In particular, random walks do not care if the group has a large subgroup with trivial action, given that they hit non-elementary loxodromic elements for a positive probability. Maher-Tiozzo's result indeed applies to all 4 group actions in Subsection 1.1.

Consequently, for the uniform measure $\mu_S$ on a finite generating set $S$ of $G$, the genericity of loxo-dromics with respect to $\mu_S^{*n}$ does not imply the genericity with respect to (uniform measure on $B_S(n)$). This is anticipated by the fact that the two measures differ by an exponential factor in $n$.

If one is allowed to pick their favorite generating set $S$ for $G$, then one can bring the estimates from random walks to the counting problem. This was indeed the strategy of [Cho24b], where the author proved that every finitely generated weakly hyperbolic group has a finite generating set $S$ for which loxodromics are generic. Since the asymptotic density may depend on the choice of $S$ (as shown in [GTT18, Example 1]), this strategy does not establish Theorem A.

1.3. **Beyond hyperbolic spaces.** The method for Theorem 1.1 does not require global hyper-bolicity of the space $X$. It only uses the *strongly contracting property* and the WPD property of $g \in G$ in $X$. For simplicity, however, we will not pursue this generality. It should be noted that the previous assumption does not imply that $g$ is strongly contracting *in* $G$, i.e., with respect to the word metric. For example, the author does not know whether fully irreducibles are weakly contracting with respect to the word metrics (cf. [BD14, Question 6.8]).

For example, the method for Theorem 1.1 applies to finitely generated groups acting on a CAT(0) space (not necessarily cocompactly) that involves a rank-1 isometry with the WPD property. The study of strongly contracting isometries and their dynamics is growing rapidly. We refer the readers to the references in [ACT15], [Yan19], [Yan20], [Cou22], [SZ23], [DMGZ25].

In fact, the very notion of acylindrically hyperbolic group was already formulated in terms of contracting elements by A. Sisto [Sis18], who generalized Maher-Tiozzo's random walk theory in [MT18] to non-hyperbolic spaces. We also note a recent construction by H. Petyt and A. Zalloum [PZS24, Theorem B] that justifies why it suffices to consider WPD action on hyperbolic spaces.

1.4. **Open questions.** The methods in [Cho24a] and this paper still do not answer:

**Question 1.4.** *Are pseudo-Anosovs exponentially generic in every word metric on* $\mathrm{Mod}(\Sigma)$*?*

There are two types of word metrics for which exponential genericity of pseudo-Anosovs is known. One comes from generating sets mostly consisting of independent pseudo-Anosovs [Cho24b]. The other recent one is due to L. Ding, D. Martínez-Granado and A. Zalloum [DMGZ25], where the authors consider the $\mathrm{Mod}(\Sigma)$-action on an injective metric space $(Y, d_Y)$ and collect orbit points in a large $d_Y$-ball. It seems hard to push either method to handle arbitrary word metric.

For non-HHGs, we can ask:

**Question 1.5.** *Are fully irreducibles exponentially generic in* $\mathrm{Out}(F_N)$ *with respect to every word metric? Or, is it at least true for some* $\alpha > 0$ *that*

$$\frac{\#B_S(n) \cap \{\text{fully irreducibles}\}}{\#B_S(n)} \lesssim n^{-\alpha}?$$

This question might be answered for a given group $G$ whenever we know the growth of the function $f(D, M)$ in Subsection 1.1.

One can ask more details about generic elements. In the random walk side we have strong law of large numbers (SLLN): for any non-elementary random walk $(Z_n)_{n>0}$ on a Gromov hyperbolic space, there exists $\lambda \in (0, +\infty]$ such that $\lim_n \frac{\|Z_n\|_X}{n} = \lim_n \frac{\tau_X(Z_n)}{n} = \lambda$ almost surely ([CM15], [MT18], [BCK21]). Here the key point is the linear growth of displacement and translation length. We pose:

**Question 1.6.** *Do generic fully irreducibles have linearly growing translation length? Namely, given a finite generating set* $S$ *of* $\mathrm{Out}(F_N)$*, does there exist a linear function* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$(1.1) \qquad \lim_{n \to +\infty} \frac{\#B_S(n) \cap \left\{g \in \mathrm{Out}(F_N) : \tau_{\mathcal{FF}}(g) \geq f(n)\right\}}{\#B_S(n)} = 1?$$

Our method does provide a diverging function $f$ for which Equation 1.1 holds, but we have no control on the growth of $f$. For the mapping class group, the author anticipates that the method in [Cho24a] guarantees $f(n) \gtrsim \sqrt{n}$. The results of [Cho24b] and [DMGZ25] imply that $f(n) \gtrsim n$ works for *certain* finite generating set $S$.

Finally, we state a question related to Question 1.4.

**Question 1.7.** *Does $G = \mathrm{Mod}(\Sigma)$ or $G = \mathrm{Out}(F_N)$ have purely exponential growth? That means, for (some or every) finite generating set $S$ of $G$, does there exist $K, \lambda > 1$ such that*

$$\frac{1}{K}\lambda^n \leq \#B_S(n) \leq K\lambda^n? \quad (\forall n > 0)$$

This question is answered by W. Yang for groups with strongly contracting elements [Yan19, Theorem B]. Meanwhile, we do not know the answer for $\mathrm{Mod}(\Sigma)$ for any finite generating set.

1.5. **Plans.** After reviewing preliminaries in Section 2, we observe a variant of A. Sisto's geometric separation lemma [Sis16, Lemma 3.3] in Section 3. We then prove the main theorem in Section 4.

## 2. Preliminaries

In this section, we collect some notions and facts about acylindrically hyperbolic groups. We refer to Gromov's seminal paper [Gro87] and standard textbooks [CDP90], [GdlH90].

A metric space is said to be *geodesic* if every pair of points can be connected by a geodesic. For two points $x$ and $y$ in this space, we denote by $[x, y]$ an arbitrary geodesic connecting $x$ to $y$. Given $\delta > 0$, we say that a geodesic metric space is $\delta$-hyperbolic if every geodesic is $\delta$-slim.

Given a geodesic $\gamma : I \rightarrow X$, we will sometimes denote the image $Im(\gamma) \subseteq X$ by $\gamma$. Based on this convention, we define the *closest point projection* $\pi_\gamma : X \rightarrow 2^\gamma$ by

$$y \in \pi_\gamma(x) \iff d_X(x, y) = \inf\{d_X(x, p) : p \in \gamma\}.$$

We say that two geodesics $\gamma : [0, L] \rightarrow X$ and $\eta : [0, L'] \rightarrow X$ are $\epsilon$-*fellow traveling* if

$$d_X(\gamma(0), \eta(0)) < \epsilon, \ d_X(\gamma(L), \eta(L')) < \epsilon \text{ and } d_{Haus}(\gamma, \eta) < \epsilon.$$

The fellow traveling property is transitive: if $\gamma_1$ and $\gamma_2$ are $\epsilon$-fellow traveling; $\gamma_2$ and $\gamma_3$ are $\epsilon'$-fellow traveling, then $\gamma_1$ and $\gamma_3$ are $(\epsilon + \epsilon')$-fellow traveling. Furthermore, we have:

**Fact 2.1.** *Let $X$ be a $\delta$-hyperbolic space and let $x, y, z, w \in X$ be such that $d_X(x, y) < \epsilon$ and $d_X(z, w) < \epsilon'$. Then $[x, z]$ and $[y, z]$ are $(\epsilon + \delta)$-fellow traveling. Moreover, $[x, z]$ and $[y, w]$ are $(\epsilon + \epsilon' + 2\delta)$-fellow traveling.*

For each $x \in X$, $\pi_\gamma(x)$ may not be a singleton. Nevertheless, its diameter is bounded and $\pi_\gamma(\cdot)$ is coarsely Lipschitz. The following is a consequence of [CDP90, Proposition 10.2.1], which follows from the tree approximation lemma [CDP90, Théorème 8.1], [GdlH90, Théoréme 2.12].

**Fact 2.2.** *Let $X$ be a $\delta$-hyperbolic space.*

    (1) *Let $x, y \in X$ and let $\gamma$ be a geodesic in $X$. Then $\pi_\gamma(x) \cup \pi_\gamma(y)$ has diameter at most $d_X(x, y) + 12\delta$.*

(2) Let $x, y \in X$, let $\gamma$ be a geodesic in $X$ and let $p \in \pi_\gamma(x)$ and $q \in \pi_\gamma(y)$. Suppose that $p$ appears earlier than $q$ on $\gamma$ and that $d_X(p, q) > 20\delta$. Then any geodesic $[x, y]$ between $x$ and $y$ contains a subsegment that is $20\delta$-fellow traveling with $[p, q]$.

**Corollary 2.3** ([Sis18, Lemma 4.1]). *Let $X$ be a $\delta$-hyperbolic space, let $\gamma$ be a geodesic in $X$, let $x, y \in X$ and let $\eta$ be a subsegment of $\gamma$ that contains $\pi_\gamma(x) \cup \pi_\gamma(y)$. Then $\pi_\gamma([x, y])$ is contained in the $60\delta$-neighborhood of $\eta$.*

*Proof.* Suppose to the contrary that there exist $z \in [x, y]$, $p \in \pi_\gamma(x)$, $q \in \pi_\gamma(y)$, $r \in \pi_\gamma(z)$ such that $d_X(p, r), d_X(q, r) \geq 60\delta$ and such that $p, q$ are to the right of $r$. Let $p_0$ be the point on $\gamma$ to the right of $r$ such that $d_X(r, p_0) = 60\delta$. Then Fact 2.2(2) implies that there exist a subsegment $[r', p']$ of $[z, x]$ and a subsegment $[r'', p'']$ of $[z, y]$ such that $d_X(r', r), d_X(r'', r) < 20\delta$ and $d_X(p', p_0), d_X(p'', p_0) < 20\delta$. We then observe that

$$40\delta > d_X(p', p_0) + d_X(p_0, p'') \geq d_X(p', p'') \geq d_X(p', r') + d_X(r'', p'')$$
$$\geq [d_X(p_0, r) - d_X(p_0, p') - d_X(r, r')] + [d_X(p_0, r) - d_X(p_0, p'') - d_X(r, r'')] > 20\delta + 20\delta,$$

a contradiction. Similar contradiction happens when $p, q$ are both to the left of $r$. $\qquad\square$

For $x, y, z \in X$, we define the *Gromov product* of $y$ and $z$ based at $x$ by

$$(y, z)_x := \frac{1}{2}\big[d_X(y, x) + d_X(x, z) - d_X(y, z)\big].$$

Gromov hyperbolicity has the following consequence.

**Fact 2.4** ([Cho24a, Lemma A.3]). *Let $X$ be a $\delta$-hyperbolic space. Let $x, y, z \in X$ and let $p \in [y, z]$ be such that $d_X(p, y) = (x, z)_y$. Then $\pi_{[y,z]}(x)$ is contained in the $8\delta$-neighborhood of $p$.*

**Definition 2.5.** *Let $G$ be a finitely generated group acting on a $\delta$-hyperbolic space $(X, d_X)$ with a basepoint $x_0 \in X$. We say that a loxodromic element $\varphi \in G$ has the* WPD *(weak proper discontinuity) property if for each $K$ there exists $N, M$ such that*

$$\#\Big(Stab_K(x_0, \varphi^N x_0) := \big\{g \in G : d_X(x_0, gx_0) < K \text{ and } d_X(\varphi^N x_0, g\varphi^N x_0) < K\big\}\Big) < M.$$

We say that a finitely generated group $G$ is *acylindrically hyperbolic* if it admits an isometric action on a $\delta$-hyperbolic space with a WPD loxodromic element $\varphi \in G$. An acylindrically hyperbolic group $G$ is said to be *non-elementary* if it is not virtually cyclic. The following fact is a consequence of [BF02, Proposition 6(1), (2)]. The proof is sketched in [Cho24a, Fact 2.2].

**Fact 2.6.** *Let $G$ be a non-virtually cyclic group with a generating set $S$. Suppose that $G$ acts on a $\delta$-hyperbolic space $X \ni x_0$ with a WPD loxodromic element $\varphi \in G$. Then there exists $E_0 > 0$ such that the following hold.*

(1) *For each $g \in G$, there exist $s, t \in S \cup \{id\}$ such that $(\varphi^i x_0, sgx_0)_{x_0} \leq E_0$ for all $i > 0$ and $(\varphi^j x_0, tgx_0)_{x_0} \leq E_0$ for all $j < 0$.*

(2) *Let $n > 0$ and $g \in G$. Let $\gamma := [x_0, \varphi^n x_0]$, let $p \in \pi_\gamma(gx_0)$ and let $q \in \pi_\gamma(g\varphi^n x_0)$. Suppose that $p$ appears earlier than $q$ along $\gamma$ and suppose that $d_X(p, q) > E_0$. Then $d_S(\varphi^i, g\varphi^j) < E_0$ for some $i, j \in \{0, 1, \ldots, n\}$.*

We now recall the notion of alignment.

**Definition 2.7.** *Let $K > 0$ and let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be finite geodesics (which can be degenerate, i.e., points). We say that $(\gamma_1, \ldots, \gamma_n)$ is $K$-aligned if for each $i = 1, \ldots, n-1$ we have*

$$\mathrm{diam}\big(\pi_{\gamma_i}(\gamma_{i+1}) \cup (\text{ending point of } \gamma_i)\big) < K \text{ and}$$
$$\mathrm{diam}\big(\pi_{\gamma_{i+1}}(\gamma_i) \cup (\text{beginning point of } \gamma_{i+1})\big) < K.$$

The following facts are straightforward, whose proofs can be found in [Cho24a, Appendix].

**Fact 2.8.** *Let $\gamma$ be a geodesic in a metric space. Let $\gamma_1$ and $\gamma_2$ be subsegments of $\gamma$, with $\gamma_1$ appearing earlier than $\gamma_2$. Let $\kappa_1$ and $\kappa_2$ be geodesics that are $K$-fellow traveling with $\gamma_1$ and $\gamma_2$, respectively. Then $(\kappa_1, \kappa_2)$ is $6K$-aligned.*

**Fact 2.9.** *The following holds for each $K > 0$ and $L \geq 12K$. Let $\gamma$ be a geodesic in a metric space and let $\gamma_1$ and $\gamma_2$ be subsegments of $\gamma$ such that $\gamma_1 \cap \gamma_2$ has length $L$. Let $[x, y]$ and $\kappa_2$ be geodesics that are $K$-fellow traveling with $\gamma_1$ and $\gamma_2$, respectively. Then $\pi_\kappa(x)$ appears earlier than $\pi_\kappa(y)$ along $\kappa$, and $d_X(\pi_\kappa(x), \pi_\kappa(y)) > L - 10K$.*

We now record a version of Behrstock's inequality [Beh06, Theorem 4.3] (cf. [Sis18, Lemma 2.5]) and its consequences. The proofs can be found in [Cho24a, Section 3, Appendix].

**Fact 2.10.** *Let $X$ be a $\delta$-hyperbolic space. Let $x \in X$ and let $(\gamma_1, \gamma_2)$ be a $K$-aligned sequence of geodesics in $X$. Then either $(x, \gamma_2)$ is $(K + 60\delta)$-aligned or $(\gamma_1, x)$ is $(K + 60\delta)$-aligned.*

**Fact 2.11.** *Let $X$ be a $\delta$-hyperbolic space. Let $n \geq 3$ and let $(\gamma_1, \ldots, \gamma_n)$ be a $K$-aligned sequence of geodesics in $X$. Suppose that $\gamma_2, \ldots, \gamma_{n-1}$ are longer than $2K + 120\delta$. Then $(\gamma_i, \gamma_j)$ is $(K + 60\delta)$-aligned for each $1 \leq i < j \leq n$.*

**Fact 2.12.** *Let $X$ be a $\delta$-hyperbolic space. Let $x, y \in X$ and let $\gamma_1, \ldots, \gamma_n$ be geodesics in $X$, longer than $2K + 140\delta$ each, such that $(x, \gamma_1, \ldots, \gamma_n, y)$ is $K$-aligned.*
*Then there exist disjoint subsegments $\eta_1, \ldots, \eta_n$ of $[x, y]$ such that*

  *(1) $\eta_1, \ldots, \eta_n$ are in order from left to right along $[x, y]$, i.e., $\eta_i$ appears earlier than $\eta_{i+1}$ along $[x, y]$ for each $i = 1, \ldots, n - 1$, and*
  *(2) $\gamma_i$ and $\eta_i$ are $(K + 80\delta)$-fellow traveling for each $i = 1, \ldots, n$.*

Let $G$ be a group and let $S \subseteq G$ be its finite generating set. The word metric $d_S$ is defined by

$$d_S(g, h) := \min \left\{ n \in \mathbb{Z}_{\geq 0} : \begin{array}{c} \exists a_1, a_2, \ldots, a_n \in S, \, \epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{1, -1\} \\ \text{such that } g^{-1}h = a_1^{\epsilon_1} a_2^{\epsilon_2} \cdots a_n^{\epsilon_n}. \end{array} \right\}$$

We use the notation for the word norm $\|g\|_S := d_S(id, g)$. We define

$$B_S(n) := \big\{ g \in \mathrm{Mod}(\Sigma) : d_S(id, g) \leq n \big\}.$$

We denote by $[g, h]_S$ an arbitrary $d_S$-geodesic between $g, h \in G$. By a $d_S$-path, we mean a sequence of group elements $P = (g_1, g_2, \ldots, g_n)$ such that $d_S(g_i, g_{i+1}) = 1$ for each $i$; we denote $n$ by $Len(P)$.

When the group $G$ acts on a metric space $X \ni x_0$, we often define

$$\|g\|_X := d_X(x_0, gx_0) \quad (g \in G),$$
$$K_{Lip} := \max_{s \in S} \|s\|_X.$$

Then we have $\|g\|_X \leq K_{Lip}\|g\|_S$ for each $g \in G$.

## 3. WPD property and contraction

It is well-known that a loxodromic isometry $\varphi$ of a $\delta$-hyperbolic space $X \ni x_0$ has strictly positive asymptotic translation length $\tau := \lim_n d_X(x_0, \varphi^n x_0)/n$. Moreover, its orbit $\{\varphi^i x_0\}_{i \in \mathbb{Z}}$ is a quasigeodesic and hence quasi-convex. In summary,

**Fact 3.1.** *Let $\varphi$ be a loxodromic isometry of a $\delta$-hyperbolic space $X \ni x_0$. Then there exists $\mathcal{G} > 0$ such that the sequence $(\varphi^i x_0, \ldots, \varphi^j x_0)$ and the geodesic $[\varphi^i x_0, \varphi^j x_0]$ are $\mathcal{G}$-fellow traveling for each $i \leq j$. Furthermore, the sequence $(\varphi^i x_0)_{i \in \mathbb{Z}}$ is a $\mathcal{G}$-coarse geodesic, i.e.,*

$$d_X(\varphi^i x_0, \varphi^l x_0) \geq d_X(\varphi^i x_0, \varphi^j x_0) + d_X(\varphi^j x_0, \varphi^l x_0) - \mathcal{G} \quad (\forall i \leq j \leq l).$$

In Subsection 1.1, we claimed that $f(D, M) < +\infty$ for each $D, M > 0$ for every acylindrically hyperbolic group. We prove a variant of this fact.

**Lemma 3.2.** *Let $G$ be a non-virtually cyclic group with a finite generating set $S \subseteq G$. Suppose that $G$ acts on a $\delta$-hyperbolic space $X \ni x_0$ with a WPD loxodromic element $\varphi \in G$. Then there exists $D_0 > 0$, and for each $k, M > 0$ there exists $R = R(k, M) > 0$, such that the following holds.*

*Let $g, h \in G$ be such that $\|g\|_S > R$ and $\|h\|_S \leq M$. Then $\pi_{[x_0, \varphi^k x_0]}(\{gx_0, ghx_0\})$ has diameter at most $D_0$.*

This lemma closely resembles [Sis16, Lemma 3.3] and [MS20, Lemma 8.1]. Here, the crucial point is that $D_0$ is uniform and is independent from $k, M$ and $R$.

*Proof.* Let $\mathcal{G} > 0$ be the constant for $\varphi$ as in Fact 3.1. For $K = 24\mathcal{G} + 130\delta$, we pick $N$ such that $Stab_K(x_0, \varphi^N x_0)$ is finite using the WPD property of $\varphi$. We then set $D_0 := 1002\mathcal{G} + ND_\varphi + 1000\delta$, where $D_\varphi := d_X(x_0, \varphi x_0)$.

To prove the lemma, let $k, M > 0$ and denote $\gamma := [x_0, \varphi^k x_0]$. Suppose to the contrary that there does not exist $R$ for $(k, M)$. That means, suppose that there exist a sequence $(g_1, g_2, \ldots)$ of distinct elements of $G$ and a sequence $(h_1, h_2, \ldots)$ in $B_S(M)$ such that

$$\text{diam}\left(\pi_\gamma(\{g_i x_0, g_i h_i x_0\})\right) \geq D_0 \quad (\forall i > 0).$$

Let $p_i, q_i$ be points in $\pi_\gamma(\{g_i x_0, g_i h_i x_0\})$ that are at least $D_0$-apart. Recall that the nearest point projection of a single point onto $\gamma$ has diameter at most $20\delta < D_0$ (Fact 2.2(1)). Hence, up to relabelling, we can say that $p_i \in \pi_\gamma(g_i x_0)$ and $q_i \in \pi_\gamma(g_i h_i x_0)$.

Since $\gamma = [x_0, \varphi^k x_0]$ is compact and $B_S(M)$ is finite, we can take a subsequence and assert that:

$$h_1 = h_2 = \ldots =: h,$$
$$d_X(p_i, p_j), d_X(q_i, q_j) < \mathcal{G} \quad (\forall i, j > 0).$$

Either $p_i$'s appear earlier than or later than $q_i$'s. In the latter case, we can replace $h$ with $h^{-1}$, $g_i$ with $g_i h$ and swap $p_i$'s with $q_i$'s. Hence, we may assume that $p_i$'s appear earlier than $q_i$'s.

By Fact 2.2(2), there exists $[\alpha_i, \beta_i] \subseteq [g_i x_0, g_i h x_0]$ that is $20\delta$-fellow traveling with $[p_i, q_i]$. Note that $0 \leq d_X(g_i x_0, \alpha_i) \leq d_X(x_0, hx_0)$. By taking further subsequence, we can obtain $T$ such that

$$\left|d_X(g_i x_0, \alpha_i) - T\right| < \delta \quad (\forall i > 0).$$

By Fact 3.1, there exist $l, m \in \mathbb{Z}$ such that $d_X(\varphi^l x_0, p_1) < \mathcal{G}$ and $d_X(\varphi^m x_0, q_1) < \mathcal{G}$. Note that

$$(3.1) \qquad D_\varphi \cdot |l - m| \geq d_X(\varphi^l x_0, \varphi^m x_0) > d_X(p_1, q_1) - 2\mathcal{G} > 1000\mathcal{G} + ND_\varphi.$$

Here, if $m \leq l$ then

$$d_X(x_0, q_1) \leq d_X(x_0, \varphi^m x_0) + \mathcal{G} \leq d_X(x_0, \varphi^l x_0) - d_X(\varphi^m x_0, \varphi^l x_0) + 2\mathcal{G}$$
$$\leq d_X(x_0, \varphi^l x_0) - 1000\mathcal{G} + 2\mathcal{G} < d_X(x_0, \varphi^l x_0) - 998\mathcal{G} \leq d_X(x_0, p_1) - 997\mathcal{G}.$$

This contradicts the fact that $p_1$ appears earlier than $q_1$ on $\gamma$. Hence, we have $l < m$.

Now Inequality 3.1 implies that $l + N$ lies between $l$ and $m$. By Fact 3.1, $\varphi^{l+N} x_0$ lies in a $\mathcal{G}$-neighborhood of $[\varphi^l x_0, \varphi^m x_0]$. Note that $d_X(\varphi^l x_0, p_i) \leq d_X(\varphi^l x_0, p_1) + d_X(p_1, p_i) \leq 2\mathcal{G}$ for each $i$, and similarly $\varphi^m x_0$ and $q_i$ are $2\mathcal{G}$-close.

By Fact 2.1 $[\varphi^l x_0, \varphi^m x_0]$ is $(4\mathcal{G} + 2\delta)$-fellow traveling with $[p_i, q_i]$, which is $20\delta$-fellow traveling with $[\alpha_i, \beta_i]$. Thus, there exists $c_i \in [\alpha_i, \beta_i]$ such that $d_X(c_i, \varphi^{l+N} x_0) < 5\mathcal{G} + 22\delta$. We have

$$\left|d_X(\alpha_i, c_i) - d_X(\varphi^l x_0, \varphi^{l+N} x_0)\right| \leq d_X(\alpha_i, \varphi^l x_0) + d_X(c_i, \varphi^{l+N} x_0)$$
$$\leq d_X(\alpha_i, p_i) + d_X(p_i, p_1) + d_X(p_1, \varphi^l x_0) + d_X(c_i, \varphi^{l+N} x_0)$$
$$\leq 20\delta + \mathcal{G} + \mathcal{G} + (5\mathcal{G} + 22\delta) \leq 7\mathcal{G} + 42\delta.$$

9

Now, for each $i$ we have four points $g_i g_1^{-1} \alpha_1$, $g_i g_1^{-1} c_1$, $\alpha_i$ on the geodesic

$$g_i \cdot g_1^{-1}([g_1 x_0, g_1 h x_0]) = [g_i x_0, g_i h x_0].$$

Recall that $d_X(g_i x_0, g_i g_1^{-1} \alpha_1) = d_X(g_1 x_0, \alpha_1)$ and $d_X(g_i x_0, \alpha_i)$ are both $\delta$-close to $T$. This implies that $g_i g_1^{-1} \alpha_1$ and $\alpha_i$ are $2\delta$-close. Hence, we have

$$\begin{aligned}
d_X(\varphi^l x_0, g_i g_1^{-1} \cdot \varphi^l x_0) &\leq d_X(\varphi^l x_0, \alpha_i) + d_X(\alpha_i, g_i g_1^{-1} \alpha_1) + d_X(g_i g_1^{-1} \alpha_1, g_i g_1^{-1} \cdot \varphi^l x_0) \\
&\leq d_X(\varphi^l x_0, p_i) + d_X(p_i, \alpha_i) + 2\delta + d_X(\varphi^l x_0, p_1) + d_X(p_1, \alpha_1) \\
&\leq (20\delta + 2\mathcal{G}) + 2\delta + (\mathcal{G} + 20\delta) = 42\delta + 3\mathcal{G}.
\end{aligned}$$

Next, $d_X(g_i x_0, c_i) = d_X(g_i x_0, \alpha_i) + d_X(\alpha_i, c_i)$ is $(7\mathcal{G} + 43\delta)$-close to $T + d_X(x_0, \varphi^N x_0)$. So is $d_X(g_i x_0, g_i g_1^{-1} c_1) = d_X(g_1 x_0, c_1)$. Hence, $c_i$ and $g_i g_1^{-1} c_1$ are $(14\mathcal{G} + 86\delta)$-close. We conclude

$$\begin{aligned}
d_X(\varphi^{l+N} x_0, g_i g_1^{-1} \cdot \varphi^{l+N} x_0) &\leq d_X(\varphi^{l+N} x_0, c_i) + d_X(c_i, g_i g_1^{-1} c_1) + d_X(g_i g_1^{-1} c_1, g_i g_1^{-1} \cdot \varphi^{l+N} x_0) \\
&\leq (5\mathcal{G} + 22\delta) + (14\mathcal{G} + 86\delta) + (5\mathcal{G} + 22\delta) < 24\mathcal{G} + 130\delta.
\end{aligned}$$

To summarize, $\varphi^{-l} g_i g_1^{-1} \varphi^l$ belongs to $Stab_K(x_0, \varphi^N x_0)$ for each $i$. Furthermore, we have

$$\varphi^{-l} g_i g_1^{-1} \varphi^l = \varphi^{-l} g_j g_1^{-1} \varphi^l \iff g_i = g_j.$$

Since $g_1, g_2, \ldots$ are distinct, it follows that $Stab_K(x_0, \varphi^N x_0)$ is infinite, a contradiction. $\square$

**Proposition 3.3.** *Let $G$ be a non-virtually cyclic group and let $S \subseteq G$ be its finite generating set. Suppose that $G$ acts on a $\delta$-hyperbolic space $X \ni x_0$ with a WPD loxodromic element $\varphi \in G$. Then for each $K > 0$ there exists $L_0 = L_0(K)$ such that, for each $L \geq L_0$ and for each $M > 0$ there exists $R_0 = R_0(L, M) > 0$ satisfying the following.*

*Let $P_l$ be a $d_S$-path connecting $a_l \in G$ to $b_l \in G$ for $l = 1, 2$. Let $g_1, \ldots, g_m \in G$ be such that*

$$(a_i x_0, g_1[x_0, \varphi^L x_0], \ldots, g_m[x_0, \varphi^L x_0], b_i x_0) \text{ is } K\text{-aligned.} \quad (i = 1, 2)$$

*Let $k \leq m$. Then one of the following happens:*

*(1) For each subset $I \subseteq \{1, \ldots, m\}$ of cardinality $k$, there exist $i \in I$ such that*

$$d_S(P_l, g_i) < R_0 \quad (l = 1, 2); \text{ or,}$$

*(2) $Len(P_1) + Len(P_2) \geq M \cdot k$.*

*Proof.* Let $D_0$ be as in Lemma 3.2. Let $K_{Lip} = \max_{s \in S} \|s\|_X$, let $\tau := \lim_n \frac{1}{n} d_X(x_0, \varphi^n x_0)$ and let

$$L_0 = \frac{1}{\tau}\left(2K + 2K_{Lip} + 1000\delta + D_0\right).$$

Now given $L > L_0$, we will declare $R_0 = R_0(L, M)$ following Lemma 3.2.

Consider $P_1, P_2$ and $g_i$'s as in the proposition. Note that

$$(3.2) \qquad \mathrm{diam}_X([x_0, \varphi^L x_0]) \geq \tau L \geq 2K + 2K_{Lip} + 1000\delta + D_0.$$

Thanks to this inequality, we can apply Fact 2.11 to the $K$-aligned translates of $[x_0, \varphi^L x_0]$.

We will negate the case (1) and prove that (2) holds. For this, let $I \subseteq \{1, \ldots, m\}$ be a $k$-element subset such that $g_i$ is not simultaneously $R_0$-close to $P_1$ and $P_2$ for each $i \in I$. Let

$$I_l := \{i : d_S(P_l, g_i) \geq R_0\}. \quad (l = 1, 2)$$

Then $I_1 \cup I_2$ has cardinality at least $k$. For convenience, we will denote $g_i[x_0, \varphi^L x_0]$ by $\gamma_i$.

For each $i \in \{1, \ldots, m\}$, let $p_i$ be the latest vertex of $P_1$ such that $(p_i x_0, \gamma_i)$ is $(K + 60\delta)$-aligned. Such a vertex exists because $(a_i x_0, \gamma_i)$ is $(K + 60\delta)$-aligned by Fact 2.11. Now let

$$V_i := \{v \in P_1 : v \text{ comes later than } p_i \text{ along } P_1 \text{ and } (\gamma_i, v x_0) \text{ is } (K + 60\delta)\text{-aligned}\};$$

10

$V_i$ is nonempty because $(\gamma_i, b_i)$ is $(K + 60\delta)$-aligned by Fact 2.11. Let $q_i$ be the earliest vertex in $V_i$. Lastly, let $p'_i$ be the vertex of $P_1$ right after $p_i$, and let $q'_i$ be the one right before $q_i$.

Note that $\pi_{\gamma_i}(\{p_i x_0, p'_i x_0\})$ has diameter at most $K_{Lip} + 12\delta$ by Fact 2.2(1). Hence, $\pi_{\gamma_i}(p'_i x_0)$ is contained in the beginning $(K_{Lip} + K + 80\delta)$-subsegment of $\gamma_i$ and does not meet the ending $(K_{Lip} + 60\delta)$-subsegment (Inequality 3.2). This implies $p'_i \notin V_i$, and $q_i$ comes later than $p'_i$. Let

$$V'_i := \{v \in P_1 : v \text{ in between } p_i \text{ and } q_i \text{ (excluding } p_i, q_i)\}.$$

We have then observed that $p'_i, q'_i \in V'_i$ and $V'_i$ is nonempty. For each $v \in V'_i$ we conclude that

$$\text{neither } (\gamma_i, vx_0) \text{ nor } (vx_0, \gamma_i) \text{ is } (K + 60\delta)\text{-aligned.}$$

Now repeated application of Fact 2.10 tells us that for $i \neq j$,

$$v \in V'_i \Rightarrow \left\{ \begin{array}{ll} (\gamma_j, vx_0) \text{ is } (K + 60\delta)\text{-aligned} & \text{if } j < i \\ (vx_0, \gamma_j) \text{ is } (K + 60\delta)\text{-aligned} & \text{if } j > i \end{array} \right\} \Rightarrow v \notin V'_j.$$

In conclusion, $V'_1, \ldots, V'_m$'s are disjoint subpaths of $P_1$.

Now observe for each $i$ that

$$\text{diam}_X(\pi_{\gamma_i}(\{p'_i, q'_i\})) \geq \text{diam}(\gamma_i) - 2(K + 60\delta) - \text{diam}(\pi_{\gamma_i}(p_i x_0, p'_i x_0)) - \text{diam}(\pi_{\gamma_i}(q_i x_0, q'_i x_0))$$
$$\geq \tau L - 2(K + 60\delta) - 2 \cdot (K_{Lip} + 20\delta) > D_0.$$

If $i \in I_1$, we additionally know that $d_S(p'_i, g_i) \geq R_0$. Lemma 3.2 then implies that $d_S(p'_i, q'_i) > M$. Hence, $Len(V'_i) \geq M$ for each $i \in I_1$. Summing up, we obtain $Len(P_1) \geq M \cdot \#I_1$.

Similar logic implies $Len(P_2) \geq M \cdot \#I_2$. We thus conclude

$$Len(P_1) + Len(P_2) \geq M \cdot (\#I_1 \cup \#I_2) \geq M \cdot k. \qquad \square$$

## 4. Proof of Theorem 1.1

Throughout, let $G$ be a non-virtually cyclic group with a finite generating set $S$. Suppose that $G$ acts on a $\delta$-hyperbolic space $X \ni x_0$ with a WPD loxodromic element $\varphi \in G$. When a constant $L$ is understood, we will use the notation

$$\Upsilon_L := [x_0, \varphi^L x_0].$$

Since $G$ contains independent loxodromics, $G$ has exponential growth. In other words,

$$\lambda_S := \liminf_n \frac{\ln \#B_S(n)}{n} > 1.$$

This immediately implies that:

**Fact 4.1.** *For each sufficiently large $n$ we have*

$$\#B_S(0.9n)/\#B_S(n) \leq \lambda_S^{0.05n}.$$

Let us fix some more constants for the proof. Let $E_0$ be as in Fact 2.6. Let $K_{Lip} := \max_{s \in S} \|s\|_X$, let $\tau := \lim_n \|\varphi^n\|_X/n$ (so that $\|\varphi^k\|_X \geq k\tau$ for each $k$). Let $F_0 := \|\varphi\|_S$. Finally, let $L_0$ be as in Proposition 3.3 for $K = 100(E_0 + 1000\delta + K_{Lip})$, and

$$L_1 := L_0 + \frac{1}{\tau} 100(E_0 + 1000\delta + K_{Lip}).$$

This choice implies that:

**Fact 4.2.** *For each $L > L_1$,*

$$d_X(x_0, \varphi^L x_0) - 40(E_0 + 1000\delta + K_{Lip}) \geq \tau L - 40(E_0 + 1000\delta + K_{Lip}) \geq 0.5\tau L + 140\delta.$$

Given $L, \epsilon > 0$, we define

$$\mathcal{V}_{L,\epsilon}(n) := \left\{ g \in B_S(n) : \begin{array}{c} \text{there exist } h_1, \ldots, h_{\epsilon n} \in G \text{ such that} \\ (x_0, h_1 \Upsilon_L, \ldots, h_{\epsilon n} \Upsilon_L, g x_0) \text{ is } (6E_0 + 300\delta)\text{-aligned} \end{array} \right\},$$

$$\mathcal{BAD}_{L,\epsilon}(n) := B_S(n) \setminus \big( B_S(0.9n) \cup \mathcal{V}_{L,\epsilon}(n) \big).$$

**Lemma 4.3.** *For each $L > L_1$ and $\epsilon > 0$, we have*

$$\limsup_n \frac{\#\mathcal{BAD}_{L,\epsilon}(n)}{\#B_S(n)} < 5 \cdot (2E_0 + 4\,L\,F_0 + 5) \cdot (\#S)^{E_0 + 3\,L\,F_0 + 4} \cdot \epsilon.$$

*Proof of Lemma 4.3.* Let us define a map

$$F : Dom(F) := \mathcal{BAD}_{L,\epsilon}(n) \times \{1, \ldots, 0.9n\} \to B_S(n)$$

as follows. Given $(g, i) \in \mathcal{BAD}_{L,\epsilon}(n) \times \{1, \ldots, 0.9n\}$, we first fix a $d_S$-geodesic representative $g = a_1 a_2 \cdots a_{\|g\|_S}$. By Fact 2.6, there exist $s = s(g, i)$ and $t = t(g, i)$ in $S \cup \{id\}$ such that

$$\left( s^{-1} \cdot (a_1 \cdots a_i)^{-1} x_0, \varphi^L x_0 \right)_{x_0} < E_0, \quad \left( \varphi^{-L} x_0, t \cdot a_{i+F_0\,L+3} \cdots a_{\|g\|} x_0 \right)_{x_0} < E_0.$$

We then define

$$h(g, i) := a_1 \cdots a_i \cdot s, \quad h'(g, i) := t \cdot a_{i+F_0\,L+3} \cdots a_{\|g\|}, \quad F(g, i) := h(g, i) \varphi^L h'(g, i).$$

Note that $F(g, i) \in B_S(n)$ because

$$\|F(g, i)\|_S \leq \|h(g, i)\|_S + \|\varphi^L\|_S + \|h'(g, i)\|_S$$
$$\leq (i + 1) + F_0\,L + (\|g\|_S - i - F_0\,L - 2) + 1 \leq \|g\|_S \leq n.$$

Before the proof, we first declare

$$T := (2E_0 + 4\,L\,F_0 + 5) \cdot \#B_S(E_0 + 2\,L\,F_0) \cdot \#B_S(F_0\,L + 4)$$

and $R_0 = R_0(L, 4/\epsilon)$ as in Proposition 3.3.

**Claim 4.4.** *Let $m$ be the maximum number of elements $(g_1, i_1), \ldots, (g_m, i_m) \in \mathcal{BAD}_{L,\epsilon}(n) \times \{1, \ldots, 0.9n\}$ such that*

(1) $F(g_1, i_1) = \ldots = F(g_m, i_m) =: U$,
(2) $\left( x_0, h(g_1, i_1) \cdot \Upsilon_L, \ldots, h(g_m, i_m) \cdot \Upsilon_L, U x_0 \right)$ *is $6(E_0 + 30\delta)$-aligned.*

*Then $\#F^{-1}(U) \leq 2T \cdot m$ holds for each $U \in B_S(n)$.*

*Proof of Claim 4.4.* Fix an arbitrary $U \in B_S(n)$. For each $(g, i) \in F^{-1}(U)$, we have:

(1) $(x_0, h(g, i) \varphi^L x_0)_{h(g,i) x_0} < E_0$; hence $\pi_{h(g,i)\Upsilon_L}(x_0)$ is $(E_0 + 8\delta)$-close to $h(g, i) x_0$ (Fact 2.4).
(2) Similarly, the projection of $U x_0$ onto $h(g, i)\Upsilon_L$ is $(E_0 + 8\delta)$-close to $h(g, i) \varphi^L x_0$.
(3) $h(g, i) x_0$ and $h(g, i) \varphi^L x_0$ are at least $\tau\,L$-apart, which is much larger than $20\delta$.

Now Fact 2.2 guarantees a subsegment $\gamma(g, i)$ of $[x_0, U x_0]$ and a subsegment $\eta = [p, q]$ of $h(g, i)\Upsilon_L$ that are $20\delta$-fellow traveling. Here, $p$ and $h_k x_0$, and $q$ and $h_k \varphi^L x_0$ are pairwise $(E_0 + 8\delta)$-close. Hence, $\gamma(g, i)$ and $h(g, i)\Upsilon_L$ are $(E_0 + 30\delta)$-fellow traveling. It follows that $\gamma(g, i)$'s are longer than $\tau\,L - 2(E_0 + 30\delta) \geq 25(E_0 + 30\delta)$.

We now pick a maximal subset $\mathcal{A}$ of $F^{-1}(U)$ such that

for any $(g, i), (g', i') \in \mathcal{A}$, $\gamma(g, i)$ and $\gamma(g', i')$ overlap for length at most $12(E_0 + 30\delta)$.

We claim that $\#F^{-1}(U) \leq T \cdot \#\mathcal{A}$. To show this, pick an arbitrary $(g, i) \in F^{-1}(U)$. Let $a_1 \cdots a_{\|g\|_S}$ be the geodesic representative for $g$ that was used when defining

$$h(g, i) := a_1 \cdots a_i \cdot s(g, i), \quad h(g, i)' := t(g, i) \cdot a_{i+F_0\,L+3} + \cdots a_{\|g\|_S}.$$

By the maximality of $\mathcal{A}$, there exists $(\mathfrak{g}, \mathfrak{i}) \in \mathcal{A}$ such that $\gamma(g, i)$ and $\gamma(\mathfrak{g}, \mathfrak{i})$ overlap for length at least $12(E_0 + 30\delta)$. Recall that $h(g, i)\Upsilon_L$ and $h(\mathfrak{g}, \mathfrak{i})\Upsilon_L$ are $(E_0 + 30\delta)$-fellow traveling $\gamma(g, i)$ and $\gamma(\mathfrak{g}, \mathfrak{i})$, respectively. By Fact 2.9, $\pi_{h(g,i)\Upsilon_L}(h(\mathfrak{g}, \mathfrak{i})x_0)$ appears earlier than $\pi_{h(g,i)\Upsilon_L}(h(\mathfrak{g}, \mathfrak{i})\varphi^L x_0)$. Moreover, they are $(12(E_0 + 30\delta) - 10(E_0 + 30\delta))$-apart and hence $E_0$-apart. By Fact 2.6, we conclude that $\varphi^{-a} \cdot h(g, i)^{-1} h(\mathfrak{g}, \mathfrak{i})\varphi^b \in B_S(E_0)$ for some $a, b \in \{0, \ldots, L\}$. We conclude that

$$h(g, i) \in h(\mathfrak{g}, \mathfrak{i}) \cdot \{\varphi^a : a = 0, \ldots, L\} \cdot B_S(E_0) \cdot \{\varphi^{-a} : a = 0, \ldots, L\} \subseteq h(\mathfrak{g}, \mathfrak{i})B_S(E_0 + 2\,L\,F_0).$$

This also implies that $\|h(g, i)\|_S$ and $\|h(\mathfrak{g}, \mathfrak{i})\|_S$ differ by at most $E_0 + 2\,L\,F_0$, and hence

$$i \in \big[\mathfrak{i} - (E_0 + 2\,L\,F_0 + 2), \mathfrak{i} + (E_0 + 2\,L\,F_0 + 2)\big].$$

Note also that

$$h(g, i)' = \varphi^{-L} h(g, i)^{-1} \cdot U$$

is determined as soon as $h(g, i)$ is determined.

Finally, in order to reconstruct $g = a_1 \cdots a_{\|g_l\|_S}$ from $h(g, i)$ and $h(g, i)$, it suffices to pick $c := s_l^{-1} a_{i_l+1} \cdots a_{i_l + F_0\,L + 2} t^{-1} \in B_S(F_0\,L + 4)$ and multiply $h(g, i)$, $c$ and $h(g', i')$. In summary, we have

$$F^{-1}(U) \subseteq \bigcup_{(\mathfrak{g},\mathfrak{i})\in\mathcal{A}} \left( \left\{ h(\mathfrak{g}, \mathfrak{i})f \cdot c \cdot \varphi^{-L} f^{-1} h(\mathfrak{g}, \mathfrak{i})^{-1} U : f \in B_S(E_0 + 2\,L\,F_0),\ c \in B_S(F_0\,L + 4) \right\} \times I(\mathfrak{i}) \right)$$

where $I(\mathfrak{i}) := [\mathfrak{i} - (E_0 + 2\,L\,F_0 + 2), \mathfrak{i} + (E_0 + 2\,L\,F_0 + 2)]$. From this, we conclude $\#F^{-1}(U) \leq T \cdot \#\mathcal{A}$.

Next, let us enumerate $\mathcal{A}$ as

$$\mathcal{A} = \{(\mathfrak{g}_1, \mathfrak{i}_1), (\mathfrak{g}_2, \mathfrak{i}_2), \ldots\}$$

so that $\gamma(\mathfrak{g}_l, \mathfrak{i}_l)$ starts earlier than $\gamma(\mathfrak{g}_{l+1}, \mathfrak{i}_{l+1})$ along $[x_0, Ux_0]$, for each $l$. Then the beginning point of $\gamma(\mathfrak{g}_2, \mathfrak{i}_2)$ is later than that of $\gamma(\mathfrak{g}_1, \mathfrak{i}_1)$ and earlier than that of $\gamma(\mathfrak{g}_3, \mathfrak{i}_3)$. (∗) Moreover, $\gamma(\mathfrak{g}_2, \mathfrak{i}_2)$ does not contain $\gamma(\mathfrak{g}_3, \mathfrak{i}_3)$, as their overlap should not be longer than $12(E_0 + 30\delta)$ while $\gamma(\mathfrak{g}_3, \mathfrak{i}_3)$ is longer than $25(E_0 + 30\delta)$. Hence, the ending point of $\gamma(\mathfrak{g}_2, \mathfrak{i}_2)$ is earlier than that of $\gamma(\mathfrak{g}_3, \mathfrak{i}_3)$. (∗∗)

At this point, if $\gamma(\mathfrak{g}_1, \mathfrak{i}_1)$ and $\gamma(\mathfrak{g}_3, \mathfrak{i}_3)$ intersect, then $\gamma(\mathfrak{g}_3, \mathfrak{i}_3)$ is completely covered by $\gamma(\mathfrak{g}_1, \mathfrak{i}_1)$ and $\gamma(\mathfrak{g}_3, \mathfrak{i}_3)$ due to (∗) and (∗∗). We would then have

$$\mathrm{diam}_X\big(\gamma(\mathfrak{g}_1, \mathfrak{i}_1) \cap \gamma(\mathfrak{g}_2, \mathfrak{i}_2)\big) + \mathrm{diam}_X\big(\gamma(\mathfrak{g}_2, \mathfrak{i}_2) \cap \gamma(\mathfrak{g}_3, \mathfrak{i}_3)\big) \geq \mathrm{diam}_X\big(\gamma(\mathfrak{g}_2, \mathfrak{i}_2)\big) \geq 25(E_0 + 30\delta),$$

which contradicts to the bound $12(E_0 + 30\delta)$ on $\mathrm{diam}_X\big(\gamma(\mathfrak{g}_1, \mathfrak{i}_1) \cap \gamma(\mathfrak{g}_2, \mathfrak{i}_2)\big)$ and $\mathrm{diam}_X\big(\gamma(\mathfrak{g}_2, \mathfrak{i}_2) \cap \gamma(\mathfrak{g}_3, \mathfrak{i}_3)\big)$. Hence, we conclude that $\gamma(\mathfrak{g}_1, \mathfrak{i}_1)$ and $\gamma(\mathfrak{g}_3, \mathfrak{i}_3)$ do not intersect.

With the same logic, we conclude that $\gamma(\mathfrak{g}_l, \mathfrak{i}_l)$'s for odd integers $l$ are disjoint subsegments of $[x_0, Ux_0]$, in order from left to right along $[x_0, Ux_0]$. Recall again that $\gamma(\mathfrak{g}_l, \mathfrak{i}_l)$ and $h(\mathfrak{g}_l, \mathfrak{i}_l)\Upsilon_L$ are $(E_0 + 30\delta)$-fellow traveling. Now Fact 2.8 tells us that

$$\big(x_0, h(\mathfrak{g}_1, \mathfrak{i}_1)\Upsilon_L, h(\mathfrak{g}_3, \mathfrak{i}_3)\Upsilon_L, \ldots, h(\mathfrak{g}_{2\lceil\#\mathcal{A}/2\rceil+1}, \mathfrak{i}_{2\lceil\#\mathcal{A}/2\rceil+1})\Upsilon_L,\ Ux_0\big)$$

is $6(E_0 + 30\delta)$-aligned. This implies that $m \geq \lceil\#\mathcal{A}/2\rceil \geq \frac{1}{2T}\#F^{-1}(U)$ as desired. $\qquad\square$

Now let $(g_1, i_1), \ldots, (g_m, i_m) \in \mathcal{BAD}_{L,\epsilon}(n) \times \{1, \ldots, 0.9n\}$ the elements as in Claim 4.4:

(1) $F(g_1, i_1) = \ldots = F(g_m, i_m) =: U$,

(2) $\big(x_0, h(g_1, i_1) \cdot \Upsilon_L, \ldots, h(g_m, i_m) \cdot \Upsilon_L, Ux_0\big)$ is $6(E_0 + 30\delta)$-aligned.

It remains to prove that $m < 2\epsilon n$ for large enough $n$. We will prove it for every $n \geq 32R_0 K_{Lip}/\tau\,L\,\epsilon$. Suppose to the contrary that $m \geq 2\epsilon n$. Let us denote the $d_S$-geodesic representative used for $g_1$ by $a_1 \cdots a_{\|g_1\|_S}$, so that $h(g_1, i_1) = a_1 \cdots a_{i_1} s(g_1, i_1)$. We will abbreviate $h(g_l, i_l)$ by $h_l$, $h'(g_l, i_l)$ by $h'_l$, $s(g_l, i_l)$ by $s_l$ and $t(g_l, i_l)$ by $t_l$.

We focus on a particular vertex on the $d_S$-geodesic $Ug_1^{-1} \cdot [x_0, g_1]_S$, namely

$$v := Ug_1^{-1} \cdot a_1 \cdots a_{i_1 + F_0\,L + 2} = a_1 \cdots a_{i_1} \cdot s_1 \cdot \varphi^L \cdot t_1 x_0 = h_1 \cdot \varphi^L \cdot t_1.$$

Then $vx_0$ is $K_{Lip}$-close to $h_1 \cdot \varphi^L x_0$. Since $(h_1 \varphi^L x_0, h_2[x_0, \varphi^L x_0])$ is $(6E_0 + 180\delta)$-aligned, Fact 2.2(1) tells us that $(vx_0, h_2 \Upsilon_L)$ is $(6E_0 + 200\delta + K_{Lip})$-aligned.

Now, Fact 2.10 tells us that either:

(1) $(U g_1^{-1} x_0, h_{\epsilon n} \Upsilon_L)$ is $(6E_0 + 240\delta)$-aligned, or
(2) $(h_{\epsilon n-1} \Upsilon_L, U g_1^{-1} x_0)$ is $(6E_0 + 240\delta)$-aligned.

In Case (1), we conclude that

$$\left(x_0, \, g_1 U^{-1} h_{\epsilon n} \Upsilon_L, \, g_1 U^{-1} h_{\epsilon n+1} \Upsilon_L, \, \ldots, \, g_1 U^{-1} h_m \Upsilon_L, \, g_1 x_0\right)$$

is $(6E_0 + 240\delta)$-aligned. This contradicts the fact that $g_1 \notin \mathcal{V}_{L,\epsilon}(n)$.

In the latter case, we have:

$$(vx_0, h_1 \Upsilon_L, \ldots, h_{\epsilon n-1} \Upsilon_L, y_i)$$

is $(6E_0 + 240\delta + K_{Lip})$-aligned for $y_1 = U g_1^{-1} x_0$ and $y_2 = U x_0$. Here, the alignment of $(h_{\epsilon n-1} \Upsilon_L, y_2)$ is due to Fact 2.11. Let $P_1$ be the first half of the geodesic $U g_1^{-1}[x_0, g_1 x_0]$ connecting $U g_1^{-1} x_0$ to $vx_0$, and let $P_2$ be the latter half connecting $vx_0$ to $U x_0$. Then $Len(P_1) + Len(P_2) \leq \|g_1\|_S \leq n$.

Recall that $R_0 = R_0(L, 4/\epsilon)$ is chosen as in Proposition 3.3 and that $L \geq L_1$ is longer than $L_0(K)$ for $K = 6E_0 + 240\delta + K_{Lip}$. Since $Len(P_1) + Len(P_2) \leq n \leq (4/\epsilon) \cdot (\epsilon n/4)$, the paths should satisfy the first alternative in Proposition 3.3 for $k = \epsilon n/4$. In particular, there exists $i \in \{0.5\epsilon n, \ldots, 0.75\epsilon n\}$ such that $d_S(P_1, h_i), d_S(P_2, h_i) \leq R_0$. Let $u_1 \in P_1$ and $u_2 \in P_2$ be the vertices realizing the distance.

Meanwhile, note that $(vx_0, h_2 \Upsilon_L, \ldots, h_{i-1} \Upsilon_L, h_i x_0)$ is $(6E_0 + 200\delta + K_{Lip})$-aligned. Fact 2.12 implies that there exist $i - 2 \geq 0.25\epsilon n$ disjoint subsegments of $[vx_0, h_i x_0]$, each longer than $\tau L - 2(6E_0 + 200\delta + K_{Lip}) - 160\delta \geq 0.5\tau L$. This implies that

$$d_S(h_i, v) \geq \frac{1}{K_{Lip}} d_X(vx_0, h_i x_0) \geq \frac{1}{K_{Lip}} \cdot 0.5\tau L \cdot 0.35\epsilon n.$$

This implies that

$$d_S(u_1, v) \geq d_S(h_i, v) - d_S(h_i, u_1) \geq \frac{\tau L \epsilon n}{8 K_{Lip}} - R_0 \geq 3R_0.$$

Meanwhile, $u_1, v$ and $u_2$ are aligned along a $d_S$-geodesic $U g_1^{-1}[x_0, g_1 x_0]$. This leads to a contradiction

$$2R_0 \geq d_S(u_1, h_i) + d_S(u_2, h_i) \geq d_S(u_1, u_2) \geq d_S(u_1, v) \geq 3R_0$$

In conclusion, $m \leq 2\epsilon n$ holds for $m$ in Claim 4.4 when $n \geq 32 R_0 K_{Lip}/\tau L \epsilon$. This implies that

$$\#\mathcal{BAD}_{L,\epsilon}(n) \times 0.9n = \#\left(\mathrm{Dom}\, F\right) = \sum_{U \in B_S(n)} \left(\#F^{-1}(U)\right) \leq 4T\epsilon n \cdot \#B_S(n).$$

We conclude

$$\frac{\#\mathcal{BAD}_{L,\epsilon}(n)}{\#B_S(n)} \leq 5T\epsilon. \quad (\forall n \geq 32 R_0 K_{Lip}/\tau L \epsilon) \qquad \square$$

Let us now define

$$\mathcal{W}_{L,\epsilon}(n) := \left\{ g \in B_S(n) : \begin{array}{l} \exists h_1, \ldots, h_{\epsilon n} \in G \text{ such that the sequences } (x_0, h_1 \Upsilon_L, \ldots, h_{\epsilon n} \Upsilon_L, g x_0), \\ (g^{-1} x_0, h_1 \Upsilon_L) \text{ and } (h_{\epsilon n} \Upsilon_L, g^2 x_0) \text{ are each } (6E_0 + 360\delta)\text{-aligned} \end{array} \right\}.$$

**Lemma 4.5.** *Let $L > L_1$. Then the following is true for $g \in \mathcal{W}_{L,\epsilon}(n)$:*

*(1) $g$ is a loxodromic isometry on $X$ with $\tau_X(g) \geq 0.5\tau L \epsilon n$.*

*(2) $g$ has the WPD property and hence is Morse ([Sis16, Theorem 1]).*

*(3) There exists a conjugate $\psi$ of $\varphi$ such that for each large enough $i$, the projections of $g^{-i} x_0$ and $g^i x_0$ onto $[\psi^{-i} x_0, \psi^i x_0]$ are at least $\tau L/2$-apart, with the former one coming first.*

14

*Proof.* For the first item, we claim that

(4.1) $$\left(\ldots, g^{-1}\gamma_1, \ldots, g^{-1}\gamma_{\epsilon n}, \gamma_1, \ldots, \gamma_{\epsilon n}, g\gamma_1, \ldots, g\gamma_{\epsilon n}, \ldots\right)$$

is $(12E_0 + 900\delta)$-aligned, where $\gamma_i := h_i\Upsilon_L$. The only nontrivial part is the $(12E_0 + 900\delta)$-alignment of $(\gamma_{\epsilon n}, g\gamma_1)$. First, observe that $(\gamma_{\epsilon n}, gx_0)$ and $(\gamma_{\epsilon n}, g^2 x_0)$ are each $(6E_0 + 360\delta)$-aligned. By Corollary 2.3, $(\gamma_{\epsilon n}, z)$ is $(6E_0 + 420\delta)$-aligned for each $z \in [gx_0, g^2 x_0]$.

Meanwhile, Fact 2.12 implies that $g\gamma_1$ is contained in the $(6E_0 + 440\delta)$-neighborhood of $[x_0, gx_0]$. Fact 2.2(1) implies that $\pi_{\gamma_{\epsilon n}}(g\gamma_1)$ is contained in the $(12E_0 + 900\delta)$-long ending subsegment of $\gamma_{\epsilon n}$.

By a symmetric argument, we can similarly observe that $\pi_{g\gamma_1}(\gamma_{\epsilon n})$ is contained in the $(12E_0 + 900\delta)$-long ending subsegment of $g\gamma_1$. This concludes the desired alignment.

Now Fact 2.12 applies to the $(12E_0 + 900\delta)$-aligned sequence $(x_0, \gamma_1, \ldots, \gamma_{\epsilon n}, g\gamma_1, \ldots, g\gamma_{\epsilon n}, \ldots, g^k x_0)$ and concludes that

$$d_X(x_0, g^k x_0) \geq \sum_{i=0}^{k-1} \sum_{j=1}^{\epsilon n} \left( \operatorname{diam}_X(g^i \gamma_j) - (2(12E_0 + 900\delta) + 160\delta) \right)$$

$$\geq \epsilon n k \cdot \left( \tau\, L - (2(12E_0 + 900\delta) + 160\delta) \right) \geq \epsilon n k \cdot \frac{1}{2}\tau\, L.$$

This implies that $\tau_X(g) \geq 0.5\tau\, L\,\epsilon n$.

In order to discuss WPD property, let $K > 0$. Because $g$ is loxodromic, there exists $N$ such that $d_X(g^{\pm N} x_0, h_1 \Upsilon_L) \geq K + 1000\delta$. We then claim that $Stab_K(x_0, g^{2N} x_0)$ is finite. Suppose to the contrary that $Stab_K(x_0, g^{2N} x_0)$ is not contained in any finite $d_S$-metric ball. Then we can take infinitely many distinct elements $g_1, g_2, \ldots \in Stab_K(x_0, g^{2N} x_0)$.

Combining the alignment of the sequence in Display 4.1 and Fact 2.11, we observe that

$$(x_0, g^N h_1 \Upsilon_L, g^{2N} x_0)$$

is $(12E_0 + 960\delta)$-aligned. Since $d_X(x_0, g_i x_0) \leq K \leq d_X(x_0, g^N h_1 \Upsilon_L) - 1000\delta$, the contraposition of Fact 2.2(2) tells us that $\pi_{g^N h_1 \Upsilon_L}(\{x_0, g_i x_0\})$ has diameter at most $20\delta$. Similarly, $\pi_{g^N h_1 \Upsilon_L}(\{g_0^{2N}, g_i g^{2N} x_0\})$ is also $20\delta$-small. Hence, $(g_i x_0, g^N h_1 \Upsilon_L, g_i g^{2N} x_0)$ is also $(12E_0 + 980\delta)$-aligned. Hence, $[g_i x_0, g_i g^{2N} x_0]$ contains a subsegment $\eta_i$ that is $(12E_0 + 1060\delta)$-fellow traveling with $g^N h_1 \Upsilon_L$.

Now, $g_i^{-1}\eta_i$'s are subsegments of $[x_0, g^{2N} x_0]$ that is longer than $\tau\, L - 2(12E_0 + 1060\delta) \geq 0.5\tau\, L$. Since $[x_0, g^{2N} x_0]$ is compact, by passing to subsequence, we may assume that $g_i^{-1}\eta_i$'s converge to a subsegment of $[x_0, g^{2N} x_0]$ of length at least $0.5\tau\, L$. Also, these subsegments are $(12E_0 + 1060\delta)$-fellow traveling with $g_i^{-1}g^N h_1 \Upsilon_L$ and $g_j^{-1}g^N h_1 \Upsilon_L$, respectively. Since $0.5\tau\, L > 12(12E_0 + 1060\delta) + E_0$, Fact 2.9 implies for large $i, j$ that $\pi_{g_i^{-1}g^N h_1 \Upsilon_L}(g_j^{-1}g^N h_1 \Upsilon_L)$ is $E_0$-large and is orientation-matching. Now Fact 2.6(2) implies that

$$h_1^{-1}g^{-N}g_i g_j^{-1}g^N h_1 \subseteq B_S(E_0 + 2\,L\,F_0).$$

In particular, $g_i g_j^{-1}$ is uniformly bounded for every pair of $g_i, g_j$. This contradicts the infinitude of $Stab_K(x_0, g^{2N} x_0)$. The WPD property of $g$ is now proven.

The third item holds for $\psi = h_1 \varphi h_1^{-1}$. Indeed, when $N$ is sufficiently large, $[g^{-N} x_0, g^N x_0]$ and $[h_1 \varphi^{-N} h_1^{-1} x_0, h_1 \varphi^N h_1 x_0]$ both contain subsegments that are $0.01\tau\, L$-fellow traveling with a $\tau\, L$-long geodesic $h_1 \Upsilon_L$. We omit the detail. $\square$

We now claim that $\mathcal{V}_{L,3\epsilon}(n) \setminus \mathcal{W}_{L,\epsilon}(n)$ is non-generic.

**Lemma 4.6.** *For each $L > L_1$ and $\epsilon > 0$, there exists $\lambda > 1$ such that*

$$\lim_{n \to +\infty} \frac{\#\mathcal{V}_{L,3\epsilon}(n) \setminus \mathcal{W}_{L,\epsilon}(n)}{\#B_S(n)} \leq \lambda^{-n}$$

*for all large enough $n$.*

15

*Proof.* Before the proof, let $R_1 = R_1(L, 12/\epsilon)$ be as in Proposition 3.3. Let us first define

$$\mathcal{K}_1 := \bigcup_{r=\epsilon n}^{n/2} \{abca^{-1} : a \in B_S(r), b \in B_S(n - 2r + 2R_1), c \in B_S(2R_1)\},$$

$$\mathcal{K}_2 := \bigcup_{r,r' \geq 0, r+r' \leq (1-\epsilon)n} \{acba^{-1} : a \in B_S(r + 2R_1), b \in B_S(r' + 2R_1), c \in B_S(2R_1)\}.$$

We also define $\mathcal{K}_i^{-1} := \{g^{-1} : g \in \mathcal{K}_i\}$ for $i = 1, 2$. Then we have

$$\#\big(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_1^{-1} \cup \mathcal{K}_2^{-1}\big) \lesssim 2 \cdot n^2 \lambda_S^{(1-0.5\epsilon)n}.$$

This is exponentially smaller than $\#B_S(n)$.

It remains to prove that $\mathcal{V}_{L,3\epsilon}(n) \setminus \big(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_1^{-1} \cup \mathcal{K}_2^{-1}\big)$ is contained in $\mathcal{W}_{L,\epsilon}(n)$. To show this, let $g \in \mathcal{V}_{L,3\epsilon}(n) \setminus \big(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_1^{-1} \cup \mathcal{K}_2^{-1}\big)$. Then there exists $h_1, \ldots, h_{3\epsilon n} \in G$ such that

(4.2) $$(x_0, h_1 \Upsilon_L, \ldots, h_{3\epsilon n} \Upsilon_L, g x_0)$$

is $(6E_0 + 300\delta)$-aligned. Then $(x_0, h_i \Upsilon_L, g x_0)$ is $(6E_0 + 360\delta)$-aligned for each $i$ by Fact 2.11.

Fact 2.10 guarantees that the following dichotomy holds: either

(1) $(g^{-1} x_0, h_{\epsilon n+1} \Upsilon_L)$ is $(6E_0 + 360\delta)$-aligned, or
(2) $(h_{\epsilon n} \Upsilon_L, g^{-1} x_0)$ is $(6E_0 + 360\delta)$-aligned.

We claim that Case (1) holds. Suppose to the contrary that Case (2) holds. That means,

$$(x_0, h_1 \Upsilon_L, \ldots, h_{\epsilon n} \Upsilon_L, g^{\pm 1} x_0) \text{ is } (6E_0 + 360\delta)\text{-aligned.}$$

Pick a $d_S$-geodesic path $P$ connecting $id$ to $g$. Then $g^{-1} P$ is a path connecting $g^{-1}$ to $id$.

We now apply Proposition 3.3. Since $Len(P) + Len(g^{-1} P) \leq 2n \leq (12/\epsilon) \cdot (\epsilon n/6)$, the first alternative in Proposition 3.3 should hold for $k = \epsilon n/6$. In particular, there exists $i \in \{\epsilon n/2, \ldots, \epsilon n\}$ such that $d_S(h_i, P), d_S(h_i, g^{-1} P) \leq R_1$. Let $v \in P$ and $g^{-1} u \in g^{-1} P$ be the vertices realizing the distance. Here, as before, the alignment of the sequence in Display 4.2 implies that $[x_0, h_i x_0]$ contains $i$ disjoint subsegments longer than $\text{diam}_X(\Upsilon_L) - 2(6E_0 + 520\delta) \geq 0.5\tau L$. Hence,

$$d_S(id, h_i) \geq \frac{1}{K_{Lip}} d_X(x_0, h_i x_0) \geq \frac{1}{4K_{Lip}} \tau L \epsilon n.$$

This implies

$$\|v\|_S \geq \|h_i\|_S - d_S(h_i, v) \geq \frac{1}{4K_{Lip}} \tau L \epsilon n - R_1 \geq \epsilon n. \quad \text{(when } n \geq R_1/\epsilon\text{)}$$

Meanwhile, since $(h_i x_0, h_{i+1} \Upsilon_L, \ldots, h_{3\epsilon n} \Upsilon_L, g x_0)$ is also aligned, we have

$$d_S(h_i, g) \geq \frac{1}{K_{Lip}} d_X(h_i x_0, g x_0) \geq \frac{1}{K_{Lip}} \tau L \epsilon.$$

This implies $d_S(v, g) \geq \epsilon n$. Note that $\|g^{-1} u\|_S = \|g\|_S - \|u\|_S$ and $\|v\|_S$ differ by at most $2R_0$. $(*)$

We now divide the cases:

(1) $\epsilon n \leq \|v\|_S \leq \|g\|_S/2$. Recall that $id, u, v, g$ are on the same $d_S$-geodesic $P$. This means

$$\|v^{-1} u\|_S = d_S(v, u) = \big|\|v\|_S - \|u\|_S\big| = \Big|\|v\|_S + \big(\|g^{-1} u\|_S - \|g\|_S\big)\Big|.$$

Thanks to $(*)$, we have

$$\Big|\|v\|_S + \big(\|g^{-1} u\|_S - \|g\|_S\big)\Big| \leq \big|2\|v\|_S - \|g\|_S\big| + 2R_0 = \|g\|_S - 2\|v\|_S + 2R_0.$$

Finally, $g^{-1} u$ and $v$ are $2R_0$-close so $u^{-1} g \cdot v \in B_S(2R_0)$. This implies the contradiction

$$g = v \cdot (v^{-1} u) \cdot (u^{-1} g v) \cdot v^{-1} \in \mathcal{K}_1.$$

16

(2) $\|g\|_S/2 \le \|v\|_S \le \|g\|_S - \epsilon n$. In this case, $(*)$ implies that

$$\|u\|_S \le \|g\|_S - \|v\|_S + 2R_0, \quad \|u^{-1}v\|_S \le \big|2\|v\|_S - \|g\|_S\big| + 2R_0 = 2\|v\|_S - \|g\|_S + 2R_0.$$

We also have $u^{-1}g \cdot v^{-1} \in B_S(2R_0)$. Note that $\|g\|_S - \|v\|_S$, $2\|v\|_S - \|g\|_S$ are positive integers whose sum is at most $\|v\|_S \le \|g\|_S - \epsilon n \le n - \epsilon n$. These facts lead to a contradiction

$$g = u \cdot (u^{-1}gv) \cdot (v^{-1}u) \cdot u^{-1} \in \mathcal{K}_2.$$

We can thus conclude that Case (1) holds. Meanwhile, Fact 2.10 asserts that either
  (a) $(h_{2\epsilon n}\Upsilon_L, g^2 x_0)$ is $(6E_0 + 360\delta)$-aligned, or
  (b) $(g^2 x_0, h_{2\epsilon n+1}\Upsilon_L)$ is $(6E_0 + 360\delta)$-aligned.
In Case (b), we are led to the alignment that

$$(g^{\pm 1}x_0, g^{-1}h_{2\epsilon n+1}\Upsilon_L, \ldots, g^{-1}h_{3\epsilon n}\Upsilon_L, x_0) \text{ is } (6E_0 + 360\delta)\text{-aligned.}$$

A similar argument as before implies $g \in \mathcal{K}_1^{-1} \cup \mathcal{K}_2^{-1}$, a contradiction. Hence, Case (a) must hold. In conclusion, the following sequence is $(6E_0 + 360\delta)$-aligned:

$$(g^{-1}x_0, h_{\epsilon n+1}\Upsilon_L, \ldots, h_{2\epsilon n}\Upsilon_L, g^2 x_0)$$

Also, $(x_0, h_{\epsilon n+1}\Upsilon_L)$ and $(h_{2\epsilon n}\Upsilon_L, gx_0)$ are $(6E_0 + 360\delta)$-aligned. Hence $g \in \mathcal{W}_{L,\epsilon}(n)$. $\qquad\square$

We can now finish the proof of Theorem 1.2.

*Proof of Theorem 1.2.* We again start by fixing the constants $E_0$, $\tau$, $K_{Lip}, F_0, L_1$. Take $L \ge L_1$ large enough such that $\tau L \ge M$.

By Lemma 4.5, it suffices to show that for each $\eta > 0$ there exists $\epsilon > 0$ such that

$$(4.3) \qquad \limsup_{n \to +\infty} \frac{\#\big(B_S(n) \setminus \mathcal{W}_{L,\epsilon}(n)\big)}{\#B_S(n)} \le \eta.$$

To this end, we take

$$\epsilon := \frac{1}{30(2E_0 + 4LF_0 + 5)(\#S)^{E_0 + 3LF_0 + 4}} \cdot \eta.$$

Then by Fact 4.1, Lemma 4.3 and Lemma 4.6,

$$\lim_{n \to +\infty} \frac{\#B_S(0.9n)}{\#B_S(n)} = \lim_{n \to +\infty} \frac{\#\big(\mathcal{V}_{L,3\epsilon}(n) \setminus \mathcal{W}_{L,\epsilon}(n)\big)}{\#B_S(n)} = 0,$$

$$\limsup_{n \to +\infty} \frac{\#\mathcal{BAD}_{L,3\epsilon}(n)}{\#B_S(n)} < \eta/2.$$

Moreover, we have

$$B_S(n) \setminus \mathcal{W}_{L,\epsilon}(n) \subseteq B_S(0.9n) \cup \Big(\big(B_S(n) \setminus B_S(0.9n)\big) \setminus \mathcal{W}_{L,\epsilon}(n)\Big)$$

$$\subseteq B_S(0.9n) \cup \Big(B_S(n) \setminus \big(B_S(0.9n) \cup \mathcal{V}_{L,3\epsilon}(n)\big)\Big) \cup \big(\mathcal{V}_{L,3\epsilon}(n) \setminus \mathcal{W}_{L,\epsilon}(n)\big)$$

$$= B_S(0.9n) \cup \mathcal{BAD}_{L,3\epsilon}(n) \cup \big(\mathcal{V}_{L,3\epsilon}(n) \setminus \mathcal{W}_{L,\epsilon}(n)\big).$$

Hence, Equation 4.3 holds. $\qquad\square$

*Proof of Theorem 1.2.* We only list additional observations needed for Theorem 1.2. For detailed explanations about the notion of principal/triangular/ageometric fully irreducible outer automorphism in $\mathrm{Out}(F_n)$, refer to [AKKP19] and [KMPT22].

By [AKKP19, Example 6.1], there exists a principal fully irreducible $\varphi \in \mathrm{Out}(F_N)$. Now [KMPT22, Remark 5.4] provides a *lone axis* $\gamma$ for $\varphi$, which is necessarily a periodic greedy folding line. Further, every fully irreducibe $g \in \mathrm{Out}(F_N)$ has a simple (periodic) folding axis.

Pick a basepoint $x_0 \in \mathcal{FF}_N$. For now, let us denote the projection map from the Outer space $CV_N$ to $\mathcal{FF}$ by $\Pi$. Then [KMPT22, Proposition 8.1] guarantees that:

**Fact 4.7.** *There exists $M_0 > 0$ such that the following holds. If $g$ is a fully irreducible and if the $d_{\mathcal{FF}}$-nearest point projections of $g^{-i}x_0$ and $g^i x_0$ onto $[\varphi^{-i}x_0, \varphi^i x_0]_{\mathcal{FF}}$ is at least $M_0$-apart, the first projection coming first, then $g$ is ageometric and triangular.*

The original [KMPT22, Proposition 8.1] is formulated in terms of $\mathrm{Pr}_\gamma$, but this can be replaced with the $d_{\mathcal{FF}}$-nearest point projection onto $\Pi(\gamma)$ by [DT17, Lemma 4.2]. Furthermore, $[\varphi^{-i}x_0, \varphi^i x_0]_{\mathcal{FF}}$ uniformly fellow travels with subsegments $\Pi(\gamma_i)$ of $\Pi(\gamma)$, where $\gamma_i$ exhausts $\gamma$ as $i$ tends to infinity. This justifies the reformulation.

Given Fact 4.7, we take $M > M_0$ and run the proof of Theorem 1.1: for each $\eta > 0$ there exists $\epsilon > 0$ such that $\mathcal{W}_{L,\epsilon}$ has asymptotic density $\geq 1 - \eta$. For this $\epsilon$, elements of $\mathcal{W}_{L,\epsilon}(n)$ for large enough $n$ satisfy the assumption of Fact 4.7 by Lemma 4.5. Hence, $\mathcal{W}_{L,\epsilon}(n)$ consists of ageometric triangular fully irreducibles for large enough $n$. By shrinking $\eta$, we conclude Theorem 1.2. $\qquad\square$

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