Counting pseudo-Anosovs and fully irreducibles

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53rd Barrett Lectures: Recent Development in Geometric Group Theory

Setup

- G: group
- $\mathcal{P}$: property about group elements

Example

- $G = \mathbb{Z}/4\mathbb{Z}$
- $\mathcal{P}$ = "having order 2"

$$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$$

Finite group

- Basically, we just count.

Infinite group

Example

- $G = \mathbb{Z}$
- $\mathcal{P} =$ "being identity"

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Infinite group

Example

- $G = \mathbb{Z}$
- $\mathcal{P} =$ “being even”

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Infinite group

Example

- $G = \mathbb{Z}$
- $\mathcal{P} =$ "being prime"

$$\mathbb{Z} = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

Infinite group

Setup

- G: group generated by $S \subseteq G$
- $\mathcal{P}$: some property
- $\|\cdot\|_S : G \to \mathbb{Z}_{\geq 0}$ gives an exhaustion
- $B_S(n) := \{g \in G : \|g\|_S \leq n\}$

Problem

$$\lim_{n \to +\infty} \frac{\# B_S(n) \cap \mathcal{P}}{\# B_S(n)} = ?$$

Disclaimer

- Only cares about *finitely generated groups*
- Focuses on the *word metrics*

Missing many important work by Maher, Yang, ...

Groups

$$F_2, \quad \mathrm{Mod}(\Sigma), \quad \mathrm{Out}(F_N), \quad F_2 \times F_3$$

Sorry for: $\pi_1(\Sigma)$, braid groups, CAT(0) (cubical) groups, 3-manifold groups, small cancellation groups, handlebody group, Torelli group, $\mathrm{Aut}(F_N)$, ...

F_2

- Elliptics, ~~parabolics~~, loxodromics

Dani '05, Gekhtman-Taylor-Tiozzo '18

Let S be a finite generating set of F_2. Then

$$\lim_{n \to +\infty} \frac{\#\big(B_S(n) \cap \{\text{loxodromics}\}\big)}{\# B_S(n)} = 1.$$

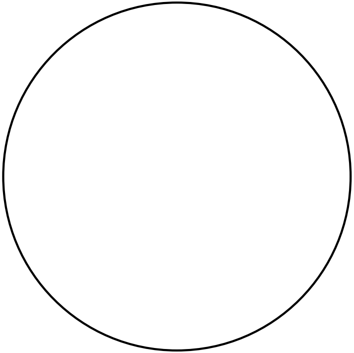
Something like F_2

- $\mathrm{PSL}(2,\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$



Hyperbolic group

- $G \curvearrowright X$, G: hyperbolic, X: hyperbolic
- Elliptics, parabolics, loxodromcis

Dani '05, Gekhtman-Taylor-Tiozzo '18

Let G be a word hyperbolic group with a non-elementary action on a Gromov hyperbolic space X. Let S be a finite generating set of G. Then

$$\lim_{n \to +\infty} \frac{\#\big(B_S(n) \cap \{\text{loxodromics}\}\big)}{\# B_S(n)} = 1.$$

In fact, there exists $\epsilon > 0$ such that

$$\frac{\#\{g \in B_S(n) : \tau_X(g) \geq \epsilon n\}}{\# B_S(n)} \geq 1 - e^{-\epsilon n} \quad (n \gg 1).$$

$F_2 \times F_3$

- $G \curvearrowright X$, G: f.g. group, X: hyperbolic
- Weakly hyperbolic groups (Maher-Tiozzo '18)
- $F_2 \times F_3 \curvearrowright F_2 : (a, b) \cdot c := a \cdot c$

Gekhtman-Taylor-Tiozzo '18 (cf. Kim)

Consider the action of $G = F_2 \times F_3$ on the Cayley graph of $Cay(F_2)$. Then $\exists$ finite generating sets S and S' of G such that

$$\lim_{n \to +\infty} \frac{\#\big(B_S(n) \cap \{\text{loxodromics}\}\big)}{\# B_S(n)} = 1,$$

$$\lim_{n \to +\infty} \frac{\#\big(B_{S'}(n) \cap \{\text{loxodromics}\}\big)}{\# B_{S'}(n)} < 1.$$

Takeaway

1. Genericity problem is not QI-invariant in general.
2. Even when the QI map is G-equivariant, or even when fixing the group and changing the word metrics.

Interlude: $\mathbb{Z} * \mathbb{Z}^2$

1. Right-angled Artin groups, or more generally relatively hyperbolic groups
2. Gekhtman-Taylor-Tiozzo '20, Yang '20
3. More generally, groups with *strongly contracting elements* in the word metric

$\mathrm{Mod}(\Sigma)$

- $\mathrm{Mod}(\Sigma) := \mathrm{Homeo}^{+}(\Sigma) / \mathrm{Homeo}_0(\Sigma)$
- Infinite
- Finitely generated
- Resembles hyperbolic group? (non-amenability, Tits alternative, etc.)
- Not (relatively) hyperbolic (thick)
- HHG?

$\mathrm{Mod}(\Sigma)$

Nielsen-Thurston '40s, '70s

Finite-order / reducible / pseudo-Anosov

Folklore conjecture (Farb '06)

In $\mathrm{Mod}(\Sigma)$, pseudo-Anosovs are generic elements.

Answered by Rivin '07, Maher '11, somehow by C. '24, ~~C. '24~~, C. '24, C. '25

$\mathrm{Out}(F_N)$

- $\mathrm{Out}(F_N) = \mathrm{Aut}(F_N)/\,\mathrm{Inn}(F_N)$
- Resembles hyperbolic group? or $\mathrm{SL}(n, \mathbb{Z})$?
- Not (relatively) hyperbolic, not an HHG

Bestvina-Handel '92

Finite-order / reducible / irreducible / fully irreducible

Out(F_N)

??

Is a typical outer automorphism fully irreducible?

Answered by Sisto '16, Maher-Tiozzo '18 , somehow by C. '25

Best scenario for some S

C. '24

Let G be a finitely generated group with a non-elementary action on a Gromov hyperbolic space X.
Then $\exists$ finite generating set S of G and $\epsilon > 0$ such that

$$\frac{\{g \in B_S(n) : \tau_X(g) \leq \epsilon n\}}{\# B_S(n)} \lesssim e^{-\epsilon n}.$$

Ingredient: ping-pong argument + Gouëzel's pivoting technique '22

$\mathrm{Mod}(\Sigma)$ with any S

C. '25

For any finite generating set S of $\mathrm{Mod}(\Sigma)$, we have

$$\frac{B_S(n) \cap \{\text{non-pseudo-Anosovs}\}}{\#B_S(n)} \lesssim n^{-k}. \quad (k = 1, 2, \ldots)$$

Ingredient: weakly contracting property of pseudo-Anosovs (Behrstock '06, Duchin-Rafi '09)

$\mathrm{Out}(F_N)$ with any S

C. '25

For any finite generating set S of $\mathrm{Out}(F_N)$, we have

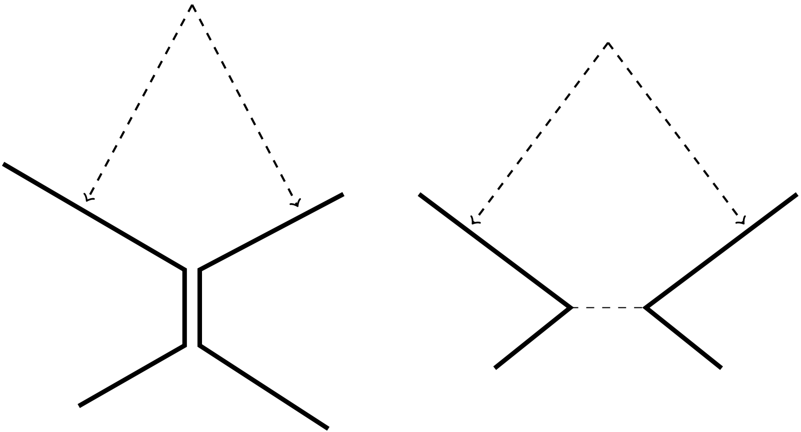
$$\lim_{n \to +\infty} \frac{B_S(n) \cap \{\text{fully irreducibles}\}}{\# B_S(n)} = 1$$

Ingredient: weak proper discontinuity (WPD property) of fully irreducibles (Bestvina-Feighn '10, '14)

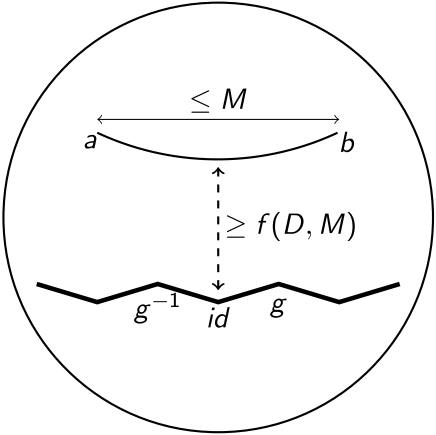
More than 3 theorems

- More specific info: principal stratum/atoroidal/triangular
- Many *subgroups*: Torelli group, handlebody group, ...
- *Group extension*: Braid group (!)
- *Quasi-isometries*: CAT(0) groups, groups QI to (some) 3-manifold groups, ...

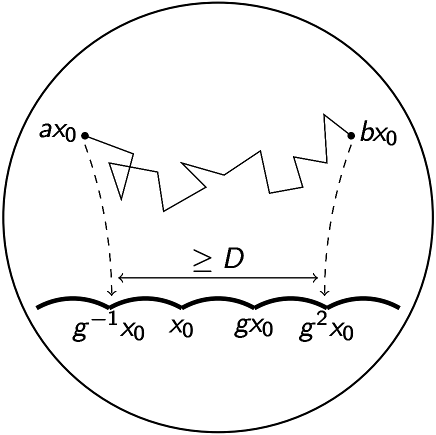
Behrstock's inequality



Geometric separation



$(Cay(G), d_S)$



(X, d_X)

Geometric separation

	X	$f(D, M)$	Density of non-loxodromics
F_2	$Cay(F_2)$	constant in M	$\lesssim \lambda^{-n}$ for some $\lambda > 1$
$\mathrm{Mod}(\Sigma)$	$\mathcal{C}(\Sigma)$	linear in M	$\lesssim n^{-k}$ $(\forall k)$
$\mathrm{Out}(F_N)$	$\mathcal{FF}_N$	finite	tends to 0
$F_2 \times F_3$	$Cay(F_2)$	$+\infty$	can be bounded away from 0

Questions

What is $f(D, M)$ for $\mathrm{Out}(F_N)$? for other AHG?

Are pseudo-Anosovs exponentially generic in $\mathrm{Mod}(\Sigma)$?

Are Morse elements generic in hyperbolic-like groups?

Thank you very much!