

# CLASSIFICATION OF HOROSPHERICAL INVARIANT MEASURES IN HIGHER RANK: THE FULL STORY

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**ABSTRACT.** In this paper, we classify horospherical invariant Radon measures for Anosov subgroups of arbitrary semisimple real algebraic groups. This generalizes the works of Burger and Roblin in rank one to higher ranks. At the same time, this extends the works of Furstenberg, Veech, and Dani, and a special case of Ratner’s theorem for finite-volume homogeneous spaces to infinite-volume Anosov homogeneous spaces.

Especially, this resolves the open problems proposed by Landesberg–Lee–Lindenstrauss–Oh and by Oh. Our measure classification is in fact for a more general class of discrete subgroups, including relatively Anosov subgroups with respect to any parabolic subgroups, not necessarily minimal. Our method is rather geometric, not relying on continuous flows or ergodic theorems.

## DECLARATION

Prior to this paper, we wrote a paper [CK25a] dealing with the case that the ambient Lie group is a product of rank one groups (in fact for products of CAT( $-1$ )-spaces), which is the setting originally asked by Landesberg–Lee–Lindenstrauss–Oh and Oh. As many parts of the argument in this paper are simplified in such a case, [CK25a] might be more accessible to readers who are not familiar with homogeneous dynamics but have some geometry background.

In this regard, we decided to keep [CK25a] public on arXiv and our websites, but **not to publish [CK25a] in any journal**. The paper [CK25a] can be considered as a teaser for this paper.

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## 1. INTRODUCTION

Given a dynamical system, classifying invariant measures is a natural and important question with many applications, as also indicated by the celebrated theorem of Ratner [Rat91] for instance. We study this question for dynamical systems given by *horospherical actions*.

Let  $G$  be a connected semisimple real algebraic group and  $P < G$  its minimal parabolic subgroup with a fixed Langlands decomposition  $P = MAN$ , where  $A$  is a maximal real split torus of  $G$ ,  $M < P$  is a maximal compact subgroup commuting with  $A$ , and  $N$  is the unipotent radical of  $P$ .

Let  $\Gamma < G$  be a Zariski dense discrete subgroup. The right multiplication of  $NM$  on  $\Gamma \backslash G$  is called (maximal) *horospherical action*. For a uniform lattice  $\Gamma < G$ , the  $NM$ -action on  $\Gamma \backslash G$  is uniquely ergodic,<sup>1</sup> with the Haar measure for  $G$  as the ergodic measure. This was first shown for  $G = \mathrm{PSL}(2, \mathbb{R})$  by Furstenberg [Fur73], and by Veech [Vee77] in general. When  $\Gamma < G$  is a non-uniform lattice, Dani classified all  $NM$ -invariant ergodic Radon measures on  $\Gamma \backslash G$  ([Dan78], [Dan81]). In fact, all results above hold for the  $N$ -action as well, and they are generalized further by Ratner [Rat91] to actions generated by unipotent elements, which is now called Ratner's theorem.

In contrast, the case that  $\Gamma < G$  is not a lattice, i.e.,  $\Gamma$  has infinite co-volume, has remained mysterious for general  $G$ . For a non-lattice Zariski dense discrete  $\Gamma < G$ , a natural  $NM$ -invariant subspace of  $\Gamma \backslash G$  that has been considered is

$$\text{the unique } P\text{-minimal set } \mathcal{E}_\Gamma \subset \Gamma \backslash G,$$

whose uniqueness for higher-rank  $G$  is due to Benoist [Ben97]. Indeed,  $\mathcal{E}_\Gamma$  consists of  $[g] \in \Gamma \backslash G$  whose “forward endpoint”  $gP$  in the Furstenberg boundary  $\mathcal{F} := G / P$  belongs to the limit set  $\Lambda(\Gamma) \subset \mathcal{F}$ , the unique  $\Gamma$ -minimal set. When  $G$  is of rank one, it is easy to see this picture and that any  $NM$ -orbit in  $(\Gamma \backslash G) \setminus \mathcal{E}_\Gamma$  is closed.

The first classification of horospherical invariant measure in this infinite-volume setting is due to Burger [Bur90], who considered the case that  $G = \mathrm{PSL}(2, \mathbb{R})$  and  $\Gamma < G$  is convex cocompact whose limit set  $\Lambda(\Gamma)$  has Hausdorff dimension strictly bigger than  $1/2$ . Burger proved the unique ergodicity of the  $NM$ -action on  $\mathcal{E}_\Gamma$  in this case. For a general rank one group  $G$  and  $\Gamma < G$  geometrically finite, Roblin classified all  $NM$ -invariant ergodic Radon measures on  $\mathcal{E}_\Gamma$  [Rob03], by showing that there exists an  $NM$ -invariant Radon measure  $\mu_\Gamma$  on  $\mathcal{E}_\Gamma$  so that any  $NM$ -invariant ergodic Radon measure on  $\mathcal{E}_\Gamma$  is either supported on a closed  $NM$ -orbit, or a constant multiple of  $\mu_\Gamma$ . The measure  $\mu_\Gamma$  is now called

$$\text{Burger-Roblin measure of } \Gamma.$$

Later, Winter showed that the Burger-Roblin measure is  $N$ -ergodic and extended the Roblin's classification to  $N$ -invariant Radon measures [Win15].

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<sup>1</sup>By unique ergodicity, we mean that there exists a unique invariant ergodic Radon measure up to a constant multiple.

For geometrically infinite cases, Babillot and Ledrappier first discovered that there may be continuous family of NM-invariant ergodic Radon measures ([Bab04], [BL98]); see also ([Sar04], [Sar10], [Led08], [LS07], [Win15], [OP19], [LL22], [Lan21], [LLLO23]) for partial classification results in the rank-one case.

We now move to the case that

$$G \text{ is of higher rank.}$$

Edwards–Lee–Oh extended the notion of Burger–Roblin measure to higher rank, introducing higher-rank Burger–Roblin measures [ELO23]. Their ergodicity with respect to horospherical actions was proved for Zariski dense Borel Anosov subgroups, higher-rank analogues of convex cocompact subgroups, by Lee–Oh [LO23]. For a larger class of discrete subgroups including relatively Anosov subgroups, higher-rank analogues of geometrically finite subgroups, the ergodicity was shown by the second author [Kim24].

On the other hand, even for Zariski dense Borel Anosov subgroups, there was no single specific example that the horospherical invariant measures are classified, except our teaser paper [CK25a]. A work related to this direction is by Landesberg–Lee–Lindenstrauss–Oh [LLLO23].

**1.1. Work of Landesberg–Lee–Lindenstrauss–Oh and Problems.** As we are answering an open problem proposed in [LLLO23] which has been motivational about the classification of horospherical invariant measures in higher rank, we briefly describe their work. They considered a specific group

$$G := \prod_{i=1}^r G_i$$

where  $G_i$  is a simple real algebraic group of rank one. In this case, we have  $r = \text{rank } G$ . (We will consider *arbitrary*  $G$ , in contrast.)

Even for the case that  $G$  is a product of rank one groups as above, the classification of horospherical invariant measure was not known before our teaser paper [CK25a]. Instead, in [LLLO23], they considered the directionally recurrent set in  $\Gamma \backslash G$  for each *1-dimensional diagonal flow* (or, directional flow). More precisely, denote by  $\mathfrak{a} := \text{Lie } A$  and fix a closed Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$  so that for each  $n \in \mathbb{N}$ , the set  $\{a^{-1}na : a \in \exp \mathfrak{a}^+\} \subset N$  is precompact. Then for each  $v \in \text{int } \mathfrak{a}^+$ , they considered the set  $\mathcal{R}_{\Gamma, v} \subset \mathcal{E}_\Gamma$  consisting of elements each of whose 1-dimensional  $\exp(\mathbb{R}_{>0}v)$ -orbit is recurrent to a compact subset.

Their main result is about the horospherical action restricted on  $\mathcal{R}_{\Gamma, v} \subset \mathcal{E}_\Gamma$ , for  $\Gamma$  Zariski dense Borel Anosov. In the setting of product of rank-one Lie groups,  $\Gamma < G$  is Borel Anosov if the projection  $\Gamma \rightarrow G_i$  has finite kernel and convex cocompact image for all  $1 \leq i \leq r$ . Based on the ergodicity results of Lee–Oh ([LO23], [LO24]) and Burger–Landesberg–Lee–Oh [BLLO23], the rigidity result of [LLLO23] is as follows:

**Theorem 1.1** ([LLLO23]). *Suppose that  $G$  is a product of rank one groups. Let  $\Gamma < G$  be a Zariski dense Borel Anosov subgroup and  $v \in \text{int } \mathfrak{a}^+$ . Let  $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$  denote the limit cone<sup>2</sup> of  $\Gamma$ .*

- (1) *For  $r \leq 3$  and  $v \in \text{int } \mathcal{L}_\Gamma$ , the  $N$ -action and the  $NM$ -action on  $\mathcal{R}_{\Gamma,v} \subset \mathcal{E}_\Gamma$  are uniquely ergodic.*
- (2) *For  $r > 3$  or  $v \notin \text{int } \mathcal{L}_\Gamma$ , there exists no non-zero,  $N$ -invariant or  $NM$ -invariant measure Radon measure supported on  $\mathcal{R}_{\Gamma,v} \subset \mathcal{E}_\Gamma$ .*

The ergodic measures in (1) above are higher-rank Burger–Roblin measures, whose ergodicity was proved in [LO24], and being supported on the directionally recurrent set was proved in [BLLO23]. Delaying their definitions, we note that in contrast to rank-one settings, they come as a family of mutually singular measures, because higher-rank Patterson–Sullivan measures do so. The reason for the rank dichotomy in Theorem 1.1(2) is that  $\mathcal{R}_{\Gamma,v}$  has zero Burger–Roblin measures when  $r > 3$  [BLLO23].

In view of Theorem 1.1, the following open problem was proposed by Landesberg–Lee–Lindenstrauss–Oh, towards classifying horospherical invariant measures.

**Problem 1.2** ([LLLO23, Open problem 1.8]). *Let  $\Gamma < G$  be a Zariski dense Borel Anosov subgroup and suppose  $\text{rank } G \leq 3$ . Is any  $N$ -invariant ergodic Radon measure on  $\mathcal{E}_\Gamma$  supported on  $\mathcal{R}_{\Gamma,v}$  for some  $v \in \text{int } \mathfrak{a}^+$ ?*

More generally, in a very recent preprint for the Proceedings of the ICM 2026, Oh asked for horospherical measure classification for Anosov subgroups without any assumption on rank  $G$ .

**Problem 1.3** ([Oh25, Section 8.2]). *Let  $\Gamma < G$  be a Zariski dense Borel Anosov subgroup. Is any  $N$ -invariant ergodic Radon measure on  $\mathcal{E}_\Gamma$  a Burger–Roblin measure?*

Main results of this paper are resolvents of Problem 1.2 and Problem 1.3, open problems proposed by Landesberg–Lee–Lindenstrauss–Oh in [LLLO23] and by Oh in [Oh25]. Indeed, we give a complete classification of horospherical invariant measures.

*Remark 1.4.* Before stating our main results, we remark the following regarding our setup:

- While products of rank one groups are considered above, we deal with general semisimple real algebraic groups. Hence, in the rest of this paper,  $G$  is an *arbitrary semisimple real algebraic group*.
- We also consider Anosov and relatively Anosov subgroups with respect to *any parabolic subgroup*, not necessarily Borel. In fact, we consider a more general class of discrete subgroups.

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<sup>2</sup>The limit cone of  $\Gamma$  is the asymptotic cone of the Cartan projections of  $\Gamma$  in  $\mathfrak{a}$ . We will revisit this later.

- Unless  $G$  is a product of rank one groups, higher-rank Burger–Roblin measures are not  $N$ -ergodic in general, as shown by ergodic decomposition results ([LO24], [Kim24]). In this perspective, we classify  $NM$ -invariant ergodic Radon measures on  $\mathcal{E}_\Gamma$ , and deduce the classification for the  $N$ -action in the case of product of rank one groups, based on the ergodic decomposition of Burger–Roblin measures ([LO24], [Kim24]).

1.2. **Main results for Borel Anosov subgroups.** From now on,

let  $G$  be an **arbitrary** connected semisimple real algebraic group.

For simplicity, we first present our main results for Borel Anosov subgroups as special cases, which resolve the open problems proposed by Landesberg–Lee–Lindenstrauss–Oh (Problem 1.2) and by Oh (Problem 1.3).

An *Anosov subgroup* is a higher-rank generalization of convex cocompact subgroups, introduced by Labourie [Lab06] for surface groups and generalized by Guichard–Wienhard [GW12] for hyperbolic groups. Delaying some precise definitions, we first state our classification as follows:

**Theorem 1.5.** *Let  $\Gamma < G$  be a Zariski dense Borel Anosov subgroup. If  $\mu$  is an  $NM$ -invariant ergodic Radon measure on  $\mathcal{E}_\Gamma$ , then*

*$\mu$  is a constant multiple of a Burger–Roblin measure of  $\Gamma$ .*

Let  $P^\circ$  denote the identity component of  $P$ . When the  $P^\circ$ -action on  $\mathcal{E}_\Gamma$  is minimal (e.g.  $G$  is a product of rank one groups), the  $NM$ -action in Theorem 1.5 can be replaced with the  $N$ -action, due to the ergodic decomposition of Burger–Roblin measures in [LO24].

**Theorem 1.6.** *Let  $\Gamma < G$  be a Zariski dense Borel Anosov subgroup and suppose that the  $P^\circ$ -action on  $\mathcal{E}_\Gamma$  is minimal. If  $\mu$  is an  $N$ -invariant ergodic Radon measure on  $\mathcal{E}_\Gamma$ , then  $\mu$  is a constant multiple of a Burger–Roblin measure of  $\Gamma$ .*

*Remark 1.7.* We note the following regarding Theorem 1.5 and Theorem 1.6.

- (1) If the  $P^\circ$ -action on  $\mathcal{E}_\Gamma$  is not minimal, then Burger–Roblin measures are not  $N$ -ergodic [LO24].
- (2) When  $G$  is a product of rank one groups, it is easy to see that any  $NM$ -orbit in  $(\Gamma \backslash G) \setminus \mathcal{E}_\Gamma$  is closed. On the other hand, this does not seem to hold for general  $G$  (see Remark 4.2).

We now define Borel Anosov subgroups and Burger–Roblin measures. A Zariski dense discrete subgroup  $\Gamma < G$  is called *Borel Anosov* if

- $\Gamma$  is a hyperbolic group, and
- there exists a  $\Gamma$ -equivariant embedding  $f : \partial\Gamma \rightarrow \mathcal{F}$  such that for any distinct  $x, y \in \partial\Gamma$ ,  $f(x)$  and  $f(y)$  are in *general position*, i.e., the diagonal  $G$ -orbit  $G \cdot (f(x), f(y))$  is open in  $\mathcal{F} \times \mathcal{F}$ , where  $\mathcal{F} = G / P$  is the Furstenberg boundary.

**Example 1.8.** A famous example of Borel Anosov subgroup is the so-called *Hitchin subgroup*, as shown by Labourie [Lab06]. For a closed connected orientable surface  $S$  of genus at least two and  $d \geq 3$ , a representation

$$\pi_1(S) \rightarrow \mathrm{PSL}(d, \mathbb{R})$$

is called a Hitchin representation if it belongs to the same component as the composition  $\pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(d, \mathbb{R})$  in the character variety  $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(d, \mathbb{R}))/\sim$ , for a cocompact  $\pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$  and irreducible  $\mathrm{PSL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(d, \mathbb{R})$ . Hitchin subgroups are images of Hitchin representations.

We note the for a Hitchin subgroups  $\Gamma < G$ , the  $P^\circ$ -action on  $\mathcal{E}_\Gamma$  is not minimal [Lab06, Theorem 1.5].

To define Burger–Roblin measures, recall that  $\mathfrak{a}^+$  is a fixed positive Weyl chamber in  $\mathfrak{a} = \mathrm{Lie} A$ , and fix a maximal compact subgroup  $K < G$  so that the Cartan decomposition  $G = K(\exp \mathfrak{a}^+)K$  holds. Then we have the Furstenberg boundary

$$\mathcal{F} = K/M = G/P.$$

Let  $\Gamma < G$  be a Zariski dense discrete subgroup. For  $\delta \geq 0$  and a linear form  $\psi \in \mathfrak{a}^*$ , a Borel probability measure  $\nu$  on  $\mathcal{F}$  is called a  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure of  $\Gamma$  if

$$(1.1) \quad \frac{dg_*\nu}{d\nu}(x) = e^{-\delta \cdot \psi(\sigma(g^{-1}, x))} \quad \text{a.e.}$$

where  $\sigma$  is the Iwasawa cocycle, i.e.,  $\sigma(g^{-1}, x) \in \mathfrak{a}$  is the unique element satisfying  $g^{-1}k \in K(\exp \sigma(g^{-1}, x))N$ , for  $k \in K$  with  $x = kM$  in  $\mathcal{F}$ . This notion of Patterson–Sullivan measures was first introduced by Quint [Qui02b], generalizing the classical Patterson–Sullivan theory to higher rank. We simply call  $\nu$  Patterson–Sullivan measure of  $\Gamma$  if it is  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure for some  $\delta \geq 0$  and  $\psi \in \mathfrak{a}^*$ .

In [ELO23], Edwards–Lee–Oh extended the classical Burger–Roblin measure to higher rank. For a  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure  $\nu$  of  $\Gamma$  on  $\mathcal{F}$ , the (higher-rank) *Burger–Roblin measure* associated to  $\nu$  is the Radon measure  $\mu_\nu^{\mathrm{BR}}$  on  $\Gamma \backslash G$  induced by the  $\Gamma$ -invariant measure  $\tilde{\mu}_\nu^{\mathrm{BR}}$  on  $G$  defined as follows: for  $g = k(\exp u)n \in K(\exp \mathfrak{a})N$  in Iwasawa decomposition of  $G$ ,

$$d\tilde{\mu}_\nu^{\mathrm{BR}}(g) := e^{\delta \cdot \psi(u)} d\tilde{\nu}(k) du dn$$

where  $\tilde{\nu}$  is the  $M$ -invariant lift of  $\nu$  to  $K$  and  $du$  and  $dn$  are Lebesgue measures on  $\mathfrak{a}$  and  $N$  respectively. The measure  $\mu_\nu^{\mathrm{BR}}$  is  $NM$ -invariant.

Recall that the limit set  $\Lambda(\Gamma) \subset \mathcal{F}$  of  $\Gamma$  is the unique  $\Gamma$ -minimal subset [Ben97] and

$$\mathcal{E}_\Gamma = \{[g] \in \Gamma \backslash G : g P \in \Lambda(\Gamma)\}.$$

Hence,  $\mu_\nu^{\mathrm{BR}}$  is supported on  $\mathcal{E}_\Gamma$  if and only if  $\nu$  is supported on  $\Lambda(\Gamma)$ , and in this case, the  $NM$ -ergodicity was proved by Lee–Oh [LO23]. As a corollary of Theorem 1.5 and Theorem 1.6, we conclude that Burger–Roblin measures are all such ergodic measures.

**Corollary 1.9.** *Let  $\Gamma < G$  be a Zariski dense Borel Anosov subgroup. Then the following sets are the same, up to constant multiples:*

- (1)  $\{\mu_\nu^{\text{BR}} : \nu \text{ is a Patterson-Sullivan measure of } \Gamma \text{ on } \Lambda(\Gamma)\}.$
- (2) *the set of all NM-invariant ergodic Radon measures on  $\mathcal{E}_\Gamma$ .*

*Suppose further that  $\mathcal{E}_\Gamma$  is  $P^\circ$ -minimal. Then the above sets are the same, up to constant multiples, as*

- (3) *the set of all N-invariant ergodic Radon measures on  $\mathcal{E}_\Gamma$ .*

The set of NM-ergodic measures in Corollary 1.9 can be described more explicitly. Denote by  $\kappa : G \rightarrow \mathfrak{a}^+$  the Cartan projection, defined by the condition  $g \in K(\exp \kappa(g))K$  for all  $g \in G$ . The *limit cone*  $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$  of  $\Gamma$  is the asymptotic cone of Cartan projections  $\kappa(\Gamma)$ . Benoist showed that if  $\Gamma$  is Zariski dense,  $\mathcal{L}_\Gamma$  is convex and has non-empty interior [Ben97]. For a Zariski dense Borel Anosov subgroup  $\Gamma < G$ , Lee–Oh classified Patterson–Sullivan measures of  $\Gamma$  on  $\Lambda(\Gamma)$  in [LO23], and provided a natural homeomorphism

$$(1.2) \quad \mathbb{P}(\text{int } \mathcal{L}_\Gamma) \longleftrightarrow \{\mu_\nu^{\text{BR}} : \nu \text{ a Patterson-Sullivan measure of } \Gamma \text{ on } \Lambda(\Gamma)\}$$

constructed using tangencies of the growth indicator of  $\Gamma$ , introduced by Quint [Qui02a]. Corollary 1.9 is now rephrased as follows:

**Corollary 1.10.** *Let  $\Gamma < G$  be a Zariski dense Borel Anosov subgroup. Then the homeomorphism in Equation (1.2) becomes a homeomorphism*

$$\text{int } \mathcal{L}_\Gamma \longleftrightarrow \left\{ \begin{array}{c} \text{NM-invariant ergodic} \\ \text{non-zero Radon measures on } \mathcal{E}_\Gamma \end{array} \right\}.$$

*Suppose further that  $\mathcal{E}_\Gamma$  is  $P^\circ$ -minimal. Then the homeomorphism in Equation (1.2) becomes homeomorphisms among*

$$\begin{array}{ccc} & \left\{ \begin{array}{c} \text{NM-invariant ergodic} \\ \text{non-zero Radon measures on } \mathcal{E}_\Gamma \end{array} \right\} & \\ \text{int } \mathcal{L}_\Gamma & \begin{array}{c} \xleftarrow{\hspace{1cm}} \\ \xleftarrow{\hspace{1cm}} \end{array} & \updownarrow \\ & \left\{ \begin{array}{c} \text{N-invariant ergodic} \\ \text{non-zero Radon measures on } \mathcal{E}_\Gamma \end{array} \right\} & \end{array}$$

*In particular, all the sets are homeomorphic to  $\mathbb{R}^{\text{rank } G}$ .*

Combining with the work of Burger–Landesberg–Lee–Oh [BLLO23], our measure classification implies the following.

**Corollary 1.11.** *Suppose  $\text{rank } G \leq 3$  and let  $\Gamma < G$  be a Zariski dense Borel Anosov subgroup. Then the following holds.*

- (1) *Any NM-invariant ergodic Radon measure on  $\mathcal{E}_\Gamma$  is supported on  $\mathcal{R}_{\Gamma,v}$  for some  $v \in \text{int } \mathcal{L}_\Gamma$ .*
- (2) *If  $\mathcal{E}_\Gamma$  is  $P^\circ$ -minimal, then any N-invariant ergodic Radon measure on  $\mathcal{E}_\Gamma$  is supported on  $\mathcal{R}_{\Gamma,v}$  for some  $v \in \text{int } \mathcal{L}_\Gamma$ .*

Note that  $\mathcal{E}_\Gamma$  is  $\mathsf{P}^\circ$ -minimal when  $\mathsf{G}$  is a product of rank one groups. Then Corollary 1.11 resolves Problem 1.2, the open problem proposed by Landesberg–Lee–Lindenstrauss–Oh [LLLO23, Open problem 1.8]. Similarly, as Corollary 1.9 and Corollary 1.10 do not have any rank assumption, they resolve Problem 1.3, which was proposed by Oh in [Oh25, Section 8.2].

**1.3. Relatively Borel Anosov subgroups.** Before discussing the most general version of our measure classification, we state the classification for relatively Borel Anosov subgroups, due to its specific feature distinguished from Anosov subgroups and the most general case.

Relatively Anosov subgroups are higher-rank generalization of geometrically finite subgroups in rank one. A discrete subgroup  $\Gamma < \mathsf{G}$  is called *relatively Borel Anosov* if

- for any infinite sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \Gamma$  and a simple root  $\alpha$ ,
- $$\alpha(\kappa(g_n)) \rightarrow +\infty \quad \text{as } n \rightarrow +\infty,$$
- $\Gamma$  is a relatively hyperbolic group with respect to some peripheral structure, and
  - there exists a  $\Gamma$ -equivariant embedding  $f : \partial_B \Gamma \rightarrow \mathcal{F}$  such that any distinct  $x, y \in \partial_B \Gamma$ ,  $f(x)$  and  $f(y)$  are in general position, where  $\partial_B \Gamma$  is the associated Bowditch boundary.

For Zariski dense relatively Borel Anosov subgroups, the NM-ergodicity of Burger–Roblin measures was proved by the second author [Kim24]. Their ergodic decomposition for the N-action was also shown in [Kim24]. As a special case of our result, we obtain that those are essentially all possible ergodic measures.

**Theorem 1.12.** *Let  $\Gamma < \mathsf{G}$  be a Zariski dense relatively Borel Anosov subgroup. If  $\mu$  is an NM-invariant ergodic Radon measure on  $\mathcal{E}_\Gamma$ , then either*

- (1)  $\mu$  is a constant multiple of a Burger–Roblin measure of  $\Gamma$ , or
- (2)  $\mu$  is supported on a closed NM-orbit in  $\mathcal{E}_\Gamma$ .

When  $\mathcal{E}_\Gamma$  is  $\mathsf{P}^\circ$ -minimal, the same holds for N-invariant ergodic Radon measures as well.

As a relative version of Hitchin subgroups in Example 1.8, there is a notion of cusped Hitchin subgroups, and they are relatively Borel Anosov subgroups of  $\mathrm{PSL}(d, \mathbb{R})$ ,  $d \geq 3$ . See the work of Canary–Zhang–Zimmer [CZZ22] for more details.

**1.4. Measure classification in the full generality.** In the rest of the introduction, we present the most general version of our horospherical invariant measure classification result.

**1.4.1. Horospherical foliations.** Let  $\Delta$  be the set of all simple roots associated to  $\mathfrak{a}^+$ . For a non-empty subset  $\theta \subset \Delta$ , let  $\mathsf{P}_\theta < \mathsf{G}$  be the standard parabolic subgroup corresponding to  $\theta$ , i.e.,  $\mathsf{P}_\theta$  is generated by MA and all

root subgroups for all positive roots and any negative root which is a  $\mathbb{Z}$ -linear combination of  $\Delta \setminus \theta$ . Let  $N_\theta < P_\theta$  be the unipotent radical of  $P_\theta$  and let  $L_\theta < P_\theta$  be a Levi subgroup with the Levi decomposition  $P_\theta = L_\theta N_\theta$ . Then  $L_\theta = (\exp \mathfrak{a}_\theta) S_\theta$ , where  $\mathfrak{a}_\theta := \bigcap_{\alpha \in \Delta \setminus \theta} \ker \alpha$  and  $S_\theta$  is an almost direct product of a connected semisimple real algebraic subgroup and a compact subgroup.

Using them, we define the  $\theta$ -horospherical foliation of  $G$  by

$$\mathcal{H}_\theta := G / N_\theta S_\theta$$

so that each element of  $\mathcal{H}_\theta$  corresponds to a single  $N_\theta S_\theta$ -orbit in  $G$ . Then  $G$  acts on  $\mathcal{H}_\theta$  by left multiplication.

When  $\theta = \Delta$ , we have  $P_\Delta = P$ ,  $N_\Delta = N$ , and  $S_\Delta = M$ . In particular, in this case, the  $\Gamma$ -action on  $\mathcal{H}_\Delta$  for a discrete  $\Gamma < G$  is dual to the  $NM$ -action on  $\Gamma \backslash G$ . Hence, the  $\theta$ -horospherical foliation of  $G$  generalizes the picture of the horospherical action on homogeneous spaces.

**1.4.2. Burger–Roblin measures.** We now define Burger–Roblin measures on  $\mathcal{H}_\theta$ , employing the following identification of  $\mathcal{H}_\theta$ . Let

$$\mathcal{F}_\theta := G / P_\theta$$

be the  $\theta$ -boundary of  $G$ . Then there exists a homeomorphism

$$\mathcal{H}_\theta \rightarrow \mathcal{F}_\theta \times \mathfrak{a}_\theta$$

so that the  $G$ -action on  $\mathcal{H}_\theta$  descends to the  $G$ -action on  $\mathcal{F}_\theta \times \mathfrak{a}_\theta$  given by

$$g \cdot (x, u) := (gx, u + \sigma_\theta(g, x)) \quad \text{for } g \in G, (x, u) \in \mathcal{F}_\theta \times \mathfrak{a}_\theta,$$

where  $\sigma_\theta$  is the partial Iwasawa cocycle. See Section 4.2 and Section 4.3 for details.

Let  $\Gamma < G$  be a discrete subgroup. For  $\delta \geq 0$  and a linear form  $\psi \in \mathfrak{a}_\theta^*$ , a  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure  $\nu$  of  $\Gamma$  on  $\mathcal{F}_\theta$  is defined similarly as in Equation (1.1) (see Equation (4.4) for the precise definition). Then the associated *Burger–Roblin measure* on  $\mathcal{H}_\theta$  is the  $\Gamma$ -invariant measure defined by

$$d\mu_\nu^{\text{BR}}(x, u) := e^{\delta \cdot \psi(u)} d\nu(x) du$$

abusing notations, using the identification  $\mathcal{H}_\theta = \mathcal{F}_\theta \times \mathfrak{a}_\theta$ . Note also that  $A_\theta$  acts on  $\mathcal{H}_\theta = G / N_\theta S_\theta$  by right multiplication. This descends to the translation action on the  $\mathfrak{a}_\theta$ -component of  $\mathcal{F}_\theta \times \mathfrak{a}_\theta$ , and the measure  $\mu_\nu^{\text{BR}}$  is quasi-invariant under this action.

In our discussion, Burger–Roblin measures associated to Patterson–Sullivan measure of divergence type are main objects. The critical exponent of  $\psi \in \mathfrak{a}_\theta^*$  is defined by  $\delta_\psi(\Gamma) := \inf\{s > 0 : \sum_{g \in \Gamma} e^{-s\psi(\kappa(g))} < +\infty\}$  where we identify  $\mathfrak{a}_\theta^*$  with a subspace of  $\mathfrak{a}^*$ , by precomposing the orthogonal projection  $\mathfrak{a} \rightarrow \mathfrak{a}_\theta$ . We call a  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure  $\nu$  on  $\mathcal{F}_\theta$  *divergence-type* if  $\delta = \delta_\psi(\Gamma)$  and  $\sum_{g \in \Gamma} e^{-\delta_\psi(\Gamma)\psi(\kappa(g))} = +\infty$ .

**1.4.3. Hypertransverse subgroups.** A class of discrete subgroups we consider in this paper is called hypertransverse subgroups, introduced in [Kim24]. This notion extends rank one discrete subgroups and includes Anosov subgroups, relatively Anosov subgroups, and their subgroups. In addition, hypertransverse subgroups are special types of transverse subgroups, which are introduced and studied by Canary–Zhang–Zimmer [CZZ24].

We denote by  $i : \Delta \rightarrow \Delta$  the opposition involution, i.e.,  $\alpha(\kappa(g^{-1})) = i(\alpha)(\kappa(g))$  for all  $g \in G$  and  $\alpha \in \Delta$ . A Zariski dense discrete subgroup  $\Gamma < G$  is called  $P_\theta$ -transverse if

- for any infinite sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \Gamma$  and  $\alpha \in \theta$ ,

$$\alpha(\kappa(g_n)) \rightarrow +\infty \quad \text{as } n \rightarrow +\infty,$$

and

- any distinct  $x, y \in \Lambda_{\theta \cup i(\theta)}(\Gamma) \subset \mathcal{F}_{\theta \cup i(\theta)}$  are in general position, i.e., the diagonal  $G$ -orbit  $G \cdot (x, y)$  is open in  $\mathcal{F}_{\theta \cup i(\theta)} \times \mathcal{F}_{\theta \cup i(\theta)}$ .

A  $P_\theta$ -transverse  $\Gamma < G$  is called  $P_\theta$ -hypertransverse if we further have that

- there exists a proper geodesic Gromov hyperbolic space  $Z$  on which  $\Gamma$  acts properly discontinuously by isometries, and
- there exists a  $\Gamma$ -equivariant homeomorphism  $\Lambda_Z(\Gamma) \rightarrow \Lambda_\theta(\Gamma)$  where  $\Lambda_Z(\Gamma) \subset \partial Z$  is the limit set of  $\Gamma$ .

When  $\Gamma$  is a hyperbolic group and  $Z$  can be taken as the Cayley graph of  $\Gamma$ ,  $\Gamma$  is called  $P_\theta$ -Anosov. When  $\Gamma$  is a relatively hyperbolic group and  $Z$  can be taken as the Gromov model of  $\Gamma$ ,  $\Gamma$  is called relatively  $P_\theta$ -Anosov.

An important property of  $P_\theta$ -hypertransverse  $\Gamma$  is that the  $\Gamma$ -action on  $\Lambda_\theta(\Gamma)$  is a convergence action, which also holds for transverse subgroups [KLP17]. Hence, the notion of *conical limit set*  $\Lambda_\theta^{\text{con}}(\Gamma) \subset \Lambda_\theta(\Gamma)$  can be defined in terms of the convergence action. If  $\Gamma$  is  $P_\theta$ -Anosov, then  $\Lambda_\theta^{\text{con}}(\Gamma) = \Lambda_\theta(\Gamma)$ . If  $\Gamma$  is relatively  $P_\theta$ -Anosov, then  $\Lambda_\theta(\Gamma)$  is a disjoint union of  $\Lambda_\theta^{\text{con}}(\Gamma)$  and its *parabolic limit set*, which we denote by  $\Lambda_\theta^P(\Gamma) \subset \Lambda_\theta(\Gamma)$ .

**1.4.4. Measure classification.** For a Zariski dense  $P_\theta$ -hypertransverse  $\Gamma < G$ , we define the *recurrence locus* by

$$\mathcal{R}_{\Gamma, \theta} := \left\{ [g] \in \mathcal{H}_\theta : \begin{array}{l} \exists \text{ infinite sequences } \{g_n\}_{n \in \mathbb{N}} \subset \Gamma, \{u_n\}_{n \in \mathbb{N}} \subset \mathfrak{a}^+ \cap \mathfrak{a}_\theta \\ \text{s.t. the sequence } g_n[g] \exp u_n \in \mathcal{H}_\theta \text{ is precompact.} \end{array} \right\}$$

Under the identification  $\mathcal{H}_\theta = \mathcal{F}_\theta \times \mathfrak{a}_\theta$ , we have

$$\mathcal{R}_{\Gamma, \theta} = \Lambda_\theta^{\text{con}}(\Gamma) \times \mathfrak{a}_\theta$$

(see [KOW25a, Lemma 5.5]), and when  $\theta = \Delta$ , the recurrence locus  $\mathcal{R}_{\Gamma, \Delta}$  corresponds to the set of elements in  $\Gamma \backslash G$  whose  $\exp \mathfrak{a}^+$ -orbits are recurrent to compact subsets, which is also a subset of  $\mathcal{E}_\Gamma$ . Note that in this sense,  $\mathcal{R}_{\Gamma, \Delta}$  contains all directional recurrent set  $\mathcal{R}_{\Gamma, v}$ ,  $v \in \text{int } \mathfrak{a}^+$ , considered in [LLLO23].

We now state the most general version of our measure classification result. By the higher-rank Hopf–Tsujii–Sullivan dichotomy for transverse subgroups

([CZZ24], [KOW25b]), the Burger–Roblin measure  $\mu_\nu^{\text{BR}}$  is supported on  $\mathcal{R}_{\Gamma,\theta}$  for a divergence-type Patterson–Sullivan measure  $\nu$  of  $\Gamma$  on  $\mathcal{F}_\theta$ . Moreover, in this case, the second author showed the  $\Gamma$ -ergodicity of  $\mu_\nu^{\text{BR}}$  in [Kim24]. It turns out that they are the only ergodic measures.

**Theorem 1.13.** *Let  $\Gamma < G$  be a Zariski dense  $P_\theta$ -hypertransverse subgroup. Then the following sets are the same, up to constant multiples:*

- (1)  $\{\mu_\nu^{\text{BR}} : \nu \text{ a divergence-type Patterson–Sullivan measure of } \Gamma \text{ on } \mathcal{F}_\theta\}.$
- (2)  $\text{the set of all } \Gamma\text{-invariant ergodic Radon measures on } \mathcal{R}_{\Gamma,\theta}.$

See Theorem 6.1 for the version without Zariski density, but with the non-arithmeticity of the Jordan spectrum, which is due to the work of Benoist [Ben00]. As a byproduct of our method, we also obtain a strengthened version of the Hopf–Tsujii–Sullivan dichotomy (Corollary 6.5).

The subset of  $\mathcal{H}_\theta$  corresponding to the  $P$ -minimal set  $\mathcal{E}_\Gamma \subset \Gamma \backslash G$  defined above is

$$\mathcal{E}_{\Gamma,\theta} := \{[g] \in \mathcal{H}_\theta : g P_\theta \in \Lambda_\theta(\Gamma)\}.$$

We finally state a corollary of Theorem 1.13 for relatively  $P_\theta$ -Anosov subgroups that we classify all horospherical invariant measures  $\mathcal{E}_{\Gamma,\theta}$ .

**Corollary 1.14.** *Let  $\Gamma < G$  be a Zariski dense relatively  $P_\theta$ -Anosov subgroup. If  $\mu$  is a  $\Gamma$ -invariant ergodic Radon measure on  $\mathcal{E}_{\Gamma,\theta} \subset \mathcal{H}_\theta$ , then either*

- (1)  $\mu$  is supported on  $\mathcal{R}_{\Gamma,\theta}$  and is a constant multiple of a Burger–Roblin measure, or
- (2)  $\mu$  is supported on a closed  $\Gamma$ -orbit  $\Gamma \cdot \xi$  in  $\mathcal{H}_\theta$ , for some  $\xi \in \mathcal{E}_{\Gamma,\theta} \setminus \mathcal{R}_{\Gamma,\theta}$ .

**1.5. On the proof.** The main part of the proof in this paper is to show that any  $\Gamma$ -invariant ergodic Radon measure on  $\mathcal{R}_{\Gamma,\theta} \subset \mathcal{H}_\theta$  is quasi-invariant under the  $\exp \mathfrak{a}_\theta$ -action. It is now well-known that such a quasi-invariance implies that the measure should be of the same form as Burger–Roblin measures ([ANSS02, 0.1 Basic Lemma], [Sar04, Lemma 1], [LO23, Proposition 10.25]).

Our proof is rather geometric, and does not make use of any continuous flow on  $\Gamma \backslash G$ , such as one-dimensional diagonal flows given by  $v \in \text{int } \mathfrak{a}^+$  or multi-dimensional action of  $\exp \mathfrak{a}^+$ , not relying on ergodic theorems. This can be compared with the work of Landesberg–Lee–Lindenstrauss–Oh [LLLO23] where the open problem in Problem 1.2 was proposed, and the dynamics of one-dimensional diagonal flow  $\exp(\mathbb{R}_{>0} v)$  played a role, and hence the measure under the consideration is restricted to the smaller subset  $\mathcal{R}_{\Gamma,v}$ . This is one difference that enables us to classify horospherical invariant measures without restricting the supports of measures to smaller subsets.

We also do not rely on the existence of Besicovitch-type covering, which was used in [LLLO23] (and other previous literatures) based on their assumption that  $G$  is a product of rank one Lie groups. This is another difference which allows us to handle arbitrary semisimple real algebraic group  $G$ .

Our proof of the measure classification is in a similar spirit to our recent work [CK25b], where we considered the action of subgroups of the mapping class group on the space of measured laminations on a surface. On the other hand, there are essential difficulties in higher-rank homogeneous settings, due to their geometric differences from Teichmüller spaces of surfaces, and to cocycles being vector-valued.

In both settings, the proof of the quasi-invariance goes through showing that the measure is quasi-invariant under the translations by Jordan projections of loxodromic elements in this paper (Theorem 6.4), and by Teichmüller translation lengths of pseudo-Anosov elements in [CK25b]. These are eventually done by carefully approximating cocycles by Jordan projections and translation lengths. Major difficulties arise at this step.

In [CK25b], it was crucial that the axis of a pseudo-Anosov mapping class in the Teichmüller space possesses a CAT( $-1$ )-property, which we called *squeezing*. More precisely, this means that the farther the projections of any two points in the Teichmüller space to the axis, the closer the geodesic segment between the two points to the axis. The squeezing property can be regarded as an effective/quantitative version of so-called contracting property, and it was deduced from Minsky's contraction theorem [Min96] in [CK25b]. Such an effective/quantitative property enabled us to do appropriate approximations of cocycles by Teichmüller translation lengths.

In contrast, having such squeezing properties in higher-rank homogeneous settings may be impossible in general, due to the presence of higher-rank flats. Indeed, in the teaser paper [CK25a] where  $G$  is a product of rank one groups, the fact that each component is CAT( $-1$ ) was helpful, while this is only possible for such a restricted case.

What we have in our setting is only Gromov hyperbolicity associated to  $\Gamma$ , which is much weaker than CAT( $-1$ ). The Gromov hyperbolicity only gives contracting property of geodesics, an ineffective/coarse version of squeezing. That is, it only says that if two points have sufficiently far projection to a geodesic, then a geodesic segment connecting those two points intersects some neighborhood of the geodesic. See Figure 1 for the comparison.

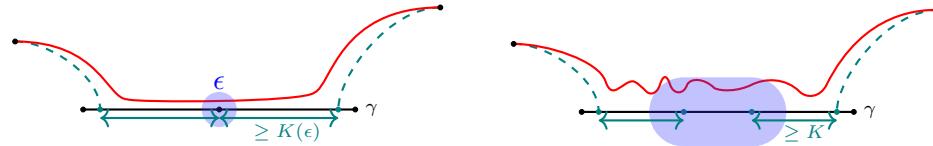


FIGURE 1. A squeezing geodesic  $\gamma$  (left) and a contracting geodesic  $\gamma$  (right)

The coarseness of the contracting property does not allow similar approximations of cocycles as in ([CK25b], [CK25a]) to work. To overcome this obstruction, we establish in this paper a robust connection between alignments of several geodesics in the Gromov hyperbolic space on which  $\Gamma$  acts

and the projective spaces obtained by Tits' representations [Tit71]. While this machinery does not produce any squeezing-like property, we could proceed an appropriate approximation of cocycles even for the coarse contracting geodesics, as desired.

The approximation argument above is also based on our observation that almost every point on the limit set is accumulated by synchronically aligned geodesic segments, along translates of an axis of a loxodromic element. We call the set of such points *guided limit set*, which is a subset of the conical limit set. As a byproduct, a part of our proof gives a strengthened version of the Hopf–Tsuji–Sullivan dichotomy that divergence-type Patterson–Sullivan measures are supported on guided limit sets (Corollary 6.5).

**1.6. Organization.** In Section 2, we review basic geometry of a Gromov hyperbolic space, and discuss specific properties of alignment of geodesics. Section 3 is devoted to guided limit sets, which are key players in the proof of our measure classification. Horospherical foliations and Iwasawa cocycles are discussed in Section 4. In Section 5, we review some dynamical properties of transverse subgroups, and prove an approximation result for Iwasawa cocycles and Jordan projections, with a certain uniform control. Our main measure classification is proved in Section 6. Statements in the introduction are deduced at the end of Section 6.

**1.7. Notation.** For  $a, b, c \in \mathbb{R}$ , we write the condition  $|a - b| \leq c$  by  $a =_c b$ .

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## 2. GEOMETRY OF GROMOV HYPERBOLIC SPACES

In this section, we review some facts about Gromov hyperbolic spaces. We refer the readers to classical references ([Gro87], [CDP90], [GdlH90], [BH99], etc.) for more comprehensive expositions.

Throughout this section, we fix a proper geodesic metric space  $(X, d)$  that is Gromov hyperbolic. This means that the following holds.

(Proper) For any  $x \in X$  and  $R > 0$ , the metric ball  $\mathcal{N}_R(x) := \{z \in X : d(x, z) < R\}$  is precompact.

- (Geodesic) For every pair of points  $x, y \in X$ , there exists a geodesic denoted by  $[x, y]$  connecting  $x$  to  $y$ .
- ( $\delta$ -hyperbolic) There exists  $\delta > 0$  such that every geodesic triangle in  $X$  is  $\delta$ -thin, i.e., for each  $x, y, z \in X$  we have

$$[x, y] \subset \mathcal{N}_\delta([x, z] \cup [z, y]).$$

For  $w, z \in [x, y]$ , we intrinsically assume  $[w, z]$  to be a segment of  $[x, y]$ . Every parametrization of a geodesic is of unit speed.

Let  $\partial X$  be the Gromov boundary of  $X$ . Instead of giving a precise definition, we note that there is a natural topology on  $X \sqcup \partial X$  so that it is a compact metrizable space where  $X$  is an open dense subset and  $\partial X$  is compact. For any distinct  $x, y \in X \cup \partial X$ , there exists a geodesic in  $X$  with endpoints  $x$  and  $y$ . We also denote by  $[x, y] \subset X$  an arbitrarily chosen geodesic connecting  $x$  to  $y$ , and continue the same convention as for geodesics connecting points in  $X$ . By enlarging  $\delta > 0$  if necessary, every geodesic triangle in  $X$  with vertices in  $X \cup \partial X$  is also  $\delta$ -thin.

Given three points  $x, y, z \in X$ , we define the Gromov product

$$(x|y)_z := \frac{1}{2} (d(x, z) + d(z, y) - d(x, y)).$$

This definition extends to the case that  $x, y \in X \cup \partial X$ , by setting

$$(x|y)_z := \sup \liminf_{i, j \rightarrow +\infty} (x_i|y_j)_z$$

where the supremum is taken over all sequences  $\{x_i\}_{i \in \mathbb{N}}, \{y_j\}_{j \in \mathbb{N}} \subset X$  converging to  $x, y$ , respectively. The Gromov product  $(x|y)_z$  measures the distance from  $z$  to  $[x, y]$  up to a uniform error, i.e.,  $|((x|y)_z - d(z, [x, y]))|$  is bounded by a constant depending only on  $\delta$ .

By enlarging  $\delta > 0$  if necessary, we can also guarantee the following Gromov's 4-point inequality: for every  $x, y, z, w \in X \cup \partial X$  we have

$$(x|y)_w \geq \min((x|z)_w, (z|y)_w) - \delta.$$

**2.1. Contracting.** Due to the Gromov hyperbolicity of  $X$ , the nearest-point projection map in  $X$  has some useful geometric properties. We note that the discussion below works for a general metric space which is not necessarily Gromov hyperbolic.

For a closed subset  $A \subset X$ , we denote by  $\pi_A(\cdot) : X \rightarrow 2^A$  the *nearest-point projection*, i.e.,

$$\pi_A(x) := \left\{ a \in A : d(x, a) = \inf_{z \in A} d(x, z) \right\}.$$

**Definition 2.1.** Let  $C \geq 0$ . We say that a closed subset  $A \subset X$  is *C-contracting* if for every geodesic  $\gamma \subset X$  with  $d(\gamma, A) \geq C$  we have  $\text{diam } \pi_A(\gamma) \leq C$ . We say that  $A$  is (*strongly*) *contracting* if it is  $C$ -contracting for some  $C \geq 0$ .

By definition, if  $A \subset X$  is  $C$ -contracting, then  $\text{diam } \pi_A(x) \leq 2C$  for all  $x \in X$ . It is also easy to see that if  $A \subset X$  is  $C$ -contracting and  $x, y \in X$  are in the  $C$ -neighborhood of  $A$ , then  $[x, y]$  is contained in the  $2.5C$ -neighborhood of  $A$ . Contracting property is relevant to our setting as every geodesic in the Gromov hyperbolic space  $X$  is contracting.

**Fact 2.2** ([CDP90, Proposition 10.2.1]). *There exists  $C = C(\delta) > 0$  such that every geodesic<sup>3</sup> in  $X$  is  $C$ -contracting.*

We now describe some geometric features of contracting geodesics. While the constants below can be made uniform depending only on  $\delta$  as we will see, we present as follows, to emphasize the power of contracting geodesics in arbitrary metric spaces. We use the version in [CCT25] due to its conciseness, but we note that similar results were already observed in ([CDP90, Proposition 10.2.1], [Sis13, Lemma 2.4, 2.5], [ACT15, Proposition 2.9, Lemma 2.10, Lemma 2.11], [Yan14, Proposition 3.1]).

We say that two geodesics  $\gamma_1, \gamma_2 \subset X$  are  $K$ -equivalent for  $K \geq 0$  if their Hausdorff distance is at most  $K$  and if their beginning/ending points are pairwise  $K$ -close.

**Lemma 2.3** ([CCT25, Lemma 2.2]). *Let  $\gamma \subset X$  be a  $C$ -contracting geodesic and  $x, y \in X$ . Suppose that  $\text{diam } \pi_\gamma([x, y]) > C$ . Then there exist points  $p, q \in [x, y]$ , with  $p$  closer to  $x$  than  $q$ , such that*

- $\pi_\gamma([x, y])$  and  $[p, q]$  are  $4C$ -equivalent,
- $\text{diam}(\pi_\gamma([x, p]) \cup \{p\}) \leq 2C$ ,
- $\text{diam}(\pi_\gamma([q, y]) \cup \{q\}) \leq 2C$ , and
- for all  $x' \in \pi_\gamma(x)$  and  $y' \in \pi_\gamma(y)$ ,  $[x', y']$  and  $[p, q]$  are  $10C$ -equivalent.

The following is immediate from Lemma 2.3.

**Corollary 2.4.** *For a  $C$ -contracting geodesic  $\gamma \subset X$ , the following holds.*

- (1) *The map  $\pi_\gamma(\cdot)$  is  $(1, 4C)$ -Lipschitz: for each  $x, y \in X$ ,*

$$\text{diam } \pi_\gamma(\{x, y\}) \leq d(x, y) + 4C.$$

- (2) *Let  $x \in X$  and  $\gamma(t) \in \pi_\gamma(x)$ . Then for every  $s \in \mathbb{R}$ , we have*

$$(2.1) \quad d(x, \gamma(s)) =_{4C} d(x, \gamma(t)) + |t - s|.$$

The  $\delta$ -thinness of geodesic triangles also implies the following:

**Fact 2.5.** *Let  $M > 0$  and let  $x, y, x', y' \in X$  be such that  $d(x, x'), d(y, y') \leq M$ . Then the geodesics  $[x, y]$  and  $[x', y']$  are  $(2\delta + M)$ -equivalent. The same is true when  $x, y$  or  $x', y'$  are the same point on  $\partial X$ .*

---

<sup>3</sup>More generally, every quasi-convex set is contracting [Sis18, Lemma 3.1].

**2.2. Alignment.** We now define alignment between geodesics and points.

**Definition 2.6** (Alignment). Let  $w, x, y, z \in X$ . For a geodesic  $[x, y] \subset X$  and  $K \geq 0$ , we say that the sequence  $(w, [x, y])$  is  $K$ -aligned if

$$\text{diam } \pi_{[x,y]}(w) \cup \{x\} < K.$$

We say that the sequence  $([x, y], z)$  is  $K$ -aligned if  $(z, [y, x])$  is  $K$ -aligned.

We say that the sequence  $(w, [x, y], z)$  is  $K$ -aligned if both sequences  $(w, [x, y])$  and  $([x, y], z)$  are  $K$ -aligned. See Figure 2.

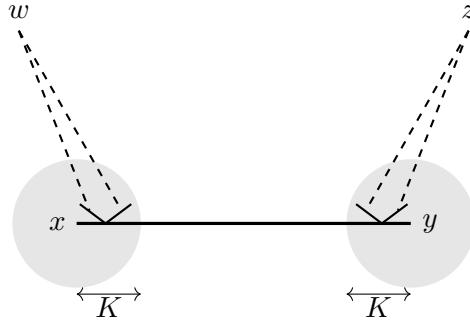


FIGURE 2. Alignment of geodesics and points.

The following is immediate.

**Lemma 2.7.** Let  $\gamma \subset X$  be a geodesic of length  $L \geq 0$ , let  $0 \leq D \leq L$  and let  $x \in X$ . Then  $(\gamma, x)$  is not  $D$ -aligned or  $(x, \gamma)$  is not  $(L - D)$ -aligned.

**Definition 2.8.** Let  $x \in \partial X$  and  $\gamma \subset X$  be a compact geodesic. For  $K \geq 0$ , we say that  $(x, \gamma)$  is  $K$ -aligned if for every sequence  $\{z_i\}_{i \in \mathbb{N}} \subset X$  converging to  $x$ ,  $(z_i, \gamma)$  is  $K$ -aligned eventually (i.e., for all large  $i \in \mathbb{N}$ ). We define the alignment for  $(\gamma, x')$  and  $(x, \gamma, x')$  similarly for  $x' \in X \cup \partial X$ .

**2.3. Shadows and alignment.** We make a useful elementary observation that the alignment can be interpreted in terms of shadows.

**Definition 2.9.** For  $x, y \in X$  and  $R > 0$ , we define the shadow  $O_R(x, y)$  of a ball of radius  $R$  centered at  $y$  viewed from  $x$ , as follows:

$$O_R(x, y) := \{w \in X \cup \partial X : d([x, w], y) < R\}.$$

In other words,  $w \in O_R(x, y)$  if there exists a geodesic  $[x, w]$  intersecting the  $R$ -neighborhood of  $y$ .

We now interpret the alignment using shadows. First, note that one can imagine that if  $x, y, z, w \in X$  satisfy

$$w \in O_R(x, y) \cap O_R(y, z),$$

then  $y$  comes earlier than  $z$  along  $[x, w]$ . Let us make this more precise.

**Lemma 2.10.** *Let  $R > 0$  and  $x, y, z, w \in X$ . Let  $C > 0$  be such that  $[y, z]$  is  $C$ -contracting.*

(1) *If  $w \in O_R(x, y) \cap O_R(y, z)$ , then*

$$(x, [y, z], w) \text{ is } (2R + 4C)\text{-aligned.}$$

(2) *If  $(x, [y, z], w)$  is  $R$ -aligned and  $d(y, z) > 2R + C$ , then*

$$w \in O_{R+2C}(x, y) \cap O_{R+2C}(y, z).$$

*Proof.* Let us prove (1). Since  $w \in O_R(y, z)$ ,

$$d(y, w) \geq d(y, z) + d(z, w) - 2R.$$

By Corollary 2.4(2), if  $p \in \pi_{[y, z]}(w)$ , then

$$\begin{aligned} d(y, w) &\leq d(y, p) + d(p, w) + 4C, \\ d(z, w) &\geq d(z, p) + d(p, w) - 4C. \end{aligned}$$

Hence,

$$d(y, z) + d(z, p) - 2R - 4C \leq d(y, p) + 4C,$$

which implies

$$d(z, p) \leq R + 4C.$$

Therefore,  $([y, z], w)$  is  $(R + 4C)$ -aligned.

Similarly, since  $w \in O_R(x, y)$ , we have

$$d(x, w) \geq d(x, y) + d(y, w) - 2R.$$

Since  $d(x, w) \leq d(x, z) + d(z, w)$  and  $w \in O_R(y, z)$ , we have

$$d(x, z) \geq d(x, y) + d(y, z) - 4R.$$

If  $q \in \pi_{[y, z]}(x)$ , then it follows from Corollary 2.4(2) that

$$\begin{aligned} d(x, z) &\leq d(x, q) + d(q, z) + 4C, \\ d(x, y) &\geq d(x, q) + d(q, y) - 4C. \end{aligned}$$

Hence,

$$d(q, y) + d(y, z) - 4R - 4C \leq d(q, z) + 4C.$$

This implies

$$d(q, y) \leq 2R + 4C,$$

and therefore  $(x, [y, z])$  is  $(2R + 4C)$ -aligned.

We now prove (2). By the assumption,  $\text{diam } \pi_{[y, z]}(\{x, w\}) > C$ . It then follows from Lemma 2.3 that  $d(y, [x, w]) < R + 2C$ , and hence  $w \in O_{R+2C}(x, y)$ . Since  $d(y, z) > R + 2C$ , we also have  $\text{diam } \pi_{[y, z]}(\{y, w\}) > C$ . By Lemma 2.3 again,  $d(z, [y, w]) < R + 2C$ , and therefore  $w \in O_{R+2C}(y, z)$  as well.  $\square$

**2.4. Isometries.** We now turn to isometries of  $X$ . The isometries can be classified in terms of their fixed points in  $X \cup \partial X$ . A non-trivial isometry  $g \in \text{Isom}(X)$  is either *elliptic* (i.e., fixes a point in  $X$ ), *parabolic* (i.e., has a unique fixed point in  $\partial X$ ), or *loxodromic* (i.e., has a unique pair of two fixed points in  $\partial X$ ). If  $g \in \text{Isom}(X)$  is of infinite order, it is either parabolic or loxodromic.

For a loxodromic  $g \in \text{Isom}(X)$ , its *stable translation length* is defined by

$$\tau_g := \lim_{n \rightarrow +\infty} \frac{d(x, g^n x)}{n} > 0 \quad \text{for any } x \in X.$$

One can observe that

$$\tau_{g^k} = |k| \tau_g \leq d(x, g^k x) \quad \text{for all } k \in \mathbb{Z} \text{ and } x \in X.$$

Let  $\gamma : \mathbb{R} \rightarrow X$  be a geodesic such that  $\gamma(t) \in X$  converges to the attracting and repelling fixed points of  $g$  as  $t \rightarrow +\infty$  and  $t \rightarrow -\infty$  respectively. Since  $g$  fixes its attracting and repelling fixed points,  $g \cdot \gamma$  is again a geodesic connecting those points. In particular,  $\gamma$  and  $g \cdot \gamma$  have Hausdorff distance  $2\delta$ .

We call  $\gamma$  an *axis* of  $g$ , which might not be unique in general. Note that for any  $t \in \mathbb{R}$ ,

$$(2.2) \quad d(g \cdot \gamma(t), \gamma(t + \tau_g)) \leq c$$

for some constant  $c = c(\delta) > 0$  that depends only on  $\delta$ , not on the choice of  $g$ .

On the other hand, we often consider points outside its axis. Regarding this, we record useful properties of  $\gamma$  in the following lemma, whose version for squeezing isometries is proved in [CK25b, Lemma 5.9].

**Lemma 2.11.** *Let  $g \in \text{Isom}(X)$  be a loxodromic isometry,  $\gamma : \mathbb{R} \rightarrow X$  its axis, and  $x_0 \in X$ . Then there exists  $C = C(g, \gamma, x_0) > 0$  such that the following holds.*

- (1) Any geodesic in  $X$  is  $C$ -contracting.
- (2)  $d(g^k x_0, \gamma(\tau_g k)) < C$  for all  $k \in \mathbb{Z}$ .
- (3) Let  $k \in \mathbb{N}$ , let  $x \in X$ , and let  $K \geq C$ . Then  $(x, [x_0, g^k x_0])$  is not  $K$ -aligned  $\implies \pi_\gamma(x) \subset \gamma([K - C, +\infty))$ .
- (4) Let  $k \in \mathbb{N}$ , let  $x \in X$ , and let  $0 \leq K \leq \tau_g k - C$ . Then  $(x, [x_0, g^k x_0])$  is  $K$ -aligned  $\implies \pi_\gamma(x) \subset \gamma((-\infty, K + C])$ .

Moreover,  $C$  can be chosen so that  $C(g, \gamma, x_0) = C(g^k, \gamma, x_0)$  for all  $k \in \mathbb{N}$  and  $C(g^{-1}, \hat{\gamma}, x_0) = C(g, \gamma, x_0)$  where  $\hat{\gamma}$  is the inversion of  $\gamma$ .

We often write  $C(g) = C(g, \gamma, x_0)$  by implicitly choosing its axis  $\gamma$ .

*Proof.* Let  $C_0 > 0$  be the constant depending only on the Gromov hyperbolicity of  $X$  so that any geodesic in  $X$  is  $C_0$ -contracting. Then Item (1) follows from the fact that any geodesic in  $X$  is  $C_0$ -contracting. Items (2) and (3) follow from the fact that  $d(g^k x_0, \gamma(\tau_g k)) \leq c$  and  $\pi_\gamma(x) \subset \gamma([K - C, +\infty))$  for some constant  $c = c(\delta) > 0$ . Item (4) follows from the fact that  $d(g^k x_0, \gamma(\tau_g k)) \leq c$  and  $\pi_\gamma(x) \subset \gamma((-\infty, K + C])$  for some constant  $c = c(\delta) > 0$ . The proof is complete.

when  $C \geq C_0$ . For the constant  $c > 0$  in Equation (2.2) which also depends only on the Gromov hyperbolicity, Item (2) holds for  $C \geq d(x_0, \gamma(0)) + c$ .

Now let

$$D = 10(C_0 + d(x_0, \gamma(0)) + \delta + c).$$

We will see that  $C = 100D$  plays the desired role.

We first show Item (3). Let  $k \in \mathbb{N}$  and  $K \geq 100D$  and suppose that  $(x, [x_0, g^k x_0])$  is not  $K$ -aligned. Recall that  $[x_0, g^k x_0]$  and  $\gamma([0, \tau_g k])$  are  $D$ -equivalent. Hence there exist  $p \in \pi_{[x_0, g^k x_0]}(x)$  and  $t_p \in [0, \tau_g k]$  such that

$$d(x_0, p) > K \quad \text{and} \quad d(p, \gamma(t_p)) \leq D.$$

Note also that  $d(x_0, \gamma(0)) \leq D$ . We then have

$$t_p \geq K - 2D.$$

If there exists  $t \in (-\infty, K - 6D]$  such that  $\gamma(t) \in \pi_\gamma(x)$ , then

$$\begin{aligned} d(x, \gamma(t_p)) &\geq d(x, \gamma(K - 6D)) + t_p - (K - 6D) - D \\ &\geq d(x, \gamma(K - 6D)) + 3D \end{aligned}$$

by Corollary 2.4(2). Since  $0 \leq K - 6D \leq t_p \leq \tau_g k$ , and since  $[x_0, g^k x_0]$  and  $\gamma([0, \tau_g k])$  are  $D$ -equivalent, we have

$$d(x, p) \geq d(x, \gamma(t_p)) - D \geq d(x, \gamma(K - 6D)) + 2D \geq d(x, [x_0, g^k x_0]) + D,$$

which contradicts  $p \in \pi_{[x_0, g^k x_0]}(x)$ . Therefore,

$$\pi_\gamma(x) \subset \gamma([K - 6D, +\infty))$$

and hence Item (3) holds for  $C = 100D$ .

For Item (4), let  $k \in \mathbb{N}$  and  $0 \leq K \leq \tau_g k - 100D$ , and suppose that  $(x, [x_0, g^k x_0])$  is  $K$ -aligned. Then  $([x_0, g^k x_0], x)$  is not  $(d(x_0, g^k x_0) - K)$ -aligned by Lemma 2.7. Note that

$$d(x_0, g^k x_0) - K \geq \tau_g k - K \geq 100D.$$

As  $(g^{-k} x, [x_0, g^{-k} x_0])$  is not  $(d(x_0, g^k x_0) - K)$ -aligned, we apply Item (3) by reversing the orientation of  $\gamma$  and get

$$\begin{aligned} \pi_\gamma(g^{-k} x) &\subset \gamma((-\infty, -(d(x_0, g^k x_0) - K) + 6D]) \\ &\subset \gamma((-\infty, -\tau_g k + K + 6D]) \end{aligned}$$

Hence,

$$\pi_{g^k \gamma}(x) \subset g^k \gamma((-\infty, -\tau_g k + K + 6D]).$$

Since  $\gamma$  and  $g^k \gamma$  have Hausdorff distance at most  $2\delta$  by Fact 2.5, if  $y \in \pi_\gamma(x)$ , then there exists  $z \in g^k \gamma$  such that  $d(y, z) \leq 2\delta$ . By Corollary 2.4(2),

$$d(x, z) \geq d(x, \pi_{g^k \gamma}(x)) + d(\pi_{g^k \gamma}(x), z) - 4C_0,$$

and hence

$$d(x, y) + 2\delta \geq d(x, y) - 2\delta + d(\pi_{g^k \gamma}(x), z) - 4C_0.$$

We then have

$$d(\pi_{g^k\gamma}(x), z) \leq 4\delta + 4C_0,$$

which implies

$$z \in g^k\gamma((-\infty, -\tau_g k + K + 6D + 4\delta + 4C_0]).$$

Since  $d(y, z) \leq 2\delta$ ,

$$y \in \gamma((-\infty, K + 6D + 4\delta + 4C_0 + c]).$$

Therefore, Item (4) holds for  $C = 100D$  as well.  $\square$

The “Moreover” part is straightforward.  $\square$

**2.5. Non-elementary subgroups of isometries.** We denote by  $\text{Isom}(X)$  the isometry group of  $(X, d)$ . We call a subgroup  $\Gamma < \text{Isom}(X)$  *discrete* if it acts properly on  $X$ . The class of subgroups of  $\text{Isom}(X)$  we are interested in is as follows:

**Definition 2.12.** A discrete subgroup  $\Gamma < \text{Isom}(X)$  is called *non-elementary* if

- $\Gamma$  is not virtually cyclic, and
- $\Gamma$  contains a loxodromic isometry.

We can characterize non-elementary subgroups in terms of their limit sets:

**Definition 2.13.** Let  $\Gamma < \text{Isom}(X)$  be a discrete subgroup. Its *limit set*  $\Lambda(\Gamma) \subset \partial X$  is the set of all accumulation points of  $\Gamma x \subset X$  on  $\partial X$ , for any fixed  $x \in X$ .

One can see that  $\Lambda(\Gamma)$  is compact and  $\Gamma$ -invariant. The  $\Gamma$ -action on  $X \cup \partial X$  is a convergence action, and the limit set  $\Lambda(\Gamma)$  is also the limit set as a convergence group. It is a fact that a discrete subgroup  $\Gamma < \text{Isom}(X)$  is non-elementary if and only if  $\#\Lambda(\Gamma) \geq 3$ , and in this case the  $\Gamma$ -action on  $\Lambda(\Gamma)$  is minimal.

We call two loxodromic isometries *independent* if they have disjoint sets of fixed points in  $\partial X$ . One can see that there are plenty of pairs of independent loxodromic isometries on a non-elementary subgroup.

**Lemma 2.14.** *Let  $\Gamma < \text{Isom}(X)$  be a non-elementary subgroup. For a loxodromic isometry  $g \in \Gamma$ , there exists  $h \in \Gamma$  such that  $hgh^{-1}$  and  $g$  are independent. Moreover, there are infinitely many pairwise independent loxodromic isometries in  $\Gamma$ .*

The following is a variant of the so-called *extension lemma* of Yang, which can be regarded as the coarse-geometric version of the Anosov closing lemma (cf. [Bow08, Lemma 3.8]).

**Lemma 2.15** (Extension lemma [Yan19, Lemma 1.13]). *Let  $\Gamma < \text{Isom}(X)$  be a non-elementary subgroup. Then for each loxodromic isometry  $\varphi \in \Gamma$ , there exist  $a_1, a_2, a_3 \in \Gamma$  and  $\beta = \beta(\varphi) > 0$  such that for each  $x, y \in X \cup \partial X$ , there exists  $a \in \{a_1, a_2, a_3\}$  that makes*

$$(x, a \cdot [x_0, \varphi^n x_0], a\varphi^n a \cdot y) \quad \beta\text{-aligned for all } n \in \mathbb{N}.$$

Moreover,  $\beta$  can be chosen so that  $\beta(\varphi^k) = \beta(\varphi)$  for all  $k \in \mathbb{Z}$ .

See also [CK25b, Lemma 5.12, Lemma 5.15].

**2.6. Conical limit sets.** We define conical limit sets using shadows. Fix a basepoint  $x_0 \in X$ , while the conical limit sets do not depend on the choice of the basepoint.

**Definition 2.16.** Let  $\Gamma < \text{Isom}(X)$  be a discrete subgroup. A point  $x \in \partial X$  is called a *conical limit point* of  $\Gamma$  if there exist  $R > 0$  and an infinite sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \Gamma$  such that

$$x \in O_R(x_0, g_n x_0) \quad \text{for all } n \in \mathbb{N}.$$

We denote the *conical limit set* by  $\Lambda^{\text{con}}(\Gamma) \subset \partial X$ .

The conical limit set  $\Lambda^{\text{con}}(\Gamma)$  is  $\Gamma$ -invariant.

### 3. GUIDED LIMIT SETS

We continue the setting of the previous section that  $(X, d)$  is a proper geodesic Gromov hyperbolic space, with the same Gromov hyperbolicity constant  $\delta > 0$ . In [CK25b], we introduced the notion of guided limit sets on the horofunction boundary, which are variants of Coulon's contracting limit sets [Cou24] and Yang's  $(L, \mathcal{F})$ -limit sets [Yan24].

In this section, we consider the guided limit sets defined on the Gromov boundary and study its properties analogous to the ones we studied in [CK25b]. They will be a key player in our proof of the main measure classification theorem.

**Definition 3.1.** Let  $\Gamma < \text{Isom}(X)$  be a non-elementary subgroup. Let  $\varphi \in \Gamma$  be a loxodromic isometry and let  $C(\varphi) > 0$  be as in Lemma 2.11 and fix  $K \geq C(\varphi)$ . We say that  $x \in \partial X$  is a  $(\varphi, K)$ -guided limit point of  $\Gamma$  if for each sufficiently large  $n \in \mathbb{N}$ , there exists  $h \in \Gamma$  such that

$$(x_0, h[x_0, \varphi^n x_0], x) \quad \text{is } K\text{-aligned.}$$

We call the set of  $(\varphi, K)$ -guided limit points of  $\Gamma$  the  $(\varphi, K)$ -guided limit set of  $\Gamma$ . We denote it by  $\Lambda^{\varphi, K}(\Gamma)$ .

By Lemma 2.10, guided limit sets are subsets of conical limit sets. The role of  $K$  in the definition of  $(\varphi, K)$ -guided limit set is quite flexible. The following was proved in [CK25b, Lemma 6.4] in a slightly different setting. We present the proof for completeness.

**Lemma 3.2.** Let  $\Gamma < \text{Isom}(X)$  be a non-elementary subgroup. Let  $\varphi \in \Gamma$  be a loxodromic isometry and let  $C = C(\varphi) > 0$  be as in Lemma 2.11. Then for each  $K > C$ ,

$$\Lambda^{\varphi, K}(\Gamma) = \Lambda^{\varphi, C}(\Gamma).$$

Moreover,  $\Lambda^{\varphi, C}(\Gamma)$  is  $\Gamma$ -invariant.

*Proof.* Let  $\gamma : \mathbb{R} \rightarrow X$  be an axis of  $\varphi$  chosen for the constant  $C = C(\varphi)$  in Lemma 2.11. Fix  $K > C$ . We then set

$$N = \lceil (K + 100C + 100\delta + 100c)/\tau_\varphi \rceil$$

where  $c > 0$  is the constant given in Equation (2.2).

Now pick an arbitrary  $x \in \Lambda^{\varphi, K}(\Gamma)$  and let  $\{z_i\}_{i \in \mathbb{N}} \subset X$  be a sequence converging to  $x$ . Since  $x$  is  $(\varphi, K)$ -guided, for each large enough  $n \in \mathbb{N}$  there exists  $h \in \Gamma$  such that  $(x_0, h[x_0, \varphi^{n+2N}x_0], z_i)$  is  $K$ -aligned for all large  $i \in \mathbb{N}$ . Since  $(n + 2N)\tau_\varphi > K + C$ , Lemma 2.11(4) tells us that

$$\begin{aligned} \pi_{h\gamma}(x_0) &\subset h\gamma((-\infty, K + C]) \quad \text{and} \\ \pi_{h\varphi^{n+2N}\gamma}(z_i) &\subset h\varphi^{n+2N}\gamma([-K - C, +\infty)) \end{aligned}$$

As in the proof of Lemma 2.11(4),  $\pi_{h\gamma}(z_i)$  and  $\pi_{h\varphi^{n+2N}\gamma}(z_i)$  have Hausdorff distance at most  $6\delta + 4C$ . Hence,

$$\pi_{h\gamma}(z_i) \subset h\gamma([(n + 2N)\tau_\varphi - c - 6\delta - 4C - K - C, +\infty)).$$

We now show that

$$(x_0, h\varphi^N[x_0, \varphi^n x_0], x) \quad \text{is } C\text{-aligned.}$$

Suppose to the contrary that  $(x_0, h\varphi^N[x_0, \varphi^n x_0])$  is not  $C$ -aligned. Then by Lemma 2.11(3), we have

$$\pi_{h\varphi^N\gamma}(x_0) \subset h\varphi^N\gamma([0, +\infty)).$$

As above, this implies

$$\pi_{h\gamma}(x_0) \subset h\gamma([N\tau_\varphi - c - 6\delta - 4C, +\infty)).$$

On the other hand, since

$$K + C < N\tau_\varphi - c - 6\delta - 4C,$$

this is a contradiction. Therefore,  $(x_0, h\varphi^N[x_0, \varphi^n x_0])$  is  $C$ -aligned.

Similarly, if  $(h\varphi^N[x_0, \varphi^n x_0], z_i)$  is not  $C$ -aligned, then

$$\pi_{h\varphi^{n+N}\gamma}(z_i) \subset h\varphi^{n+N}\gamma((-\infty, 0]).$$

Hence as above,

$$\pi_{h\gamma}(z_i) \subset h\gamma((-\infty, (n + N)\tau_\varphi + c + 6\delta + 4C]).$$

Since

$$(n + N)\tau_\varphi + c + 6\delta + 4C < (n + 2N)\tau_\varphi - c - 6\delta - 4C - K - C,$$

this yields a contradiction, and therefore  $(h\varphi^N[x_0, \varphi^n x_0], z_i)$  is  $C$ -aligned. Since this is the case for arbitrary sequence  $\{z_i\}_{i \in \mathbb{N}} \subset X$  converging to  $x$ ,  $(x_0, h\varphi^N[x_0, \varphi^n x_0], x)$  is  $C$ -aligned. We conclude  $x \in \Lambda^{\varphi, C}(\Gamma)$ , proving the first statement.

We now show that  $\Lambda^{\varphi, C}(\Gamma)$  is  $\Gamma$ -invariant. Fix  $x \in \Lambda^{\varphi, C}(\Gamma)$  and  $g \in \Gamma$ . Then for each sufficiently large  $n \in \mathbb{N}$ , there exists  $h \in \Gamma$  such that  $(x_0, h[x_0, \varphi^n x_0], x)$  is  $C$ -aligned. In other words,  $(gx_0, gh[x_0, \varphi^n x_0], gx)$  is  $C$ -aligned. By Corollary 2.4(1), this implies that  $(x_0, gh[x_0, \varphi^n x_0], gx)$  is

$(5C + d(x_0, gx_0))$ -aligned. Hence,  $gx$  is a  $(\varphi, 5C + d(x_0, gx_0))$ -guided limit point of  $\Gamma$ , and therefore  $gx \in \Lambda^{\varphi, C}(\Gamma)$  by the first statement. This shows the desired  $\Gamma$ -invariance.  $\square$

For a non-elementary subgroup  $\Gamma < \text{Isom}(X)$ , isometries  $g, \varphi \in \Gamma$ , constants  $C > 0$ , and  $n \in \mathbb{N}$ , we set

$$U_C(g; \varphi, n) := \{x \in \Lambda(\Gamma) : (x_0, g[x_0, \varphi^n x_0], x) \text{ is } C\text{-aligned}\}.$$

In [CK25b, Lemma 7.9], we observed that they form a basis for the topology on the guided limit set, which is defined on the horofunction boundary for a general metric space. We prove the following version of it for the guided limit set defined on the Gromov boundary of  $X$ .

**Lemma 3.3.** *Let  $\Gamma < \text{Isom}(X)$  be a non-elementary subgroup. Let  $\varphi \in \Gamma$  be loxodromic and let  $C = C(\varphi) > 0$  be as in Lemma 2.11. Then for each  $x \in \Lambda^{\varphi, C}(\Gamma)$ , for each open set  $O \subset \partial X$  with  $x \in O$  and for each  $N \in \mathbb{N}$ , there exist  $g \in \Gamma$  and  $n > N$  such that*

$$x \in U_C(g; \varphi, n) \subset O.$$

As the collection of open sets

$$\{y \in \partial X : (x|y)_{x_0} > D\}$$

for  $x \in \partial X$  and  $D > 0$  forms a basis of the topology on  $\partial X$ , Lemma 3.3 is a consequence of the following.

**Lemma 3.4.** *Let  $\Gamma < \text{Isom}(X)$  be a non-elementary subgroup. Let  $\varphi \in \Gamma$  be loxodromic and let  $C = C(\varphi) > 0$  be as in Lemma 2.11. Then for any  $D > 0$ , there exists  $N > 0$  such that*

$$(x|y)_{x_0} \geq D \quad \text{for all } n > N, g \in \Gamma, \text{ and } x, y \in U_C(g; \varphi, n).$$

*Proof.* Let  $D_0 > 0$  be a variable which will be determined later. Let  $N \in \mathbb{N}$  be such that  $d(x_0, \varphi^n x_0) > 2D_0$  for all  $n > N$ .

Fix  $g \in \Gamma$  and  $n > N$ . Taking  $D_0$  large enough, we can apply Lemma 2.10, and have some  $R > 0$  depending on  $C$  such that

$$U_C(g; \varphi, n) \subset O_R(x_0, gx_0) \cap O_R(gx_0, g\varphi^n x_0).$$

Hence there exists  $D' > 0$  depending on  $R$  so that for any  $x, y \in U_C(g; \varphi, n)$ ,

$$(x|y)_{x_0} \geq \max(d(x_0, gx_0), d(x_0, g\varphi^n x_0)) - D'.$$

If  $d(x_0, gx_0) \geq d(x_0, g\varphi^n x_0)$ , then

$$d(x_0, gx_0) \geq d(x_0, \varphi^n x_0) - d(x_0, gx_0) > 2D_0 - d(x_0, gx_0),$$

and hence  $d(x_0, gx_0) \geq D_0$ . This implies

$$(x|y)_{x_0} \geq D_0 - D'.$$

Otherwise,

$$(x|y)_{x_0} \geq d(x_0, g\varphi^n x_0) - D'.$$

It follows from  $O_R(x_0, gx_0) \cap O_R(gx_0, g\varphi^n x_0) \neq \emptyset$  that for some  $D'' > 0$  depending on  $R > 0$ , we have

$$d(x_0, g\varphi^n x_0) \geq d(x_0, gx_0) + d(x_0, \varphi^n x_0) - D'' \geq D_0 - D''.$$

Hence,

$$(x|y)_{x_0} \geq D_0 - D'' - D'.$$

In any case, we can take  $D_0 = D + D'' + D'$ , and the claim follows.  $\square$

#### 4. HOROSPHERICAL FOLIATIONS OF LIE GROUPS

Let  $G$  be a connected semisimple real algebraic group. In this section, we define horospherical foliations of  $G$  and Burger–Roblin measures.

Let  $P < G$  be a minimal parabolic subgroup with a fixed Langlands decomposition  $P = MAN$  where  $A$  is a maximal real split torus of  $G$ ,  $M < P$  is the maximal compact subgroup commuting with  $A$ , and  $N$  is the unipotent radical of  $P$ .

Let  $\mathfrak{a} := \text{Lie } A$ . Fix a positive closed Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$  and set  $A^+ := \exp \mathfrak{a}^+$  so that for each  $n \in \mathbb{N}$ , the set  $\{a^{-1}na : a \in A^+\} \subset N$  is precompact. Let  $K < G$  be a maximal compact subgroup such that the Cartan decomposition  $G = KA^+K$  holds. The *Cartan projection* is the map  $\kappa : G \rightarrow \mathfrak{a}^+$  satisfying

$$g \in K(\exp \kappa(g))K \quad \text{for all } g \in G.$$

The bi- $K$ -invariant norm  $\|\cdot\|$  on  $\mathfrak{a}$  induced by the Killing form gives a left  $G$ -invariant Riemannian metric  $d(\cdot, \cdot)$  on  $G$ : for  $g, h \in G$ ,  $d(g, h) := \|\kappa(g^{-1}h)\|$ . This is right  $K$ -invariant as well, and hence induces a left  $G$ -invariant Riemannian metric on the associated symmetric space  $G/K$ .

Fix an element  $w_0 \in K$  which normalizes  $A$  and represents the longest Weyl element so that  $\text{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$ . This induces the map

$$i := -\text{Ad}_{w_0} : \mathfrak{a} \rightarrow \mathfrak{a}$$

which is an involution preserving  $\mathfrak{a}^+$ , called the opposition involution. Then we have  $\kappa(g^{-1}) = i(\kappa(g))$  for all  $g \in G$ .

Let  $\Delta$  denote the set of all simple roots associated to  $\mathfrak{a}^+$ . The opposition involution on  $\mathfrak{a}$  induces an involution  $\Delta \rightarrow \Delta$ ,  $\alpha \mapsto \alpha \circ i$ . We also denote this involution by  $i : \Delta \rightarrow \Delta$ .

For a non-empty subset  $\theta \subset \Delta$ , we set

$$\mathfrak{a}_\theta := \bigcap_{\alpha \in \Delta \setminus \theta} \ker \alpha \quad \text{and} \quad \mathfrak{a}_\theta^+ := \mathfrak{a}_\theta \cap \mathfrak{a}^+.$$

We denote by  $p_\theta : \mathfrak{a} \rightarrow \mathfrak{a}_\theta$  the projection invariant under all Weyl elements fixing  $\mathfrak{a}_\theta$  pointwise. Let  $\mathfrak{a}^*$  and  $\mathfrak{a}_\theta^*$  be spaces of all  $\mathbb{R}$ -linear forms on  $\mathfrak{a}$  and  $\mathfrak{a}_\theta$  respectively. Via  $p_\theta$ ,  $\mathfrak{a}_\theta^*$  can be regarded as the subspace of  $\mathfrak{a}^*$  invariant under the precomposition of  $p_\theta$ .

For  $g \in G$ , its *Jordan projection* is defined as

$$\lambda(g) := \lim_{n \rightarrow +\infty} \frac{\kappa(g^n)}{n} \in \mathfrak{a}^+$$

and is invariant under conjugations. We also set

$$\lambda_\theta(g) := p_\theta(\lambda(g)) \in \mathfrak{a}_\theta^+.$$

For Zariski dense subsemigroups, Benoist proved the non-arithmeticity of Jordan projections.

**Theorem 4.1** ([Ben00]). *Let  $H \subset G$  be a Zariski dense subsemigroup. Then  $\lambda(H) \subset \mathfrak{a}^+$  generates a dense additive subgroup of  $\mathfrak{a}$ .*

**4.1. Furstenberg boundary and  $\theta$ -boundary.** The *Furstenberg boundary* of  $G$  is defined as the quotient

$$\mathcal{F} := G / P = K / M$$

where the last equality is due to the Iwasawa decomposition  $G = K P = KAN$ .

In a similar spirit, each non-empty subset of  $\Delta$  gives rise to a boundary of  $G$ . For a non-empty subset  $\theta \subset \Delta$ , let  $P_\theta < G$  be the standard parabolic subgroup corresponding to  $\theta$ . That is,  $P_\theta$  is generated by  $MA$  and all root subgroups for all positive roots and any negative root which is a  $\mathbb{Z}$ -linear combination of  $\Delta \setminus \theta$ . Then we have  $P_\Delta = P$  and for  $\theta \subset \theta'$ ,  $P_{\theta'} < P_\theta$ . This is the reason why  $P$  is a minimal parabolic subgroup.

Then  $\theta$ -boundary is defined as the quotient

$$\mathcal{F}_\theta := G / P_\theta$$

and the natural projection  $\pi_\theta : \mathcal{F} \rightarrow \mathcal{F}_\theta$ ,  $gP \mapsto gP_\theta$ , has compact fibers. Note that  $w_0 P_{i(\theta)} w_0^{-1}$  is a parabolic subgroup opposite to  $P_\theta$ . Two points  $x \in \mathcal{F}_\theta$  and  $y \in \mathcal{F}_{i(\theta)}$  are said to be *in general position* if there exists  $g \in G$  such that

$$x = gP_\theta \quad \text{and} \quad y = gw_0 P_{i(\theta)}.$$

Equivalently, a pair  $(x, y) \in \mathcal{F}_\theta \times \mathcal{F}_{i(\theta)}$  is in general position if its  $G$ -orbit  $G \cdot (x, y) \subset \mathcal{F}_\theta \times \mathcal{F}_{i(\theta)}$  under the diagonal action is open, which turns out to be a unique open  $G$ -orbit.

The convergence of a sequence in  $G$  to  $\mathcal{F}_\theta$  is defined as follows: given a sequence  $\{g_n\}_{n \in \mathbb{N}} \subset G$ , let  $\{k_n\}_{n \in \mathbb{N}} \subset K$  be a sequence such that  $g_n = k_n A^+ K$  in Cartan decomposition, for each  $n \in \mathbb{N}$ . Then we have the convergence  $g_n \rightarrow x \in \mathcal{F}_\theta$  if, as  $n \rightarrow +\infty$ ,

$$(4.1) \quad k_n P_\theta \rightarrow x \text{ in } \mathcal{F}_\theta \quad \text{and} \quad \alpha(\kappa(g_n)) \rightarrow +\infty \text{ for each } \alpha \in \theta.$$

While  $k_n \in K$  may not be well-defined, the point  $k_n P_\theta \in \mathcal{F}_\theta$  is well-defined, independent of the choice of  $k_n \in K$ , for each  $n \in \mathbb{N}$ .

Moreover, if  $g_n = k_n (\exp \kappa(g_n)) \ell_n \in KA^+ K$  in Cartan decomposition for  $n \in \mathbb{N}$ , then

$$g_n^{-1} = (\ell_n^{-1} w_0) (\exp i(\kappa(g_n))) (w_0^{-1} k_n^{-1}) \in KA^+ K$$

is a Cartan decomposition as well. Hence, if  $\alpha(\kappa(g_n)) \rightarrow +\infty$  for each  $\alpha \in \theta$  and  $\ell_n \rightarrow \ell \in K$  as  $n \rightarrow +\infty$ , then the sequence  $\{g_n^{-1}\}_{n \in \mathbb{N}}$  converges to  $\ell^{-1}w_0 P_{i(\theta)} \in \mathcal{F}_{i(\theta)}$ .

For  $\alpha \in \Delta$ , we simply write  $P_\alpha := P_{\{\alpha\}}$  and  $\mathcal{F}_\alpha := \mathcal{F}_{\{\alpha\}}$ , and similarly to other places.

**4.2. Iwasawa cocycles.** The Iwasawa decomposition is the diffeomorphism

$$\begin{aligned} K \times A \times N &\rightarrow G \\ (k, a, n) &\mapsto kan. \end{aligned}$$

The *Iwasawa cocycle*  $\sigma : G \times \mathcal{F} \rightarrow \mathfrak{a}$  is defined as follows: for  $g \in G$  and  $x \in \mathcal{F}$ , let  $\sigma(g, x) \in \mathfrak{a}$  be the unique element such that

$$gk \in K(\exp \sigma(g, x))N$$

where  $k \in K$  be such that  $x = kM \in \mathcal{F}$ .

For non-empty  $\theta \subset \Delta$ , the *partial Iwasawa cocycle*  $\sigma_\theta : G \times \mathcal{F}_\theta \rightarrow \mathfrak{a}_\theta$  is defined as

$$\sigma_\theta(g, x) := p_\theta(\sigma(g, \tilde{x}))$$

for some (any)  $\tilde{x} \in \pi_\theta^{-1}(x) \in \mathcal{F}$ . This does not depend on the choice of  $\tilde{x}$  [Qui02b, Lemma 6.1]. Then for  $g, h \in G$  and  $x \in \mathcal{F}_\theta$ , the cocycle relation holds:

$$(4.2) \quad \sigma_\theta(gh, x) = \sigma_\theta(g, hx) + \sigma_\theta(h, x).$$

See [Qui02b, Lemma 6.2]. In particular,

$$(4.3) \quad \sigma_\theta(g^{-1}, x) = -\sigma_\theta(g, g^{-1}x).$$

**4.3. Horospherical foliations.** We now define the horospherical foliation of  $G$  associated to  $\theta \subset \Delta$ . Let  $L_\theta$  be the centralizer of  $A_\theta := \exp \mathfrak{a}_\theta$ . Then  $L_\theta$  is a Levi subgroup of  $P_\theta$  with the Levi decomposition  $P_\theta = L_\theta N_\theta$ , where  $N_\theta$  is the unipotent radical of  $P_\theta$ . We also have  $L_\theta = A_\theta S_\theta$  where  $S_\theta$  is an almost direct product of a connected semisimple real algebraic subgroup and a compact subgroup. Then  $M_\theta := K \cap P_\theta$  is a maximal compact subgroup of  $S_\theta$ . Note that  $M_\Delta = M = S_\Delta$  and  $N_\Delta = N$ .

The  $\theta$ -horospherical foliation of  $G$  is the foliation whose leaves are right  $N_\theta S_\theta$ -orbits. Hence, it can be represented as the quotient space  $G / N_\theta S_\theta$ , on which  $G$  acts by left multiplications.

Set

$$\mathcal{H}_\theta := \mathcal{F}_\theta \times \mathfrak{a}_\theta,$$

and define the  $G$ -action on  $\mathcal{H}_\theta$  by

$$g \cdot (x, u) := (gx, u + \sigma_\theta(g, x))$$

for  $g \in G$  and  $(x, u) \in \mathcal{H}_\theta$ . Then the map

$$\begin{aligned} G / N_\theta S_\theta &\longrightarrow \mathcal{H}_\theta = \mathcal{F}_\theta \times \mathfrak{a}_\theta \\ g N_\theta S_\theta &\longmapsto (g P_\theta, \sigma_\theta(g, P_\theta)) \end{aligned}$$

is a  $G$ -equivariant homeomorphism. Therefore, we identify

$$\mathcal{H}_\theta = \mathcal{F}_\theta \times \mathfrak{a}_\theta = G / N_\theta S_\theta$$

and regard  $\mathcal{H}_\theta$  as the  $\theta$ -horospherical foliation of  $G$  as well.

Since  $A_\theta$  normalizes  $N_\theta S_\theta$ , we have the  $A_\theta$ -action on  $G / N_\theta S_\theta$  given by the right multiplication. This  $A_\theta$ -action commutes with the left  $G$ -action, and induces an action on  $\mathcal{H}_\theta$  given by the translation on the  $\mathfrak{a}_\theta$ -component: for  $a \in A_\theta$  and  $(x, u) \in \mathcal{H}_\theta$ ,

$$(x, u) \cdot a := (x, u + \log a).$$

*Remark 4.2.* When  $\text{rank } G = 1$ , if  $\{g_n\}_{n \in \mathbb{N}} \subset G$  is a sequence such that  $\kappa(g_n)$  diverges while  $\sigma(g_n, x)$  is bounded for some  $x \in \mathcal{F}$ , then  $g_n^{-1} \rightarrow x$  in the sense of Equation (4.1). This can be easily seen especially when  $G = \text{PSL}(2, \mathbb{R}) = \text{Isom}^+ \mathbb{H}^2$ , by interpreting  $\kappa(\cdot)$  as the displacement and  $\sigma(\cdot, \cdot)$  as the Busemann function in  $\mathbb{H}^2$ . In particular, if  $\{g_n\}_{n \in \mathbb{N}} \subset G$  is a sequence such that  $\kappa(g_n)$  diverges and  $g_n^{-1}$  does not converge to  $x \in \mathcal{F}$ , then for any  $u \in \mathfrak{a}$ , the sequence  $g_n \cdot (x, u)$  diverges in  $\mathcal{H}_\Delta$ .

On the other hand, this does not hold in higher-rank  $G$  in general, unless  $G$  is a product of rank one groups. For example, consider the case that  $G = \text{PSL}(3, \mathbb{R})$  where  $P < G$  is the subgroup of upper triangular matrices, so that  $\mathcal{F}$  is the full flag variety  $\{(V_1, V_2) : 0 < V_1 < V_2 < \mathbb{R}^3, \dim V_i = i\}$ . Then set  $x = P \in \mathcal{F}$  which corresponds to the flag  $((\mathbb{R}, 0, 0), (\mathbb{R}, \mathbb{R}, 0))$  and let  $\{g_n\}_{n \in \mathbb{N}} \subset G$  be the sequence given by

$$g_n := \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N \quad \text{for all } n \in \mathbb{N}.$$

One can see that  $\sigma_\Delta(g_n^\pm, x) = 0$  for all  $n \in \mathbb{N}$  and  $\alpha(\kappa(g_n^\pm)) \rightarrow +\infty$  for all  $\alpha \in \Delta$ .

On the other hand,  $\{g_n\}_{n \in \mathbb{N}}$  or  $\{g_n^{-1}\}_{n \in \mathbb{N}}$  cannot converge to  $x$ . Indeed, if both sequences converge to  $x$ , then for any flag  $y \in \mathcal{F}$  transverse to  $x$ ,  $g_n^\pm y \rightarrow x$  as  $n \rightarrow +\infty$ . Now taking  $y$  to be the flag  $((0, 0, \mathbb{R}), (0, \mathbb{R}, \mathbb{R}))$  yields a contradiction.

If one seeks a sequence of loxodromic elements, the sequence

$$g_n := \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

also works.

**4.4. Patterson–Sullivan measures and Burger–Roblin measures.** In [Qui02b], Quint introduced higher-rank Patterson–Sullivan measures on  $\mathcal{F}_\theta$  using the partial Iwasawa cocycle. For  $\delta \geq 0$ ,  $\psi \in \mathfrak{a}_\theta^*$ , and  $\Gamma < G$ , a Borel probability measure  $\nu$  on  $\mathcal{F}_\theta$  is called a  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan

measure of  $\Gamma$  if for any  $g \in \Gamma$ ,

$$(4.4) \quad \frac{dg_*\nu}{d\nu}(x) = e^{-\delta \cdot \psi(\sigma_\theta(g^{-1}, x))} \quad \text{for } \nu\text{-a.e. } x \in \mathcal{F}_\theta.$$

For a  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure  $\nu$  of  $\Gamma$ , the associated *Burger–Roblin measure*  $\mu_\nu^{\text{BR}}$  of  $\Gamma$  on  $\mathcal{H}_\theta$  is defined by

$$d\mu_\nu^{\text{BR}}(x, u) := e^{\delta \cdot \psi(u)} d\nu(x) du$$

where  $du$  is the Lebesgue measure on  $\mathfrak{a}_\theta$ . The measure  $\mu_\nu^{\text{BR}}$  is a  $\Gamma$ -invariant Radon measure on  $\mathcal{H}_\theta$ .

In a special case that  $\theta = \Delta$ , we have  $N_\Delta = N$  and  $S_\Delta = M$ , and the Burger–Roblin measure of  $\Gamma$  associated to  $\nu$  is also defined on  $\Gamma \backslash G$ . Recalling that  $\mathcal{F} = K/M$ , we denote by  $\widehat{\nu}$  the  $M$ -invariant lift of  $\nu$  to  $K$ . Using the Iwasawa decomposition  $G = KAN$ , we define the measure  $\tilde{\mu}_\nu^{\text{BR}}$  on  $G$  as follows: for  $g = k(\exp u)n \in KAN$ ,

$$d\tilde{\mu}_\nu^{\text{BR}}(g) := e^{\delta \cdot \psi(u)} d\widehat{\nu}(k) du dn$$

where  $dn$  is the Haar measure on  $N$ . Then the Radon measure  $\tilde{\mu}_\nu^{\text{BR}}$  is left  $\Gamma$ -invariant and right  $NM$ -invariant, and therefore it induces the  $NM$ -invariant Radon measure

$$\mu_\nu^{\text{BR}} \quad \text{on} \quad \Gamma \backslash G,$$

abusing notations, which is also called the *Burger–Roblin measure* of  $\Gamma$  associated to  $\nu$ .

**4.5. Tits representations and shadows.** To deduce a property of Iwasawa cocycles, which is an important player in this paper, we employ Tits representations and associated shadows. We refer to ([Qui02b], [BQ16, Section 6]) for a comprehensive discussion on this topic.

For each  $\alpha \in \Delta$ , let  $\omega_\alpha \in \mathfrak{a}^*$  be the (restricted) fundamental weight associated to  $\alpha$ . Then for a non-empty subset  $\theta \subset \Delta$ ,  $\{\omega_\alpha : \alpha \in \theta\}$  is a basis of  $\mathfrak{a}_\theta^*$ . The following is a theorem of Tits [Tit71], and representations given below are referred to as *Tits representations*.

**Theorem 4.3** ([Tit71, Theorem 7.2]). *For each  $\alpha \in \Delta$ , there exists an irreducible  $\mathbb{R}$ -representation  $\rho_\alpha : G \rightarrow \text{GL}(V_\alpha)$  whose highest (restricted) weight  $\chi_\alpha$  is equal to  $k_\alpha \omega_\alpha$  for some  $k_\alpha \in \mathbb{N}$  and whose highest weight space is one-dimensional.*

For each  $\alpha \in \Delta$ , we fix a Tits representation  $\rho_\alpha : G \rightarrow \text{GL}(V_\alpha)$  throughout the paper. We denote by  $V_\alpha^+$  the highest weight space of  $\rho_\alpha$  which is one-dimensional, and by  $V_\alpha^-$  its unique complementary  $\rho_\alpha(A)$ -invariant subspace in  $V_\alpha$ . Let  $\langle \cdot, \cdot \rangle_\alpha$  be a  $\rho_\alpha(K)$ -invariant inner product on  $V_\alpha$  with respect to which  $\rho_\alpha(A)$  is symmetric, so that  $V_\alpha^+$  is perpendicular to  $V_\alpha^-$ . We denote by  $\|\cdot\|_\alpha$  the norm on  $V_\alpha$  induced by  $\langle \cdot, \cdot \rangle_\alpha$ , and use the same notation  $\|\cdot\|_\alpha$  for the operator norm of elements of  $\text{GL}(V_\alpha)$ . Then we have

$$\log \|\rho_\alpha(g)\|_\alpha = \chi_\alpha(\kappa(g)) \quad \text{for all } g \in G.$$

The norm  $\|\cdot\|_\alpha$  induces a metric on the projective space  $\mathbb{P}(V_\alpha)$ : for  $v, w \in V_\alpha$  with  $\|v\|_\alpha = \|w\|_\alpha = 1$ ,

$$d_{\mathbb{P}(V_\alpha)}([v], [w]) := \min \{\|v - w\|_\alpha, \|v + w\|_\alpha\}.$$

Through  $\rho_\alpha$ , the stabilizers of  $V_\alpha^+, V_\alpha^< \subset V_\alpha$  in  $\mathsf{G}$  are parabolic subgroups  $\mathsf{P}_\alpha, w_0 \mathsf{P}_{i(\alpha)} w_0^{-1} < \mathsf{G}$ , respectively. Hence, denoting by  $V_\alpha^*$  the dual of  $V_\alpha$  (i.e.,  $\mathbb{P}(V_\alpha^*)$  consists of hyperplanes in  $V_\alpha$ ), maps  $\mathsf{G} \rightarrow \mathbb{P}(V_\alpha)$ ,  $g \mapsto \rho_\alpha(g)V_\alpha^+$ , and  $\mathsf{G} \rightarrow \mathbb{P}(V_\alpha^*)$ ,  $g \mapsto \rho_\alpha(g)V_\alpha^<$ , induce  $\rho_\alpha$ -equivariant embeddings:

$$\begin{aligned} \Phi_\alpha : \mathcal{F}_\alpha &\longrightarrow \mathbb{P}(V_\alpha) & \text{and} & \Phi_\alpha^* : \mathcal{F}_{i(\alpha)} &\longrightarrow \mathbb{P}(V_\alpha^*) \\ g \mathsf{P}_\alpha &\longmapsto \rho_\alpha(g)V_\alpha^+ & g w_0 \mathsf{P}_{i(\alpha)} &\longmapsto \rho_\alpha(g)V_\alpha^< \end{aligned}$$

The induced maps have the property that  $x \in \mathcal{F}_\alpha$  and  $y \in \mathcal{F}_{i(\alpha)}$  are in general position if and only if the line  $\Phi_\alpha(x)$  and hyperplane  $\Phi_\alpha^*(y)$  are transverse as subspaces of  $V_\alpha$ , i.e.,

$$\Phi_\alpha(x) \oplus \Phi_\alpha^*(y) = V_\alpha.$$

See [GGKW17, Section 3.2] for details.

Now more generally, fix a non-empty subset  $\theta \subset \Delta$ . Similar to the above, the map  $\mathsf{G} \rightarrow \prod_{\alpha \in \theta} \mathbb{P}(V_\alpha)$ ,  $g \mapsto (\rho_\alpha(g)V_\alpha^+)_{\alpha \in \theta}$ , factors through a  $(\rho_\alpha)_{\alpha \in \theta}$ -equivariant embedding

$$\Phi_\theta : \mathcal{F}_\theta \longrightarrow \prod_{\alpha \in \theta} \mathbb{P}(V_\alpha).$$

Then the metric on  $\prod_{\alpha \in \theta} \mathbb{P}(V_\alpha)$  induced by  $d_{\mathbb{P}(V_\alpha)}$ 's equips  $\mathcal{F}_\theta$  with a  $\mathsf{K}$ -invariant metric via the above map. Note that the following diagram commutes for each  $\alpha \in \theta$ :

$$\begin{array}{ccc} \mathcal{F}_\theta & \xrightarrow{\Phi_\theta} & \prod_{\alpha' \in \theta} \mathbb{P}(V_{\alpha'}) \\ \downarrow & & \downarrow \\ \mathcal{F}_\alpha & \xrightarrow{\Phi_\alpha} & \mathbb{P}(V_\alpha) \end{array}$$

where the vertical maps are canonical projections. For this reason, we also denote by  $\Phi_\alpha(x)$  the  $\alpha$ -component of  $\Phi_\theta(x)$  for  $x \in \mathcal{F}_\theta$ .

Tits representations are useful to study Iwasawa cocycles, as in the following lemma due to Quint.

**Lemma 4.4** ([Qui02b, Lemma 6.4]). *Let  $\alpha \in \theta$ . For any  $g \in \mathsf{G}$  and  $x \in \mathcal{F}_\theta$ , if  $v \in \Phi_\alpha(x) \setminus \{0\}$ , then*

$$\chi_\alpha(\sigma_\theta(g, x)) = \log \frac{\|\rho_\alpha(g)v\|_\alpha}{\|v\|_\alpha}.$$

Quint defined shadows in  $\mathcal{F}_\theta$  using Tits representations as follows: for  $g \in \mathsf{G}$ , write its Cartan decomposition

$$g = k_g(\exp \kappa(g))\ell_g \in \mathsf{KA}^+ \mathsf{K}.$$

Note that while  $k_g, \ell_g \in \mathsf{K}$  may not be well-defined, the corresponding points  $k_g \mathsf{P}_\theta \in \mathcal{F}_\theta$  and  $\ell_g^{-1} w_0 \mathsf{P}_{i(\theta)} \in \mathcal{F}_{i(\theta)}$  are well-defined. For  $0 < \epsilon \leq 1$ , let

$$B_{\theta,g}^\epsilon := \{x \in \mathcal{F}_\theta : d_{\mathbb{P}(V_\alpha)}(\Phi_\alpha(x), \mathbb{P}(\rho_\alpha(\ell_g^{-1}) V_\alpha^<)) \geq \epsilon \text{ for all } \alpha \in \theta\}.$$

In other words,  $B_{\theta,g}^\epsilon$  consists of points  $x \in \mathcal{F}_\theta$  such that the line  $\Phi_\alpha(x) \subset V_\alpha$  has uniform transversality to the hyperplane  $\Phi_\alpha^*(\ell_g^{-1} w_0 \mathsf{P}_{i(\alpha)}) \subset V_\alpha$  for all  $\alpha \in \theta$ .

For  $\epsilon > 0$  and  $g \in \mathsf{G}$ , the  $\epsilon$ -shadow of  $g$  is defined as the set

$$g \cdot B_{\theta,g}^\epsilon \subset \mathcal{F}_\theta.$$

Within this shadow, Iwasawa cocycles behave nicely due to the following estimate.

**Lemma 4.5** ([Qui02b, Lemma 4.2]). *Let  $0 < \epsilon \leq 1$  and  $g \in \mathsf{G}$ . For any  $x \in B_{\theta,g}^\epsilon$  and  $\alpha \in \theta$ , if  $v \in \Phi_\alpha(x) \setminus \{0\}$ , then*

$$\frac{\|\rho_\alpha(g)v\|_\alpha}{\|v\|_\alpha} \geq \epsilon \cdot \|\rho_\alpha(g)\|_\alpha.$$

Indeed, together with Lemma 4.4, this implies the compatibility of Cartan projections and Iwasawa cocycles in shadows.

**Lemma 4.6** ([Qui02b, Lemma 6.5]). *For any  $0 < \epsilon \leq 1$  and  $g \in \mathsf{G}$ , if  $x \in B_{\theta,g}^\epsilon$ , then*

$$\chi_\alpha(\kappa(g)) + \log \epsilon \leq \chi_\alpha(\sigma_\theta(g, x)) \leq \chi_\alpha(\kappa(g)) \quad \text{for all } \alpha \in \theta$$

**4.6. Local regularity of Iwasawa cocycles.** We deduce locally Lipschitz property of Iwasawa cocycles. We emphasize that the Lipschitz constant below is independent of the choice of a group element.

**Proposition 4.7.** *Let  $0 < \epsilon \leq 1$  and  $g \in \mathsf{G}$ . Then for any  $x, y \in g \cdot B_{\theta,g}^\epsilon$  and  $\alpha \in \theta$ ,*

$$|\chi_\alpha(\sigma_\theta(g^{-1}, x) - \sigma_\theta(g^{-1}, y))| \leq \frac{1}{\epsilon} \cdot d_{\mathbb{P}(V_\alpha)}(\Phi_\alpha(g^{-1}x), \Phi_\alpha(g^{-1}y)).$$

*Proof.* By Equation (4.3), it suffices to show that for  $x', y' \in B_{\theta,g}^\epsilon$  and  $\alpha \in \theta$ ,

$$|\chi_\alpha(\sigma_\theta(g, x') - \sigma_\theta(g, y'))| \leq \frac{1}{\epsilon} \cdot d_{\mathbb{P}(V_\alpha)}(\Phi_\alpha(x'), \Phi_\alpha(y')).$$

Fix vectors  $v \in \Phi_\alpha(x')$  and  $w \in \Phi_\alpha(y')$  such that  $\|v\|_\alpha = \|w\|_\alpha = 1$  and  $d_{\mathbb{P}(V_\alpha)}(\Phi_\alpha(x'), \Phi_\alpha(y')) = \|v - w\|_\alpha$ .

Now by Lemma 4.4,

$$|\chi_\alpha(\sigma_\theta(g, x') - \sigma_\theta(g, y'))| = |\log \|\rho_\alpha(g)v\|_\alpha - \log \|\rho_\alpha(g)w\|_\alpha|.$$

Then by the mean value theorem for  $\log : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  and Lemma 4.5,

$$\begin{aligned} |\log \|\rho_\alpha(g)v\|_\alpha - \log \|\rho_\alpha(g)w\|_\alpha| &\leq \frac{|\|\rho_\alpha(g)v\|_\alpha - \|\rho_\alpha(g)w\|_\alpha|}{\epsilon \cdot \|\rho_\alpha(g)\|_\alpha} \\ &\leq \frac{\|\rho_\alpha(g)(v-w)\|_\alpha}{\epsilon \cdot \|\rho_\alpha(g)\|_\alpha} \\ &\leq \frac{1}{\epsilon} \cdot \|v-w\|_\alpha. \end{aligned}$$

Combining altogether finishes the proof.  $\square$

## 5. DYNAMICS OF DISCRETE SUBGROUPS

In this section, let  $\Gamma < G$  be a discrete subgroup and fix a non-empty subset  $\theta \subset \Delta$ . Major objects in this paper are transverse subgroups, or Anosov or relatively Anosov subgroups, which we define in this section. An important observation in this section is a local regularity of Iwasawa cocycles for transverse subgroups, with a certain uniform control (Theorem 5.5). We continue the notations and terminologies introduced in Section 4.

The *limit set*  $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$  of  $\Gamma$  is defined as the set of all accumulation points of  $\Gamma$  in the sense of Equation (4.1). With respect to the natural  $\Gamma$ -action on  $\mathcal{F}_\theta = G / P_\theta$  given by the left multiplication,  $\Lambda_\theta(\Gamma)$  is  $\Gamma$ -invariant. When  $\Gamma < G$  is Zariski dense,  $\Lambda_\theta(\Gamma)$  is the unique  $\Gamma$ -minimal set and is Zariski dense subset of  $\mathcal{F}_\theta$  by the work of Benoist [Ben97].

**Definition 5.1.** A discrete subgroup  $\Gamma < G$  is called  $P_\theta$ -transverse if

- for any infinite sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \Gamma$  and  $\alpha \in \theta$ ,

$$\alpha(\kappa(g_n)) \rightarrow +\infty \quad \text{as } n \rightarrow +\infty$$

and

- for any two distinct  $x, y \in \Lambda_{\theta \cup i(\theta)}(\Gamma)$ ,

$x$  and  $y$  are in general position.

We call a  $P_\theta$ -transverse subgroup *non-elementary* if  $\#\Lambda_{\theta \cup i(\theta)}(\Gamma) \geq 3$ .

Note that points in general position are also said to be transverse, and hence the name.

Since  $i(\kappa(g)) = \kappa(g^{-1})$  for all  $g \in G$ ,  $P_\theta$ -transversality of  $\Gamma$  is equivalent to  $P_{i(\theta)}$ -transversality, and to  $P_{\theta \cup i(\theta)}$ -transversality. For a  $P_\theta$ -transverse subgroup  $\Gamma < G$  and for any  $\theta' \subset \theta \cup i(\theta)$ , the canonical projection  $\mathcal{F}_{\theta \cup i(\theta)} \rightarrow \mathcal{F}_{\theta'}$  is restricted to a  $\Gamma$ -equivariant homeomorphism  $\Lambda_{\theta \cup i(\theta)}(\Gamma) \rightarrow \Lambda_{\theta'}(\Gamma)$  (cf. [KOW25b, Lemma 9.5]).

An important property of transverse subgroups is that their actions on limit sets are convergence actions, which was proved by Kapovich–Leeb–Porti.

**Theorem 5.2** ([KLP17, Theorem 4.16]). *Let  $\Gamma < G$  be a  $P_\theta$ -transverse subgroup. Then the  $\Gamma$ -action on  $\Lambda_\theta(\Gamma)$  is a convergence action.*

Hence, the notion of being non-elementary coincides with the one for convergence groups. In addition, we employ the notion of *conical limit set* and the classification of elements of a convergence group in our setting.

**5.1. Shadows for transverses subgroups.** In [BCZZ24b], Blayac–Canary–Zhu–Zimmer developed Patterson–Sullivan theory for a general convergence group. We apply their framework to a  $\mathsf{P}_\theta$ -transverse subgroup  $\Gamma < \mathsf{G}$  and its action on  $\Lambda_\theta(\Gamma)$ , which was discussed in further details in [BCZZ24a, Section 10].

For a non-elementary  $\mathsf{P}_\theta$ -transverse subgroup  $\Gamma < \mathsf{G}$ , they showed that the compactification  $\Gamma \sqcup \Lambda_\theta(\Gamma)$  in the sense of Equation (4.1) is metrizable by a metric  $d_{BCZZ}$  on  $\Gamma \sqcup \Lambda_\theta(\Gamma)$  so that the natural  $\Gamma$ -action on  $\Gamma \sqcup \Lambda_\theta(\Gamma)$  by left multiplications is a convergence action [BCZZ24b, Proposition 2.3] (see also [BCZZ24a, Section 10]).

Using this metric, shadows were constructed in [BCZZ24b] as follows: for  $g \in \Gamma$  and  $\epsilon > 0$ , let  $\mathcal{N}_\epsilon(g^{-1}) \subset \Gamma \sqcup \Lambda_\theta(\Gamma)$  denote the open ball of radius  $\epsilon$  centered at  $g^{-1}$  with respect to the metric  $d_{BCZZ}$ . Then the  $\epsilon$ -shadow of  $g$  is defined as the set

$$\mathcal{S}_\epsilon(g) := g \cdot (\Lambda_\theta(\Gamma) \setminus \mathcal{N}_\epsilon(g^{-1})) \subset \Lambda_\theta(\Gamma).$$

The *conical limit set*

$$\Lambda_\theta^{\text{con}}(\Gamma) \subset \mathcal{F}_\theta$$

then can be defined as the set of points  $x \in \Lambda_\theta(\Gamma)$  such that there exist  $\epsilon > 0$  and an infinite sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \Gamma$  such that  $x \in \mathcal{S}_\epsilon(g_n)$  for all  $n \in \mathbb{N}$ . This notion of conical limit set coincides with the conical limit set for the convergence action  $\Gamma \curvearrowright \Lambda_\theta(\Gamma)$ , as shown in [BCZZ24b, Lemma 5.4].

The shadows defined as above are in fact comparable to the shadows defined in terms of Tits representations in Section 4.5. Note that in item (1) below, excluding finitely many elements might be necessary, because for a fixed  $g \in \Gamma$ ,  $\mathcal{S}_\epsilon(g)$  can be the whole limit set  $\Lambda_\theta(\Gamma)$  when  $\epsilon$  is small enough while  $g \cdot B_{\theta,g}^\epsilon$  always excludes hyperplanes in  $\mathbb{P}(V_\alpha)$ 's.

**Proposition 5.3.** *Let  $\Gamma < \mathsf{G}$  be a non-elementary  $\mathsf{P}_\theta$ -transverse subgroup.*

- (1) *For any  $\epsilon > 0$ , there exists  $\epsilon' = \epsilon'(\epsilon) > 0$  such that*

$$\mathcal{S}_\epsilon(g) \subset g \cdot B_{\theta,g}^{\epsilon'} \quad \text{for all but finitely many } g \in \Gamma.$$

- (2) *For any  $\epsilon > 0$ , there exists  $\epsilon' = \epsilon'(\epsilon) > 0$  such that*

$$g \cdot B_{\theta,g}^\epsilon \cap \Lambda_\theta(\Gamma) \subset \mathcal{S}_{\epsilon'}(g) \quad \text{for all } g \in \Gamma.$$

*Proof.* We first prove (1). Suppose to the contrary that for some  $\epsilon > 0$ , there exists an infinite sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \Gamma$  such that

$$\mathcal{S}_\epsilon(g_n) \not\subset g_n \cdot B_{\theta,g_n}^{1/n} \quad \text{for all } n \in \mathbb{N}.$$

In other words,

$$\Lambda_\theta(\Gamma) \setminus \mathcal{N}_\epsilon(g_n^{-1}) \not\subset B_{\theta,g_n}^{1/n} \quad \text{for all } n \in \mathbb{N}.$$

Let  $\{\ell_n\}_{n \in \mathbb{N}} \subset K$  be a sequence such that  $g_n \in KA^+ \ell_n$  in Cartan decomposition, for each  $n \in \mathbb{N}$ . Then there exists a sequence  $\{x_n\}_{n \in \mathbb{N}} \subset \Lambda_\theta(\Gamma)$  such that for each  $n \in \mathbb{N}$ ,

- (a)  $d_{BCZZ}(x_n, g_n^{-1}) \geq \epsilon$  and
- (b)  $d_{\mathbb{P}(V_{\alpha_n})}(\Phi_{\alpha_n}(x_n), \Phi_{\alpha_n}^*(\ell_n^{-1} w_0 P_{i(\alpha_n)})) < \frac{1}{n}$  for some  $\alpha_n \in \theta$ .

After passing to a subsequence, we may assume that  $x_n \rightarrow x \in \Lambda_\theta(\Gamma)$  and  $\ell_n \rightarrow \ell \in K$  as  $n \rightarrow +\infty$ , and  $\alpha_n = \alpha \in \theta$  for all  $n \in \mathbb{N}$ .

By the  $P_\theta$ -transversality of  $\Gamma$  and that  $\{g_n\}_{n \in \mathbb{N}}$  is infinite,  $g_n^{-1} \in \Gamma$  converges to  $\ell^{-1} w_0 P_\theta \in \mathcal{F}_\theta$ . Hence, it follows from (a) that  $x \neq \ell^{-1} w_0 P_\theta$ . Recall that we have a  $\Gamma$ -equivariant homeomorphism  $\Lambda_{\theta \cup i(\theta)}(\Gamma) \rightarrow \Lambda_\theta(\Gamma)$  which is a restriction of the projection  $\mathcal{F}_{\theta \cup i(\theta)} \rightarrow \mathcal{F}_\theta$ . Since we also have a convergence  $g_n^{-1} \rightarrow \ell^{-1} w_0 P_{\theta \cup i(\theta)} \in \mathcal{F}_{\theta \cup i(\theta)}$  as  $n \rightarrow +\infty$ , if  $\tilde{x} \in \Lambda_{\theta \cup i(\theta)}(\Gamma)$  is the point mapped to  $x \in \Lambda_\theta(\Gamma)$  by the above homeomorphism, then  $\tilde{x} \neq \ell^{-1} w_0 P_{\theta \cup i(\theta)}$ . Therefore,

$$(5.1) \quad \tilde{x} \in \mathcal{F}_{\theta \cup i(\theta)} \quad \text{and} \quad \ell^{-1} w_0 P_{\theta \cup i(\theta)} \in \mathcal{F}_{\theta \cup i(\theta)} \quad \text{are in general position}$$

by the  $P_\theta$ -transversality of  $\Gamma$ .

On the other hand, by (b), the line  $\Phi_\alpha(x) \subset V_\alpha$  is contained in the hyperplane  $\Phi_\alpha^*(\ell^{-1} w_0 P_{i(\alpha)}) \subset V_\alpha$ . This implies that if  $x_\alpha \in \Lambda_\alpha(\Gamma)$  is the projection of  $x \in \Lambda_\theta(\Gamma)$ , then  $x_\alpha \in \mathcal{F}_\alpha$  and  $\ell^{-1} w_0 P_{i(\alpha)} \in \mathcal{F}_{i(\alpha)}$  are not in general position. This contradicts Equation (5.1), and (1) follows.

We now show (2). Suppose to the contrary that for some  $\epsilon > 0$ , there exists a sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \Gamma$  such that

$$g_n \cdot B_{\theta, g_n}^\epsilon \cap \Lambda_\theta(\Gamma) \not\subset \mathcal{S}_{1/n}(g_n) \quad \text{for all } n \in \mathbb{N}.$$

In particular,

$$B_{\theta, g_n}^\epsilon \cap \Lambda_\theta(\Gamma) \not\subset \Lambda_\theta(\Gamma) \setminus \mathcal{N}_{1/n}(g_n^{-1}) \quad \text{for all } n \in \mathbb{N}.$$

If  $\{g_n\}_{n \in \mathbb{N}}$  is a finite sequence, then the right hand side above eventually becomes  $\Lambda_\theta(\Gamma)$ , yielding a contradiction.

Hence, the sequence  $\{g_n\}_{n \in \mathbb{N}} \subset K$  is infinite. Let  $\{\ell_n\}_{n \in \mathbb{N}} \subset K$  be a sequence such that  $g_n \in KA^+ \ell_n$  in Cartan decomposition, for each  $n \in \mathbb{N}$ . Then there exists a sequence  $\{x_n\}_{n \in \mathbb{N}} \subset \Lambda_\theta(\Gamma)$  such that for each  $n \in \mathbb{N}$ ,

$$x_n \in \mathcal{N}_{1/n}(g_n^{-1}) \quad \text{and} \quad d_{\mathbb{P}(V_\alpha)}(\Phi_\alpha(x_n), \Phi_\alpha^*(\ell_n^{-1} w_0 P_{i(\alpha)})) \geq \epsilon \quad \text{for all } \alpha \in \theta.$$

After passing to a subsequence, we may assume that  $x_n \rightarrow x \in \Lambda_\theta(\Gamma)$  and  $\ell_n \rightarrow \ell \in K$  as  $n \rightarrow +\infty$ . Then as in the proof of (1),  $g_n^{-1} \in \Gamma$  converges to  $\ell^{-1} w_0 P_{\theta \cup i(\theta)} \in \mathcal{F}_{\theta \cup i(\theta)}$  as  $n \rightarrow +\infty$ , and if  $\tilde{x} \in \Lambda_{\theta \cup i(\theta)}(\Gamma)$  projects to  $x \in \Lambda_\theta(\Gamma)$ , then

$$\tilde{x} = \ell^{-1} w_0 P_{\theta \cup i(\theta)}.$$

In particular, for each  $\alpha \in \theta$ , the line  $\Phi_\alpha(x) = \rho_\alpha(\ell^{-1} w_0) V_\alpha^+$  is contained in the hyperplane  $\rho_\alpha(\ell^{-1}) V_\alpha^- = \Phi_\alpha^*(\ell^{-1} w_0 P_{i(\alpha)})$ .

On the other hand, we have

$$d_{\mathbb{P}(V_\alpha)}(\Phi_\alpha(x), \Phi_\alpha^*(\ell^{-1} w_0 P_{i(\alpha)})) \geq \epsilon \quad \text{for all } \alpha \in \theta,$$

which is a contradiction.  $\square$

**5.2. Iwasawa cocycles for transverse subgroups.** For a  $P_\theta$ -transverse subgroup  $\Gamma < G$ , we call  $\varphi \in \Gamma$  *loxodromic* if it is a loxodromic element for a convergence action  $\Gamma \curvearrowright \Lambda_\theta(\Gamma)$ . For a loxodromic  $\varphi \in \Gamma$ , we denote by  $\varphi^+ \in \Lambda_\theta(\Gamma)$  its attracting fixed point.

**Lemma 5.4.** *Let  $\Gamma < G$  be a non-elementary  $P_\theta$ -transverse subgroup. For any loxodromic  $\varphi \in \Gamma$ ,*

$$\sigma_\theta(\varphi, \varphi^+) = \lambda_\theta(\varphi).$$

*Proof.* Fix  $\epsilon > 0$  such that  $\varphi^+ \in S_\epsilon(\varphi^n)$  for all  $n \in \mathbb{N}$ . Then by Proposition 5.3 and Lemma 4.6, there exists  $C > 0$  such that

$$\|\sigma_\theta(\varphi^n, \varphi^+) - p_\theta(\kappa(\varphi^n))\| \leq C \quad \text{for all large } n \in \mathbb{N}.$$

By Equation (4.2),  $\sigma_\theta(\varphi^n, \varphi^+) = n \cdot \sigma_\theta(\varphi, \varphi^+)$  for all  $n \in \mathbb{N}$ . Therefore,

$$\sigma_\theta(\varphi, \varphi^+) = \lim_{n \rightarrow +\infty} \frac{\sigma_\theta(\varphi^n, \varphi^+)}{n} = \lim_{n \rightarrow +\infty} \frac{p_\theta(\kappa(\varphi^n))}{n} = \lambda_\theta(\varphi)$$

as desired.  $\square$

The following is one of the key observations on transverse subgroups in this paper. It is important that  $N$  below does not depend on  $g \in \Gamma$ .

**Theorem 5.5.** *Let  $\Gamma < G$  be a non-elementary  $P_\theta$ -transverse subgroup. Let  $\varphi \in \Gamma$  be loxodromic and  $\epsilon > 0$ . Then there exists  $N = N(\varphi, \epsilon) \in \mathbb{N}$  such that for any  $g \in \Gamma$  and  $n \geq N$ , if  $x, g\varphi g^{-1}x \in S_\epsilon(g) \cap g \cdot S_\epsilon(\varphi^n)$ , then*

$$\|\sigma_\theta(g\varphi g^{-1}, x) - \lambda_\theta(\varphi)\| < \epsilon.$$

*Proof.* Suppose to the contrary that there exist sequences  $\{g_n\}_{n \in \mathbb{N}} \subset \Gamma$ ,  $\{x_n\}_{n \in \mathbb{N}} \subset \Lambda_\theta(\Gamma)$ , and  $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$  such that  $k_n \rightarrow +\infty$  as  $n \rightarrow +\infty$  and for all large  $n \in \mathbb{N}$ ,

$$x_n, g_n \varphi g_n^{-1} x_n \in S_\epsilon(g_n) \cap g_n \cdot S_\epsilon(\varphi^{k_n}) \quad \text{and} \quad \|\sigma_\theta(g_n \varphi g_n^{-1}, x_n) - \lambda_\theta(\varphi)\| \geq \epsilon.$$

If the sequence  $\{g_n\}_{n \in \mathbb{N}}$  is finite, then after passing to a subsequence,  $g_n = g \in \Gamma$  for all  $n \in \mathbb{N}$ , and hence  $x_n$  converges to  $g\varphi^+$ . By Lemma 5.4, this is a contradiction.

Hence, the sequence  $\{g_n\}_{n \in \mathbb{N}}$  is infinite. By Equation (4.2) and Equation (4.3), we have for each  $n \in \mathbb{N}$  that

$$\begin{aligned} \sigma_\theta(g_n \varphi g_n^{-1}, x_n) &= \sigma_\theta(g_n, \varphi g_n^{-1} x_n) + \sigma_\theta(\varphi g_n^{-1}, x_n) \\ &= \sigma_\theta(g_n, \varphi g_n^{-1} x_n) + \sigma_\theta(\varphi, g_n^{-1} x_n) + \sigma_\theta(g_n^{-1}, x_n) \\ &= \sigma_\theta(g_n, \varphi g_n^{-1} x_n) + \sigma_\theta(\varphi, g_n^{-1} x_n) - \sigma_\theta(g_n, g_n^{-1} x_n). \end{aligned}$$

Since  $g_n^{-1} x_n \in S_\epsilon(\varphi^{k_n})$  for all  $n \in \mathbb{N}$ , we have  $g_n^{-1} x_n \rightarrow \varphi^+$  as  $n \rightarrow +\infty$ . This implies  $\sigma_\theta(\varphi, g_n^{-1} x_n) \rightarrow \lambda_\theta(\varphi)$  as  $n \rightarrow +\infty$ , by Lemma 5.4. Therefore, for all large  $n \in \mathbb{N}$ , we have

$$(5.2) \quad \|\sigma_\theta(g_n, \varphi g_n^{-1} x_n) - \sigma_\theta(g_n, g_n^{-1} x_n)\| > \frac{\epsilon}{2}.$$

On the other hand, since  $\{g_n\}_{n \in \mathbb{N}}$  is an infinite sequence it follows from Proposition 5.3 that for some  $\epsilon' = \epsilon'(\epsilon)$ , we have after passing to a subsequence that

$$\mathcal{S}_\epsilon(g_n) \subset g_n \cdot B_{\theta, g_n}^{\epsilon'} \quad \text{for all } n \in \mathbb{N}.$$

Since  $x_n, g_n \varphi g_n^{-1} x_n \in \mathcal{S}_\epsilon(g_n)$ , we have for each  $\alpha \in \theta$  and all  $n \in \mathbb{N}$  that

$$\begin{aligned} & |\chi_\alpha(\sigma_\theta(g_n^{-1}, x_n) - \sigma_\theta(g_n^{-1}, g_n \varphi g_n^{-1} x_n))| \\ & \leq \frac{1}{\epsilon'} \cdot d_{\mathbb{P}(V_\alpha)}(\Phi_\alpha(g_n^{-1} x_n), \Phi_\alpha(\varphi g_n^{-1} x_n)) \end{aligned}$$

by Proposition 4.7. By Equation (4.3), this is same ase

$$\begin{aligned} & |\chi_\alpha(\sigma_\theta(g_n, g_n^{-1} x_n) - \sigma_\theta(g_n, \varphi g_n^{-1} x_n))| \\ & \leq \frac{1}{\epsilon'} \cdot d_{\mathbb{P}(V_\alpha)}(\Phi_\alpha(g_n^{-1} x_n), \Phi_\alpha(\varphi g_n^{-1} x_n)). \end{aligned}$$

Since  $\chi_\alpha = k_\alpha \omega_\alpha$  for some  $k_\alpha \in \mathbb{N}$  and  $\{\omega_\alpha : \alpha \in \theta\}$  is a basis of  $\mathfrak{a}_\theta^*$ , there exists  $C > 0$  such that

$$\|\cdot\| \leq C \sum_{\alpha \in \theta} |\chi_\alpha(\cdot)|.$$

Together with Equation (5.2),

$$\frac{\epsilon}{2} < \frac{C}{\epsilon'} \sum_{\alpha \in \theta} d_{\mathbb{P}(V_\alpha)}(\Phi_\alpha(g_n^{-1} x_n), \Phi_\alpha(\varphi g_n^{-1} x_n))$$

for all  $n \in \mathbb{N}$ .

On the other hand, since  $x_n, g_n \varphi g_n^{-1} x_n \in g_n \cdot \mathcal{S}_\epsilon(\varphi^{k_n})$  and hence we have  $g_n^{-1} x_n, \varphi g_n^{-1} x_n \in \mathcal{S}_\epsilon(\varphi^{k_n})$  for all  $n \in \mathbb{N}$ , both  $g_n^{-1} x_n$  and  $\varphi g_n^{-1} x_n$  converge to  $\varphi^+$  as  $n \rightarrow +\infty$ . Therefore,

$$\frac{\epsilon}{2} < 0,$$

contradiction.  $\square$

**5.3. Patterson–Sullivan measures of transverse subgroups.** Canary–Zhang–Zimmer initiated the study of Patterson–Sullivan theory for transverse subgroups in [CZZ24]. Later, this was extended further to general convergence groups with coarse cocycles by Blayac–Canary–Zhu–Zimmer [BCZZ24b]. We briefly review some of their results.

Let  $\Gamma < \mathsf{G}$  be a  $\mathbb{P}_\theta$ -transverse subgroup and  $\psi \in \mathfrak{a}_\theta^*$ . The associated Poincaré series is defined by

$$\mathcal{P}_{\Gamma, \psi}(s) := \sum_{g \in \Gamma} e^{-s\psi(\kappa(g))} \quad \text{for } s \in \mathbb{R},$$

and the *critical exponent* is its abscissa of convergence:

$$\delta_\psi(\Gamma) := \inf\{s > 0 : \mathcal{P}_{\Gamma, \psi}(s) < +\infty\} \in [0, +\infty].$$

At the dimension of critical exponent, Patterson–Sullivan measure always exists. The following is a special case of [CZZ24, Proposition 3.2].

**Theorem 5.6** ([CZZ24, Proposition 3.2]). *Let  $\Gamma < G$  be a non-elementary  $P_\theta$ -transverse subgroup and  $\psi \in \mathfrak{a}_\theta^*$ . If  $\delta_\psi(\Gamma) < +\infty$ , then there exists a  $\delta_\psi(\Gamma)$ -dimensional  $\psi$ -Patterson–Sullivan measure of  $\Gamma$  on  $\Lambda_\theta(\Gamma)$ .*

Indeed, existence of Patterson–Sullivan measure is equivalent to finiteness of the critical exponent. Under an extra assumption that  $\delta_\psi(\Gamma) < +\infty$ , the following was proved earlier in [CZZ24]. Later, under  $\delta_\psi(\Gamma) < +\infty$  but for measures not necessarily supported on limit sets, it was shown in [KOW25b].

**Theorem 5.7** ([BCZZ24a, Proposition 10.1]). *Let  $\Gamma < G$  be a non-elementary  $P_\theta$ -transverse subgroup and  $\psi \in \mathfrak{a}_\theta^*$ . If there exists a  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure of  $\Gamma$  on  $\Lambda_\theta(\Gamma)$ , then*

$$\delta_\psi(\Gamma) \leq \delta.$$

*In particular,  $\delta_\psi(\Gamma) < +\infty$ .*

Whether a Poincaré series diverges or not at the critical exponent turns out to be important in the study of related dynamical systems.

**Definition 5.8.** For a  $P_\theta$ -transverse  $\Gamma < G$ , we say that  $\Gamma$  is of  $\psi$ -divergence type for  $\psi \in \mathfrak{a}_\theta^*$  if  $\delta_\psi(\Gamma) < +\infty$  and  $\mathcal{P}_{\Gamma,\psi}(\delta_\psi(\Gamma)) = +\infty$ . A Patterson–Sullivan measure  $\nu$  of  $\Gamma$  on  $\mathcal{F}_\theta$  is also said to be of divergence type if  $\nu$  is a  $\delta_\psi(\Gamma)$ -dimensional  $\psi$ -Patterson–Sullivan measure of  $\Gamma$  such that  $\Gamma$  is of  $\psi$ -divergence type.

As part of their generalization of Hopf–Tsuji–Sullivan dichotomy, Canary–Zhang–Zimmer proved the following for Patterson–Sullivan measures supported on limit sets, which was extended further by Kim–Oh–Wang, removing an assumption on their supports:

**Theorem 5.9** ([CZZ24], [KOW25b]). *Let  $\Gamma < G$  be a non-elementary  $P_\theta$ -transverse subgroup and  $\psi \in \mathfrak{a}_\theta^*$ . Let  $\nu$  be a  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure of  $\Gamma$  on  $\mathcal{F}_\theta$ .*

*Then the following are equivalent:*

- (1)  $\delta = \delta_\psi(\Gamma) < +\infty$  and  $\Gamma$  is of  $\psi$ -divergence type.
- (2)  $\nu(\Lambda_\theta^{\text{con}}(\Gamma)) = 1$ .

*Moreover, in this case, the  $\Gamma$ -action on  $(\Lambda_\theta(\Gamma), \nu)$  is ergodic and  $\nu$  is the unique  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure on  $\Lambda_\theta(\Gamma)$ .*

**5.4. Hypertransverse subgroups and (relatively) Anosov subgroups.** We consider subclasses of transverse subgroups, whose action on limit sets can be realized by nice actions on Gromov hyperbolic spaces. More precisely, a  $P_\theta$ -transverse subgroup  $\Gamma < G$  is called  $P_\theta$ -hypertransverse if there exists a proper geodesic Gromov hyperbolic space  $(Z, d_Z)$  such that

- $\Gamma$  acts on  $Z$  properly discontinuously by isometries and
- there exists a  $\Gamma$ -equivariant homeomorphism

$$(5.3) \quad \Psi : \Lambda_Z(\Gamma) \rightarrow \Lambda_\theta(\Gamma),$$

where  $\Lambda_Z(\Gamma) \subset \partial Z$  is the limit set of  $\Gamma$ .

We call  $(Z, d_Z)$  a model space and the map in Equation (5.3) boundary map. The notion of hypertransverse subgroups was introduced in the work of the second author [Kim24] to study ergodicity of horospherical foliations and its relation to rigidity theory.

*Remark 5.10.* Canary–Zhang–Zimmer showed that a general transverse subgroup is realized as a projectively visible subgroup of isometries on a properly convex domain equipped with the Hilbert metric, with an equivariant homeomorphisms between limit sets [CZZ24]. It is unknown whether this Hilbert metric can be chosen to be Gromov hyperbolic on the convex hull of the limit set. It is also unknown whether there exists a transverse subgroup which is not hypertransverse.

One can see that the shadows for  $\Gamma$ -orbits in  $(Z, d_Z)$  are comparable to the shadows defined in Section 5.1, by the same argument as in [CK25a, Lemma 3.17]. Hence, the conical limit set of  $\Gamma$  in  $\partial Z$  is identified with the conical limit set of  $\Gamma$  in  $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$ , via the boundary map  $\Psi$  in Equation (5.3).

**Proposition 5.11.** *Let  $\Gamma < G$  be a non-elementary  $\mathsf{P}_\theta$ -hypertransverse subgroup with a model space  $(Z, d_Z)$  and boundary map  $\Psi : \Lambda_Z(\Gamma) \rightarrow \Lambda_\theta(\Gamma)$ .*

(1) *For any  $R > 0$  and  $z \in Z$ , there exists  $\epsilon = \epsilon(R, z) > 0$  such that*

$$\Psi(O_R(z, gz) \cap \Lambda_Z(\Gamma)) \subset \mathcal{S}_\epsilon(g) \quad \text{for all } g \in \Gamma.$$

(2) *For any  $\epsilon > 0$  and  $z \in Z$ , there exists  $R = R(\epsilon, z) > 0$  such that*

$$\mathcal{S}_\epsilon(g) \subset \Psi(O_R(z, gz) \cap \Lambda_Z(\Gamma)) \quad \text{for all } g \in \Gamma.$$

Together with Proposition 5.3(1), Lemma 4.6, and Theorem 5.5, we have the following estimates on Iwasawa cocycles.

**Corollary 5.12.** *Let  $\Gamma < G$  be a non-elementary  $\mathsf{P}_\theta$ -hypertransverse subgroup with a model space  $(Z, d_Z)$  and boundary map  $\Psi : \Lambda_Z(\Gamma) \rightarrow \Lambda_\theta(\Gamma)$ . Fix  $z \in Z$ .*

(1) *For any  $R > 0$ , there exists  $c = c(R, z) > 0$  such that for any  $g \in \Gamma$  and  $x \in O_R(z, gz) \cap \Lambda_Z(\Gamma)$ ,*

$$\chi_\alpha(\kappa(g)) - c \leq \chi_\alpha(\sigma_\theta(g, \Psi(g^{-1}x))) \leq \chi_\alpha(\kappa(g)) \quad \text{for all } \alpha \in \theta.$$

(2) *Let  $\varphi \in \Gamma$  be loxodromic and  $R > 0$ . For any  $\epsilon > 0$ , there exists  $N = N(\varphi, R, \epsilon, z) \in \mathbb{N}$  such that for any  $g \in \Gamma$  and  $n \geq N$ , if  $x, g\varphi g^{-1}x \in O_R(z, gz) \cap O_R(gz, g\varphi^n z) \cap \Lambda_Z(\Gamma)$ , then*

$$\|\sigma_\theta(g\varphi g^{-1}, \Psi(x)) - \lambda_\theta(\varphi)\| < \epsilon.$$

Important examples of hypertransverse subgroups are Anosov subgroups and relatively Anosov subgroups.

**Definition 5.13.** Let  $\Gamma < G$  be a non-elementary  $\mathsf{P}_\theta$ -transverse subgroup.

- $\Gamma$  is  $\mathsf{P}_\theta$ -Anosov if  $\Gamma$  is a hyperbolic group and there exists a  $\Gamma$ -equivariant homeomorphism  $\partial\Gamma \rightarrow \Lambda_\theta(\Gamma)$ , where  $\partial\Gamma$  is the Gromov boundary of  $\Gamma$ .

- $\Gamma$  is *relatively  $P_\theta$ -Anosov* if  $\Gamma$  is a relatively hyperbolic group with some choice of a peripheral structure and there exists a  $\Gamma$ -equivariant homeomorphism  $\partial_B\Gamma \rightarrow \Lambda_\theta(\Gamma)$ , where  $\partial_B\Gamma$  is the Bowditch boundary of  $\Gamma$  with respect to the chosen peripheral structure.

Hence, Anosov subgroups and relatively Anosov subgroups are hypertransverse with their Cayley graphs and Gromov models as model spaces, respectively. Note that Anosov subgroups are also special cases of relatively Anosov subgroups.

Importantly, Canary–Zhang–Zimmer proved that relatively Anosov subgroups are of divergence type. Therefore, in view of Theorem 5.9, they possess interesting dynamical properties.

**Theorem 5.14** ([CZZ25, Theorem 1.1]). *Let  $\Gamma < G$  be a non-elementary relatively  $P_\theta$ -Anosov subgroup and  $\psi \in \mathfrak{a}_\theta^*$ . If  $\delta_\psi(\Gamma) < +\infty$ , then  $\Gamma$  is of  $\psi$ -divergence type.*

They also characterized finiteness of critical exponents in terms of limit cones.

**Theorem 5.15** ([CZZ25, Theorem 10.1]). *Let  $\Gamma < G$  be a non-elementary relatively  $P_\theta$ -Anosov subgroup and  $\psi \in \mathfrak{a}_\theta^*$ . Then  $\delta_\psi(\Gamma) < +\infty$  if and only if  $\psi > 0$  on  $\mathcal{L}_\Gamma \setminus \{0\}$ .*

**5.5. Closed orbits in  $\mathcal{H}_\theta$ .** For a relatively  $P_\theta$ -Anosov subgroup  $\Gamma < G$ , we denote by  $\Lambda_\theta^P(\Gamma) \subset \Lambda_\theta(\Gamma)$  the *parabolic limit set* of  $\Gamma$ , which is defined as the image of parabolic limit points of  $\Gamma$  on its Bowditch boundary  $\partial_B\Gamma$  under the equivariant homeomorphism  $\partial_B\Gamma \rightarrow \Lambda_\theta(\Gamma)$ .

Noting that  $\Lambda_\theta^P(\Gamma)$  is  $\Gamma$ -invariant and  $\Lambda_\theta(\Gamma) = \Lambda_\theta^{\text{con}}(\Gamma) \sqcup \Lambda_\theta^P(\Gamma)$ , we record that  $\Gamma$ -orbits of horospheres based at  $\Lambda_\theta^P(\Gamma)$  are closed in  $\mathcal{H}_\theta$ .

**Proposition 5.16.** *Let  $\Gamma < G$  be a non-elementary relatively  $P_\theta$ -Anosov subgroup. For any  $(x, u) \in \Lambda_\theta^P(\Gamma) \times \mathfrak{a}_\theta \subset \mathcal{H}_\theta$ , its orbit  $\Gamma \cdot (x, u)$  is closed in  $\mathcal{H}_\theta$ , i.e.,*

$$\overline{\Gamma \cdot (x, u)} = \Gamma \cdot (x, u) \quad \text{in } \mathcal{H}_\theta.$$

To prove Proposition 5.16, we introduce a result of the second author joint with Oh and Wang [KOW25b]. For a non-elementary  $P_\theta$ -transverse subgroup  $\Gamma < G$ , set

$$\Lambda_\theta^{(2)}(\Gamma) := \{(x, y) \in \Lambda_\theta(\Gamma) \times \Lambda_{i(\theta)}(\Gamma) : x \text{ and } y \text{ are in general position}\},$$

noting that  $x \in \Lambda_\theta(\Gamma)$  and  $y \in \Lambda_{i(\theta)}(\Gamma)$  are in general position if and only if  $y$  is not the image of  $x$  under the  $\Gamma$ -equivariant homeomorphism  $\Lambda_\theta(\Gamma) \rightarrow \Lambda_{i(\theta)}(\Gamma)$ , as  $\Gamma$  is  $P_\theta$ -transverse.

Now consider the space

$$\Omega_\theta(\Gamma) := \Lambda_\theta^{(2)}(\Gamma) \times \mathfrak{a}_\theta$$

which can be regarded as a subspace of  $G/S_\theta$ . Then the  $G$ -action on  $G/S_\theta$  given by left multiplication induces the  $\Gamma$ -action on  $\Omega_\theta(\Gamma)$ . The induced  $\Gamma$ -action can be written as follows: for  $g \in \Gamma$  and  $(x, y, u) \in \Omega_\theta(\Gamma)$ ,

$$g \cdot (x, y, u) = (gx, gy, u + \sigma_\theta(g, x)).$$

Note that the action on the first and the third components are the same as the action on  $\mathcal{H}_\theta$ .

It is not true in general that the  $\Gamma$ -action on  $G/S_\theta$  is properly discontinuous, due to the non-compactness of  $S_\theta$ . On the other hand, its restriction on  $\Omega_\theta(\Gamma)$  is indeed properly discontinuous.

**Theorem 5.17** ([KOW25b, Theorem 1.7]). *Let  $\Gamma < G$  be a non-elementary  $P_\theta$ -transverse subgroup. Then the  $\Gamma$ -action on  $\Omega_\theta(\Gamma)$  is properly discontinuous.*

Using Theorem 5.17, we prove Proposition 5.16.

*Proof of Proposition 5.16.* Let  $(x, u) \in \Lambda_\theta^P(\Gamma) \times \mathfrak{a}_\theta$ . Suppose to the contrary that there exists a sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \Gamma$  such that the sequence  $g_n(x, u)$  converges in  $\mathcal{H}_\theta$ , with the limit in  $\mathcal{H}_\theta \setminus \Gamma \cdot (x, u)$ .

Let  $\Gamma_x < \Gamma$  denote the stabilizer of  $x$ . By the relative hyperbolicity of  $\Gamma$ , the  $\Gamma_x$ -action on  $\Lambda_\theta(\Gamma) \setminus \{x\}$  is properly discontinuous and cocompact. Let  $\iota : \Lambda_\theta(\Gamma) \rightarrow \Lambda_{i(\theta)}(\Gamma)$  be the  $\Gamma$ -equivariant homeomorphism. Then the  $\Gamma_x$ -action on  $\Lambda_{i(\theta)}(\Gamma) \setminus \{\iota(x)\}$  is properly discontinuous and cocompact as well. Hence, there exists a compact subset  $K \subset \Lambda_{i(\theta)}(\Gamma) \setminus \{\iota(x)\}$  such that  $\Gamma_x \cdot K = \Lambda_{i(\theta)}(\Gamma) \setminus \{\iota(x)\}$ .

Let  $y := \lim_{n \rightarrow +\infty} g_n \iota(x)$  and let  $U \subset \Lambda_{i(\theta)}(\Gamma)$  be an open neighborhood of  $y$  such that  $\#(\Lambda_{i(\theta)}(\Gamma) \setminus U) \geq 2$ . Then for each  $n \in \mathbb{N}$ , we have that  $(\Lambda_{i(\theta)}(\Gamma) \setminus \{\iota(x)\}) \setminus g_n^{-1}U \neq \emptyset$ , and hence there exist  $w_n \in K$  and  $h_n \in \Gamma_x$  such that  $h_n w_n \notin g_n^{-1}U$ . In particular,  $g_n h_n w_n \notin U$  for all  $n \in \mathbb{N}$ . After passing to a subsequence, we may assume that the sequence  $g_n h_n w_n$  converges in  $\Lambda_{i(\theta)}(\Gamma)$ , whose limit is different from  $y$ . Hence, the sequence

$$g_n h_n \cdot (x, w_n) = (g_n x, g_n h_n w_n) \in \Lambda_\theta^{(2)}(\Gamma)$$

converges in  $\Lambda_\theta^{(2)}(\Gamma)$ .

We now consider the sequence

$$(5.4) \quad g_n h_n \cdot (x, w_n, u) = (g_n x, g_n h_n w_n, u + \sigma_\theta(g_n h_n, x)) \in \Omega_\theta(\Gamma).$$

By Equation (4.2) and that  $h_n \in \Gamma_x$ , we have for all  $n \in \mathbb{N}$  that

$$\sigma_\theta(g_n h_n, x) = \sigma_\theta(g_n, x) + \sigma_\theta(h_n, x).$$

Since the sequence  $g_n \cdot (x, u) = (g_n x, u + \sigma_\theta(g_n, x))$  is convergent in  $\mathcal{H}_\theta$ , we also have that  $\sigma_\theta(g_n, x) \in \mathfrak{a}_\theta$  is bounded. Moreover, by [CZZ25, Theorem 4.4, Section 4.4], we may assume that  $\Gamma_x$  is a cocompact lattice in a closed Lie subgroup  $H < G$  with finitely many components such that  $H = L \ltimes U$  and  $H^\circ = L^\circ \times U^\circ$ , where  $L$  is compact,  $U$  is the unipotent radical of  $H$ , and the superscript  $\circ$  denotes the identity component. Therefore, we

have  $\sigma_\theta(g_n h_n, x) = \sigma_\theta(g_n, x)$  for all  $n \in \mathbb{N}$ , and in particular the sequence  $\sigma_\theta(g_n h_n, x) = \sigma_\theta(g_n, x) \in \mathfrak{a}_\theta$  is bounded as well.

Combining altogether, we have that the sequence

$$g_n h_n \cdot (x, w_n, u) \in \Omega_\theta(\Gamma)$$

in Equation (5.4) converges in  $\Omega_\theta(\Gamma)$ , after passing to a subsequence. On the other hand, the sequence  $(x, w_n, u) \in \Omega_\theta(\Gamma)$  converges in  $\Omega_\theta(\Gamma)$  as well after passing to a subsequence, since  $w_n \in K$  and  $K \subset \Lambda_{i(\theta)}(\Gamma)$  is a compact subset disjoint from  $\iota(x)$ .

By the proper discontinuity of the  $\Gamma$ -action on  $\Omega_\theta(\Gamma)$  (Theorem 5.17), this implies that the sequence  $\{g_n h_n\}_{n \in \mathbb{N}} \subset \Gamma$  is a finite sequence. After passing to a subsequence, we may assume that for some  $g \in \Gamma$ ,

$$g_n h_n = g \quad \text{for all } n \in \mathbb{N}.$$

Then we have

$$g_n \cdot (x, u) = g h_n^{-1} \cdot (x, u) = g \cdot (x, u) \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, the sequence  $g_n \cdot (x, u)$  is assumed to converge to a point in  $\mathcal{H}_\theta \setminus \Gamma \cdot (x, u)$ , which is a contradiction. This finishes the proof.  $\square$

## 6. MEASURE CLASSIFICATIONS

In this section, we prove the main theorem of this paper. For  $\Gamma < \mathsf{G}$ , we set

$$\mathrm{Spec}_\theta(\Gamma) := \{\lambda_\theta(\gamma) : \gamma \in \Gamma \text{ loxodromic}\}.$$

We call  $\mathrm{Spec}_\theta(\Gamma)$  *non-arithmetic* if it generates a dense additive subgroup of  $\mathfrak{a}_\theta$ . The non-arithmeticity holds when  $\Gamma$  is Zariski dense by the work of Benoist [Ben00] (Theorem 4.1).

Recalling the spaces

$$\begin{array}{ccccccc} \mathcal{R}_{\Gamma, \theta} & \subset & \mathcal{E}_{\Gamma, \theta} & \subset & \mathcal{H}_\theta \\ \parallel & & \parallel & & \parallel \\ \Lambda_\theta^{\mathrm{con}}(\Gamma) \times \mathfrak{a}_\theta & & \Lambda_\theta(\Gamma) \times \mathfrak{a}_\theta & & \mathcal{F}_\theta \times \mathfrak{a}_\theta \end{array}$$

we now state our main theorem.

**Theorem 6.1.** *Let  $\Gamma < \mathsf{G}$  be a non-elementary  $\mathsf{P}_\theta$ -hypertransverse subgroup with non-arithmetic  $\mathrm{Spec}_\theta(\Gamma)$ . If  $\mu$  is a  $\Gamma$ -invariant ergodic Radon measure on  $\mathcal{H}_\theta$  supported on  $\mathcal{R}_{\Gamma, \theta}$ , then*

$$\mu \text{ is a constant multiple of } \mu_\nu^{\mathrm{BR}}$$

for some divergence-type Patterson–Sullivan measure  $\nu$  of  $\Gamma$ .

Together with Proposition 5.16, we have the following corollary of Theorem 6.1 for relatively Anosov subgroups.

**Corollary 6.2.** *Let  $\Gamma < G$  be a non-elementary relatively  $P_\theta$ -Anosov subgroup with non-arithmetic  $\text{Spec}_\theta(\Gamma)$ . If  $\mu$  is a  $\Gamma$ -invariant ergodic Radon measure on  $\mathcal{E}_{\Gamma,\theta} \subset \mathcal{H}_\theta$ , then either*

- (1)  $\mu$  is supported on a closed  $\Gamma$ -orbit in  $\mathcal{H}_\theta \setminus \mathcal{R}_{\Gamma,\theta}$ , or
- (2)  $\mu$  is a constant multiple of  $\mu_\nu^{\text{BR}}$  for some divergence-type Patterson-Sullivan measure  $\nu$  of  $\Gamma$ .

The rest of this section is devoted to the proof of Theorem 6.1. We deduce statements in the introduction from Theorem 6.1 and Corollary 6.2 at the end of this section. We prove the theorem by establishing a robust relation between invariant Radon measures and guided limit sets. Note that due to ergodic decompositions, Theorem 6.1 can be regarded as the classification of  $\Gamma$ -invariant Radon measures on  $\mathcal{R}_{\Gamma,\theta} \subset \mathcal{H}_\theta$ .

**6.1. Concentration on guided limit sets.** In the rest of this section, let  $\Gamma < G$  be a non-elementary  $P_\theta$ -hypertransverse subgroup with a model space  $(Z, d_Z)$  and the boundary map  $\Psi : \Lambda_Z(\Gamma) \rightarrow \Lambda_\theta(\Gamma)$ . Since  $\Psi$  is a  $\Gamma$ -equivariant homeomorphism, we identify  $\Lambda_Z(\Gamma)$  and  $\Lambda_\theta(\Gamma)$  via  $\Psi$  for simplicity. We also fix a basepoint  $z_0 \in Z$ .

For a loxodromic  $\varphi \in \Gamma$  and  $C > 0$ , denote by  $\Lambda_Z^{\varphi,C}(\Gamma) \subset \Lambda_Z(\Gamma)$  the  $(\varphi, C)$ -guided limit set of  $\Gamma$ . We set

$$\Lambda_\theta^{\varphi,C}(\Gamma) := \Psi(\Lambda_Z^{\varphi,C}(\Gamma))$$

and call it the  $(\varphi, C)$ -guided limit set of  $\Gamma$  as well. We first show that invariant ergodic Radon measures are charged on guided limit sets.

**Theorem 6.3.** *Let  $\varphi \in \Gamma < \text{Isom}(Z)$  be a loxodromic element, and let  $C = C(\varphi) > 0$  be as in Lemma 2.11. Let  $\mu$  be a  $\Gamma$ -invariant ergodic Radon measure on  $\mathcal{H}_\theta$  supported on  $\mathcal{R}_{\Gamma,\theta}$ . Then  $\mu$  is supported on*

$$\Lambda_\theta^{\varphi,C}(\Gamma) \times \mathfrak{a}_\theta \subset \mathcal{H}_\theta.$$

*Proof.* Applying Lemma 2.15 to  $\varphi \in \Gamma < \text{Isom}(Z)$ , we get  $\beta(\varphi) > 0$  and  $a_1, a_2, a_3 \in \Gamma$  satisfying the conclusion of Lemma 2.15 as isometries on  $Z$ . Let  $C(\varphi) > 0$  be as in Lemma 2.11 for  $g = \varphi$ . We set  $C_0 := 10(\beta(\varphi) + C(\varphi))$ .

For each  $K > 0$  let

$$\Lambda_K := \left\{ x \in \Lambda_\theta(\Gamma) : \begin{array}{l} \exists \text{ an infinite sequence } \{g_j\}_{j \in \mathbb{N}} \subset \Gamma \text{ s.t.} \\ x \in O_K(z_0, g_j z_0) \text{ for all } j \in \mathbb{N} \end{array} \right\}.$$

Then  $\Gamma \cdot (\Lambda_K \times \mathfrak{a}_\theta) \subset \mathcal{H}_\theta$  is  $\Gamma$ -invariant and we have

$$\Lambda_\theta^{\text{con}}(\Gamma) \times \mathfrak{a}_\theta = \bigcup_{K>0} \Gamma \cdot (\Lambda_K \times \mathfrak{a}_\theta)$$

by Proposition 5.11. Since  $\mu$  is supported on  $\mathcal{R}_{\Gamma,\theta} = \Lambda_\theta^{\text{con}}(\Gamma) \times \mathfrak{a}_\theta$ ,

$$\mu(\Lambda_K \times \mathfrak{a}_\theta) > 0 \text{ for all large } K > 0.$$

We fix such  $K > 100C_0 + 2 \sum_{j=1}^3 d_Z(z_0, a_j z_0)$ .

We identify  $\mathfrak{a}_\theta = \mathbb{R}^{\# \theta}$  via the map  $u \mapsto (\chi_\alpha(u))_{\alpha \in \theta}$ . For each  $R > 0$ , we set

$$\mathcal{H}_{K,R} := \Lambda_K \times [-R, R]^{\# \theta}.$$

Since  $\Lambda_K \times \mathfrak{a}_\theta = \bigcup_{R=1}^{\infty} \mathcal{H}_{K,R}$ ,

$$\mu(\mathcal{H}_{K,R}) > 0 \quad \text{for all large } R > 0.$$

We fix such  $R > 0$ .

Now we fix  $n > \frac{1000(C_0+K+1)}{\tau_\varphi}$  and  $k > 0$ . We define a map

$$F = F_{n,k} : \mathcal{H}_{K,R} \rightarrow \mathcal{H}_\theta$$

as follows. For each  $\xi = (x, u) \in \mathcal{H}_{K,R}$ , there exists  $g \in \Gamma$  such that

$$(6.1) \quad d_Z(z_0, gz_0) > k \quad \text{and} \quad x \in O_K(z_0, gz_0).$$

Among many such  $g$ 's, take the one with minimal  $d_Z(z_0, gz_0)$  and call it  $g_\xi$ .<sup>4</sup> Then the map  $\xi \in \mathcal{H}_{K,R} \mapsto g_\xi$  is Borel measurable. By Lemma 2.15, there exists  $a_\xi \in \{a_1, a_2, a_3\}$  such that<sup>5</sup>

$$(6.2) \quad (z_0, g_\xi \cdot a_\xi[z_0, \varphi^n z_0], g_\xi \cdot a_\xi \varphi^n a_\xi \cdot g_\xi^{-1} x) \text{ is } C_0\text{-aligned in } Z \cup \partial Z$$

where  $x \in \Lambda_\theta(\Gamma)$  is considered as a point  $\Lambda_Z(\Gamma) \subset \partial Z$  via  $\Psi$ . This map  $\xi \mapsto a_\xi$  is also Borel measurable. We now set

$$F(\xi) := g_\xi \cdot a_\xi \varphi^n a_\xi \cdot g_\xi^{-1} \xi.$$

Let

$$D := 100 \left( C_0 + d_Z(z_0, \varphi^n z_0) + \sum_{j=1}^3 d_Z(z_0, a_j z_0) \right).$$

**Claim.** *We have that*

$$(6.3) \quad F \text{ is at most } 3 \cdot \#\{g \in \Gamma : d_Z(z_0, gz_0) \leq D\}\text{-to-one.}$$

To prove this claim, suppose that we have  $\xi, \xi' \in \mathcal{H}_{K,R}$  with the same image  $F(\xi) = F(\xi') =: (x_0, v_0) \in \Lambda_\theta(\Gamma) \times \mathfrak{a}_\theta$ .

Regarding  $x_0$  as a point in  $\partial Z$ , let  $\{u_i\}_{i \in \mathbb{N}} \subset Z$  be a sequence converging to  $x_0$ . Up to a subsequence,

$$(z_0, g_\xi a_\xi[z_0, \varphi^n z_0], u_i) \text{ is } C_0\text{-aligned for all } i \in \mathbb{N}.$$

Here, note that  $d_Z(z_0, \varphi^n z_0) \geq \tau_\varphi n \geq 100C_0$ . It follows from Lemma 2.3 that for each  $i \in \mathbb{N}$ , there exists  $P \in [z_0, u_i]$  such that  $d_Z(g_\xi a_\xi z_0, P) \leq 3C_0$ , and similarly, there exists  $Q \in [z_0, u_i]$  such that  $d_Z(g_{\xi'} a_{\xi'} z_0, Q) \leq 3C_0$ . We have three cases.

<sup>4</sup>There exists a technicality when several candidates tie. An easy rescue is to first enumerate  $\Gamma = \{g^{(1)}, g^{(2)}, \dots\}$ , and we choose the earliest whenever there is a tie.

<sup>5</sup>Again, when more than one of  $\{a_1, a_2, a_3\}$  do the job we choose the earliest.

- (1) If  $d_Z(z_0, g_\xi z_0) < d_Z(z_0, g_{\xi'} z_0) - 0.5D$ : In this case, along  $[z_0, u_i]$ ,  $P$  is closer to  $z_0$  than  $Q$  is, and  $d_Z(P, Q) \geq 0.5D - 6C_0$ . For convenience we write  $\eta := g_\xi a_\xi [z_0, \varphi^n z_0]$ . Note that

$$\begin{aligned} d_Z(u_i, \eta) &\geq d_Z(u_i, g_\xi a_\xi z_0) - d_Z(g_\xi a_\xi z_0, g_\xi a_\xi \varphi^n z_0) \\ &\geq d_Z(u_i, P) - 3C_0 - d_Z(z_0, \varphi^n z_0) \\ &\geq d_Z(u_i, Q) + 0.2D \\ &> d_Z(u_i, g_{\xi'} a_{\xi'} z_0) + d_Z(a_{\xi'} z_0, z_0) + C_0 \\ &\geq d_Z(u_i, g_{\xi'} z_0) + C_0. \end{aligned}$$

This implies that  $d_Z(\eta, [u_i, g_{\xi'} z_0]) \geq C_0$ . Since  $\eta$  is  $C_0$ -contracting,  $\text{diam } \pi_\eta([u_i, g_{\xi'} z_0]) \leq C_0$ . Since  $(z_0, \eta, u_i)$  is  $C_0$ -aligned, we also have that

$$(6.4) \quad (z_0, \eta, g_{\xi'} z_0) \text{ is } 2C_0\text{-aligned.}$$

We denote by  $x' \in \partial Z$  the first component of  $\xi'$ . Since  $x' \in O_K(z_0, g_{\xi'} z_0)$ , there exists a geodesic ray  $[z_0, x'] \subset Z$  that intersects  $\mathcal{N}_K(g_{\xi'} z_0)$ . Let  $\{u'_i\}_{i \in \mathbb{N}} \subset [z_0, x']$  be a sequence converging to  $x'$ . We claim that

$$(6.5) \quad (\eta, u'_i) \text{ is } (K + 30C_0)\text{-aligned for all large } i \in \mathbb{N}.$$

Suppose to the contrary that, passing to a subsequence,  $(\eta, u'_i)$  is never  $(K + 30C_0)$ -aligned. We choose

$$p \in \pi_\eta(z_0), \quad q \in \pi_\eta(u'_i), \quad \text{and} \quad r \in \pi_\eta(g_{\xi'} z_0).$$

By Equation (6.4) and the assumption that  $(\eta, u'_i)$  is not  $(K + 30C_0)$ -aligned,  $r$  is closer to  $g_\xi a_\xi \varphi^n z_0$  than  $p$  and  $q$  are. Moreover, we have  $d_Z(p, r), d_Z(q, r) \geq K + 25C_0$ . Now, it follows from Lemma 2.3 that

$$\begin{aligned} d_Z(z_0, u'_i) &\leq d_Z(z_0, \eta) + d_Z(p, q) + d_Z(\eta, u'_i), \\ d_Z(g_{\xi'} z_0, u'_i) &= 8C_0 d_Z(g_{\xi'} z_0, \eta) + d_Z(r, q) + d_Z(\eta, u'_i), \quad \text{and} \\ d_Z(z_0, g_{\xi'} z_0) &= 8C_0 d_Z(z_0, \eta) + d_Z(p, r) + d_Z(\eta, g_{\xi'} z_0). \end{aligned}$$

Then we have

$$\begin{aligned} d_Z(z_0, g_{\xi'} z_0) + d_Z(g_{\xi'} z_0, u'_i) - d_Z(z_0, u'_i) \\ &\geq 2d_Z(g_{\xi'} z_0, \eta) + 2 \min(d_Z(p, r), d_Z(q, r)) - 16C_0 \\ &\geq 2K + 34C_0 \end{aligned}$$

In particular,  $d_Z(z_0, u'_i) - d_Z(g_{\xi'} z_0, u'_i) \leq d_Z(z_0, g_{\xi'} z_0) - 2K - C_0$ . This contradicts that  $[z_0, u'_i]$  intersects  $\mathcal{N}_K(g_{\xi'} z_0)$  for all large  $i \in \mathbb{N}$ , and therefore Equation (6.5) follows.

By Lemma 2.3, it follows from Equation (6.4) and Equation (6.5) that  $[z_0, u'_i]$  passes through the  $4C_0$ -neighborhood of  $g_\xi a_\xi z_0$ , for all large  $i \in \mathbb{N}$ . This implies

$$x' \in O_{4C_0 + d_Z(z_0, g_\xi z_0)}(z_0, g_\xi z_0) \subset O_K(z_0, g_\xi z_0).$$

Meanwhile, we also have  $k < d_Z(z_0, g_\xi z_0) < d_Z(z_0, g_{\xi'} z_0) - 0.5D$ . This contradicts the definition of  $g_{\xi'}$  that  $d_Z(z_0, g_{\xi'} z_0)$  is minimal among the elements of  $\Gamma$  satisfying Equation (6.1).

- (2) If  $d_Z(z_0, g_{\xi'} z_0) < d_Z(z_0, g_\xi z_0) - 0.5D$ : In this case, one can obtain a similar contradiction as in (1).
- (3) If  $d_Z(z_0, g_\xi z_0) = 0.5D$ : Recall that for each fixed  $i \in \mathbb{N}$ , we have  $P, Q \in [z_0, u_i]$  such that  $d_Z(g_\xi a_\xi z_0, P), d_Z(g_{\xi'} a_{\xi'} z_0, Q) \leq 3C_0$ . Hence, we have

$$d_Z(g_\xi z_0, P), d_Z(g_{\xi'} z_0, Q) \leq 0.1D.$$

Since both  $P$  and  $Q$  belong to the geodesic  $[z_0, u_i]$ , this, together with  $d_Z(z_0, g_\xi z_0) = 0.5D$ , implies

$$d_Z(g_\xi z_0, g_{\xi'} z_0) \leq D.$$

Now when  $\xi$  is given (and hence  $a_\xi, g_\xi$  are given as well),

$$\xi' = g_{\xi'} a_{\xi'}^{-1} \varphi^{-n} a_{\xi'}^{-1} (g_{\xi'}^{-1} g_\xi) a_\xi \varphi^n a_\xi g_\xi^{-1} \xi$$

is determined by  $g_{\xi'}^{-1} g_\xi$  and  $a_{\xi'} \in \{a_1, a_2, a_3\}$ . The number of these choices is at most  $3 \cdot \#\{g \in \Gamma : d_Z(z_0, g z_0) \leq D\}$ .

Therefore, Equation (6.3) follows.

We simply write  $M := 3 \cdot \#\{g \in \Gamma : d_Z(z_0, g z_0) \leq D\}$ , which is finite. Then we have

$$\begin{aligned} \mu(F(\mathcal{H}_{K,R})) &= \mu \left( \bigcup_{g \in \Gamma, a \in \{a_1, a_2, a_3\}} F(\{\xi \in \mathcal{H}_{K,R} : g_\xi = g, a_\xi = a\}) \right) \\ &\geq \frac{1}{M} \sum_{g \in \Gamma, a \in \{a_1, a_2, a_3\}} \mu(F(\{\xi \in \mathcal{H}_{K,R} : g_\xi = g, a_\xi = a\})) \\ &= \frac{1}{M} \sum_{g \in \Gamma, a \in \{a_1, a_2, a_3\}} \mu(ga\varphi^n ag^{-1} \{\xi \in \mathcal{H}_{K,R} : g_\xi = g, a_\xi = a\}) \\ &= \frac{1}{M} \sum_{g \in \Gamma, a \in \{a_1, a_2, a_3\}} \mu(\{\xi \in \mathcal{H}_{K,R} : g_\xi = g, a_\xi = a\}) \\ &= \frac{1}{M} \mu(\mathcal{H}_{K,R}). \end{aligned}$$

Now to see the image of  $F$ , let  $\xi = (x, u) \in \mathcal{H}_{K,R}$ . For simplicity, write  $g := g_\xi$  and  $a := a_\xi$ . Then

$$F(\xi) = (ga\varphi^n ag^{-1} x, u + \sigma_\theta(ga\varphi^n ag^{-1}, x)).$$

We first estimate the  $\mathfrak{a}_\theta$ -component. By Equation (4.2),

$$\begin{aligned} \sigma_\theta(ga\varphi^n ag^{-1}, x) &= \sigma_\theta(ga\varphi^n, ag^{-1}x) + \sigma_\theta(ag^{-1}, x) \\ &= \sigma_\theta(ga, \varphi^n ag^{-1}x) + \sigma_\theta(\varphi^n, ag^{-1}x) \\ &\quad + \sigma_\theta(a, g^{-1}x) + \sigma_\theta(g^{-1}, x). \end{aligned}$$

Recall from Equation (6.2) that  $(z_0, ga[z_0, \varphi^n z_0], ga\varphi^n ag^{-1}x)$  is  $C_0$ -aligned. By Lemma 2.10(2), there exists  $r > 0$  depending on  $C_0$  such that

$$ga\varphi^n ag^{-1}x \in O_r(z_0, gaz_0) \cap O_r(gaz_0, ga\varphi^n z_0).$$

By Corollary 5.12, there exists  $c > 0$  depending on  $r$  so that for each  $\alpha \in \theta$ ,

$$\begin{aligned} \chi_\alpha(\kappa(ga)) - c &\leq \chi_\alpha(\sigma_\theta(ga, \varphi^n ag^{-1}x)) \leq \chi_\alpha(\kappa(ga)) \\ \chi_\alpha(\kappa(\varphi^n)) - c &\leq \chi_\alpha(\sigma_\theta(\varphi^n, ag^{-1}x)) \leq \chi_\alpha(\kappa(\varphi^n)). \end{aligned}$$

Note also that by Lemma 4.4,

$$-\chi_\alpha(\kappa(a^{-1})) \leq \chi_\alpha(\sigma_\theta(a, g^{-1}x)) \leq \chi_\alpha(\kappa(a)).$$

Hence,

$$\begin{aligned} &\chi_\alpha(\sigma_\theta(ga\varphi^n ag^{-1}, x)) \\ &=_{2c+\chi_\alpha(\kappa(a))+\chi_\alpha(\kappa(a^{-1}))} \chi_\alpha(\kappa(ga)) + \chi_\alpha(\kappa(\varphi^n)) + \chi_\alpha(\sigma_\theta(g^{-1}, x)) \\ &=_{\chi_\alpha(\kappa(a))+\chi_\alpha(\kappa(a^{-1}))} \chi_\alpha(\kappa(g)) + \chi_\alpha(\kappa(\varphi^n)) - \chi_\alpha(\sigma_\theta(g, g^{-1}x)) \end{aligned}$$

where the second equality is due to

$$\chi_\alpha(\kappa(g)) - \chi_\alpha(\kappa(a^{-1})) \leq \chi_\alpha(\kappa(ga)) \leq \chi_\alpha(\kappa(g)) + \chi_\alpha(\kappa(a)).$$

Moreover, since  $x \in O_K(z_0, gz_0)$ , it follows from Corollary 5.12(1) that there exists  $c' > 0$  depending on  $K$  such that

$$|\chi_\alpha(\kappa(g)) - \chi_\alpha(\sigma_\theta(g, g^{-1}x))| < c'.$$

Hence, setting

$$\widehat{D} := 2c + 2 \sum_{j=1}^3 \sum_{\alpha \in \theta} \chi_\alpha(\kappa(a_j)) + \chi_\alpha(\kappa(a_j^{-1})) + \chi_\alpha(\kappa(\varphi^n)) + c',$$

we have that

$$u + \sigma_\theta(ga\varphi^n ag^{-1}, x) \in [-R - \widehat{D}, R + \widehat{D}]^{\#\theta}.$$

We note that  $\widehat{D}$  is independent of  $g$ .

In addition, by Equation (6.1), we have  $d_Z(z_0, gaz_0) > k - \sum_{j=1}^3 d_Z(z_0, a_j z_0)$  and that  $(z_0, ga[z_0, \varphi^n z_0], ga\varphi^n ag^{-1}x)$  is  $C_0$ -aligned by Equation (6.2).

This implies that  $F(\mathcal{H}_{K,R})$  is contained in

$$B_{k;n} := \left\{ (y, v) \in \mathcal{H}_\theta : \begin{array}{l} v \in [-R - \widehat{D}, R + \widehat{D}]^{\#\theta} \text{ and } \exists h \in \Gamma \text{ such that} \\ d_Z(z_0, hz_0) > k - \sum_{j=1}^3 d_Z(z_0, a_j z_0) \text{ and} \\ (z_0, h[z_0, \varphi^n z_0], y) \text{ is } C_0\text{-aligned} \end{array} \right\}.$$

Hence, we have

$$\mu(B_{k;n}) \geq \mu(\mathcal{H}_{K,R})/M > 0.$$

Note that the set  $B_{k;n}$  is decreasing in  $k$ . Since  $\mu$  is a Radon measure and  $B_{k;n} \subset \partial Z \times [-R - \widehat{D}, R + \widehat{D}]^{\# \theta}$  which is *compact*, we have  $\mu(B_{k;n}) < +\infty$ . Therefore, setting

$$(6.6) \quad B_n := \bigcap_{k>0} B_{k;n},$$

we have

$$(6.7) \quad \mu(B_n) = \lim_{k \rightarrow +\infty} \mu(B_{k;n}) \geq \mu(\mathcal{H}_{K,R})/M > 0,$$

noting that  $M$  does not depend on  $k$ .

Now,  $\Gamma \cdot B_n$  is a  $\Gamma$ -invariant set of positive  $\mu$ -measure. Hence, by the  $\Gamma$ -ergodicity of  $\mu$ , we have that  $\Gamma \cdot B_n$  is  $\mu$ -conull, and therefore

$$\bigcap_n \Gamma \cdot B_n \text{ is } \mu\text{-conull.}$$

We then show that for each  $(y, v) \in \bigcap_n \Gamma \cdot B_n$ , we have  $y \in \Lambda_\theta^{\varphi, 2C_0}(\Gamma)$ . This finishes the proof by Lemma 3.2.

Let  $(y, v) \in \bigcap_n \Gamma \cdot B_n$ . Then for each large enough  $n \in \mathbb{N}$ , there exists  $h_0 \in \Gamma$  so that  $(z_0, h[z_0, \varphi^n z_0], h_0^{-1}y)$  is  $C_0$ -aligned for infinitely many  $h \in \Gamma$ . In other words,

$$(h_0 z_0, h_0 h[z_0, \varphi^n z_0], y) \\ \text{is } C_0\text{-aligned for infinitely many } h \in \Gamma.$$

Among infinitely many such  $h \in \Gamma$ , we can choose one such that

$$d_Z(h_0 z_0, h_0 h[z_0, \varphi^n z_0]) > d_Z(z_0, h_0 z_0) + C_0$$

and hence

$$d_Z([h_0 z_0, z_0], h_0 h[z_0, \varphi^n z_0]) > C_0.$$

Since  $h_0 h[z_0, \varphi^n z_0]$  is  $C_0$ -contracting,  $\pi_{h_0 h[z_0, \varphi^n z_0]}([h_0 z_0, z_0])$  has diameter at most  $C_0$ . Therefore,

$$(z_0, h_0 h[z_0, \varphi^n z_0], y) \text{ is } 2C_0\text{-aligned.}$$

Since this holds for all large  $n \in \mathbb{N}$ , we conclude  $y \in \Lambda_\theta^{\varphi, 2C_0}(\Gamma)$ .  $\square$

**6.2. Quasi-invariance under translations.** We rephrase the  $\mathsf{A}_\theta$ -action on  $\mathcal{H}_\theta$  in terms of the  $\mathfrak{a}_\theta$ -action as follows: for  $u \in \mathfrak{a}_\theta$ , consider a map  $T_u : \mathcal{H}_\theta \rightarrow \mathcal{H}_\theta$  given by  $(x, v) \mapsto (x, v + u)$ . For a Radon measure  $\mu$  on  $\mathcal{H}_\theta$ , we consider its pullback measure  $T_u^* \mu$ , which is given by

$$T_u^* \mu(E) := \mu(T_u E)$$

for each Borel subset  $E \subset \mathcal{H}_\theta$ . For a loxodromic  $\varphi \in \Gamma$ , we simply write  $T_\varphi := T_{\lambda_\theta(\varphi)}$ . We show that  $\Gamma$ -invariant ergodic measures on  $\mathcal{H}_\theta$  are quasi-invariant under this translation.

**Theorem 6.4.** *Let  $\Gamma < \mathbf{G}$  be a non-elementary  $P_\theta$ -hypertransverse subgroup. Let  $\mu$  be a  $\Gamma$ -invariant ergodic Radon measure on  $\mathcal{H}_\theta$  supported on  $\mathcal{R}_{\Gamma,\theta}$ . Then for a loxodromic  $\varphi \in \Gamma$ , there exists  $s \geq 0$  such that*

$$\frac{dT_\varphi^*\mu}{d\mu} = e^s \quad a.e.$$

*Proof.* Let  $\varphi \in \Gamma$  be a loxodromic element and let  $C = C(\varphi) > 0$  be the constant satisfying Lemma 2.11 regarding  $\varphi$  as an isometry on  $Z$ , with the choice of axis  $\gamma : \mathbb{R} \rightarrow X$ . As in [CK25b, Proof of Theorem 7.10], we may assume that

$$\tau_\varphi > 100(\delta + C + c)$$

where  $\delta > 0$  is the Gromov hyperbolicity constant of  $Z$  and  $c > 0$  is the constant in Equation (2.2).

It is a standard ergodicity argument that it suffices to show

$$(6.8) \quad (T_\varphi^*\mu)(E) \geq \mu(E)$$

for each Borel subset  $E \subset \mathcal{H}_\theta$  (see [CK25b, Proof of Theorem 7.10], for instance).

By Theorem 6.3,  $\mu$  is supported on  $\Lambda_\theta^{\varphi,C}(\Gamma) \times \mathfrak{a}_\theta$ . Hence, it suffices to consider the case

$$E = K \times I$$

for a compact subset  $K \subset \Lambda_\theta^{\varphi,C}(\Gamma)$  and a compact box  $I \subset \mathfrak{a}_\theta$ , where  $\mathfrak{a}_\theta$  is identified with  $\mathbb{R}^{\#\theta}$  via  $u \mapsto (\chi_\alpha(u))_{\alpha \in \theta}$ . One can refer to [CK25a, Proof of Theorem 4.6] or [CK25b, Proof of Theorem 7.10] for detailed deductions.

As before, we identify two limit sets  $\Lambda_Z(\Gamma) \subset \partial Z$  and  $\Lambda_\theta(\Gamma)$  via the  $\Gamma$ -equivariant homeomorphism  $\Psi : \Lambda_Z(\Gamma) \rightarrow \Lambda_\theta(\Gamma)$ . We fix some open subset  $O \subset \Lambda_Z(\Gamma)$  such that  $K \subset O$  and  $\epsilon > 0$ . Let  $N = N(\varphi, 4C, \epsilon, z_0) \in \mathbb{N}$  be given in Corollary 5.12(2). By Lemma 3.3, for each  $x \in K$ , there exist  $g(x) \in \Gamma$  and  $n(x) > 100 \left( \frac{C}{\tau_\varphi} + N \right)$  such that

$$x \in U_C(g(x); \varphi, n(x)) \subset O.$$

Let  $\mathcal{U} := \{U_C(g(x); \varphi, n(x)) : x \in K\}$ , which is a countable collection of sets. For convenience, let us enumerate  $\mathcal{U}$  based on their  $d_Z$ -distances from  $z_0$ , i.e., let

$$\mathcal{U} = \{U_1, U_2, \dots\}$$

where  $U_j := U_C(g_j; \varphi, n_j)$  for each  $j \in \mathbb{N}$  so that

$$d_Z(z_0, g_1 \varphi^{n_1} z_0) \leq d_Z(z_0, g_2 \varphi^{n_2} z_0) \leq \dots$$

We will now define a subcollection

$$\mathcal{V} := \{U_{i(1)}, U_{i(2)}, \dots\} \subset \mathcal{U}$$

by inductively defining  $i(1), i(2), \dots$  as follows: let  $i(1) = 1$ . Having defined  $i(1), \dots, i(k)$ , define  $i(k+1)$  as the smallest  $j \in \mathbb{N}$  such that  $U_j$  is disjoint from  $U_{i(1)} \cup \dots \cup U_{i(k)}$ .

For each  $l \in \mathbb{N}$ , we set

$$(6.9) \quad C_l := U_{i(l)} \cup \bigcup \{U_k : k \geq i(l), U_k \cap U_{i(l)} \neq \emptyset\}.$$

Then  $\{C_l : l \in \mathbb{N}\}$  is a covering of  $K$  contained in  $O$ .

A very similar argument as in [CK25b, First claim in the proof of Theorem 7.10] gives that for each  $l \in \mathbb{N}$ ,

$$(6.10) \quad C_l \subset U_C(g_{i(l)}; \varphi, n_{i(l)} - 1).$$

Since some modifications are required, we present its proof as follows, for the sake of completeness.

Let  $k \geq i(l)$  be such that  $U_k = U_C(g_k; \varphi, n_k)$  and  $U_{i(l)}$  have a common element  $x$ . As  $\Lambda_\theta$  and  $\Lambda_Z(\Gamma)$  are identified,  $x$  is also regarded as a point in  $\partial Z$ . Then for every  $z \in Z$  close enough to  $x$  in  $Z \cup \partial Z$ ,

$$(6.11) \quad (z_0, g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0], z) \text{ is } C\text{-aligned.}$$

By Lemma 2.3, there exists  $q \in [z_0, z]$  that is  $3C$ -close to  $g_{i(l)}\varphi^{n_{i(l)}} z_0$ . Similarly, by the condition  $x \in U_k$ ,  $[z_0, z]$  contains a point  $p$  that is  $3C$ -close to  $g_k\varphi^{n_k} z_0$ . Since  $k \geq i(l)$ , our enumerating convention tells us that

$$d(z_0, p) \geq d(z_0, g_k\varphi^{n_k} z_0) - 3C \geq d(z_0, g_{i(l)}\varphi^{n_{i(l)}} z_0) - 3C \geq d(z_0, q) - 6C.$$

In other words,

$$d(z, p) \leq d(z, q) + 6C.$$

If  $d(z, p) \leq 10C$ , then we have  $d(z, g_k\varphi^{n_k} z_0) \leq 13C$ . By the  $(1, 4C)$ -Lipschitzness of  $\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(\cdot)$  in Corollary 2.4(1), we have

$$\text{diam } \pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(\{z, g_k\varphi^{n_k} z_0\}) \leq 17C.$$

If  $d(z, p) > 10C$ , then we take a point  $p^\dagger \in [z, p]$  such that  $d(p, p^\dagger) = 10C$ . Then  $\text{diam}[p^\dagger, z] = d(p, z) - 10C$ , and hence

$$\begin{aligned} d(g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0], [p^\dagger, z]) &\geq d(g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0], z) - \text{diam}[p^\dagger, z] \\ &\geq d(z, q) - 3C - d(z, p) + 10C \\ &\geq C. \end{aligned}$$

By the  $C$ -contracting property of  $g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]$ , we have

$$\text{diam } \pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(\{z, p^\dagger\}) \leq C.$$

Since  $d(p, g_k\varphi^{n_k} z_0) \leq 3C$ , we have  $d(p^\dagger, g_k\varphi^{n_k} z_0) \leq 13C$ , and hence

$$\text{diam } \pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(\{z, g_k\varphi^{n_k} z_0\}) \leq 18C$$

by Corollary 2.4(1).

Hence, in any case, together with Equation (6.11), we have that

$$(6.12) \quad (z_0, g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0], g_k\varphi^{n_k} z_0) \text{ is } 19C\text{-aligned.}$$

Now let  $y \in U_k$  be arbitrary. Then for every  $z' \in Z$  close to  $y$  in  $Z \cup \partial Z$ ,

$$(z_0, g_k[z_0, \varphi^{n_k} z_0], z') \text{ is } C\text{-aligned}$$

and hence  $[z_0, z']$  passes through the  $3C$ -neighborhood of  $g_k \varphi^{n_k} z_0$  as before. Hence, we have

$$\begin{aligned} \text{diam } \pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}([z_0, z']) &\geq d(g_{i(l)} z_0, g_{i(l)} \varphi^{n_{i(l)}} z_0) \\ &\quad - \text{diam } (\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(z_0) \cup \{g_{i(l)} z_0\}) \\ &\quad - (d([z_0, z'], g_k \varphi^{n_k} z_0) + 4C) \\ &\quad - \text{diam } (\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(g_k \varphi^{n_k} z_0) \cup \{g_{i(l)} \varphi^{n_{i(l)}} z_0\}) \\ &\geq 100C - C - (3C + 4C) - 19C \\ &\geq 70C \end{aligned}$$

where we applied Corollary 2.4(1) in the first inequality, and Equation (6.11), Equation (6.12), and that  $d([z_0, z'], g_k \varphi^{n_k} z_0) \leq 3C$  in the second.

We then apply Lemma 2.3 and obtain two points  $u, v \in [z_0, z']$  such that

- (1)  $[u, v]$  and  $\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}([z_0, z'])$  are within Hausdorff distance  $4C$ ,
- (2)  $\text{diam } (\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}([z_0, u]) \cup \{u\}) \leq 2C$ ,
- (3)  $\text{diam } (\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}([v, z']) \cup \{v\}) \leq 2C$ , and
- (4) for each  $u' \in \pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(z_0)$  and  $v' \in \pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(z')$ ,  $[u', v']$  and  $[u, v]$  are within Hausdorff distance  $10C$ .

Since  $d([z_0, z'], g_k \varphi^{n_k} z_0) \leq 3C$ , it follows from Equation (6.12) and Corollary 2.4(1) that there exists  $w \in [z_0, z']$  such that

$$(6.13) \quad \text{diam } (\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(w) \cup \{g_{i(l)} \varphi^{n_{i(l)}} z_0\}) \leq 26C.$$

By the condition (2) above, Equation (6.12), and  $d(z_0, \varphi^{n_{i(l)}} z_0) > 100C$ , we must have  $w \notin [z_0, u]$ , and hence  $w \in [v, z']$  or  $w \in [u, v]$ .

- If  $w \in [v, z']$ , then it follows from (3) above that

$$\text{diam } (\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(z') \cup \{g_{i(l)} \varphi^{n_{i(l)}} z_0\}) \leq 28C.$$

- If  $w \in [u, v]$ , then it follows from (4) above that for each pair of two points  $u' \in \pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(z_0)$  and  $v' \in \pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(z')$ , there exists  $w' \in [u', v']$  such that  $d(w, w') \leq 10C$ . We then have

$$\text{diam } (\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(w) \cup \{w'\}) \leq 20C.$$

Together with Equation (6.13),  $\text{diam}\{w', g_{i(l)} \varphi^{n_{i(l)}} z_0\} \leq 46C$ . Since  $v'$  is inbetween  $w'$  and  $g_{i(l)} \varphi^{n_{i(l)}} z_0$ , and  $v' \in \pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(z')$  is arbitrary, we have

$$\text{diam } (\pi_{g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0]}(z') \cup \{g_{i(l)} \varphi^{n_{i(l)}} z_0\}) \leq 46C.$$

Therefore,  $(g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0], z')$  is  $46C$ -aligned.

We now apply Lemma 2.11(4) to  $(\varphi^{-n_{i(l)}} g_{i(l)}^{-1} z', [z_0, \varphi^{-n_{i(l)}} z_0])$  and have

$$\pi_{g_{i(l)} \varphi^{n_{i(l)}} \gamma}(z') \subset g_{i(l)} \varphi^{n_{i(l)}} \gamma([-47C, +\infty)).$$

As in the proof of Lemma 2.11(4), this implies

$$(6.14) \quad \begin{aligned} \pi_{g_{i(l)} \gamma}(z') &\subset g_{i(l)} \gamma([n_{i(l)} \tau_\varphi - 6\delta - 4C - c - 47C, +\infty)) \\ &\subset g_{i(l)} \gamma([(n_{i(l)} - 1) \tau_\varphi + 6\delta + 4C + c + 1, +\infty)) \end{aligned}$$

Hence,  $(g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0], z')$  is  $C$ -aligned. To see this, suppose to the contrary that  $(\varphi^{-(n_{i(l)}-1)} g_{i(l)}^{-1} z', [z_0, \varphi^{-(n_{i(l)}-1)} z_0])$  is not  $C$ -aligned. Then it follows from Lemma 2.11(3) and the similar argument as above that

$$\pi_{g_{i(l)} \gamma}(z') \subset g_{i(l)} \gamma((-\infty, (n_{i(l)} - 1) \tau_\varphi + 6\delta + 4C + c])$$

which is a contradiction.

Therefore,  $(g_{i(l)}[z_0, \varphi^{n_{i(l)}} z_0], z')$  is  $C$ -aligned. Since this holds for every  $z'$  close to  $y \in U_k$ , we have

$$y \in U_C(g_{i(l)}; \varphi, n_{i(l)} - 1).$$

Equation (6.10) is now established.

Now for each  $l \in \mathbb{N}$ , we define a map  $F_l : C_l \times I \rightarrow \mathcal{H}_\theta$  as follows: for  $g = g_{i(l)}$ , we set

$$(6.15) \quad F_l : \xi \mapsto g \varphi g^{-1} \xi.$$

Then we have  $\mu(F_l(C_l \times I)) = \mu(C_l \times I)$  as  $\mu$  is  $\Gamma$ -invariant.

**Claim.** *We have*

$$(6.16) \quad F_l(C_l \times I) \subset U_{i(l)} \times (\epsilon\text{-neighborhood of } I + \lambda_\theta(\varphi)).$$

To see this, we simply write  $g = g_{i(l)}$  and  $n = n_{i(l)} - 1$ . We then fix  $\xi = (x, u) \in C_l \times I$ . Note that

$$F_l(\xi) = (g \varphi g^{-1} x, u + \sigma_\theta(g \varphi g^{-1}, x)).$$

The inclusion for the first component can be shown by a very similar argument to [CK25b, Second claim in the proof of Theorem 7.10]. Again, as this requires modifications of involved constants in our setting, we present the proof.

Let  $z \in Z$  be a close point to  $x$  in  $Z \cup \partial Z$ . It suffices to show that

$$(z_0, g[z_0, \varphi^{n+1} z_0], g \varphi g^{-1} z) \text{ is } C\text{-aligned for all large } i \in \mathbb{N},$$

Since  $U_{i(l)} \neq \emptyset$ , we already have that  $(z_0, g[z_0, \varphi^{n+1} z_0])$  is  $C$ -aligned. Now suppose to the contrary that  $(g[z_0, \varphi^{n+1} z_0], g \varphi g^{-1} z)$  is not  $C$ -aligned. In other words,  $(\varphi^{-n} g^{-1} z, [z_0, \varphi^{-(n+1)} z_0])$  is not  $C$ -aligned. Then by Lemma 2.11(3) and as in the proof of Lemma 2.11(4), we have

$$\pi_{g\gamma}(z) \subset g\gamma((-\infty, n\tau_\varphi + 6\delta + 4C + c]).$$

This contradicts Equation (6.14). Therefore,  $g \varphi g^{-1} x \in U_{i(l)}$ .

We now show the inclusion for the second component. Since  $U_{i(l)} \subset C_l$ , it follows from the inclusion for the first component that  $g\varphi g^{-1}x \in C_l$ . Hence, we have

$$x, g\varphi g^{-1}x \in C_l \subset U_C(g; \varphi, n)$$

where the last inclusion is due to Equation (6.10). By Lemma 2.10(2),

$$x, g\varphi g^{-1}x \in O_{4C}(z_0, gz_0) \cap O_{4C}(gz_0, g\varphi^n z_0).$$

Now by Corollary 5.12,

$$\|\sigma_\theta(g\varphi g^{-1}, x) - \lambda_\theta(\varphi)\| < \epsilon.$$

Therefore, the inclusion for the second component follows.

Now by the above claim and disjointness of  $U_{i(l)}$ 's, we have

$$\begin{aligned} \mu(O \times (\epsilon\text{-neighborhood of } I + \lambda_\theta(\varphi))) &\geq \mu\left(\bigcup_l F_l(C_l \times I)\right) \\ &= \sum_l \mu(F_l(C_l \times I)) \\ &= \sum_l \mu(C_l \times I) \\ &\geq \mu(K \times I). \end{aligned}$$

Note that  $\mu(O \times (\epsilon\text{-neighborhood of } I + \lambda_\theta(\varphi))) < +\infty$  since  $\mu$  is Radon. Since  $\epsilon > 0$  and an open set  $O \supset K$  are arbitrary, we have

$$(T_\varphi^*\mu)(K \times I) = \mu(K \times (I + \lambda_\theta(\varphi))) \geq \mu(K \times I).$$

This finishes the proof.  $\square$

**6.3. Proof of the rigidity.** Theorem 6.1 is now deduced from Theorem 6.4 and Theorem 5.9. This deduction is originally due to ([ANSS02, 0.1 Basic Lemma], [Sar04, Lemma 1]).

To elaborate more, if  $\mu$  is a  $\Gamma$ -invariant ergodic Radon measure on  $\mathcal{H}_\theta$  supported on  $\mathcal{R}_{\Gamma, \theta}$ , then

$$\text{Spec}_\theta(\Gamma) \subset \{u \in \mathfrak{a}_\theta : T_u^*\mu \text{ and } \mu \text{ are equivalent}\}$$

by Theorem 6.4. Since the right hand side is an additive subgroup of  $\mathfrak{a}_\theta$ , it follows from the non-arithmeticity of  $\text{Spec}_\theta(\Gamma)$  that the set on the right hand side is dense in  $\mathfrak{a}_\theta$ .

By ([ANSS02, 0.1 Basic Lemma], [Sar04, Lemma 1]), this implies that there exists  $\psi \in \mathfrak{a}_\theta^*$ ,  $\delta > 0$ , and  $c > 0$  so that

$$d\mu(x, u) = c \cdot e^{\delta\psi(u)} d\nu(x) du$$

for a  $\delta$ -dimensional  $\psi$ -Patterson–Sullivan measure  $\nu$  of  $\Gamma$ , where  $du$  is the Lebesgue measure on  $\mathfrak{a}_\theta$ . Since  $\mu$  is supported on  $\mathcal{R}_{\Gamma, \theta} = \Lambda_\theta^{\text{con}}(\Gamma) \times \mathfrak{a}_\theta$ , it follows from Theorem 5.9 that  $\delta = \delta_\psi(\Gamma)$  and  $\nu$  is of divergence type. This finishes the proof.  $\square$

**6.4. Strengthening the Hopf–Tsuji–Sullivan dichotomy.** One important portion of the Hopf–Tsuji–Sullivan dichotomy is that divergence-type Patterson–Sullivan measures are supported on the conical limit sets, as in Theorem 5.9 ([CZZ24], [KOW25b]).

As a corollary of Theorem 6.3, we present a strengthened version of this, by showing that such Patterson–Sullivan measures are in fact supported on guided limit sets. We also refer to ([BLLO23], [Sam24], [KOW25a]) for different type of the strengthening, in terms of so-called directional limit sets.

**Corollary 6.5.** *Let  $\Gamma < G$  be a non-elementary  $P_\theta$ -hypertransverse subgroup and  $\nu$  a divergence-type Patterson–Sullivan measure of  $\Gamma$ . Let  $\varphi \in \Gamma$  be loxodromic and let  $C = C(\varphi) > 0$  be as in Lemma 2.11. Then*

$$\nu(\Lambda_\theta^{\varphi,C}(\Gamma)) = 1.$$

*Proof.* By [Kim24], the associated Burger–Roblin measure  $\mu_\nu^{\text{BR}}$  on  $\mathcal{H}_\theta$  is  $\Gamma$ -ergodic. Moreover, by the Hopf–Tsuji–Sullivan dichotomy for transverse subgroups (Theorem 5.9),  $\mu_\nu^{\text{BR}}$  is supported on  $\Lambda_\theta^{\text{con}}(\Gamma) \times \mathfrak{a}_\theta$ . Then applying Theorem 6.3 finishes the proof.  $\square$

**6.5. Theorems in the introduction.** We now deduce theorems stated in the introduction from Theorem 6.1. First, note that  $\text{Spec}_\theta(\Gamma)$  is non-arithmetic when  $\Gamma$  is Zariski dense, by the result of Benoist [Ben00] (Theorem 4.1). Then Theorem 6.1 implies the inclusion (2)  $\subset$  (1) in Theorem 1.13, and the inclusion (1)  $\subset$  (2) there is due to [Kim24] (see [LO23] for Borel Anosov cases). Corollary 1.14 is Corollary 6.2.

When  $\theta = \Delta$ , the  $\Gamma$ -action on  $\mathcal{H}_\theta$  is dual to the NM-action on  $\Gamma \backslash G$  (cf. [LO23, Proposition 10.25]). In addition, if  $\mathcal{E}_\Gamma \subset \Gamma \backslash G$  is  $P^\circ$ -minimal, then the Burger–Roblin measure on  $\mathcal{E}_\Gamma$  associated to a divergence-type Patterson–Sullivan measure of  $\Gamma$  is N-ergodic [Kim24] (see [LO24] for Borel Anosov cases). Then as in [Win15, Corollary 6.5], NM-ergodic measure classification implies the N-ergodic measure classification, with the same set of ergodic measures. Hence, Theorem 1.12 follows from Corollary 1.14. Since Anosov subgroups are special cases of relatively Anosov subgroups, Theorem 1.5 and Theorem 1.6 are particular cases of Theorem 1.12.

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