RANDOM WALKS AND CONTRACTING ELEMENTS II: TRANSLATION LENGTH AND QUASI-ISOMETRIC EMBEDDING

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ABSTRACT. Continuing from [Cho22b], we study random walks on metric spaces with contracting elements. We prove that random subgroups of the isometry group of a metric space is quasi-isometrically embedded into the space. We discuss this problem in two ways, namely, in the sense of random walks and counting problem. We also establish the genericity of contracting elements and the CLT and its converse for translation length.

Keywords. Random walk, Outer space, Teichmüller space, CAT(0) space, Central limit theorem, Contracting property, Quasi-isometric embedding

MSC classes: 20F67, 30F60, 57M60, 60G50

1. INTRODUCTION

This is the second in a series of articles concerning random walks on metric spaces with contracting elements. This series is a reformulation of the previous preprint [Cho22a] announced by the author, aiming for clearer and more concise exposition.

Let X be a geodesic metric space and let $o \in X$. We say that a subset A of X is *contracting* if the closest point projection of a geodesic γ onto A is uniformly bounded whenever γ is far away from A. An isometry g of X is *contracting* if the orbit $\{g^n o\}_{n \in \mathbb{Z}}$ is a contracting quasigeodesic (see Definition 2.1). Two contracting isometries g and h of X are *independent* if there orbits have unbounded Hausdorff distance.

Convention 1.1. *Throughout, we assume that:*
- *(X, d) is a geodesic metric space;*
- *G is a countable group of isometries of X, and*
- *G contains two independent contracting isometries.*

We also fix a basepoint $o \in X$.

This type of contraction takes place in various non-positively curved spaces including (relatively) hyperbolic spaces, CAT(0) spaces, Teichmüller space, Culler-Vogtmann Outer space and small cancellation groups. We refer the readers to [Sis13], [BF09], [Min96], [AK11] and [ACGH19] for the treatment for each cases.

We wish to sample a random isometry g in G. For this we fix i.i.d. random variables g_i's distributed according to a probability measure μ and define $Z_n := g_1 \cdots g_n$. The sequence of RVs $(Z_n)_{n \geq 0}$ then constitute the random walk generated by μ. We say that the random walk ω is *non-elementary* if the support of μ generates a semigroup that contains two independent contracting isometries (see Subsection 2.2).

Given a non-elementary random walk on G, one can ask the asymptotic behavior of the displacement $d(o, go)$ of a random isometry g. This was indeed pursued in [Cho22b], discussing the strong law of large numbers (SLLN) with an exponential bound, central limit theorem (CLT) and the law of the iterated logarithm (LIL). In this article, we aim to investigate yet another quantity, the translation length $\tau(g) := \lim_i d_X(o, g^i o)/i$, of a random isometry g. The translation length of an element g of G often encodes interesting dynamical phenomena. For example, the translation length of a deck transformation of a negatively curved manifold is the length of the corresponding closed geodesic. The translation length of a pseudo-Anosov mapping class (irreducible outer automorphism, resp.) on Teichmüller space (Outer space, resp.) equals its stretch factor as a self-map on the surface (expansion factor as a train-track map on the graph, resp.).

Our first main result is companion to the corresponding result for displacement ([Cho22b, Theorem 6.4]).

Theorem A. *Let (X, G, o) be as in Convention 1.1 and let $(Z_n)_{n \geq 0}$ be the random walk generated by a non-elementary measure μ on G. Let $\lambda(\mu)$ be the escape rate of μ, i.e., $\lambda(\mu) := \lim_{n \to \infty} \mathbb{E}_{\mu^{*n}}[d(o, go)]/n$. Then for each $0 < L < \lambda(\mu)$, there exists $K > 0$ such that for each n we have*

$$\mathbb{P}\left(\omega_n \text{ is contracting and } \tau(Z_n) \geq Ln\right) \geq 1 - Ke^{-n/K}.$$

This result has been observed by Sisto for simple random walks on various spaces in [Sis18]. In the absence of moment conditions, Maher and Tiozzo observed in [MT18] that non-elementary random walks on Gromov hyperbolic spaces favor loxodromic elements in probability. Their methods and Benoist-Quint's estimates in [BQ16b] also lead to the stronger SLLN for translation length under finite second moment condition, as noted by Dahmani and Horbez [DH18]. Dahmani and Horbez also deduced the same SLLN on Teichmüller space. Later, Baik, Choi and Kim obtained the same SLLN under finite first moment assumption using ergodic theorems and Maher-Tiozzo's notion of persistent joint [BCK23]. We also note Le Bars' recent result [LB22] that non-elementary random walks on a proper CAT(0) space favor contracting isometries in probability. Theorem A generalizes these results by obtaining an exponential bound from below without any moment condition. Finally, Goldsborough and Sisto recently presented a QI-invariant theory for exponential genericity of positive translation length [GS21]. Their theory applies to (relatively) hyperbolic groups, acylindrically hyperbolic 3-manifold groups, right-angled Artin groups and many more.

Genericity of contracting elements is a recurring theme that has been investigated in various settings. For example, Yang describes genericity of contracting elements in counting problem for proper actions on a metric space [Yan20]. The novelty of Theorem A is that it discusses genericity of contracting elements for possibly non-WPD actions on spaces and for non-elementary random walks without moment condition.

We also provide a quantitative comparison between the displacement and the translation length of a random isometry.

Theorem B. *Let (X, G, o) be as in Convention 1.1, and $(Z_n)_{n \geq 0}$ be the random walk generated by a non-elementary measure μ on G.*

(1) If μ has finite p-th moment for some $p > 0$, then

$$\lim_{n \to \infty} \frac{1}{n^{1/2p}}[d(o, Z_n o) - \tau(Z_n)] = 0 \quad a.s.$$

(2) If μ has finite first moment, then there exists $K > 0$ such that

$$\limsup_{n \to \infty} \frac{1}{\log n}[d(o, Z_n o) - \tau(Z_n)] \leq K \quad a.s.$$

There have been many results that capture the sublinear discrepancy between the displacement and translation length for random walks with bounded support or with finite exponential moment: see [Mah12], [MT18]. Moreover, using Benoist-Quint's strategy in [BQ16b], [BQ16a] and its application to other spaces ([Hor18], [DH18]), one can achieve sublinear discrepancy for random walks with finite first moment. We improve these observations by proving that random walks with finite $(1/2)$-th moment exhibit sublinear discrepancy between displacement and translation length.

Combining the CLT for displacement proved in [Cho22b] and Theorem B, we deduce the CLT for translation length:

Theorem C (CLT and its converse)**.** *Let (X, G, o) be as in Convention 1.1, and $(Z_n)_{n \geq 0}$ be the random walk generated by a non-elementary measure μ on G. If μ has finite second moment, then $\frac{1}{\sqrt{n}}(d(o, Z_n o) - n\lambda)$ and $\frac{1}{\sqrt{n}}(\tau(Z_n) - n\lambda)$ converge to the same Gaussian distribution in law.*

Conversely, if μ has infinite second moment, then for any sequence $(c_n)_{n \geq 0}$, neither $\frac{1}{\sqrt{n}}(d(o, Z_n o) - c_n)$ nor $\frac{1}{\sqrt{n}}(\tau(Z_n) - c_n)$ converges in law.

CLT for translation length and the converses of CLT for Gromov hyperbolic spaces and Teichmüller space have been observed in [Cho23]. Here, we establish the same result for general spaces with contracting isometries.

Meanwhile, Taylor and Tiozzo proved in [TT16] that random subgroups of a weakly hyperbolic group is quasi-isometrically embedded into the ambient Gromov hyperbolic space, in the sense that such event happens for eventual probability 1. See also [MS19] and [MT21] for additional conclusions under geometric assumptions, e.g., acylindricity or WPD. These results are linked to a deeper understanding of convex-cocompact subgroup of mapping class groups and random extensions of surface groups and free groups.

The following theorem strengthens the conclusion of Taylor-Tiozzo's theorem and generalizes it to more general spaces.

Theorem D. *Let (X, G, o) be as in Convention 1.1, and $(Z_n^{(1)}, \ldots, Z_n^{(k)})_{n \geq 0}$ be k independent random walks generated by a non-elementary measure μ on G. Then there exists $K > 0$ such that*

$$\mathbb{P}\left[\langle Z_n^{(1)}, \ldots, Z_n^{(k)} \rangle \text{ is q.i. embedded into a quasi-convex subset of } X\right] \geq 1 - Ke^{-n/K}.$$

Thanks to concrete control of the decay rate, we can deduce the analogous conclusion for counting problems.

Theorem E. *Let G be a finitely generated group acting on a metric space X with at least two independent contracting elements. Then for each $k > 0$, there exists a finite generating set S of G such that*

$$\frac{\# \left\{ (g_1, \ldots, g_k) \in \left(B_n(e)\right)^k : \begin{array}{c} \langle g_1, \ldots, g_k \rangle \text{ is q.i. embedded into} \\ \text{a quasi-convex subset of } X \end{array} \right\}}{\left(\# B_n(e)\right)^k}$$

converges to 1 exponentially fast.

Theorem E has been previously observed for a vast number of group actions by Han and Yang. First, Yang describes the genericity of contracting elements for counting problem in [Yan20], which corresponds to the case of $k = 1$ in Theorem E. Yang considered groups admitting a statistically convex-cocompact (SCC) actions on proper spaces, which include relatively hyperbolic group, groups with non-trivial Floyd boundary, non-splitting RAAGs and RACGs, small cancellation groups, mapping class groups acting on Teichmüller spaces and many more. This was later generalized to arbitrary k by Han and Yang in [HY21].

Meanwhile, mapping class groups are not known to possess a contracting element with respect to the actions on their Cayley graphs. This necessitates a different approach for the counting problem in the Cayley graph. Theorem E accomplishes this purpose by not requiring the action of G on X be proper or SCC, while deducing a result for the counting problem in the Cayley graph of G. For Gromov hyperbolic spaces and Teichmüller space, Theorem E for $k = 1$ has been observed in [Cho24] as an affirmative answer to a version of Farb's conjecture in [Far06].

1.1. **Structure of the paper.** In Section 2, we recall the basic notions and facts about contracting isometries, alignment of paths and Schottky sets that were discussed in [Cho22b].

All the theorems of this paper eventually follow from Gouëzel's pivoting technique [Gou22]. Among them, Theorem A, B and the first half of Theorem C (CLT) use pivoting technique implicitly via referring to the results of [Cho22b]. This part is explained in Section 3, which only assumes the content of Section 2 and does not require any knowledge about pivotal times.

The other theorems are deduced using pivoting technique that we explain in Section 4. In Subsection 4.1, we recall the pivotal time construction for a discrete model. In Subsection 4.2, we consider two lemmata to reduce a random walk into the discrete model. Given this preparation, we relate pivotal times with translation length and genericity of contracting isometries in Subsection 4.3. In Subsection 4.4, we combine these ingredients and deduce the converse of CLT. In Subsection 4.5, we generalize the discussion in Subsection 4.2 to discuss the quasi-isometric embedding of k independent random walks.

It remains to discuss counting problem. Our strategy is to compare the exponential bound for simple random walks on G and a counting estimate on G. For this, we establish a quantitative version of Subsection 4.2 in Section 5. We prove Theorem E using this in Section 6.

2. Preliminaries

In this section, we summarize the language and the setting of [Cho22b]. Throughout the paper, X is a geodesic metric space with basepoint o and G is a countable isometry group of X.

2.1. **Contracting isometries and alignment.**

Definition 2.1 (contracting sets)**.** *For a subset $A \subseteq X$ and $\epsilon > 0$, we define the* closest point projection *of $x \in X$ to A by*

$$\pi_A(x) := \big\{ a \in A : d_X(x, a) = d_X(x, A) \big\}.$$

A is said to be K-contracting if:

(1) $\pi_A(z) \neq \emptyset$ for all $z \in X$ and
(2) for all $x, y \in X$ such that $d_X(x, y) \leq d_X(x, A)$ we have

$$\operatorname{diam}_X \big(\pi_A(x) \cup \pi_A(y) \big) \leq K.$$

A K-contracting K-quasigeodesic is called a K-contracting axis. An isometry $g \in G$ is said to be K-contracting if its orbit $\{g^n o\}_{n \in \mathbb{Z}}$ is a K-contracting axis.

Definition 2.2 (Translation length). *For $g \in G$, the* (asymptotic) *translation length of g is defined by*

$$\tau(g) := \liminf_{n \to \infty} \frac{1}{n} d(o, g^n o).$$

Given an isometry $g \in G$, the orbit $\{g^n o\}_{n \in \mathbb{Z}}$ is a quasi-geodesic if and only if g has strictly positive translation length. However, an isometry with positive translation length may not be contracting.

A *path* is a map from either an interval or a set of consecutive integers to X. Given a path $\gamma : [a, b] \to X$, we define its *reversal* $\bar{\gamma} : [a, b] \to X$ by the map $\bar{\gamma}(t) := \gamma(a + b - t)$.

A subset $A \subseteq X$ is said to be K-*quasi-convex* if any geodesic $[x, y]$ connecting two points $x, y \in A$ is contained in the K-neighborhood of A. Two paths κ and η on X are said to be K-*fellow traveling* if their beginning points and ending points are pairwise K-near and $d_{Hauss}(\kappa, \eta) < K$.

Let us now recall some alignment lemmata established in [Cho22b].

Definition 2.3 ([Cho22b, Definition 3.5]). *For $i = 1, \ldots, n$, let γ_i be a path on X whose beginning and ending points are x_i and y_i, respectively. We say that $(\gamma_1, \ldots, \gamma_n)$ is C-aligned if*

$$\mathrm{diam}_X \left(y_i \cup \pi_{\gamma_i}(\gamma_{i+1}) \right) < C, \quad \mathrm{diam}_X \left(x_{i+1} \cup \pi_{\gamma_{i+1}}(\gamma_i) \right) < C$$

hold for $i = 1, \ldots, n - 1$.

Here, we regard points as degenerate paths. For example, for a point x and a path γ, we say that (x, γ) is C-aligned if

$$\mathrm{diam}_X(\text{beginning point of } \gamma \cup \pi_\gamma(x)) < C$$

holds. The following observation is immediate:

Observation 2.4. *Let g be an isometry of X. Let n be a positive integer and let k be an integer in $\{1, \ldots, n\}$. Let $K > 0$ and let $\gamma_1, \ldots, \gamma_n$ be paths on X.*

  (1) If $(\gamma_1, \ldots, \gamma_k)$ and $(\gamma_k, \ldots, \gamma_n)$ are K-aligned, then their concatenation $(\gamma_1, \ldots, \gamma_n)$ is also K-aligned.
  (2) If $(\gamma_1, \ldots, \gamma_n)$ is K-aligned, then $(\bar{\gamma}_n, \ldots, \bar{\gamma}_1)$ is also K-aligned.
  (3) If $(\gamma_1, \ldots, \gamma_n)$ is K-aligned, then $(g\gamma_1, \ldots, g\gamma_n)$ is also K-aligned.

Lemma 2.5 ([Cho22b, Lemma 3.7]). *For each $C > 0$ and $K > 1$, there exists $D = D(K, C) > K, C$ that satisfies the following.*

Let γ and γ' be K-contracting axes whose ending points are y and y', respectively. If (y, γ') and (γ, y') are C-aligned, then (γ, γ') is D-aligned.

Lemma 2.6 ([Cho22b, Proposition 3.11]). *For each $D > 0$ and $K > 1$, there exist $E = E(K, D) > K, D$ and $L = L(K, D) > K, D$ that satisfy the following.*

Let x and y be points in X and let $\gamma_1, \ldots, \gamma_n$ be K-contracting axes whose domains are longer than L and such that $(x, \gamma_1, \ldots, \gamma_n, y)$ is D-aligned. Then the geodesic $[x, y]$ has subsegments $\eta_1, \ldots, \eta_n$, in order from left to right, that are longer than $100E$ and such that η_i and γ_i are $0.1E$-fellow traveling for each i. In particular, (x, γ_i, y) are E-aligned for each i.

Lemma 2.7 ([Yan19, Proposition 2.9]). *For each $D, M > 0$ and $K > 1$, there exist $E = E(K, D, M) > D$ and $L = L(K, D) > D$ that satisfies the following.*

Let $\gamma_1, \ldots, \gamma_n$ be K-contracting axes whose domains are longer than L. Suppose that $(\gamma_1, \ldots, \gamma_n)$ is D-aligned and $d(\gamma_i, \gamma_{i+1}) < M$ for each i. Then the concatenation $\gamma_1 \cup \ldots \cup \gamma_n$ of $\gamma_1, \ldots, \gamma_n$ is an E-contracting axis.

For the proofs of these lemmata, refer to Subsection 3.1 and 3.2 of [Cho22b].

2.2. **Random walks.** Let μ be a probability measure on G. We denote by $\check{\mu}$ the *reflected version of μ*, which is defined by $\check{\mu}(g) := \mu(g^{-1})$. The *random walk* generated by μ is the Markov chain on G with the transition probability $p(g, h) := \mu(g^{-1}h)$.

Consider the *step space* $(G^{\mathbb{Z}}, \mu^{\mathbb{Z}})$, the product space of (G, μ). Each element $(g_n)_{n \in \mathbb{Z}} \in G^{\mathbb{Z}}$ is called a *step path*, and there is a corresponding *(bi-infinite) sample path* $(Z_n)_{n \in \mathbb{Z}}$ under the correspondence

$$
Z_n = \begin{cases} g_1 \cdots g_n & n > 0 \\ id & n = 0 \\ g_0^{-1} \cdots g_{n+1}^{-1} & n < 0. \end{cases}
$$

We also introduce the notation $\check{g}_n = g_{-n+1}^{-1}$ and $\check{Z}_n = Z_{-n}$. Note that we have an isomorphism $(G^{\mathbb{Z}}, \mu^{\mathbb{Z}}) \to (G^{\mathbb{Z}>0}, \check{\mu}^{\mathbb{Z}>0}) \times (G^{\mathbb{Z}>0}, \mu^{\mathbb{Z}>0})$ by $(g_n)_{n \in \mathbb{Z}} \mapsto ((\check{g}_n)_{n>0}, (g_n)_{n>0})$. We will frequently use the latter parametrization: for each $(\omega, \check{\omega}) \in (G^{\mathbb{Z}>0}, \check{\mu}^{\mathbb{Z}>0}) \times (G^{\mathbb{Z}>0}, \mu^{\mathbb{Z}>0})$ we have

$$
g_n(\check{\omega}, \omega) := g_n(\omega), \quad Z_n(\check{\omega}, \omega) := Z_n(\omega),
$$
$$
\check{g}_n(\check{\omega}, \omega) := g_n(\check{\omega}), \quad \check{Z}_n(\check{\omega}, \omega) := Z_n(\check{\omega})
$$

We define the *support* of μ, denoted by $\operatorname{supp} \mu$, as the set of elements in G that are assigned nonzero values of μ. We denote by μ^N the product measure of N copies of μ, and by μ^{*N} the N-th convolution measure of μ.

Now suppose that G is acting on the metric space (X, d). A probability measure μ on G is said to be *non-elementary* if the semigroup generated by the support of μ contains two independent contracting isometries g, h of X.

2.3. **Schottky set.** We now introduce the notion of Schottky set. Given a sequence $\alpha = (a_1, \ldots, a_n) \in G^n$, we employ the following notations:

$$\Pi(\alpha) := a_1 a_2 \cdots a_n,$$

$$\Gamma(\alpha) := \big(o, a_1 o, a_1 a_2 o, \ldots, \Pi(\alpha)o\big).$$

The following definition is an adaptation of Gouëzel's notion of Schottky sets on hyperbolic groups in [Gou22].

Definition 2.8 (cf. [Gou22, Definition 3.11], [Cho22b, Definition 3.15]). *Let $K_0 > 0$ and define:*

- $D_0 = D(K_0, K_0)$ *be as in Lemma 2.5,*
- $D_1 = E(D_0, K_0)$, $L = L(D_0, K_0)$ *be as in Lemma 2.6,*
- $E_0 = E(D_1, K_0)$, $L' = L(D_1, K_0)$ *be as in Lemma 2.6,*
- $E_1 = E(D_1, K_0)$, $L'' = L(D_1, K_0)$ *be as in Lemma 2.7.*

We say that a set of sequences $S \subseteq G^n$ is a fairly long K_0-Schottky set *if:*

(1) $n > \max\{L, L', L''\}$;

(2) $\Gamma(\alpha)$ is K_0-contracting axis for all $\alpha \in S$;

(3) $d(o, \Pi(\alpha)o) \geq 10E_0$ for all $\alpha \in S$;

(4) for each $x \in X$ we have

$$\#\Big\{\alpha \in S : \big(x, \Gamma(\alpha)\big) \text{ and } \big(\Gamma(\alpha), \Pi(s)x\big) \text{ are } K_0\text{-aligned}\Big\} \geq \#S - 1;$$

(5) for each $\alpha \in S$, $\big(\Gamma(\alpha), \Pi(\alpha)\Gamma(\alpha)\big)$ is K_0-aligned.

When the Schottky parameter K_0 is understood, the constants D_0, D_1, E_0, E_1 always denote the ones defined above. Further, we sometimes omit the Schottky parameter and just say that S is a fairly long Schottky set.

Once a fairly long Schottky set S is understood, its element α is called a Schottky sequence *and the translates of $\Gamma(\alpha)$ are called* Schottky axes. *When a probability measure μ on G is given in addition such that $S \subseteq (\operatorname{supp}\mu)^n$, we say that S is a* fairly long Schottky set for μ.

Some facts regarding Schottky sets are in order.

Lemma 2.9 ([Cho22b, Proposition 3.18]). *Let μ be a non-elementary probability measure on G. Then for each $N > 0$ there exists a fairly long Schottky set for μ with cardinality N.*

Definition 2.10. *Let S be a fairly long Schottky set and let $K > 0$. We say that a sequence of Schottky axes is* K-semi-aligned *if it is a subsequence of a K-aligned sequence of Schottky axes.*

More precisely, for Schottky axes $\gamma_1, \ldots, \gamma_n$, we say that $(\gamma_1, \ldots, \gamma_n)$ is K-semi-aligned if there exist Schottky axes $\eta_1, \ldots, \eta_m$ such that $(\eta_1, \ldots, \eta_m)$ is K-aligned and there exists a subsequence $\{i(1) < \ldots < i(n)\} \subseteq \{1, \ldots, m\}$ such that $\gamma_l = \eta_{i(l)}$ for $l = 1, \ldots, n$.

Similarly, for points $x, y \in X$ and Schottky axes $\gamma_1, \ldots, \gamma_n$, we say that $(x, \gamma_1, \ldots, \gamma_n, y)$ is K-semi-aligned if it is a subsequence of a K-aligned sequence $(x, \eta_1, \ldots, \eta_m, y)$ for some Schottky axes $\eta_1, \ldots, \eta_n$.

Lemma 2.6 implies the following corollary.

Corollary 2.11. *Let S be a fairly long K_0-Schottky set, with constants D_0, D_1 as in Definition 2.8. Let $x, y \in X$ and $\gamma_1, \ldots, \gamma_N$ be Schottky axes. Then $(x, \gamma_1, \ldots, \gamma_N, y)$ is D_1-aligned whenever it is D_0-semi-aligned.*

3. THE FIRST METHOD: DEVIATION INEQUALITIES

Our first approach to the limit laws rely on the pivoting technique implicitly via deviation inequalities. Throughout, we fix (X, G, o) as in Convention 1.1, fix a non-elementary probability measure μ on G and fix a fairly long Schottky set $S \subseteq (\operatorname{supp} \mu)^{M_0}$ for μ.

Employing the notations defined in Subsection 2.2, let $(\check{\omega}, \omega) \in (G^{\mathbb{Z}_{>0}}, \check{\mu}^{\mathbb{Z}_{>0}}) \times (G^{\mathbb{Z}_{>0}}, \mu^{\mathbb{Z}_{>0}})$. We briefly recall the random variable $v = v(\check{\omega}, \omega)$ defined in [Cho22b, Section 5]. For each $k \geq M_0$, we ask whether there exists $M_0 \leq i \leq k$ such that:

(1) $\gamma := (Z_{i-M_0}o, Z_{i-M_0+1}o, \ldots, Z_i o)$ is a Schottky axis, and
(2) $(\check{Z}_m o, \gamma, Z_n o)$ is D_0-semi-aligned for all $n \geq k$ and $m \geq 0$.

Note that, by Corollary 2.11, Item (2) forces that $(\check{Z}_m o, \gamma, Z_n o)$ is D_1-aligned for all $n \geq k$ and $m \geq 0$. We define $v(\check{\omega}, \omega)$ as the minimal index k that possesses such an auxiliary index $i \leq k$.

Similarly, we defined $\check{v} = \check{v}(\check{\omega}, \omega)$ as the minimal index $k \geq M_0$ such that there exists $M_0 \leq i \leq k$ for which:

(1) $\gamma := (\check{Z}_i o, \check{Z}_{i+1}o, \ldots, \check{Z}_{i-M_0}o)$ is a Schottky axis, and
(2) $(\check{Z}_n o, \gamma, Z_m o)$ is D_0-semi-aligned for all $n \geq k$ and $m \geq 0$.

In [Cho22b] we proved the following results:

Lemma 3.1 ([Cho22b, Lemma 4.9]). *Let (X, G, o) be as in Convention 1.1 and let μ be a non-elementary probability measure on G. Then there exist $\kappa, K > 0$ such that the following estimate holds for all k and all choices of $g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1} \in G$:*

$$\mathbb{P}_{\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}} \left(v(\check{\omega}, \omega) \geq k \,\Big|\, g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1} \right) \leq K e^{-\kappa k},$$

and for all k and for all choices of $g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1} \in G$:

$$\mathbb{P}_{\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}} \left(\check{v}(\check{\omega}, \omega) \geq k \,\Big|\, \check{g}_{k+1}, g_1, \ldots, g_{k+1} \right) \leq K e^{-\kappa k}.$$

By integrating for g_{k+1} and $\check{g}_1, \ldots, \check{g}_{k+1}$ over G, we deduce that

$$\mathbb{P}_{\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}} \left(v(\check{\omega}, \omega) \geq k \right)$$
$$= \sum_{\substack{\check{g}_1, \ldots, \check{g}_{k+1}, \\ g_{k+1} \in G}} \mathbb{P}_{\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}} \left(v(\check{\omega}, \omega) \geq k \,\Big|\, g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1} \right) \cdot \mu(g_{k+1}) \check{\mu}(\check{g}_1) \cdots \check{\mu}(\check{g}_{k+1}) \leq K e^{-\kappa k}.$$

Similarly, we have $\mathbb{P}_{\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}} \left(\check{v}(\check{\omega}, \omega) \geq k \right) \leq K e^{-\kappa k}$ for each k.

Lemma 3.2 ([Cho22b, Corollary 4.11]). *Let $p > 0$ and suppose that μ has finite p-th moment. Then there exists $K > 0$ such that*

$$\mathbb{E}_{\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}} \left[\min\{d(o, Z_v o), d(o, \check{Z}_{\check{v}} o)\}^{2p} \right] < K.$$

Let F_p be the distribution of $\min\{d(o, Z_v o), d(o, \check{Z}_{\check{v}} o)\}^{2p}$, i.e.,

$$F_p(u) = \mathbb{P}_{\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}} \left(\min\{d(o, Z_v o), d(o, \check{Z}_{\check{v}} o)\}^{2p} \geq u \right).$$

By Lemma 3.2, $\int_0^\infty F_p(u) du$ is finite whenever μ has finite p-th moment.

Using this, we can prove Theorem B.

Proof of Theorem B. Recall that μ be a non-elementary probability measure on G. Let $\kappa, K > 0$ be the constants for μ as in Lemma 3.1.

Recall also our notations introduced in Subsection 2.2: $(g_i)_{i \in \mathbb{Z}}$'s are i.i.d. RVs distributed according to μ and $Z_i := g_1 \cdots g_i$ for $i \geq 0$. Now for each $n > 0$, we bring the n-step initial subpath $(g_1, \ldots, g_n)$ of $(g_i)_{i>0}$ that is distributed according to μ^n, together with another RV $(h_i)_{i \in \mathbb{Z}}$ distributed according to $\mu^{\mathbb{Z}}$ that is independent from $(g_i)_{i \in \mathbb{Z}}$. We then define

$$g_{i;0} := \begin{cases} g_i & i = 1, \ldots, \lfloor n/2 \rfloor \\ h_i & i > \lfloor n/2 \rfloor, \end{cases} \qquad \check{g}_{i;0} := \begin{cases} g_{n-i+1}^{-1} & i = 1, \ldots, n - \lfloor n/2 \rfloor \\ h_{-i}^{-1} & i > n - \lfloor n/2 \rfloor, \end{cases}$$

$$g_{i;1} := \begin{cases} g_{\lfloor n/2 \rfloor + i} & i = 1, \ldots, n - \lfloor n/2 \rfloor \\ h_i & i > n - \lfloor n/2 \rfloor, \end{cases} \qquad \check{g}_{i;1} := \begin{cases} g_{\lfloor n/2 \rfloor - i + 1}^{-1} & i = 1, \ldots, \lfloor n/2 \rfloor \\ h_{-i}^{-1} & i > \lfloor n/2 \rfloor. \end{cases}$$

Then $\left((\check{g}_{i;t})_{i>0}, (g_{i;t})_{i>0} \right)$ is distributed according to $\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}$ for $t = 0, 1$. Using them, we similarly define other RVs such as

$$Z_{i;t} := g_{1;t} \cdots g_{i;t}, \qquad \check{Z}_{i;t} := \check{g}_{1;t} \cdots \check{g}_{i;t},$$
$$v_t := v \left((\check{g}_{i;t})_{i>0}, (g_{i;t})_{i>0} \right), \qquad \check{v}_t := \check{v} \left((\check{g}_{i;t})_{i>0}, (g_{i;t})_{i>0} \right).$$

Note that v_t, $\check{v}_t$ are RVs that depend on the choice of n. For this reason, we will also denote v_0 by $v(n)$ and $\check{v}_0$ by $\check{v}(n)$.

We now define

$$A_n := \left\{ \max\{v_0, \check{v}_0, v_1, \check{v}_1\} \geq n/10 \right\}.$$

Since $\left((\check{g}_{i;0})_{i>0}, (g_{i;0})_{i>0} \right)$ and $\left((\check{g}_{i;1})_{i>0}, (g_{i;1})_{i>0} \right)$ are both distributed according to $\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}$, Lemma 3.1 implies
(3.1)
$$\mathbb{P}(A_n) \leq 2 \, \mathbb{P}_{\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}} \left(v(\check{\omega}, \omega) \geq n/10 \right) + 2 \, \mathbb{P}_{\check{\mu}^{\mathbb{Z}_{>0}} \times \mu^{\mathbb{Z}_{>0}}} \left(\check{v}(\check{\omega}, \omega) \geq n/10 \right)$$
$$\leq 4K e^{-\kappa n/10}.$$

Furthermore, Lemma 3.2 implies

$$\mathbb{P} \left(\min \left\{ d\left(o, Z_{v_t;t} \, o\right), d\left(o, \check{Z}_{\check{v}_t;t} \, o\right) \right\}^{2p} \geq u \right) = F_p(u)$$

for each $u \geq 0$ and $t = 0, 1$. We aim to show:

Claim 3.3.

$$(3.2) \qquad [d(o, Z_n o) - \tau(Z_n)] \leq 2 \min \left\{ d\left(o, Z_{v_0;0}\, o\right), d\left(o, \check{Z}_{\check{v}_0;0}\, o\right) \right\}$$

holds outside A_n.

For the proof of Claim 3.3, we discuss everything outside A_n. That means, we assume from now on that

$$(3.3) \qquad\qquad\qquad v_0, \check{v}_0, v_1, \check{v}_1 \leq n/10.$$

First, by the definition of v_0, there exists an integer $i(0)$ such that

(1) $M_0 \leq i(0) \leq v_0$;
(2) $\gamma_0 := (Z_{i(0)-M_0;0}\, o, Z_{i(0)-M_0+1;0}\, o, \ldots, Z_{i(0);0}\, o)$ is a Schottky axis;
(3) $\left(\check{Z}_{j;0}\, o, \gamma_0, Z_{k;0}\, o\right)$ is D_1-aligned for $j \geq 0$ and $k \geq v_0$.

Recall our definition that:

$$(3.4)$$
$$Z_{i;0} = g_{1;0} \cdots g_{i;0} = g_1 \cdots g_i = Z_i \qquad\qquad (i = 0, 1, \ldots, \lfloor n/2 \rfloor),$$
$$\check{Z}_{i;0} = \check{g}_{1;0} \cdots \check{g}_{i;0} = g_n^{-1} \cdots g_{n-i+1}^{-1} = Z_n^{-1} Z_{n-i} \quad (i = 0, 1, \ldots, n - \lfloor n/2 \rfloor).$$

Item (3) of the condition for $i(0)$, Display 3.4 and Display 3.3 together imply:

Observation 3.4. *The sequence*

$$\left(Z_n^{-1} Z_{n-j} o, \gamma_0, Z_k o\right) = \left(Z_n^{-1} Z_{n-j} o, (Z_{i(0)-M_0} o, \ldots, Z_{i(0)} o), Z_k o\right)$$

is D_1*-aligned for* $0 \leq j \leq n - \lfloor n/2 \rfloor$ *and* $v_0 \leq k \leq \lfloor n/2 \rfloor$.

Meanwhile, by the definition of $\check{v}_0$, there exists an integer $j(0)$ such that

(1) $M_0 \leq j(0) \leq \check{v}_0$;
(2) $(\check{Z}_{j(0);0}\, o, \check{Z}_{j(0)-1;0}\, o, \ldots, \check{Z}_{j(0)-M_0;0}\, o)$ is a Schottky axis, and
(3) $\left(\check{Z}_{j;0}\, o, (\check{Z}_{j(0);0}\, o, \ldots, \check{Z}_{j(0)-M_0}\, o), Z_{k;0}\, o\right)$ is D_1-aligned for $k \geq 0$ and $j \geq v_0$.

We now define

$$\check{\gamma}_0 := (Z_{n-j(0)} o, Z_{n-j(0)+1} o, \ldots, Z_{n-j(0)+M_0} o).$$

Item (3) of the above, Display 3.4 and Display 3.3 together imply:

Observation 3.5.

$$\left(Z_n^{-1} Z_{n-j} o, Z_n^{-1} \check{\gamma}_0, Z_k o\right) = \left(Z_n^{-1} Z_{n-j} o, (Z_n^{-1} Z_{n-j(0)} o, \ldots, Z_n^{-1} Z_{n-j(0)+M_0} o), Z_k o\right)$$

is D_1*-aligned for* $\check{v}_0 \leq j \leq n - \lfloor n/2 \rfloor$ *and* $0 \leq k \leq \lfloor n/2 \rfloor$.

Similarly, we have indices $i(1), j(1)$ such that

(1) $M_0 \leq i(1) \leq v_1$ and $M_0 \leq j(1) \leq \check{v}_1$;
(2) $(Z_{i(1)-M_0;1} o, \ldots, Z_{i(1);1} o)$ and $(\check{Z}_{j(1);1} o, \ldots, \check{Z}_{j(1)-M_0;1} o)$ are Schottky axes,
(3) $\left(\check{Z}_{j;1} o, (Z_{i(1)-M_0;1} o, \ldots, Z_{i(1);1} o), Z_{k;1} o\right)$ is D_1-aligned for $j \geq 0$ and $k \geq v_1$, and

(4) $\big(\check{Z}_{j;1}o, (\check{Z}_{j(1);1}o, \ldots, \check{Z}_{j(1)-M_0;1}o), Z_{k;1}o\big)$ is D_1-aligned for $k \geq 0$ and $j \geq \check{v}_1$.

This time, we define

$$\gamma_1 := (Z_{\lfloor n/2 \rfloor + i(1) - M_0}o, \ldots, Z_{\lfloor n/2 \rfloor + i(1)}o), \quad \check{\gamma}_1 := (Z_{\lfloor n/2 \rfloor - j(1)}o, \ldots, Z_{\lfloor n/2 \rfloor - j(1) + M_0}o).$$

and recall the following:
(3.5)
$$Z_{i;1} = g_{1;1} \cdots g_{i;1} = g_{\lfloor n/2 \rfloor + 1} \cdots g_{\lfloor n/2 \rfloor + i} = Z_{\lfloor n/2 \rfloor}^{-1} Z_{\lfloor n/2 \rfloor + i} \quad (i = 0, 1, \ldots, n - \lfloor n/2 \rfloor),$$
$$\check{Z}_{i;1} = \check{g}_{1;1} \cdots \check{g}_{i;1} = g_{\lfloor n/2 \rfloor}^{-1} \cdots g_{\lfloor n/2 \rfloor - i + 1}^{-1} = Z_{\lfloor n/2 \rfloor}^{-1} Z_{\lfloor n/2 \rfloor - i} \qquad (i = 0, 1, \ldots, \lfloor n/2 \rfloor).$$

Combining Item (3) and (4) of the conditions for $i(1)$ and $j(1)$, Display 3.4 and Display 3.3, we obtain:

Observation 3.6. *The sequence*

$$\big(Z_{\lfloor n/2 \rfloor - j}o, \gamma_1, Z_{\lfloor n/2 \rfloor + k}o\big) = \big(Z_{\lfloor n/2 \rfloor - j}o, (Z_{\lfloor n/2 \rfloor + i(1) - M_0}o, \ldots, Z_{\lfloor n/2 \rfloor + i(1)}o), Z_{\lfloor n/2 \rfloor + k}o\big)$$

is D_0-aligned for $0 \leq j \leq \lfloor n/2 \rfloor$ and $v_1 \leq k \leq n - \lfloor n/2 \rfloor$. Moreover,

$$\big(Z_{\lfloor n/2 \rfloor - j}o, \check{\gamma}_1, Z_{\lfloor n/2 \rfloor + k}o\big) = \big(Z_{\lfloor n/2 \rfloor - j}o, (Z_{\lfloor n/2 \rfloor - j(1)}o, \ldots, Z_{\lfloor n/2 \rfloor - j(1) + M_0}o), Z_{\lfloor n/2 \rfloor + k}o\big)$$

is D_0-aligned for $\check{v}_1 \leq j \leq \lfloor n/2 \rfloor$ and $0 \leq k \leq n - \lfloor n/2 \rfloor$.

A special case of Observation 3.4 is that $(o, \gamma_0, Z_{v_0}o)$ is D_1-aligned. By Lemma 2.6 (cf. Definition 2.8), there exists a subsegment $[p.q]$ of $[o, Z_{v_0}o]$ that is $0.1E_0$-fellow traveling with γ_0. We deduce that

$$
\begin{aligned}
(3.6) \qquad d(o, Z_{i(0)-M_0}o) &\leq d(o, p) - 0.1E_0 = d(o, q) - d(p, q) - 0.1E_0 \\
&\leq d(o, Z_{v_0}o) - d(Z_{i(0)-M_0+1}o, Z_{i(0)}o) - 0.3E_0 \\
&\leq d(o, Z_{v_0}) - 90E_0.
\end{aligned}
$$

Similarly, from the fact that $(Z_{n-\check{v}_0}o, \check{\gamma}_0, Z_n o)$ is D_1-aligned we deduce

$$d(o, Z_n^{-1} Z_{n-j(0)+M_0}o) \leq d(Z_{n-\check{v}_0}o, Z_n o) - 90E_0 = d(\check{Z}_{\check{v}_0}o, o) - 90E_0.$$

Next, Display 3.3 implies that

$$\gamma_0 \subseteq \big\{ Z_k o : 0 \leq k \leq \lfloor n/2 \rfloor - \check{v}_1 \big\}, \quad \check{\gamma}_1 \subseteq \big\{ Z_k o : v_0 \leq k \leq \lfloor n/2 \rfloor \big\},$$
$$\gamma_1 \subseteq \big\{ Z_k o : \lfloor n/2 \rfloor \leq k \leq n - \check{v}_0 \big\}, \quad \check{\gamma}_0 \subseteq \big\{ Z_k o : \lfloor n/2 \rfloor + v_1 \leq k \leq n \big\}.$$

Combining this with Observation 3.4, 3.5 and 3.6, we deduce that $(\gamma_0, \check{\gamma}_1, \gamma_1, \check{\gamma}_0, Z_n \gamma_0)$ is D_1-aligned, and consequently,

$$\big(o, \gamma_0, \check{\gamma}_1, \gamma_1, \check{\gamma}_0, Z_n \gamma_0, Z_n \check{\gamma}_1, Z_n \gamma_1, Z_n \check{\gamma}_0, \ldots, Z_n^{k-1} \gamma_0, Z_n^{k-1} \check{\gamma}_1, Z_n^{k-1} \gamma_1, Z_n^{k-1} \check{\gamma}_0, Z_n^k o\big)$$

is also D_1-aligned. By Lemma 2.6, there exist points $p_0, q_0, \ldots, p_{k-1}, q_{k-1}$ on $[o, Z_n^k o]$, from left to right, so that

$$d(p_i, Z_n^i \cdot Z_{i(0)-M_0}o) < 0.1E_0, \quad d(q_i, Z_n^i \cdot Z_{n-j(0)+M_0}o) < 0.1E_0.$$

This implies that

$$d(o, Z_n^k o) \geq \sum_{i=1}^{k-1} d(p_{i-1}, p_i)$$

$$\geq \sum_{i=1}^{k-1} \left(\begin{array}{c} d(Z_n^{i-1} o, Z_n^i o) - d(Z_n^{i-1} o, Z_n^{i-1} Z_{i(0)-M_0} o) - d(Z_n^i o, Z_n^i Z_{i(0)-M_0} o) \\ -d(p_{i-1}, Z_n^{i-1} Z_{i(0)-M_0} o) - d(p_i, Z_n^i Z_{i(0)-M_0} o) \end{array} \right)$$

$$\geq (k-1) \Big(d(o, Z_n o) - 2d(o, Z_{i(0)-M_0} o) - E_0 \Big).$$

By taking the limit and applying Inequality 3.6, we deduce that

$$\tau(Z_n) \geq d(o, Z_n o) - 2d(o, Z_{i(0)-M_0} o) - 2E_0 \geq d(o, Z_n o) - 2d(o, Z_{v_0} o).$$

By a similar argument using q_i's and $Z_n^{i-1} Z_{n-j(0)+M_0}$'s, we also observe that $\tau(Z_n) \geq d(o, Z_n o) - 2d(o, \check{Z}_{\check{v}_0} o)$. Claim 3.3 is now established.

Given the claim, we obtain

$$\mathbb{P}(d(o, Z_n o) - \tau(Z_n) \geq C n^{1/2p})$$

$$= \mathbb{P}\left([d(o, Z_n o) - \tau(Z_n)]^{2p} \geq C^{2p} n \right)$$

$$\leq \mathbb{P}(A_n) + \mathbb{P}\left(2^{2p} \min \left\{ d\left(o, Z_{v_0;o} o\right), d\left(o, \check{Z}_{\check{v}_0;0} o\right) \right\}^{2p} \geq C^{2p} n \right)$$

$$\leq F_p(C^{2p} n / 2^{2p}) + 8K e^{-\kappa n/10}.$$

When μ has finite p-th moment, $F_p(u)$ is integrable and the above probability is summable. The Borel-Cantelli lemma implies Item (1) of Theorem B.

Now suppose that μ has finite first moment and let λ be its escape rate. We denote v_0 by $v(n)$ to clarify its dependence on n. This time, we define

$$A_n' := \{v(n) \geq K' \log n\}$$

for some large K' such that $\sum_n K e^{-\kappa K' \log n} < +\infty$. Recall that Lemma 3.1 tells us that

$$\mathbb{P}(A_n \cup A_n') \leq 4K e^{-\kappa k} + \mathbb{P}_{\check{\mu}^{\mathbb{Z}>0} \times \mu^{\mathbb{Z}>0}}(v(\check{\omega}, \omega) \geq K' \log n)$$

$$\leq 4K e^{-\kappa k} + K e^{-\kappa K' \log n}.$$

The Borel-Cantelli lemma implies that almost every sample path $(\check{\omega}, \omega)$ eventually lies outside $A_n \cup A_n'$, which implies that $d(o, Z_n o) - \tau(Z_n) \leq d(o, Z_{v(n)} o)$ for all large enough n (Claim 3.3) and $\limsup v(n)/\log n \leq K'$. Moreover, by subadditive ergodic theorem, we have $\lim_n d(o, Z_n o)/n = \lambda$ for almost every sample path.

It remains to show that $\limsup_n \frac{d(o, Z_n o) - \tau(Z_n)}{\log n} \leq 4\lambda K'$ whenever

$$\limsup_n \frac{d(o, Z_n o) - \tau(Z_n)}{d(o, Z_{v(n)} o)} \leq 1, \quad \limsup_n \frac{v(n)}{\log n} \leq K', \quad \lim_n \frac{d(o, Z_n o)}{n} = \lambda.$$

To show this, take N large enough such that $d(o, Z_n o)/n \le 2\lambda$ for $n \ge 2K' \log N$ and $v(n)/\log n \le 2K'$ for $n \ge N$. For $n \ge N$, we have

$$d(o, Z_{v(n)} o) \le \begin{cases} \max\{d(o, Z_i o) : 0 \le i \le 2K' \log N\} & \text{when } v(n) \le 2K' \log N, \\ 2\lambda v(n) \le 4K'\lambda \log n & \text{otherwise.} \end{cases}$$

It is clear that $d(o, Z_{v(n)} o)/\log n \le 4\lambda K'$ for all large enough n. Consequently, we have

$$\limsup_n \frac{d(o, Z_n o) - \tau(Z_n)}{\log n} \le \limsup_n \frac{d(o, Z_n o) - \tau(Z_n)}{d(o, Z_{v(n)} o)} \cdot \frac{d(o, Z_{v(n)} o)}{\log n} \le 4K'\lambda.$$

$\square$

When μ has finite first moment, Theorem B implies that
(3.7)
$$0 \le \lim_{n \to +\infty} \frac{1}{n} |d(o, Z_n o) - \tau(Z_n)| \le \lim_{n \to +\infty} \frac{1}{\sqrt{n \log \log n}} |d(o, Z_n o) - \tau(Z_n)|$$

$$\le \lim_{n \to +\infty} \frac{1}{\sqrt{n}} |d(o, Z_n o) - \tau(Z_n)| = 0 \quad \text{almost surely.}$$

Combining Inequality 3.7 with the SLLN for displacement (cf. [Cho22b, Theorem A]), we conclude:

Corollary 3.7 (SLLN for finite first moment). *Let (X, G, o) be as in Convention 1.1, and let $(Z_n)_{n \ge 0}$ be the random walk generated by a non-elementary measure μ on G with finite first moment. Then*

(3.8)
$$\lim_n \frac{1}{n} \tau(Z_n) = \lambda$$

holds almost surely, where $\lambda = \lambda(\mu)$ is the escape rate of μ.

Also, combining Inequality 3.7 with the CLT and LIL for displacement ([Cho22b, Theorem 4.13, 4.16]), we deduce:

Corollary 3.8 (CLT). *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. If μ has finite second moment, then there exists $\sigma(\mu) \ge 0$ such that $\frac{1}{\sqrt{n}}(\tau(Z_n) - n\lambda)$ and $\frac{1}{\sqrt{n}}(d(o, Z_n o) - n\lambda)$ converge to the same Gaussian distribution $\mathcal{N}(0, \sigma(\mu)^2)$ in law. We also have*

$$\limsup_{n \to \infty} \frac{\tau(Z_n) - \lambda n}{\sqrt{2n \log \log n}} = \sigma(\mu) \quad \text{almost surely.}$$

In fact, Theorem B implies Corollary 3.7 for measures with finite $(1/2)$-th moment, and the converse of CLT for measures with finite $(1/4)$-th moment. For general non-elementary measures, Theorem B is not sufficient for the SLLN and the converse of CLT. Towards a different approach, let us recall:

Theorem F ([Cho22b, Theorem 6.4]). *Let (X, G, o) be as in Convention 1.1 let $(Z_n)_{n \ge 0}$ be the random walk generated by a non-elementary measure*

μ on G. Let $\lambda(\mu)$ be the escape rate of μ. Then for any $0 < L < \lambda(\mu)$, there exists $K > 0$ such that for each n we have

$$\mathbb{P}[d(o, Z_n o) \leq Ln] \leq Ke^{-n/K}.$$

Using this theorem, let us prove Theorem A.

Proof of Theorem A. Let μ be a non-elementary probability measure on G and $\lambda(\mu) \in (0, +\infty]$ be its escape rate. Given $0 < L < \lambda(\mu)$, we fix $0 < \epsilon < 1/10$ such that $L' = L/(1 - 2\epsilon)$ is still smaller than λ.

In the proof of Theorem B, we defined $v_0 = v(n)$ and $\check{v}_0 = \check{v}(n)$ for each n. Given $\epsilon > 0$, we now define

$$A_n'' := \Big\{ \max\{v(n), \check{v}(n)\} \geq \epsilon n \Big\}.$$

As explained in Inequality 3.1, with ϵn in place of $n/10$, $\mathbb{P}(A_n'')$ decays exponentially as n increases. In the sequel, we discuss everything outside A_n''.

Observation 3.4, 3.5 and 3.6 guarantees 4 Schottky axes $\gamma_0, \check{\gamma}_0, \gamma_1, \check{\gamma}_1$, such that γ_0 contains a point $Z_{i(0)} o$ for some $i(0) \in \{M_0, M_0 + 1, \ldots, v(n)\} \subseteq \{0, 1, \ldots, \lfloor \epsilon n \rfloor\}$, $\check{\gamma}_0$ contains a point $Z_{n-j(0)} o$ for some $j(0) \in \{M_0, M_0 + 1, \ldots, \check{v}(n)\} \subseteq \{0, 1, \ldots, \lfloor \epsilon n \rfloor\}$, and such that

$$\Big(o, \gamma_0, \check{\gamma}_1, \gamma_1, \check{\gamma}_0, Z_n \gamma_0, Z_n \check{\gamma}_1, Z_n \gamma_1, Z_n \check{\gamma}_0, \ldots, Z_n^{k-1} \gamma_0, Z_n^{k-1} \check{\gamma}_1, Z_n^{k-1} \gamma_1, Z_n^{k-1} \check{\gamma}_0, Z_n^k o\Big)$$

is D_1-aligned for each k. By Lemma 2.6, there exist points $p_0, q_0, \ldots, p_{k-1}, q_{k-1}$ on $[o, Z_n^k o]$, from left to right, so that

$$d(p_i, Z_n^i \cdot Z_{i(0)-M_0} o) < 0.1 E_0, \quad d(q_i, Z_n^i \cdot Z_{n-j(0)+M_0} o) < 0.1 E_0.$$

This implies that

$$d(o, Z_n^k o) \geq \sum_{i=0}^{k-1} d(p_i, q_i)$$

$$\geq \sum_{i=0}^{k-1} \left(\begin{array}{c} d(Z_n^i Z_{i(0)-M_0} o, Z_n^i Z_{n-j(0)+M_0} o) - d(p_i, Z_n^i Z_{i(0)-M_0} o) \\ -d(q_i, Z_n^i Z_{n-j(0)+M_0} o) \end{array} \right)$$

$$\geq k \min \Big\{ d(Z_i o, Z_{n-j} o) : 0 \leq i, j \leq \epsilon n \Big\} - E_0.$$

By dividing by k and taking the limit, we conclude that

$$\tau(Z_n) \geq \min \Big\{ d(Z_i o, Z_{n-j} o) : 0 \leq i, j \leq \epsilon n \Big\}$$

outside A_n''. Now, thanks to Theorem F, there exists $K > 0$ such that

$$\mathbb{P}[d(o, Z_m o) \leq L'm] \leq Ke^{-m/K}$$

holds for all m. This implies
(3.9)
$$\mathbb{P}\left(\tau(Z_n) \le Ln\right) \le \mathbb{P}(A_n'') + \mathbb{P}[\min\{d(\omega_i\, o, \omega_{n-j}\, o) : 0 \le i, j \le \epsilon n\} \le Ln]$$
$$\le \mathbb{P}(A_n'') + \sum_{0 \le i, j \le \epsilon n} \mathbb{P}[d(\omega_i\, o, \omega_{n-j}\, o) \le Ln]$$
$$\le \mathbb{P}(A_n'') + (\epsilon n)^2 \cdot K e^{-(1-2\epsilon)n/K}$$

for large enough n, which decays exponentially.

Meanwhile, outside A_n'',

$$(\dots, \quad Z_n^{k-1}\gamma_0, Z_n^{k-1}\check{\gamma}_1, Z_n^{k-1}\gamma_1, Z_n^{k-1}\check{\gamma}_0, Z_n^k\gamma_0, Z_n^k\check{\gamma}_1, \dots)$$

is a D_1-aligned sequence of Schottky axes. Since $d(\gamma, \gamma')$ is uniformly bounded for $\gamma, \gamma' \in \{\gamma_0, \check{\gamma}_0, \gamma_1, \check{\gamma}_1, Z_n\gamma_0\}$, Lemma 2.7 tells us that the concatenation of sequences $(Z_n^k\gamma_0, Z_n^k\check{\gamma}_1, Z_n^k\gamma_1, Z_n^k\check{\gamma}_0)_{k \in \mathbb{Z}}$ is a contracting axis. Since this axis and the orbit $\{Z_n^k o\}_{k \in \mathbb{Z}}$ are fellow traveling, Z_n is also contracting. Hence we have $\mathbb{P}(Z_n \text{ is not contracting}) \le \mathbb{P}(A_n'')$, which decays exponentially. $\square$

The previous proof did not assume that the initial displacement $d(o, Z_i o)$ or the final displacement $d(Z_{n-j}o, Z_n o)$ of a random path is shorter than the middle one $d(Z_i, Z_{n-j}o)$; indeed, one cannot expect such a phenomenon for high probability if the random walk has no moment condition. Instead, the proof explicitly used the fact that the middle segment will catch up the escape rate regardless of the moment condition, which is proven using the pivoting technique.

In order to discuss the converse of CLT for general measures, one should perform the pivoting more explicitly. For this purpose, we will recall the basics of the pivotal time construction in [Cho22b].

4. The second method: pivoting technique

4.1. **Pivotal times and pivoting.** This subsection is a summary of results in [Cho22b, Subsection 5.1], which is an adaptation of Gouëzel's work in [Gou22, Subsection 5.A]; for complete proofs, refer to the explanation there.

We keep employing Convention 1.1 for the metric space X with basepoint o and the isometry group G. We fix a fairly long K_0-Schottky set $S \subseteq G^{M_0}$ with cardinality N_0 for some K_0, M_0 and $N_0 > 400$. Recall that we have associated an isometry and a contracting axis for each $\alpha \in S$:

$$\Pi(\alpha) := a_1 a_2 \cdots a_n, \quad \Gamma(\alpha) := \left(o, a_1 o, a_1 a_2 o, \dots, \Pi(\alpha)o\right).$$

We first fix sequences of isometries $(w_i)_{i=0}^{\infty}$, $(v_i)_{i=1}^{\infty}$ in G. Then we draw a sequence of Schottky sequences

$$\mathbf{s} = (\alpha_1, \beta_1, \gamma_1, \delta_1, \dots, \alpha_n, \beta_n, \gamma_n, \delta_n) \in S^{4n},$$

with respect to the uniform measure on S^{4n} and construct words

$$W_k := w_0\Pi(\alpha_1)\Pi(\beta_1)v_1\Pi(\gamma_1)\Pi(\delta_1)w_1 \cdots \Pi(\alpha_k)\Pi(\beta_k)v_k\Pi(\gamma_k)\Pi(\delta_k)w_k.$$

Following [Gou22], we constructed a set $P_n = P_n(\mathbf{s}; (w_i)_{i=0}^n, (v_i)_{i=1}^n) \subseteq \{1, \ldots, n\}$, called the *set of pivotal times*, that satisfy the properties described in Lemma 4.1, 4.3 and 4.4. First, pivotal times are (partial) recording of those Schottky axes that are aligned along the eventual progress.

Lemma 4.1 ([Cho22b, Lemma 5.3]). *Let* $\mathbf{s} \in S^{4n}$, $(w_i)_{i=0}^n \in G^{n+1}$ *and* $(v_i)_{i=1}^n \in G^n$, *Let us enumerate the elements of* $P_n(\mathbf{s}; (w_i)_{i=0}^n, (v_i)_{i=1}^n)$ *as* $\{i(1) < \ldots < i(m)\}$ *and define*

$$\gamma_{4k-3} := W_{i(k)-1} \cdot \Gamma(\alpha_{i(k)}),$$

$$\gamma_{4k-2} := W_{i(k)-1}\Pi(\alpha_{i(k)}) \cdot \Gamma(\beta_{i(k)}),$$

$$\gamma_{4k-1} := W_{i(k)-1}\Pi(\alpha_{i(k)})\Pi(\beta_{i(k)})v_{i(k)} \cdot \Gamma(\gamma_{i(k)}),$$

$$\gamma_{4k} := W_{i(k)-1}\Pi(\alpha_{i(k)})\Pi(\beta_{i(k)})v_{i(k)}\Pi(\gamma_{i(k)}) \cdot \Gamma(\delta_{i(k)}).$$

for $k = 1, \ldots, m$. *Then* $(o, \gamma_1, \gamma_2, \ldots, \gamma_{4m}, W_m o)$ *is* D_0-*semi-aligned.*

Another important feature of the pivotal times is that we are allowed to change our choices of β_i's, γ_i's and v_i's at the pivotal times. To formulate, we define a subset $\tilde{S}$ of $S^2 \times G$:

Definition 4.2 ([Cho22b, Lemma 5.5]). *We define*

$$\tilde{S} := \left\{ (\beta, \gamma, v) \in S^2 \times G : \left(\Gamma(\beta), \Pi(\beta)v\Pi(\gamma)o\right) \text{ and } \left(v^{-1}o, \Gamma(\gamma)\right) \text{ are } K_0\text{-aligned sequences} \right\}.$$

Furthermore, we define for each $v \in G$ *its section:*

$$\tilde{S}(v) := \left\{ (\beta, \gamma) \in S^2 : \left(\Gamma(\beta), \Pi(\beta)v\Pi(\gamma)o\right) \text{ and } \left(v^{-1}o, \Gamma(\gamma)\right) \text{ are } K_0\text{-aligned sequences} \right\}.$$

Motivated by [Gou22, Lemma 4.6], we observed in [Cho22b] that $\#\tilde{S}(v) \geq (\#S)^2 - 2\#S$ for each $v \in G$. The role of $\tilde{S}$ is captured by the following.

Lemma 4.3 ([Cho22b, Lemma 5.5]). *Let* $\mathbf{s} = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n)$ *be a choice drawn from* S^{4n} *and* $\mathbf{w} = (w_i)_{i=0}^n, \mathbf{v} = (v_i)_{i=1}^n$ *be auxiliary sequences in* G.
Let $i \in P_n(\mathbf{s}; \mathbf{w}, \mathbf{v})$ *and let* $(\bar{\mathbf{s}}; \mathbf{w}, \bar{\mathbf{v}})$ *be obtained from* $(\mathbf{s}; \mathbf{w}, \mathbf{v})$ *by replacing* (β_i, γ_i, v_i) *with some* $(\bar{\beta}_i, \bar{\gamma}_i, \bar{v}_i)$ *such that* $(\bar{\beta}_i, \bar{\gamma}_i, \bar{v}_i) \in \tilde{S}$.
Then $P_l(\mathbf{s}; \mathbf{w}, \mathbf{v}) = P_l(\bar{\mathbf{s}}; \mathbf{w}, \bar{\mathbf{v}})$ *for any* $1 \leq l \leq n$.

Given $\mathbf{s}, \bar{\mathbf{s}} \in S^{4n}$ and G-valued sequences $\mathbf{w}, \mathbf{v}$ and $\bar{\mathbf{v}}$, we say that $(\bar{\mathbf{s}}; \mathbf{w}, \bar{\mathbf{v}})$ is *pivoted from* $(\mathbf{s}; \mathbf{w}, \mathbf{v})$ if:

- $\alpha_i = \bar{\alpha}_i$, $\delta_i = \bar{\delta}_i$ for all $i \in \{1, \ldots, n\}$;
- $(\bar{\beta}_i, \bar{\gamma}_i, \bar{v}_i) \in \tilde{S}$ for each $i \in P_n(\mathbf{s}; \mathbf{w}, \mathbf{v})$, and
- $(\beta_i, \gamma_i, v_i) = (\bar{\beta}_i, \bar{\gamma}_i, \bar{v}_i)$ for each $i \in \{1, \ldots, n\} \setminus P_n(\mathbf{s}; \mathbf{w}, \mathbf{v})$.

By Lemma 4.3, being pivoted from each other is an equivalence relation. Using this fact, we proved in [Cho22b] that the set of pivotal times grows linearly to the step number in probability.

Lemma 4.4 ([Cho22b, Corollary 5.8]). *Fix* $\mathbf{w} \in G^{n+1}$ *and* $\mathbf{v} \in G^n$. *When* $\mathbf{s} = (\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^n$ *is chosen from* S^{4n} *with the uniform measure,* $\#P_n(\mathbf{s})$ *is greater in distribution than the sum of* n *i.i.d.* X_i, *whose distribution is*

$$(4.1) \qquad \mathbb{P}(X_i = j) = \begin{cases} (N_0 - 4)/N_0 & \text{if } j = 1, \\ (N_0 - 4)4^{-j}/N_0^{-j+1} & \text{if } j < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 4.5. *In the setting of Lemma 4.4, for each* n *we have*

$$\mathbb{P}(\#P_n(\mathbf{s}) \le n/2) \le \left(3\sqrt[4]{4/N_0}\right)^n.$$

Proof. This is a consequence of Chernoff-Hoeffding type inequality ([Hoe63, Theorem 1]). We provide here a version of proof using Chebyshev's inequality for convenience.

Let X_i's be the RVs in the conclusion of Lemma 4.4. By Equation 4.1, we have

$$\mathbb{E}\left[\sqrt{4/N_0}^{X_i}\right] = \left(1 - \frac{4}{N_0}\right) \cdot \left[\sqrt{\frac{4}{N_0}} + \sum_{j=1}^{\infty} \sqrt{\frac{N_0}{4}}^{-j} \cdot \left(\frac{4}{N_0}\right)^j\right]$$

$$= \left(1 - \frac{4}{N_0}\right)\sqrt{\frac{4}{N_0}}\left(1 + \frac{1}{1 - \sqrt{4/N_0}}\right)$$

$$= 2\sqrt{4/N_0} + \sqrt{4/N_0}^2 - \sqrt{4/N_0}^3 \le 3\sqrt{4/N_0}.$$

Here, the last inequality is due to the fact $\sqrt{4/N_0} \le 1$. Now using the conclusion of Lemma 4.4 and the independence of X_i's, we deduce

$$\mathbb{E}\left[\sqrt{4/N_0}^{\#P_n(\mathbf{s})}\right] \le \mathbb{E}\left[\sqrt{4/N_0}^{\sum_{i=1}^n X_i}\right] = \prod_{i=1}^n \mathbb{E}\left[\sqrt{4/N_0}^{X_i}\right] \le \left(3\sqrt{4/N_0}\right)^n.$$

By Chebyshev's inequality, we also have

$$\mathbb{E}\left[\sqrt{4/N_0}^{\#P_n(\mathbf{s})}\right] \ge \mathbb{P}(\#P_n(\mathbf{s}) \le n/2) \cdot \sqrt{4/N_0}^{n/2}.$$

Combining these two inequalities leads to the desired conclusion. $\square$

4.2. **Reduction for random walks.** In this subsection, we modify random walks on G into suitable combinatorial model and then apply pivoting technique. As a result, we obtain a random path which is an alternation of fixed subsegments and i.i.d. random subsegments, all aligned on X.

Lemma 4.6. *Let* μ *be a non-elementary probability measure on* G *and let* $S \subseteq G^{M_0}$ *be a fairly long* K_0-*Schottky set for* μ. *Then for each* n *there exist an integer* $m(n)$, *a probability space* Ω_n, *a measurable subset* $B_n \subseteq \Omega_n$, *a*

measurable partition $\mathcal{Q}_n$ of B_n, and random variables

$$Z \in G,$$

$$\{w_i, i = 0, \ldots, m(n)\} \in G^{m(n)+1},$$

$$\{v_i : i = 1, \ldots, m(n)\} \in G^{m(n)},$$

$$\{\alpha_i, \beta_i, \gamma_i, \delta_i : i = 1, \ldots, m(n)\} \in S^{4m(n)}$$

such that the following hold:

(1) $\lim_{n \to +\infty} \mathbb{P}(B_n) = 1$ *and* $\lim_{n \to +\infty} m(n)/n > 0$.

(2) *On each equivalence class $\mathcal{F} \in \mathcal{Q}_n$, $(w_i)_{i=0}^{m(n)}$ are constant and $(\alpha_i, \beta_i, \gamma_i, \delta_i, v_i)_{i=1}^{m(n)}$ are i.i.d.s distributed according to (uniform measure on S^4) $\times \mu$.*

(3) *Z is distributed according to μ^{*n} on Ω_n and*

$$Z = w_0 \Pi(\alpha_1)\Pi(\beta_1)v_1\Pi(\gamma_1)\Pi(\delta_1)w_1 \cdots \Pi(\alpha_{m(n)})\Pi(\beta_{m(n)})v_{m(n)}\Pi(\gamma_{m(n)})\Pi(\delta_{m(n)})w_{m(n)}$$

holds on A_n.

Proof. Let us denote the uniform measure on S by μ_S. Since S is a fairly long Schottky set for μ, μ_S is absolutely continuous with respect to μ. Consequently, there exists $0 < p < 1$ that admits a decomposition

$$\mu^{4M_0+1} = p\left(\mu_S^2 \times \mu \times \mu_S^2\right) + (1-p)\nu$$

for some (nonnegative) probability measure ν. Let $n' = \lfloor n/(4M_0 + 1) \rfloor$.

We consider Bernoulli RVs $(\rho_i)_{i=0}^\infty$ with average p, $(\eta_i)_{i=0}^\infty$ with the law $\mu_S^2 \times \mu \times \mu_S^2$ and $(\nu_i)_{i=0}^\infty$ with the law ν, all independent, and define

$$(g_{(4M_0+1)k+1}, \ldots, g_{(4M_0+1)(k+1)}) = \begin{cases} \nu_k & \text{when } \rho_k = 0, \\ \eta_k & \text{when } \rho_k = 1. \end{cases} \quad (k = 0, \ldots, n'-1).$$

Let $g_{(4M_0+1)n'+1}, \ldots, g_n$ be i.i.d.s distributed according to μ that are also independent from $(\rho_i, \nu_i, \eta_i)_{i=0}^{n'-1}$. Then $(g_1, \ldots, g_n)$ is distributed according to μ^n. We denote by Ω_n the ambient probability space on which $(\rho_i, \nu_i, \eta_i)_i, (g_i)_i$ are all measurable. We set $Z = g_1 \cdots g_n$, whose distribution on Ω_n is μ^{*n}.

Recall that ρ_i's are i.i.d. Bernoulli RVs with average p. Chernoff-Hoeffding's inequality implies that $\mathbb{P}(\sum_{i=0}^{n'-1} \rho_i \leq \epsilon n)$ decays exponentially for each $0 < \epsilon < \frac{p}{4M_0+1}$. Considering this, we define $m(n) := \lfloor pn/8M_0 \rfloor$ and

$$B_n := \left\{ \omega \in \Omega_n : \sum_{i=1}^{n'-1} \rho_i \geq m(n) \right\}.$$

The first item of the conclusion follows.

For $\omega \in B_n$, we collect those indices i at which $\rho_i = 1$. Then we denote the $m(n)$ smallest ones among them, in the increasing order, by $\vartheta(1), \ldots, \vartheta(m(n))$. In other words, $\{i : \rho_i(\omega) = 1\} = \{\vartheta(1) < \vartheta(2) < \ldots < \vartheta(m(n)) < \ldots\}$. We now define $\mathcal{Q}_n$ of B_n to be the partition determined by

the values of $\{\rho_i, \nu_i : i \geq 0\}$, and define

$$w_{i-1} := g_{(4M_0+1)[\vartheta(i-1)+1]+1} \cdots g_{(4M_0+1)\,\vartheta(i)} \quad (i = 1, \ldots, m(n)),$$

$$w_{m(n)} := g_{(4M_0+1)(\vartheta(m(n))+1)+1} \cdots g_n,$$

$$\alpha_i := \big(g_{(4M_0+1)\,\vartheta(i)+1}, \ldots, g_{(4M_0+1)\,\vartheta(i)+M_0}\big),$$

$$\beta_i := \big(g_{(4M_0+1)\,\vartheta(i)+M_0+1}, \ldots, g_{(4M_0+1)\,\vartheta(i)+2M_0}\big),$$

$$v_i := g_{(4M_0+1)\,\vartheta(i)+2M_0+1},$$

$$\gamma_i := \big(g_{(4M_0+1)\,\vartheta(i)+2M_0+2}, \ldots, g_{(4M_0+1)\,\vartheta(i)+3M_0+1}\big),$$

$$\delta_i := \big(g_{(4M_0+1)\,\vartheta(i)+3M_0+2}, \ldots, g_{(4M_0+1)\,\vartheta(i)+4M_0+1}\big).$$

On each equivalence class $\mathcal{F}$ of $\mathcal{Q}_n$, w_i's are constants (since these are determined by the values of $\{\rho_i, \nu_i : i \geq 0\}$) and $(\alpha_i, \beta_i, v_i, \gamma_i, \delta_i)$ are i.i.d.s with distribution $\mu_S^2 \times \mu \times \mu_S^2$. Moreover, on A_n we have

$$g_1 \cdots g_n = \prod_{i=1}^{m(n)} \big(g_{(4M_0+1)[\vartheta(i-1)+1]+1} \cdots g_{(4M_0+1)\,\vartheta(i)}\big) \cdot \big(g_{(4M_0+1)\,\vartheta(i)+1} \cdots g_{(4M_0+1)(\vartheta(i)+1)}\big)$$

$$\cdot\, g_{(4M_0+1)(\vartheta(m(n))+1)+1} \cdots g_n$$

$$= w_0 \Pi(\alpha_1)\Pi(\beta_1)v_1\Pi(\gamma_1)\Pi(\delta_1)w_1 \cdots \Pi(\alpha_{m(n)})\Pi(\beta_{m(n)})v_{m(n)}\Pi(\gamma_{m(n)})\Pi(\delta_{m(n)})w_{m(n)}.$$

This is the third item of the conclusion and the proof is completed. $\qquad\square$

Before the next reduction, we need the following definition.

Definition 4.7. *Let $n > 0$ and $K > 0$. We say that a sequence $(w_i)_{i=0}^n$ in G is K-pre-aligned if, for the isometries $W_0 := w_0$ and*

$$V_k := W_k\Gamma(\beta_{k+1})v_{k+1}, \ W_{k+1} := V_k\Gamma(\gamma_{k+1})w_{k+1} \quad (k = 0, \ldots, n-1),$$

the sequence

$$\Big(o, \ W_0\Gamma(\beta_1), \ V_0\Gamma(\gamma_1), \ \ldots, \ W_{n-1}\Gamma(\beta_n), \ V_{n-1}\Gamma(\gamma_n), \ W_n o\Big)$$

is K-semi-aligned for any choices of $(\beta_i, \gamma_i, v_i) \in \tilde{S}$ $(i = 1, \ldots, n)$.
   We say that an isometry $\phi \in G$ is K-pre-aligned if

$$\Big(\Gamma(\gamma'), \ \Pi(\gamma)\phi \cdot \Gamma(\beta)\Big)$$

is K-semi-aligned for any choices of $(\beta, \gamma, v), (\beta', \gamma', v') \in \tilde{S}$.

We now describe the alignment of Schottky axes at the pivotal times.

Lemma 4.8. *For each integer n, the following holds for $m(n) := 2\lfloor 0.25n \rfloor$.*
   Let μ be a probability measure on G, let $S \subseteq G^{M_0}$ be a fairly long K_0-Schottky set with cardinality $N_0 \geq 400$. Fix a sequence $(w_i)_{i=0}^n$ in G, and let $(\alpha_i, \beta_i, \gamma_i, \delta_i, v_i)_{i=1}^n$ be i.i.d.s on $(S^4 \times G)^n$, distributed according to $(\text{uniform measure on } S)^4 \times \mu$. Then there exist a measurable subset

$B'_n \subseteq (S^4 \times G)^{\mathbb{Z}_{>0}}$, a measurable partition $\mathcal{Q}'_n$ of B'_n and RVs

$$\{w'_i, i = 0, \dots, m(n)\} \in G^{m(n)+1},$$
$$\{v'_i : i = 1, \dots, m(n)\} \in G^{m(n)},$$
$$\{\beta'_i, \gamma'_i : i = 1, \dots, m(n)\} \in S^{2m(n)}$$

such that the following hold:

(1) $\mathbb{P}(B'_n) \geq 1 - \left(3 \cdot \sqrt[4]{4/N_0}\right)^n$.

(2) On each $\mathcal{F}' \in \mathcal{Q}'_n$, $(w'_i)_{i=0}^{m(n)}$ is a fixed D_0-pre-aligned sequence in G and $(\beta'_i, \gamma'_i, v'_i)_{i=1}^{m(n)}$ are i.i.d.s distributed according to (uniform measure on S^2) $\times \mu$ conditioned on $\tilde{S}$.

(3) On B'_n we have the equality

$$w'_0 \Pi(\beta'_1) v'_1 \Pi(\gamma'_1) w'_1 \cdots \Pi(\beta'_{m(n)}) v'_{m(n)} \Pi(\gamma'_{m(n)}) w'_{m(n)}$$
$$= w_0 \Pi(\alpha_1) \Pi(\beta_1) v_1 \Pi(\gamma_1) \Pi(\delta_1) w_1 \cdots \Pi(\alpha_n) \Pi(\beta_n) v_n \Pi(\gamma_n) \Pi(\delta_n) w_n.$$

Proof. Following the discussion in Subsection 4.1, we define

$$(4.2) \qquad P_n(\omega) = P_n\big((\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^n; (w_i)_{i=0}^n, (v_i)_{i=1}^n\big).$$

We take $m(n) = 2 \cdot \lfloor 0.25n \rfloor$ and let $B'_n := \{\omega \in \Omega : \#P_n(\omega) \geq n/2\}$. By Corollary 4.5, we have

$$\mathbb{P}(B'_n) \geq 1 - \left(3 \cdot \sqrt[4]{4/N_0}\right)^n \geq 1 - (3/\sqrt{10})^n.$$

We now define

$$W_k := w_0 \Pi(\alpha_1) \Pi(\beta_1) v_1 \Pi(\gamma_1) \Pi(\delta_1) w_1 \cdots \Pi(\alpha_k) \Pi(\beta_k) v_k \Pi(\gamma_k) \Pi(\delta_k) w_k \quad (k = 0, \dots, n).$$

Given $\omega \in B'_n$ with $P_n(\omega) = \{i(1) < i(2) < \dots < i(m(n)) < \dots\}$, we define

$$w'_0 := W_{i(1)-1} \Pi(\alpha_{i(1)}), \quad w'_{m(n)} := \Pi(\delta_{i(m(n))}) w_{i(m(n))} \cdot W^{-1}_{i(m(n))} \cdot W_n,$$
$$w'_k := \Pi(\delta_{i(k)}) w_{i(k)} \cdot W^{-1}_{i(k)} \cdot W_{i(k+1)-1} \Pi(\alpha_{i(1)}) \quad (k = 1, \dots, m(n) - 1).$$

Furthermore, we record the Schottky axes at the pivotal times: we define $\beta'_k := \beta_{i(k)}$, $v'_k := v_{i(k)}$ and $\gamma'_k := \gamma_{i(k)}$ for $k = 1, \dots, m(n)$. We then observe the following relation:

$$W_{i(k+1)-1} \Pi(\alpha_{i(k+1)}) = W_{i(k)-1} \Pi(\alpha_{i(k)}) \cdot \Pi(\beta_{i(k)}) v_{i(k)} \cdot \Pi(\gamma_{i(k)}) w'_k.$$

By induction, we observe that
(4.3)
$$w'_0 \Pi(\beta'_1) v'_1 \Pi(\gamma'_1) w'_1 \cdots \Pi(\beta'_k) v'_k \Pi(\gamma'_k) w'_k = W_{i(k+1)-1} \Pi(\alpha_{i(k+1)}) \quad (k = 0, \dots, m(n) - 1),$$
$$w'_0 \Pi(\beta'_1) v'_1 \Pi(\gamma'_1) w'_1 \cdots \Pi(\beta'_{m(n)}) v'_{m(n)} \Pi(\gamma'_{m(n)}) w'_{m(n)} = W_n$$

on B'_n, settling Item 3 of the conclusion.

We define the measurable partition $\mathcal{Q}'_n$ of B'_n by pivoting at the first $m(n)$ pivotal times. More explicitly, two elements $\omega, \omega' \in B'_n$ belong to the same

equivalence class if and only if:

$$\alpha_i'(\omega) = \alpha_i'(\omega'),\ \delta_i'(\omega) = \delta_i'(\omega'), w_i'(\omega) = w_i'(\omega') \qquad\qquad (i \in \{0,\ldots,n\}),$$
$$P_n(\omega) = P_n(\omega') =: \{i(1) < i(2) < \ldots\},$$
$$(\beta_i'(\omega), \gamma_i'(\omega), v_i'(\omega)) \in \tilde{S} \qquad\qquad (i \in \{i(1),\ldots,i(m(n))\}),$$
$$(\beta_i'(\omega), \gamma_i'(\omega), v_i'(\omega)) = (\beta_i'(\omega'), \gamma_i'(\omega'), v_i'(\omega')) \quad (i \in \{1,\ldots,n\} \setminus \{i(1),\ldots,i(m(n))\}).$$

Lemma 4.3 describes the structure of the equivalence class in $\mathcal{Q}_n'$. To elaborate, let us fix an equivalence class $\mathcal{F}' \in \mathcal{Q}_n'$. Then $\mathcal{F}'$ is associated with a set of indices $P_n(\mathcal{F}') = \{i(1) < \ldots < i(m(n)) < \ldots\}$. Moreover, an element $\omega \in (S^4 \times G)^n$ belongs to $\mathcal{F}'$ if and only if the following holds:

(I) the following isometries/Schottky sequences coincide with the fixed ones associated to $\mathcal{F}'$.

$$\left(w_0'(\omega), \alpha_1'(\omega), \beta_1'(\omega), \ldots, w_{i(1)-1}'(\omega), \alpha_{i(1)}'(\omega)\right),$$
$$\left(\delta_{i(1)}'(\omega), w_{i(1)}'(\omega), \ldots, w_{i(2)-1}'(\omega), \alpha_{i(2)}'(\omega)\right),$$
$$\vdots,$$
$$\left(\delta_{i(m(n)-1)}'(\omega), w_{i(m(n)-1)}'(\omega), \ldots, w_{i(m(n))-1}'(\omega), \alpha_{i(m(n))}'(\omega)\right),$$
$$\left(\delta_{i(m(n))}'(\omega), w_{i(m(n))}'(\omega), \ldots, \delta_n'(\omega), w_n'(\omega)\right).$$

(II) $\left(\beta_{i(k)}'(\omega), \gamma_{i(k)}'(\omega), v_{i(k)}'(\omega)\right) \in \tilde{S}$ for $k = 1, \ldots, m(n)$.

Moreover, for all $\omega \in \mathcal{F}'$ we have $P_n(\omega) = P_n(\mathcal{F}')$.

Item (I) says that $(w_k')_{k=0}^{m(n)}$ are constant on $\mathcal{F}'$. Item (II) says that, on each $\mathcal{F}'$, the RVs $(\beta_k', \gamma_k', v_k')_{k=1}^{m(n)}$ are $(\beta_{i(k)}, \gamma_{i(k)}, v_{i(k)})$'s restricted on the set $\left\{\left(\beta_{i(k)}, \gamma_{i(k)}, v_{i(k)}\right) \in \tilde{S}\right\}$. Hence, $\left(\beta_{i(k)}, \gamma_{i(k)}, v_{i(k)}\right)_{k=1}^{m(n)}$ conditioned on $\mathcal{F}'$ are i.i.d.s whose distribution is the restriction of (uniform measure on S^2) $\times$ μ. This establishes Item (3) of the conclusion.

Next, Lemma 4.1 tells us that

$$\begin{pmatrix} o,\ W_{i(1)-1}\Gamma(\alpha_{i(1)}),\ W_{i(1)-1}\Pi(\alpha_{i(1)})\Gamma(\beta_{i(1)}),\ W_{i(1)-1}\Pi(\alpha_{i(1)})\Pi(\beta_{i(1)})v_{i(1)}\Gamma(\gamma_{i(1)}), \\ W_{i(1)-1}\Pi(\alpha_{i(1)})\Pi(\beta_{i(1)})v_{i(1)}\Pi(\gamma_{i(1)})\Gamma(\delta_{i(1)}), \\ W_{i(2)}\Gamma(\alpha_{i(2)}),\ W_{i(2)-1}\Pi(\alpha_{i(2)})\Gamma(\beta_{i(2)}),\ W_{i(2)-1}\Pi(\alpha_{i(2)})\Pi(\beta_{i(2)})v_{i(2)}\Gamma(\gamma_{i(2)}), \\ W_{i(2)-1}\Pi(\alpha_{i(2)})\Pi(\beta_{i(2)})v_{i(2)}\Pi(\gamma_{i(2)})\Gamma(\delta_{i(2)}), \\ \vdots \\ W_{i(m(n))}\Gamma(\alpha_{i(m(n))}),\ W_{i(m(n))-1}\Pi(\alpha_{i(m(n))})\Gamma(\beta_{i(m(n))}), \\ W_{i(m(n))-1}\Pi(\alpha_{i(1)})\Pi(\beta_{i(m(n))})v_{i(1)}\Gamma(\gamma_{i(m(n))}), \\ W_{i(m(n))-1}\Pi(\alpha_{i(m(n))})\Pi(\beta_{i(m(n))})v_{i(m(n))}\Pi(\gamma_{i(m(n))})\Gamma(\delta_{i(m(n))}),\ \ldots,\ W_n o \end{pmatrix}$$

is D_0-semi-aligned on $\mathcal{F}'$. Consequently, for each $\mathcal{F}' \in \mathcal{Q}_n'$ with the associated $\left(w_i'(\mathcal{F})\right)_{i=0}^{m(n)}$, and for each $\omega \in \mathcal{F}'$ that determines $(\beta_i'(\omega), \gamma_i'(\omega), v_i'(\omega))_{i=1}^{m(n)}$,

$$\left(o, W_0'\Gamma(\beta_1'), V_0'\Gamma(\gamma_1'), \ldots, W_{m(n)-1}'\Gamma(\beta_{m(n)}'), V_{m(n)-1}'\Gamma(\gamma_{m(n)}'), W_{m(n)} o\right)$$

is D_0-semi-aligned for $W_0' := w_0$ and

$$V_i' := W_i'\Gamma(\beta_{i+1}')v_{i+1}', \quad W_{i+1}' := V_i'\Gamma(\gamma_{i+1}')w_{i+1}' \quad (i = 0, \ldots, m(n) - 1).$$

By Item (II), $(\beta_i', \gamma_i', v_i')$ ranges all over $\tilde{S}$ for each $i = 1, \ldots, m(n)$. This implies that for each $\mathcal{F}'$, $\left(w_i'(\mathcal{F})\right)_{i=0}^{m(n)}$ is D_0-pre-aligned. Since $\mathcal{B}_n'$ is composed of these equivalence classes, we can conclude Item (2) of the conclusion. $\quad\square$

4.3. **Pivoting and self-repulsion.** We discuss the heart of this paper, the pivoting for translation length. The ingredient to begin with is a D_0-pre-aligned seqeuence $(w_i)_{i=0}^n$ in G. This means that for every choice of $(\beta_i, \gamma_i, v_i) \in \tilde{S}$ for $i = 1, \ldots, n$ the sequence

$$\left(o, W_0\Gamma(\beta_1), V_0\Gamma(\gamma_1), \ldots, W_{n-1}\Gamma(\beta_n), V_{n-1}\Gamma(\gamma_n), W_n o\right)$$

is D_0-semi-aligned, where $W_0 := w_0$ and

(4.4) $\quad V_k := W_k\Pi(\beta_{k+1})v_{k+1}, \quad W_{k+1} := V_k\Pi(\gamma_{k+1})w_{k+1} \quad (k = 0, \ldots, n-1).$

Our aim is to show that W_n is contracting whose axis is fellow traveling with $W_k\Gamma(\beta_k)$ and $V_k\Gamma(\gamma_k)$ for intermediate k's. For this let us define

$$\begin{aligned}
\phi_k &= \phi_k\left((w_i)_{i=0}^n, (\beta_i, \gamma_i, v_i)_{i=1}^n\right) \\
&:= (V_{n-k}\Pi(\gamma_{n-k+1}))^{-1} W_n W_{k-1} \\
&= w_{n-k+1}\Pi(\beta_{n-k+2})v_{n-k+2}\Pi(\gamma_{n-k+2})w_{n-k+2}\cdots\Pi(\beta_n)v_n\Pi(\gamma_n)w_n \\
&\quad \cdot w_0\Pi(\beta_1)v_1\Pi(\gamma_1)w_1\cdots\Pi(\beta_{k-1})v_{k-1}\Pi(\gamma_{k-1})w_{k-1}.
\end{aligned}$$

Then ϕ_k depends only on $(w_l, \beta_l, \gamma_l, v_l)$'s for $1 \le l \le k - 1$ and $n - k + 2 \le l \le n$, together with w_0, w_{n-k+1}. For $1 \le k \le n/2$ we define

$$\begin{aligned}
S_k^* &:= \left\{(\beta_k, \gamma_k) \in \tilde{S}(v_k) : \left(W_n^{-1}V_{n-k}\Pi(\gamma_{n-k+1})o, W_{k-1}\Gamma(\beta_k)\right) \text{ is } K_0\text{-aligned}\right\} \\
&= \left\{(\beta_k, \gamma_k) \in \tilde{S}(v_k) : \left(\phi_k^{-1}o, \Gamma(\beta_k)\right) \text{ is } K_0\text{-aligned}\right\},
\end{aligned}$$

$$\begin{aligned}
S_{n-k+1}^* &:= \left\{(\beta_{n-k+1}, \gamma_{n-k+1}) \in \tilde{S}(v_{n-k+1}) : \left(W_n^{-1}V_{n-k}\Gamma(\gamma_{n-k+1}), W_{k-1}\Pi(\beta_k)o\right) \text{ is } K_0\text{-aligned}\right\} \\
&= \left\{(\beta_{n-k+1}, \gamma_{n-k+1}) \in \tilde{S}(v_{n-k+1}) : \left(\Gamma(\gamma_{n-k+1}), \Pi(\gamma_{n-k+1})\phi_k\Pi(\beta_k)o\right) \text{ is } K_0\text{-aligned}\right\}.
\end{aligned}$$

These sets are devised to capture the match between displacement and translation length.

Lemma 4.9. *Let* $1 \le l \le k \le n/2$, *let* $(w_i)_{i=0}^n$ *be a* D_0-*pre-aligned sequence in* G *and let* $(\beta_i, \gamma_i, v_i)_{i=1,\ldots,l,n-l+1,\ldots,n} \in \tilde{S}^{2l}$ *be choices such that*

(4.5) $\qquad (\beta_l, \gamma_l) \in S_l^* \text{ and } (\beta_{n-l+1}, \gamma_{n-l+1}) \in S_{n-l+1}^*.$

Then for any additional choice $(\beta_i, \gamma_i, v_i)_{i=l+1,\ldots,k,n-k+1,\ldots,n-l} \in \tilde{S}^{2(k-l)}$, *the isometry*

$$\begin{aligned}
\phi_{k+1} &:= w_{n-k}\Pi(\beta_{n-k+1})v_{n-k+1}\Pi(\gamma_{n-k+1})w_{n-2k+1}\cdots\Pi(\beta_n)v_n\Pi(\gamma_n)w_n \\
&\quad \cdot w_0\Pi(\beta_1)v_1\Pi(\gamma_1)w_1\cdots\Pi(\beta_k)v_k\Pi(\gamma_k)w_k
\end{aligned}$$

is a D_0-*pre-aligned isometry.*

Proof. Pick an arbitrary further choices $(\beta_i, \gamma_i, v_i) \in \tilde{S}$ for $i = k+1, \ldots, n-k$. Recall again the notation in Display 4.4 (with $W_0 := w_0$). For notational purpose, let

$$\kappa_{2i-1} = W_{i-1}\Gamma(\beta_i), \kappa_{2i} = V_{i-1}\Gamma(\gamma_i)(i = 1, \ldots, n).$$

Since $(w_i)_{i=0}^n$ is D_0-pre-aligned sequence, any subsequence of

$$(o, \kappa_1, \ldots, \kappa_{2n}, W_n o)$$

is D_0-semi-aligned.

Meanwhile, note that $W_n^{-1}V_{n-l}\Pi(\gamma_{n-l+1})o$ is the ending point of $W_n^{-1}V_{n-l}\Gamma(\gamma_{n-l+1}) = W_n^{-1}\kappa_{2(n-l)+2}$, while $W_{l-1}\Pi(\beta_l)o$ is the ending point of $W_{l-1}\Gamma(\beta_l)o = \gamma_{2l-1}$. Since we are assuming Condition 4.5, Lemma 2.5 tells us that $(W_n^{-1}\kappa_{2(n-l)+2}, \kappa_{2l-1})$ is D_0-aligned.

Overall, we observe that

$$(W_n^{-1}\kappa_{2(n-k)}, \ldots, W_n^{-1}\kappa_{2(n-l)+2}, \kappa_{2l-1}, \ldots, \kappa_{2k+1})$$

is D_0-semi-aligned. This implies that

$$(W_n^{-1}\kappa_{2(n-k)}, \kappa_{2k-1}) = \left(\left(W_n^{-1}V_{n-k-1}\right)\cdot\Gamma(\gamma_{n-k}), \left(W_n^{-1}V_{n-1}\right)\cdot\Pi(\gamma_{n-k})\phi_{k+1}\cdot\Gamma(\beta_{k+1})\right)$$

is D_0-semi-aligned. Also recall that the choices of $(\beta_{k+1}, \gamma_{k+1}, v_{k+1})$ and $(\beta_{n-k}, \gamma_{n-k}, v_{n-k})$ in $\tilde{S}$ were arbitrary. Hence, we conclude that ϕ_k is D_0-semi-aligned. $\square$

Lemma 4.10. *Let $1 \leq k \leq n/2$, let $(w_i)_{i=0}^n$ be a D_0-pre-aligned sequence in G and let $(\beta_i, \gamma_i, v_i)_{i=1}^n \in \tilde{S}^n$ be choices such that ϕ_k is a D_0-pre-aligned isometry. Then W_n is a contracting isometry, and moreover, $(\kappa_i)_{i\in\mathbb{Z}}$ is D_0-semi-aligned for the Schottky axes*

$$\left(\kappa_{2(n-k+1)t+1}, \kappa_{2(n-k+1)t+2}, \ldots, \kappa_{2(n-k+1)(t+1)-1}, \kappa_{2(n-k+1)(t+1)}\right)$$
$$:= \left(W_n^t W_{k-1}\Gamma(\beta_k), W_n^t W_{k-1}\Gamma(\gamma_k), \ldots, W_n^t W_{n-k}\Gamma(\beta_{n-k+1}), W_n^t V_{n-k}\Gamma(\gamma_{n-k+1})\right) \ (t \in \mathbb{Z}).$$

Proof. Since ϕ_k is D_0-pre-aligned, we have that

$$\left(W_n^{-1}V_{n-k}\Pi(\gamma_{n-k+1})o, W_{k-1}\Gamma(\beta_k)\right), \ \left(W_n^{-1}V_{n-k}\Gamma(\gamma_{n-k+1}), W_{k-1}\Pi(\beta_k)o\right)$$

is D_0-semi-aligned, as well as $(\kappa_{2(n-k+1)t-1}, \kappa_{2(n-k+1)t})$ for $t \in \mathbb{Z}$.

Moreover, since $(w_i)_{i=0}^n$ is also D_0-pre-aligned, we have that

$$\left(W_{k-1}\Gamma(\beta_k), V_{k-1}\Gamma(\gamma_k), \ldots, W_{n-k}\Gamma(\beta_{n-k+1}), V_{n-k}\Gamma(\gamma_{n-k+1})\right)$$

is D_0-semi-aligned. This implies that $(\kappa_{2(n-k+1)t+1}, \ldots, \kappa_{2(n-k+1)(t+1)})$ is D_0-aligned for each $t \in \mathbb{Z}$. Combining these facts, we conclude that $(\kappa_i)_{i\in\mathbb{Z}}$ is D_0-semi-aligned.

Since Schottky axes are K_0-contracting and sufficiently long (as described in Definition 2.8), we can apply Lemma 2.7 to conclude that the concatenation of $\{\kappa_{2(n-k+1)l}\}_{l\in\mathbb{Z}}$ is a contracting axis. Since the orbit $\{w^l o\}_{l\in\mathbb{Z}}$ is fellow traveling with this concatenation, w is contracting as desired. $\square$

We now estimate the probability for the event described in Lemma 4.9.

Lemma 4.11. *Fix $(w_i)_{i=0}^n \in G^{n+1}$ and let $1 \le k \le n/2$. Picking $(\beta_i, \gamma_i, v_i)_{i=1}^n$ from $\tilde{S}^n$, the condition*

(4.6) $(\beta_l, \gamma_l) \in \tilde{S}_l^*$ *and* $(\beta_{n-l+1}, \gamma_{n-l+1}) \in \tilde{S}_{n-l+1}^*$ *for some* $1 \le l \le k$

depends only on $\{(\beta_l, \gamma_l, v_l) : l = 1, \ldots, k, n - k + 1, \ldots, n\}$.

Moreover, fixing any choice of $\{v_l : l = 1, \ldots, k, n - k + 1, \ldots, n\}$, the condition holds for probability at least $1 - (6\#S)^k$ with respect to the uniform measure on $\tilde{S}(v_1) \times \cdots \times \tilde{S}(v_k) \times \tilde{S}_{n-k+1}(v_{n-k+1}) \times \cdots \times \tilde{S}(v_n)$.

Proof. Recall that $\phi_1, \ldots, \phi_k$ depend only on (β_i, γ_i, v_i)'s for $i = 1, \ldots, k - 1, n - k + 2, \ldots, n$. Moreover, Condition 4.6 involves $\phi_1, \ldots, \phi_k$ and (β_i, γ_i) for $i = 1, \ldots, k, n - k + 1, \ldots, n$. Combining these leads to the first assertion.

Before proving the second assertion, we observe the following claim:

Claim 4.12.   *(1) For each $1 \le l \le n/2$ and for each choices of*

$$(\beta_1, \gamma_1, \ldots, \beta_{l-1}, \gamma_{l-1}, \beta_{n-l+2}, \gamma_{n-l+2}, \ldots, \beta_n, \gamma_n) \in S^{4l-4},$$
$$(v_1, \ldots, v_l, v_{n-l+1}, \ldots, v_n) \in G^{2l},$$

the following holds with respect to the uniform measure on $\tilde{S}(v_l)$:

$$\mathbb{P}\left((\beta_l, \gamma_l) \in \tilde{S}(v_l) : \left(\phi_l^{-1}o, \Gamma(\beta_l)\right) \text{ is } K_0\text{-aligned}\right) \ge 1 - 3/\#S.$$

(2) For each $1 \le l \le n/2$ and for each choice of

$$(\beta_1, \gamma_1, \ldots, \beta_l, \gamma_l, \beta_{n-l+2}, \gamma_{n-l+2}, \ldots, \beta_n, \gamma_n) \in S^{4l-2},$$
$$(v_1, \ldots, v_l, v_{n-l+1}, \ldots, v_n) \in G^{2l},$$

the following holds with respect to the uniform measure on $\tilde{S}(v_{n-l+1})$:

$$\mathbb{P}\left(\begin{array}{c} (\beta_{n-l+1}, \gamma_{n-l+1}) \in \tilde{S}(v_{n-l+1}) : \\ \left(\Gamma(\gamma_{n-l+1}), \Pi(\gamma_{n-l+1})\phi_l\Pi(\beta_l)o\right) \text{ is } K_0\text{-aligned} \end{array}\right) \ge 1 - 3/\#S.$$

Proof of Claim 4.12. Recall that ϕ_l is determined by (β_t, γ_t, v_t)'s for $t = 1, \ldots, l - 1, n - l + 2, \ldots, n$. With this in mind, the property of the Schottky set S (cf. Definition 2.8) implies that the condition in Item (1) is violated by at most 1 choice of $\beta_l \in S$; for any other choice of β_l and for any choice of γ_l, the condition holds. Hence, at most $\#S$ choices in $S^2 \supseteq \tilde{S}(v_l)$ violate the condition. Meanwhile, since $\#\tilde{S}(v_l) \ge (\#S)^2 - 2\#S$ holds for any v_l, we conclude that

$$\mathbb{P}\left((\beta_l, \gamma_l) \in \tilde{S}(v_l) : \left(\phi_l^{-1}o, \Gamma(\beta_l)\right) \text{ is not } K_0\text{-aligned}\right) \le \frac{\#S}{(\#S)^2 - 2\#S} \le \frac{3\#S}{(\#S)^2}.$$

We similarly deduce Item (2), and Claim 4.12 is now established.   □

We now prove the second assertion. For this, we first fix $\{v_l : l = 1, \ldots, k, n - k + 1, \ldots, n\}$ and define

$$\mathcal{A}_k := \left\{(\beta_l, \gamma_l)_{l=1}^n : \begin{array}{c} (\beta_l, \gamma_l) \notin S_l^* \text{ or } (\beta_{n-l+1}, \gamma_{n-l+1}) \notin S_{n-l+1}^* \\ \text{for each } 1 \le l \le k \end{array}\right\}.$$

Then $\mathcal{A}_k \subseteq \mathcal{A}_{k+1}$ holds for each k. Further, the first assertion implies that the membership of $(\beta_i, \gamma_i)_{i=1}^n$ in $\mathcal{A}_k$ depends only on (β_i, γ_i) for $i = 1, \ldots, k, n-k+1, \ldots, n$. Finally, for each choice of $(\beta_i, \gamma_i)_{i=1,\ldots,k-1,n-k+2,\ldots,n}$ that makes $(\beta_i, \gamma_i)_{i=1}^n$ belong to $\mathcal{A}_{k-1}$, Claim 4.12 implies that

$$\mathbb{P}\left(\mathcal{A}_k \,\Big|\, \beta_1, \gamma_1, \ldots, \beta_{k-1}, \gamma_{k-1}, \beta_{n-k+2}, \gamma_{n-k+2}, \ldots, \beta_n, \gamma_n\right) \leq 6/\#S.$$

Summing up these conditional probabilities, we deduce that $\mathbb{P}(\mathcal{A}_k) \leq (6/\#S) \cdot \mathbb{P}(\mathcal{A}_{k-1})$. This leads to the conclusion of the lemma. $\square$

4.4. **Converse of CLT.** In this subsection, we prove the converse of CLT. We first assemble Lemma 4.6, 4.8 4.9 and 4.11.

Corollary 4.13. *Let μ be a non-elementary probability measure on G and let S be a fairly long K_0-Schottky set for μ with cardinality at least 400. Then for each n there exist $m(n) \in \{2^s : s > 0\}$, a probability space Ω_n with a measurable subset $A_n \subseteq \Omega_n$, a measurable partition $\mathcal{P}_n$ of A_n and RVs*

$$Z_n \in G,$$
$$\{w_i : i = 0, \ldots, m(n)\} \in G^{m(n)+1},$$
$$\{v_i : i = 1, \ldots, m(n)\} \in G^{m(n)},$$
$$\{\beta_i, \gamma_i : i = 1, \ldots, m(n)\} \in S^{2m(n)}$$

such that the following hold:
  (1) $\lim_{n \to +\infty} \mathbb{P}(A_n) = 1$ and $\lim_{n \to +\infty} m(n)/n > 0$.
  (2) On A_n, $(w_i)_{i=0}^{m(n)}$ is a D_0-pre-aligned sequence in G and $w_{m(n)}^{-1} w_0$ is a D_0-pre-aligned isometry.
  (3) On each equivalence class $\mathcal{E} \in \mathcal{P}_n$, $(w_i)_{i=0}^{m(n)}$ are constant and $(\beta_i, \gamma_i, v_i)_{i=1}^{m(n)}$ are i.i.d.s distributed according to the measure $\big(\text{uniform measure on } S^2\big) \times \mu$ conditioned on $\tilde{S}$.
  (4) Z_n is distributed according to μ^{*n} on Ω_n and

$$Z_n = w_0 \Pi(\beta_1) v_1 \Pi(\gamma_1) w_1 \cdots \Pi(\beta_{m(n)}) v_{m(n)} \Pi(\gamma_{m(n)}) w_{m(n)}$$

  holds on A_n.

Proof. Given $n > 0$, Lemma 4.6 provides an integer $N_1 = m(n)$, a probability space $(\Omega_n, \mathbb{P})$ with a subset $B = B_n$, a measurable partition $\mathcal{Q}_n = \{\mathcal{F}_\xi\}_\xi$ of B_n, random variables $Z_n \in G$, $(w_i)_{i=0}^{N_1} \in G^{N_1+1}$, $(v_i'')_{i=1}^{N_1} \in G^{N_1}$ and $(\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^{N_1} \in S^{4N_1}$ such that the following holds:
  (1) $\lim_n \mathbb{P}(B) = 1$ and $\lim_{n \to +\infty} N_1/n > 0$.
  (2) On each equivalence class $\mathcal{G}_\gamma \in \mathcal{R}$, $(w_i)_i$ are constant and $(\alpha_i, \beta_i, \gamma_i, \delta_i, v_i)_i$ are i.i.d.s distributed according to (uniform measure on S^4) $\times \mu$.
  (3) Z_n is distributed according to μ^{*n} on Ω_n and

$$Z_n = w_0 \Pi(\alpha_1) \Pi(\beta_1) v_1 \Pi(\gamma_1) \Pi(\delta_1) w_1 \cdots \Pi(\alpha_{N_1}) \Pi(\beta_{N_1}) v_{N_1} \Pi(\gamma_{N_1}) \Pi(\delta_{N_1}) w_{N_1}$$

  holds on $B = \cup_\xi \mathcal{F}_\xi$.

Next, we take $N_2(n) := 2\lfloor 0.25N_1 \rfloor$ as in Lemma 4.8, with input N_1 in place of n. Since N_1 grows linearly, $\lim_n N_2/n > 0$ also holds. We now endow each $\mathcal{F}_\xi \in \mathcal{Q}_n$ with the conditional measure $\mathbb{P}|_{\mathcal{F}_\xi}$ of $\mathbb{P}$.

We note that $(w_i, v_i, \alpha_i, \beta_i, \gamma_i, \delta_i)_i$ satisfy the assumption of Lemma 4.8 on each $\mathcal{F}_\xi \in \mathcal{Q}_n$. For each $\mathcal{F}_\xi \in \mathcal{Q}_n$, Lemma 4.8 then provides a measurable subset $B'_\xi \subseteq \mathcal{F}_\xi$, a measurable partition $\mathcal{Q}'_\xi = \{\mathcal{F}'_{\zeta;\xi}\}_\zeta$ of B'_ξ, and RVs $(w'_i)_{i=0}^{N_2}, (v'_i, \beta'_i, \gamma'_i)_{i=1}^{N_2}$ on $\mathcal{F}_\xi$ such that the following holds:

(1) $\mathbb{P}\left(B'_\xi | \mathcal{F}_\xi\right) \geq 1 - \left(3\sqrt[4]{4/N_0}\right)^{N_1} \geq 1 - \left(3/\sqrt{10}\right)^{N_1}$.

(2) On B_ξ we have

$$Z_n = w_0 \Pi(\alpha_1) \Pi(\beta_1) v_1 \Pi(\gamma_1) \Pi(\delta_1) w_1 \cdots \Pi(\alpha_{N_1}) \Pi(\beta_{N_1}) v_{N_1} \Pi(\gamma_{N_1}) \Pi(\delta_{N_1}) w_{N_1}$$
$$= w'_0 \Pi(\beta'_1) v'_1 \Pi(\gamma'_1) w'_1 \cdots \Pi(\beta'_{N_2}) v'_{N_2} \Pi(\gamma'_{N_2}) w'_{N_2}.$$

(3) On each $\mathcal{F}'_{\zeta;\xi} \in \mathcal{Q}'_\xi$, $(w'_i)_{i=0}^{N_2}$ is a fixed D_0-pre-aligned sequence in G and $(\beta'_i, \gamma'_i, v'_i)_{i=1}^{N_2}$ are i.i.d.s distributed according to the restriction of (uniform measure on S^2) $\times \mu$ to $\tilde{S}$.

We then take $B' := \sqcup_\xi B'_\xi$ and $\mathcal{Q}'_n := \sqcup_\xi \mathcal{Q}'_\xi$. Note that

$$\mathbb{P}(B') = \sum_\xi \mathbb{P}\left(B'_\xi | \mathcal{F}_\xi\right) \cdot \mathbb{P}(\mathcal{F}_\xi) \geq \left(1 - (3/\sqrt{10})^{N_1}\right) \cdot \mathbb{P}(B),$$

and the RHS converges to 1 as n tends to infinity; hence $\lim_n \mathbb{P}(B') = 1$.

We now set $m(n) = 2^{\lfloor \log_2 N_2 \rfloor - 1}$. Note that $m(n)$ is a power of 2 and $N_2 - m(n)$ is even. Moreover, we have $m(n) \geq N_2/4$ and $N_2 - m(n) \geq N_2/2$.

We now consider a refinement $\mathcal{Q}''_n$ of $\mathcal{Q}'_n$ by first fixing the values of the RVs

$$\left\{ v'_i : i = 1, 2, \ldots, 0.5(N_2 - m(n)), N_2 - 0.5(N_2 - m(n)) + 1, \ldots, N_2 - 1, N_2 \right\}.$$

We also consider a refinement $\mathcal{P}_n$ of $\mathcal{Q}''_n$ by further fixing the values of the RVs

$$\left\{ \beta'_i, \gamma'_i : i = 1, 2, \ldots, 0.5(N_2 - m(n)), N_2 - 0.5(N_2 - m(n)) + 1, \ldots, N_2 - 1, N_2 \right\}.$$

We consider the following property:

$$P = \text{``}(\beta'_l, \gamma'_l) \in \tilde{S}^*_l \text{ and } (\beta'_{N_2 - l + 1}, \gamma'_{N_2 - l + 1}) \in \tilde{S}^*_{N_2 - l + 1} \text{ for some } 1 \leq l \leq 0.5(N_2 - m(n))\text{''}.$$

The first part of Lemma 4.11 implies that, for each $\mathcal{E} \in \mathcal{P}_n$, either P holds on the entire $\mathcal{E}$ or P does not hold on the entire $\mathcal{E}$. The second part of Lemma 4.11 tells us that $\mathbb{P}(P|\mathcal{F}'') \geq 1 - (6/N_0)^{0.5(N_2 - m(n))} \geq 1 - (6/N_0)^{N_2/4}$ for each $\mathcal{F}'' \in \mathcal{Q}''_n$. If we take

$$A_n := \bigcup \{\mathcal{E} \in \mathcal{P} : P \text{ holds on } \mathcal{E}\},$$

then we get

$$\mathbb{P}(A_n) \geq \sum_{\mathcal{F}'' \in \mathcal{Q}''_n} \mathbb{P}(P|\mathcal{F}'') \cdot \mathbb{P}(\mathcal{F}'') \geq (1 - (6/N_0)^{N_2/4}) \cdot \mathbb{P}(B'_n),$$

which converges to 1 as n tends to infinity. Item (1) of the conclusion is now proven. By abuse of notation, from now on, we let $\mathcal{P}_n$ be the restriction of $\mathcal{P}_n$ on A_n.

Now, on A_n we define RVs

$$w_0'' := w_0'\Pi(\beta_1')v_1'\Pi(\gamma_1')w_1'\cdots\Pi(\beta_k')v_k'\Pi(\gamma_k')w_k',$$
$$w_{m(n)}'' := w_{N_2-k}'\Pi(\beta_{N_2-k+1}')v_{N_2-k+1}'\Pi(\gamma_{N_2-k+1}')w_{N_2-k+1}'\cdots\Pi(\beta_{N_2}')v_{N_2}'\Pi(\gamma_{N_2}')w_{N_2}',$$
$$(v_i'',\beta_i'',\gamma_i'')_{i=1}^{m(n)} := (v_{i+k}',\beta_{i+k}',\gamma_{i+k}')_{i=1}^{m(n)},$$
$$(w_i'')_{i=1}^{m(n)-1} := (w_{i+k}')_{i=1}^{m(n)-1} \quad \left(k := \frac{N_2 - m(n)}{2}\right).$$

Our goal is to prove that $(w_i'')_{i=0}^{m(n)}$ and $(\beta_i'',\gamma_i'',v_i'')_{i=1}^{m(n)}$ (in place of $(w_i)_{i=0}^{m(n)}$ and $(\beta_i,\gamma_i,v_i)_{i=1}^{m(n)}$) satisfy the desired conclusion. First, we have

$$w_0''\Pi(\beta_1'')v_1''\Pi(\gamma_1'')w_1''\cdots\Pi(\beta_{m(n)}'')v_{m(n)}''\Pi(\gamma_{m(n)}'')w_{m(n)}''$$
$$= w_0'\Pi(\beta_1')v_1'\Pi(\gamma_1')w_1'\cdots\Pi(\beta_{N_2}')v_{N_2}'\Pi(\gamma_{N_2}')w_{N_2}'$$

holds on A_n by definition, and the RHS equals Z_n on B_n. This establishes Item (4) of the conclusion.

By definition of $\mathcal{P}_n$, $(w_i'')_{i=0}^{m(n)}$ are constant and $(\beta_i'',\gamma_i'',v_i'')_{i=1}^{m(n)}$ are i.i.d.s distributed according to (uniform measure on S^2) $\times \mu$ on each $\mathcal{E}_\alpha \in \mathcal{P}$. Item (3) of the conclusion follows.

Meanwhile, pick an arbitrary $\omega \in A_n$, which determines $(w_i')_{i=0}^{N_2}$ and $(w_i'')_{i=0}^{m(n)}$. Since ω belongs to B_n', $(w_i')_{i=1}^{N_2}$ is a D_0-pre-aligned sequence. This in turn implies that $(w_i'')_{i=0}^{m(n)}$ is also D_0-pre-aligned. Moreover, Lemma 4.9 implies that

$$(w_{m(n)}'')^{-1} \cdot w_0'' = \left(\phi_{k+1} \text{ for } (w_i')_{i=0}^{N_2} \text{ and } (\beta_i',\gamma_i',\delta_i')_{i=1}^{N_2}\right)$$

is a D_0-pre-aligned isometry. Item (2) of the conclusion is now established and the proof is complete. $\square$

We now investigate the displacement and the translation length of Z_n in Corollary 4.13.

Lemma 4.14. *Let S be a fairly long K_0-Schottky set for μ with cardinality at least 400, and fix a D_0-pre-aligned sequence $(w_i)_{i=0}^n$ in G. Given $(\beta_i,\gamma_i,v_i) \in \tilde{S}$ for $i = 1,\ldots,n$, define $W_0 := w_0$ and*

$$V_k := W_k\Gamma(\beta_{k+1})v_{k+1}, \ W_{k+1} := V_k\Gamma(\gamma_{k+1})w_{k+1} \quad (k = 0,\ldots,n-1),$$
$$(x_{2k-1}, x_{2k}) := \left(W_{k-1}o, \ V_{k-1}\Pi(\gamma_k)o\right) \qquad\qquad (k = 1,\ldots,n),$$
$$x_0 := 0, \ x_{2n+1} := W_no.$$

Let μ be a probability measure on G with infinite second moment, and let $(\beta_i,\gamma_i,v_i)_{i=1}^n$ be i.i.d.s distributed according to (uniform measure on S^2) $\times \mu$ conditioned on $\tilde{S}$. Then the following holds:

(1) $\{d(x_{2l-1}, x_{2l}) : l = 1, \ldots, n\}$ are i.i.d.s with infinite second moment, whose distribution depends on S and μ but is independent of w_i's.
(2) $\{d(x_{2l}, x_{2l+1}) : l = 0, \ldots, n\}$ are constant RVs;
(3) for each $0 \leq i \leq j \leq k \leq 2n + 1$, $(x_i, x_k)_{x_j}$ is bounded by E_0.
(4) for each $0 \leq i \leq j \leq k \leq i' \leq j' \leq k' \leq 2n + 1$, $(x_i, x_k)_{x_j}$ and $(x_{i'}, x_{k'})_{x_{j'}}$ are independent.

Proof. Note that $[x_{2l-1}, x_{2l}]$'s are translates of $[o, \Pi(\beta_l)v_l\Pi(\gamma_l)o]$'s, which are i.i.d.s.

We denote by μ_S the uniform measure on S. Recall that $\#\tilde{S}(v) \geq (\#S)^2 - 2\#S$ holds for each $v \in G$. As a consequence, for each $g \in G$ we have

$$0.995 \cdot \mu(g) \leq \frac{(\#S)^2 - 2\#S}{(\#S)^2} \mu(g) \leq (\mu_S^2 \times \mu)\big(\{(\beta, \gamma, v) \in \tilde{S} : v = g\}\big) \leq \mu(g).$$

Also, we have

$$(\mu_S^2 \times \mu)(\tilde{S}) \geq \frac{(\#S)^2 - 2\#S}{(\#S)^2} \leq 0.995.$$

Hence, if we denote by μ' the common distribution of v_i's, then

$$0.99\mu(g) \leq \mu'(g) \leq 1.01\mu(g)$$

holds for each $g \in G$. Since μ is assumed to have infinite second moment, μ' also has infinite second moment. Now note that

$$\big|d(x_{2l-1}, x_{2l}) - d(o, v_l o)\big| = \big|d\big(\Pi(\beta_l)^{-1}o, v_l\Pi(\gamma_l)o\big) - d(o, v_l o)\big|$$
$$\leq 2\max\{d(o, \Pi(\mathbf{s})o) : \mathbf{s} \in S\}.$$

This implies that $d(x_{2l-1}, x_{2l})$ also has infinite second moment.

Next, recall that $(w_i)_{i=0}^n$ is a fixed D_0-pre-aligned sequence. Hence, $d(x_{2l}, x_{2l+1}) = d(o, w_l)$ is constant. Moreover, the sequence

$$(4.7) \qquad (o, W_0\Gamma(\beta_1), V_0\Gamma(\gamma_1), \ldots, W_{n-1}\Gamma(\beta_n), V_{n-1}\Gamma(\gamma_{n-1}), W_n o)$$

is D_0-semi-aligned for any choices of (β_i, γ_i, v_i)'s. Also, note that (x_{2l-1}, x_{2l}) are the beginning and ending points of the i-th Schottky axes in Display 4.7. By Lemma 2.6, there exist points $y_1, \ldots, y_{2n}$ on $[x_0, x_{2n+1}]$, from left to right, so that $d(x_i, y_i) \leq 0.1E_0$ for $i = 1, \ldots, n$. It follows that

$$(x_i, x_k)_{x_j} \leq (y_i, y_k)_{y_j} + 0.15E_0 < E_0 \quad (i \leq j \leq k).$$

Finally, for any $k \in \{0, 1, \ldots, 2n + 1\}$, the RVs $\{d(x_i, x_k) : i \leq k\}$ depend only on the choices of $\{(\beta_l, \gamma_l, v_l) : l \leq k/2\}$, while $\{d(x_i, x_k) : i \geq k\}$ depend on the choices of $\{(\beta_l, \gamma_l, v_l) : l > k/2\}$. It follows that the two collections of RVs are independent, and this settles item (4) of the claim. $\square$

Lemma 4.15. *Let S be a fairly long K_0-Schottky set for μ with cardinality at least 400, and fix a D_0-pre-aligned sequence $(w_i)_{i=0}^n$ in G such that $w_n^{-1}w_0$*

is a D_0-pre-aligned isometry. Given $(\beta_i, \gamma_i, v_i) \in \tilde{S}$ for $i = 1, \ldots, n$, define $x_0 := w_n^{-1} o$, $W_0 := w_0$ and

$$V_k := W_k \Gamma(\beta_{k+1}) v_{k+1}, \ W_{k+1} := V_k \Gamma(\gamma_{k+1}) w_{k+1} \quad (k = 0, \ldots, n-1),$$

$$(x_{2k-1}, x_{2k}) := \big(W_{k-1} o, \ V_{k-1} \Pi(\gamma_k) o\big) \qquad\qquad (k = 1, \ldots, n).$$

Let μ be a probability measure on G with infinite second moment, and let $(\beta_i, \gamma_i, v_i)_{i=1}^n$ be i.i.d.s distributed according to (uniform measure on S^2) $\times \mu$ conditioned on $\tilde{S}$. Then, in addition to all the conclusions of Lemma 4.14, the following also holds:

(5) the translation length $\tau(W_n)$ and $d(x_0, x_{2n})$ differ by at most E_0.

Proof. The single change is that x_0 is now $w_n^{-1} o$ instead of o. Since $w_n^{-1} w_0$ is assumed to be D_0-pre-aligned, we have that

$$\big(w_n^{-1} \Pi(\gamma_n)^{-1} \Gamma(\gamma_n), w_0 \Gamma(\beta_1)\big) = \big(W_n^{-1} V_{n-1} \Gamma(\gamma_n), W_0 \Gamma(\beta_1)\big)$$

is D_0-semi-aligned for any choice of $(\beta_1, \gamma_1, v_1), (\beta_n, \gamma_n, v_n) \in \tilde{S}$. Moreover, since $(w_i)_{i=0}^n$ is assumed to be D_0-pre-aligned, we still have that

$$(o, W_0 \Gamma(\beta_1), V_0 \Gamma(\gamma_1), \ldots, W_{n-1} \Gamma(\beta_n), V_{n-1} \Gamma(\gamma_n), W_n o)$$

is D_0-semi-aligned for any choices of $(\beta_i, \gamma_i, v_i) \in \tilde{S}$. Combining these, we observe that

$$(x_0, W_0 \Gamma(\beta_1), V_0 \Gamma(\gamma_1), \ldots, W_{n-1} \Gamma(\beta_n), V_{n-1} \Gamma(\gamma_n), W_n o)$$

is always D_0-semi-aligned. Item (1), (2), (3), (4) of the conclusion follows.

It remains to prove the additional conclusion. For this, define
(4.8)
$$(\kappa_1, \kappa_2, \ldots, \kappa_{2n-1}, \gamma_{2n}) := \big(W_0 \Gamma(\beta_1), V_0 \Gamma(\gamma_1), \ldots, W_{n-1} \Gamma(\beta_n), V_{n-1} \Gamma(\gamma_n)\big)$$

and $\kappa_{2nt+k} := W_n^t \cdot \kappa_k$ for each $t \in \mathbb{Z}$ and $k = 1, \ldots, 2n$. Since (κ_0, κ_1) and $(\kappa_1, \ldots, \kappa_{2n})$ are always D_0-semi-aligned, we conclude that $(\ldots, \kappa_{-1}, \kappa_0, \kappa_1, \kappa_2, \ldots)$ is D_0-semi-aligned for any $t > 0$. At this point, note that $W_n^t x_0 = W_n^{t-1} V_{n-1} \Pi(\gamma_n) o$ is the ending point of κ_{2nt} for each t. By applying Lemma 2.6 to the D_0-semi-aligned sequence

$$(\kappa_0 \ni x_0, \kappa_1, \kappa_2, \ldots, \kappa_{2nt-1}, W_n^t x_0 \in \kappa_{2nt}),$$

we deduce that there exist points $y_1, \ldots, y_{t-1}$ on $[x_0, W_n^t x_0]$, in order from closest to farthest from x_0, such that $d(W_n^k x_0, y_k) \le 0.1 E_0$ for each k. This implies that

$$d(x_0, W_n^t x_0) = d(x_0, y_1) + \sum_{k=1}^{t-2} d(y_k, y_{k+1}) + d(y_{t-1}, W_n^t x_0)$$

$$\ge d(x_0, W_n x_0) + \sum_{k=1}^{t-2} d(W_n^k x_0, W_n^{k+1} x_0) + d(W_n^{t-1} x_0, W_n^t x_0) - t E_0$$

$$\ge t\big(d(x_0, x_{2n}) - E_0\big).$$

After dividing by t and sending t to infinity, we obtain $\tau(W_n) \geq d(x_0, x_{2n}) - E_0$. Meanwhile, the triangle inequality implies that

$$d(x_0, W_n^t x_0) \leq \sum_{k=1}^{t} d(W_n^{k-1} x_0, W_n^k x_0) \leq t d(x_0, x_{2n})$$

and hence $\tau(W_n) \leq d(x_0, x_{2n})$. The conclusion is now established. $\square$

We are now ready to prove:

Proposition 4.16. *Let (X, G, o) be as in Convention 1.1 and let $(Z_n)_{n \geq 0}$ be the random walk on G generated by a non-elementary measure μ with infinite second moment. Then for any sequence $(c_n)_n$ of real numbers, neither of $\frac{1}{\sqrt{n}}(d(o, Z_n o) - c_n)$ and $\frac{1}{\sqrt{n}}(\tau(Z_n) - c_n)$ converges in law.*

Proof. We follow the proof given in [Cho23, Section 6.1]. Let S be a fairly long K_0-Schottky set for μ with cardinality at least 400. Since we are concerned with the distribution of Z_n, we may replace Z_n with the one in Corollary 4.13 that is defined on a probability space Ω_n, constructed together with the integer $m(n)$, the subset $A_n \subseteq \Omega_n$, partition $\mathcal{P}_n$ of A_n and the RVs $(w_i, v_i, \beta_i, \gamma_i)_i$.

The idea is to construct the product space $\Omega_n \times \dot{\Omega}_n$ with the product measure $\mathbb{P}_n$, where $\dot{\Omega}_n$ carries an i.i.d. copy $\dot{Z}_n$ of Z_n, as well as the corresponding notions $\dot{A}_n$, $\dot{\mathcal{P}}_n$ and $(\dot{w}_i, \dot{v}_i, \dot{\beta}_i, \dot{\gamma}_i)_i$. If $\frac{1}{\sqrt{n}} d(o, Z_n o) - c_n$ converges in law for some sequence $(c_n)_{n>0}$, the difference with its i.i.d. copy

$$\frac{1}{\sqrt{n}} [d(o, Z_n o) - d(o, \dot{Z}_n o)]$$

also converges in law. We will deduce a contradiction to this convergence.

On $A_n \subseteq \Omega_n$, we define W_i's and V_i's as in the setting of Lemma 4.14: $W_0 := w_0$ and

$$V_k := W_k \Pi(\beta_{k+1}) v_{k+1}, \quad W_{k+1} := V_k \Pi(\gamma_{k+1}) w_{k+1} \quad (k = 0, \ldots, m(n) - 1).$$

We also define

$$(x_{2k-1}, x_{2k}) := (W_{k-1} o, V_{k-1} \Pi(\gamma_k) o)$$

for $k = 1, \ldots, m(n)$ and let $x_0 := o$, $x_{2m(n)+1} := W_{m(n)} o = Z_n o$.

Let $\mathcal{E} \in \mathcal{P}_n$ be an arbitrary equivalence class. By Item (2) of the conclusion of Corollary 4.13, conditioned on $\mathcal{E}$, $(w_i)_i$ are fixed and $(\beta_i, \gamma_i, v_i))_i$ are i.i.d.s distributed according to (uniform measure on S^2) $\times \mu$ conditioned on $\tilde{S}$. Moreover, by Item (3) of the conclusion of Corollary 4.13,

$$(o, W_0 \Gamma(\beta_1), V_0 \Gamma(\gamma_1), \ldots, W_{m(n)-1} \Gamma(\beta_{m(n)}), V_{m(n)-1} \Gamma(\gamma_{m(n)}), W_{m(n)} o)$$

is D_0-semi-aligned on $\mathcal{E}$. We then apply Lemma 4.14 to the equality

$$d(o, Z_n o) = \underbrace{\sum_{i=1}^{m(n)+1} d(x_{2i-2}, x_{2i-1})}_{I_1} + \underbrace{\sum_{i=1}^{m(n)} d(x_{2i-1}, x_{2i})}_{I_2}$$

$$- 2 \underbrace{\sum_{l=0}^{\log_2 m(n)} \sum_{k=1}^{m(n)/2^l} (x_{2^l(2k-2)}, x_{2^l \cdot 2k})_{x_{2^l(2k-1)}}}_{I_3} - 2 \underbrace{\cdot (x_0, x_{2 \cdot m(n)+1})_{x_{2m(n)}}}_{I_4}.$$

Let $J_l := \sum_{k=1}^{m(n)/2^l} (x_{2^l(2k-2)}, x_{2^l \cdot 2k})_{x_{2^l(2k-1)}}$ be the $(l-1)$-th summand of I_3. Conditioned on $\mathcal{E}$, J_l is the sum of $m(n)/2^l$ independent RVs bounded by E_0. By Chebyshev's inequality, we have

$$\mathbb{P}_n \left(\left| J_l - \mathbb{E}[J_l | \mathcal{E}] \right| \geq 2^{-l/4} \cdot 100 E_0 \sqrt{m(n)} \,\Big|\, \mathcal{E} \right) \leq \frac{1}{\left(2^{-l/4} \cdot 100 E_0 \cdot \sqrt{m(n)} \right)^2} Var\left(J_l \,\big|\, \mathcal{E} \right)$$

$$\leq \frac{2^{l/2}}{10000 E_0^2 m(n)} \cdot \frac{m(n)}{2^l} \cdot E_0^2 \leq 10^{-4} \cdot 2^{-l/2}.$$

By summing this probability, we have

$$(4.9) \qquad \mathbb{P}_n \left(\left| I_3 - \mathbb{E}[I_3 \,|\, \mathcal{E}] \right| > 800 E_0 \cdot \sqrt{m(n)} \,\Big|\, A_n \right) \leq 1/2000.$$

Note also I_1 is constant on each $\mathcal{E} \in \mathcal{P}_n$ and I_4 is bounded by E_0 on A_n.

Let us now bring an arbitrary equivalence class $\dot{\mathcal{E}}$ from the partition $\dot{\mathcal{P}}_n$. Similarly, we get

$$d(o, \dot{Z}_n o) = \underbrace{\sum_{i=1}^{2^s+1} d(\dot{x}_{2i-2}, \dot{x}_{2i-1})}_{\dot{I}_1} + \underbrace{\sum_{i=1}^{2^s} d(\dot{x}_{2i-1}, \dot{x}_{2i})}_{I_2}$$

$$- 2 \underbrace{\sum_{l=0}^{s} \sum_{k=1}^{2^{s-l}} (\dot{x}_{2^l(2k-2)}, \dot{x}_{2^l \cdot 2k})_{\dot{x}_{2^l(2k-1)}}}_{\dot{I}_3} - 2 \underbrace{\cdot (\dot{x}_0, \dot{x}_{2 \cdot 2^s+1})_{\dot{x}_{2 \cdot 2^s}}}_{\dot{I}_4}.$$

We have the following $(*)$:

(1) Since the situation is symmetric, for large enough n we have

$$\mathbb{P}_n \left(\bigcup \left\{ (\mathcal{E}, \dot{\mathcal{E}}) \in \mathcal{P}_n \times \dot{\mathcal{P}}_n : I_1 + \mathbb{E}[I_3 \,|\, \mathcal{E}] \geq \dot{I}_1 + \mathbb{E}[\dot{I}_3 \,|\, \dot{\mathcal{E}}] \right\} \right) \geq \frac{1}{2} \mathbb{P}_n(A_n \times \dot{A}_n)$$

$$\geq 0.5(1 - \mathbb{P}_n(A_n) - \mathbb{P}_n(\dot{A}_n)) \geq 0.4.$$

(2) By Inequality 4.9, we have

$$\mathbb{P}_n \left(\max \left(\left| I_3 - \mathbb{E}[I_3 \,|\, \mathcal{E}] \right|, \left| \dot{I}_3 - \mathbb{E}[\dot{I}_3 \,|\, \dot{\mathcal{E}}] \right| \right) \geq 800 E_0 \cdot \sqrt{m(n)} \,\Big|\, \mathcal{E} \times \dot{\mathcal{E}} \right) \leq 1/1000$$

for each $\mathcal{E} \in \mathcal{P}_n$ and $\dot{\mathcal{E}} \in \dot{\mathcal{P}}_n$ for all n.

(3) Since $I_2 - \dot{I}_2$ is a sum of $m(n)$ i.i.d.s of *symmetric* distribution with infinite second moment, for any $K' > 0$ we have

$$\mathbb{P}_n \left(I_2 - \dot{I}_2 \geq K'\sqrt{m(n)} \,\middle|\, \mathcal{E} \times \dot{\mathcal{E}} \right) \geq 1/5$$

for each $\mathcal{E} \in \mathcal{P}$ and $\dot{\mathcal{E}} \in \dot{\mathcal{P}}$ for sufficiently large n.

(4) I_4 and $\dot{I}_4$ are bounded by E_0 on $A_n \times \dot{A}_n$.

Let $C := \lim_n m(n)/n > 0$. The 4 items in $(*)$ imply that for each $K' > 10000 E_0$, for large enough n, we have

$$d(o, Z_n o) - d(o, \dot{Z}_n o) \geq K'\sqrt{m(n)} - 1600 E_0 \sqrt{m(n)} - 2E_0 \geq 0.25 K'\sqrt{Cn}$$

with probability at least $0.4 \cdot (1/5 - 1/1000) \geq 0.05$. Hence we have

$$\liminf_n \mathbb{P}_n \left(\frac{1}{\sqrt{n}} \big(d(o, Z_n o) - d(o, \dot{Z}_n o)\big) \geq 0.25 K'\sqrt{C} \right) \geq 0.05 \quad (\forall\, K' > 10000 E_0).$$

This contradicts to the convergence of $\frac{1}{\sqrt{n}}[d(o, Z_n o) - d(o, \dot{Z}_n o)]$ in law.

We now derive a contradiction to the convergence in law of $\frac{1}{\sqrt{n}}[\tau(Z_n) - \tau(\dot{Z}_n)]$. We again prepare the outputs of Corollary 4.13, namely, $m(n), \Omega_n, A_n, \mathcal{P}_n, (w_i)_{i=0}^{m(n)}, (\beta_i, \gamma_i, v_i)_{i=1}^{m(n)}$, and their independent copies. We define $(W_i, V_i)_{i=0}^{m(n)-1}$ and $(x_i)_{i=1}^{2m(n)}$ as before, and let $x_0 := w_{m(n)}^{-1} o$.

This time, Lemma 4.15 tells us that

$$\tau(Z_n) = d(x_0, x_{2n}) + [\tau(Z_n) - d(x_0, x_{2n})]$$

$$= \underbrace{\sum_{i=1}^{m(n)+1} d(x_{2i-2}, x_{2i-1})}_{I_1} + \underbrace{\sum_{i=1}^{m(n)} d(x_{2i-1}, x_{2i})}_{I_2}$$

$$- 2 \underbrace{\sum_{l=0}^{\log_2 m(n)} \sum_{k=1}^{m(n)/2^l} (x_{2^l(2k-2)}, x_{2^l \cdot 2k})_{x_{2^l(2k-1)}}}_{I_3} + \underbrace{\big(\tau(Z_n) - d(x_0, x_{2m(n)})\big)}_{I_4}.$$

Here, I_1, I_2, I_3 and I_4 satisfy the exact same properties as before. Hence, we observe the same consequences as in $(*)$. This leads to the same contradiction to the convergence in law of $\frac{1}{\sqrt{n}}[\tau(Z_n) - \tau(\dot{Z}_n)]$. Hence, for any choice of the sequence $(c_n)_{n>0}$, $\frac{1}{\sqrt{n}}(\tau(Z_n) - c_n)$ cannot converge in law. $\qquad\square$

4.5. Pivoting and repulsion among independent random walks.

In this subsection, we consider two random walks $(Z_n^{(1)})_n$ and $(Z_n^{(2)})_n$ generated by non-elementary measures $\mu^{(1)}$ and $\mu^{(2)}$ on G. Let $S^{(1)}$ and $S^{(2)}$ be fairly long K_0-Schottky sets for $\mu^{(1)}$ and $\mu^{(2)}$. We define the subset $\tilde{S}^{(t)}$ following Definition 4.2, with $S^{(t)}$ in place of S.

We first fix D_0-semi-aligned sequences $\left(w_i^{(t)}\right)_{i=0}^n$ in G for $t = 0, 1$. Given the choices of $\left(\beta_i^{(t)}, \gamma_i^{(t)}, v_i^{(t)}\right) \in \tilde{S}^{(t)}$ for $i = 1, \ldots, n$, we define $W_i^{(t)}$'s, $V_i^{(t)}$'s as before: $W_0^{(t)} := w_0^{(t)}$ and

$$V_k^{(t)} := W_k^{(t)} \Pi\big(\beta_{k+1}^{(t)}\big) v_{k+1}^{(t)}, \quad W_{k+1}^{(t)} := V_k^{(t)} \Pi\big(\gamma_{k+1}^{(t)}\big) w_{k+1}^{(t)}.$$

This time, we want not only that $W_n^{(t)}$ and $(W_n^{(t)})^{-1}$ head in different directions for $t = 1, 2$, but also that the directions made by $W_n^{(1)}$, $W_n^{(2)}$, $(W_n^{(1)})^{-1}$, $(W_n^{(2)})^{-1}$ are all distinct. For this purpose, we define the following:

(1) (front-front repulsion) For each $1 \le k \le n/2$ we let

$$\phi_k^{(1,2),front} := \big(W_{k-1}^{(2)}\big)^{-1} \cdot W_{k-1}^{(1)}$$
$$= \big(w_{k-1}^{(2)}\big)^{-1}\big(\Pi\big(\gamma_{k-1}^{(2)}\big)\big)^{-1} \cdots \big(w_0^{(2)}\big)^{-1} \cdot w_0^{(1)} \Pi\big(\beta_1^{(1)}\big) \cdots w_{k-1}^{(1)},$$

$$\tilde{S}_k^{(1,2),front} := \left\{ \big(\beta_k^{(1)}, \gamma_k^{(1)}\big) \in \tilde{S}^{(1)}(v_k^{(1)}) : \left(\big(\phi_k^{(1,2),front}\big)^{-1}o, \, \Gamma\big(\beta_k^{(1)}\big)\right) \text{ is } K_0\text{-aligned} \right\},$$

$$\tilde{S}_k^{(2,1),front} := \left\{ \big(\beta_k^{(2)}, \gamma_k^{(2)}\big) \in \tilde{S}^{(2)}(v_k^{(1)}) : \left(\phi_k^{(1,2),front} \Pi\big(\beta_k^{(1)}\big)o, \, \Gamma\big(\beta_k^{(2)}\big)\right) \text{ is } K_0\text{-aligned} \right\}.$$

(2) (back-back repulsion) For each $1 \le k \le n/2$ we let

$$\phi_k^{(1,2),back} := \left(V_{n-k}^{(2)} \Pi\big(\gamma_{n-k+1}^{(2)}\big)\right)^{-1} W_n^{(2)} \cdot (W_n^{(1)})^{-1} V_{n-k}^{(1)} \Pi\big(\gamma_{n-k+1}^{(1)}\big)$$
$$= w_{n-k+1}^{(2)} \Pi\big(\beta_{n-k+2}^{(2)}\big) \cdots w_n^{(2)} \cdot \big(w_n^{(1)}\big)^{-1}\big(\Pi\big(\gamma_n^{(1)}\big)\big)^{-1} \cdots \big(w_{n-k+1}^{(1)}\big)^{-1},$$

$$\tilde{S}_k^{(1,2),back} := \left\{ \begin{array}{c} \big(\beta_{n-k+1}^{(1)}, \gamma_{n-k+1}^{(1)}\big) \in \tilde{S}^{(1)}(v_{n-k+1}^{(1)}) : \\ \left(\Gamma\big(\gamma_{n-k+1}^{(1)}\big), \, \Pi\big(\gamma_{n-k+1}^{(1)}\big) \cdot \big(\phi_k^{(1,2),back}\big)^{-1}o\right) \text{ is } K_0\text{-aligned} \end{array} \right\},$$

$$\tilde{S}_k^{(2,1),back} := \left\{ \begin{array}{c} \big(\beta_{n-k+1}^{(2)}, \gamma_{n-k+1}^{(2)}\big) \in \tilde{S}^{(2)}(v_{n-k+1}^{(2)}) : \\ \left(\Gamma\big(\gamma_{n-k+1}^{(2)}\big), \, \Pi\big(\gamma_{n-k+1}^{(2)}\big)\phi_k^{(1,2),back}\big(\Pi\big(\gamma_{n-k+1}^{(1)}\big)\big)^{-1} \cdot o\right) \text{ is } K_0\text{-aligned} \end{array} \right\}.$$

(3) (front-back repulsion) For each $s, t \in \{1, 2\}$ and $1 \le k \le n/2$ we let

$$\phi_k^{(s,t),mixed} := \left(V_{n-k}^{(t)} \Pi\big(\gamma_{n-k+1}^{(t)}\big)\right)^{-1} W_n^{(t)} \cdot W_{k-1}^{(s)}$$
$$= w_{n-k+1}^{(t)} \Pi\big(\beta_{n-k+2}^{(t)}\big) \cdots w_n^{(t)} \cdot w_0^{(s)} \Pi\big(\beta_1^{(s)}\big) \ldots w_{k-1}^{(s)},$$

$$\tilde{S}_k^{(s,t),\rightarrow} := \left\{ \begin{array}{c} \big(\beta_k^{(s)}, \gamma_k^{(s)}\big) \in \tilde{S}^{(s)}(v_k^{(s)}) : \\ \left(\big(\phi_k^{(s,t),mixed}\big)^{-1}o, \, \Gamma\big(\beta_k^{(s)}\big)\right) \text{ is } K_0\text{-aligned} \end{array} \right\},$$

$$\tilde{S}_k^{(s,t),\leftarrow} := \left\{ \begin{array}{c} \big(\beta_{n-k+1}^{(t)}, \gamma_{n-k+1}^{(t)}\big) \in \tilde{S}^{(t)}(v_{n-k+1}^{(t)}) : \\ \left(\Gamma\big(\gamma_{n-k+1}^{(t)}\big), \, \Pi\big(\gamma_{n-k+1}^{(t)}\big)\phi_k^{(s,t),mixed}\Pi\big(\beta_k^{(s)}\big) \cdot o\right) \text{ is } K_0\text{-aligned} \end{array} \right\}.$$

We now establish an analogue of Lemma 4.9.

Lemma 4.17. *Let $1 \le k \le n$. Let $\left(w_i^{(0)}\right)_{i=0}^n$ and $\left(w_i^{(1)}\right)_{i=0}^n$ be D_0-pre-aligned sequences in G. Let $\left(\beta_i^{(t)}, \gamma_i^{(t)}, v_i^{(t)}\right)_{i=1,\ldots,k,n-k+1,\ldots,n} \in \left(\tilde{S}^{(t)}\right)^{2k}$ for $t = 1, 2$*

be choices such that

$$(4.10) \quad \begin{aligned} (\beta_k^{(1)}, \gamma_k^{(1)}) &\in \tilde{S}_k^{(1,2),front} \cap \tilde{S}_k^{(1,1),\rightarrow} \cap \tilde{S}_k^{(1,2),\rightarrow}, \\ (\beta_k^{(2)}, \gamma_k^{(2)}) &\in \tilde{S}_k^{(2,1),front} \cap \tilde{S}_k^{(2,1),\rightarrow} \cap \tilde{S}_k^{(2,2),\rightarrow}, \\ (\beta_{n-k+1}^{(1)}, \gamma_{n-k+1}^{(1)}) &\in \tilde{S}_k^{(1,2),back} \cap \tilde{S}_k^{(1,1),\leftarrow} \cap \tilde{S}_k^{(2,1),\leftarrow}, \\ (\beta_{n-k+1}^{(2)}, \gamma_{n-k+1}^{(2)}) &\in \tilde{S}_k^{(1,2),back} \cap \tilde{S}_k^{(2,1),\leftarrow} \cap \tilde{S}_k^{(2,2),\leftarrow}. \end{aligned}$$

Then for any further choices $\left(\beta_i^{(t)}, \gamma_i^{(t)}, v_i^{(t)}\right)_{i=k+1,\ldots,n-k} \in \left(\tilde{S}^{(t)}\right)^{2(n-k)}$ *for* $t = 1, 2$, $W_n^{(1)}$ *and* $W_n^{(2)}$ *are contracting isometries that generate a free group of order 2. Moreover, the orbit map is a quasi-isometric embedding of* $\langle W_n^{(1)}, W_n^{(2)} \rangle$ *into a quasi-convex subset of* X.

Proof. Recall the following notation: given a path $\Gamma = (x_0, x_1 \ldots, x_m)$, we defined its *reversal* by

$$\bar{\Gamma} := (x_m, x_{m-1}, \ldots, x_0).$$

For convenience, let us define $\kappa^{(t)} := W_{k-1}^{(t)} \Gamma(\beta_k^{(t)})$ and let $\bar{\kappa}^{(t)}$ be its reversal for $s = 1, 2$. Also, let $\eta^{(t)} := (W_n^{(t)})^{-1} V_{n-k} \Gamma(\gamma_{n-k+1}^{(t)})$ and let $\bar{\eta}^{(t)}$ be its reversal.

By Lemma 2.5, the assumption of the lemma implies that the following sequences of Schottky axes are K_0-aligned:

$$\begin{aligned} (\bar{\kappa}_{(1)}, \kappa_{(2)}), \quad & (\eta_{(1)}, \bar{\eta}_{(1)}), \quad (\eta_{(1)}, \kappa_{(1)}), \\ (\eta_{(2)}, \kappa_{(1)}), \quad & (\eta_{(1)}, \kappa_{(2)}), \quad (\eta_{(2)}, \kappa_{(2)}). \end{aligned}$$

Moreover, since $(w_i^{(t)})_{i=0}^n$'s are D_0-pre-aligned, we have that

$$(o, \kappa_{(t)}, W_n^{(t)} \eta_{(t)}, W_n^{(t)} o) \quad (t \in \{1, 2\})$$

is D_0-semi-aligned.

We now prove that the orbit map from $\langle W_n^{(1)}, W_n^{(2)} \rangle$ to X is bi-Lipshictz and the orbit set is quasiconvex. It is clear that the map is $\max(d(o, W_n^{(1)} o), d(o, W_n^{(2)} o))$-Lipschitz. We define

$$K_1 := \min\{\operatorname{diam}(\kappa_{(1)} \cup W_n^{(1)} \eta_{(1)}), \operatorname{diam}(\kappa_{(2)} \cup W_n^{(2)} \eta_{(2)})\} - E_0,$$

$$K_2 := 4 \sum_{t \in \{1,2\}} \left(\operatorname{diam}(\eta_{(t)}, o) + \operatorname{diam}(\kappa_{(t)} o) + \operatorname{diam}(\kappa_{(t)} \cup W_n^{(t)} \eta_{(t)}) + E_0 \right).$$

Since $\kappa_{(1)}$ and $\kappa_{(2)}$ are both fairly long Schottky axes, we have $\operatorname{diam}(\kappa_{(1)}), \operatorname{diam}(\kappa_{(2)}) \geq 10E_0$ and consequently K_1 is positive.

The conclusion is established once we show that

$$(4.11) \quad \begin{aligned} d(o, a_1 \cdots a_m o) &\geq m K_1, \\ [o, a_1 \cdots a_m o] &\subseteq N_{K_2} \left(\{o, a_1 o, a_1 a_2 o, \ldots, a_1 \cdots a_m o\}\right) \end{aligned}$$

for each m and each choice of $a_1, \ldots, a_m \in \{W_n^{(1)}, W_n^{(2)}, (W_n^{(1)})^{-1}, (W_n^{(2)})^{-1}\}$ such that $a_i \neq a_{i+1}^{-1}$ for each i. Let us show this for the choice $a_1 a_2 a_3 =$

$W_n^{(1)}(W_n^{(2)})^{-1}W_n^{(1)}$ as an example; the same argument applies to all other cases. Combining the alignment mentioned above, we deduce that

$$\left(\begin{array}{c} o, \kappa_{(1)}, W_n^{(1)}\eta_{(1)}, W_n^{(1)}\bar{\eta}_{(2)}, W_n^{(1)}(W_n^{(2)})^{-1}\bar{\kappa}_{(2)}, W_n^{(1)}(W_n^{(2)})^{-1}\kappa_{(1)}, \\ W_n^{(1)}(W_n^{(2)})^{-1}W_n^{(1)}\eta_{(1)}, W_n^{(1)}(W_n^{(2)})^{-1}W_n^{(1)}o \end{array} \right)$$

is D_0-semi-aligned. By Lemma 2.6, there exist disjoint subsegments $[x_1, y_1]$, $\ldots$, $[x_6, y_6]$ of $[o, W_n^{(1)}(W_n^{(2)})^{-1}W_n^{(1)}o]$, in order from closest to farthest from o, that $0.1E_0$-fellow travel with $\kappa_{(1)}, \ldots, W_n^{(1)}(W_n^{(2)})^{-1}W_n^{(1)}\eta_{(1)}$, respectively. In particular, we deduce that

$$d(x_1, y_2) \geq \mathrm{diam}(\kappa_{(1)} \cup W_n^{(1)}\eta_{(1)}) - E_0,$$
$$d(x_3, y_4) \geq \mathrm{diam}(W_n^{(1)}\bar{\eta}_{(2)} \cup W_n^{(1)}(W_n^{(2)})^{-1}\bar{\kappa}_{(2)}) - E_0$$
$$= \mathrm{diam}(\kappa_{(2)} \cup W_n^{(2)}\eta_{(2)}) - E_0,$$
$$d(x_5, y_6) \geq \mathrm{diam}(W_n^{(1)}(W_n^{(2)})^{-1}\kappa_{(1)} \cup W_n^{(1)}(W_n^{(2)})^{-1}W_n^{(1)}\eta_{(1)})) - E_0$$
$$= \mathrm{diam}(\kappa_{(1)} \cup W_n^{(1)}\eta_{(1)}) - E_0.$$

Since $d(o, W_n^{(1)}(W_n^{(2)})^{-1}W_n^{(1)}o)$ is greater than $d(x_1, y_2)+d(x_3, y_4)+d(x_5, y_6)$, we can conclude the first inequality in Display 4.11. Moreover, note that

$$d(x_2, W_n^{(1)}o) \leq \mathrm{diam}(\eta_{(1)}o \cup o) + E_0,$$
$$d(x_4, W_n^{(1)}(W_n^{(2)})^{-1}o) \leq \mathrm{diam}(\kappa_{(1)}o \cup o) + E_0.$$

Now for each point y on $[o, x_2]$ we have

$$d(y, W_n^{(1)}o) \leq d(o, x_2) + d(x_2, W_n^{(1)}o)$$
$$\leq \left(\mathrm{diam}(o \cup \kappa_{(1)}) + \mathrm{diam}(\kappa_{(1)} \cup W_n^{(1)}\eta_{(1)}) + E_0 \right) + \left(\mathrm{diam}(\eta_{(1)}o \cup o) + E_0 \right) \leq K_2.$$

Similarly, for any point y on $[x_2, x_4]$ we have

$$d(y, W_n^{(1)}(W_n^{(2)})^{-1}o) \leq d(x_2, x_4) + d(x_4, W_n^{(1)}(W_n^{(2)})^{-1}o)$$
$$\leq \mathrm{diam}(o \cup \eta_{(1)}) + \mathrm{diam}(o \cup \eta_{(2)}) + \mathrm{diam}(\kappa_{(2)} \cup W_n^{(2)}\eta_{(2)}) + 2E_0$$
$$+ \mathrm{diam}(\eta_{(2)}o \cup o) + E_0 \leq K_2.$$

For a similar reason, $[x_4, W_n^{(1)}(W_n^{(2)})^{-1}W_n^{(1)}o]$ is contained in the K_2-neighborhood of $W_n^{(1)}(W_n^{(2)})^{-1}W_n^{(1)}o$. This settles the second line of Display 4.11. $\square$

We now establish an analogue of Lemma 4.11.

Lemma 4.18. *Let $1 \leq k \leq n/2$, let $(w_i^{(t)})_{i=1}^n \in G^{n+1}$ and let $(\beta_i^{(t)}, \gamma_i^{(t)}, v_i^{(t)})_{i=1}^n \in \tilde{S}_n^{(t)}$ for $t = 1, 2$. Then Condition 4.10 depends only on $\{(\beta_l^{(t)}, \gamma_l^{(t)}, v_l^{(t)}) : l = 1, \ldots, k, n-k+1, \ldots, n, t = 1, 2\}$.*

Moreover, fixing any choice of $\{v_l^{(t)} : l = 1, \ldots, k, n-k+1, \ldots, n, t = 1, 2\}$, the condition holds for probability at least $1 - (10\#S^{(1)})^k - (10\#S^{(2)})^k$

with respect to the uniform measures on $\tilde{S}^{(t)}(v_l)$ for $t = 1, 2$ and for $l = 1, \ldots, k, n - k + 1, \ldots, n$, all independent.

Proof. Note that the statement

$$(4.12) \qquad (\beta_k^{(1)}, \gamma_k^{(1)}) \in S_l^{(1,2),front} \cap \tilde{S}_k^{(1,1),\rightarrow} \cap \tilde{S}_k^{(1,2),\rightarrow}$$

is really a condition for $\beta_k^{(1)}$ that involves $\phi_k^{(1,2),front}$, $\phi_k^{(1,1),mixed}$ and $\phi_k^{(1,2),mixed}$, which depend only on $\beta_l^{(t)}$ and $\gamma_l^{(t)}$ for $l = 1, \ldots, k-1, n-k+2, \ldots, n$ and for $t = 1, 2$. Fixing a choice of $(\beta_k^{(1)}, \gamma_k^{(1)})$ satisfying condition 4.12, the statement

$$(4.13) \qquad (\beta_k^{(2)}, \gamma_k^{(2)}) \in S_l^{(2,1),front} \cap \tilde{S}_k^{(2,1),\rightarrow} \cap \tilde{S}_k^{(2,2),\rightarrow}$$

is now a condition for $\beta_k^{(2)}$ that is determined only by $\beta_l^{(t)}$ and $\gamma_l^{(t)}$ for $l = 1, \ldots, k-1, n-k+2, \ldots, n$ and for $t = 1, 2$, and $\beta_k^{(1)}$. Fixing a $(\beta_k^{(2)}, \gamma_k^{(2)})$ further, the condition

$$(4.14) \qquad (\beta_{n-k+1}^{(1)}, \gamma_{n-k+1}^{(1)}) \in \tilde{S}_k^{(1,2),back} \cap \tilde{S}_k^{(1,1),\leftarrow} \cap \tilde{S}_k^{(2,1),\leftarrow}$$

then depends on $\beta_l^{(t)}$ and $\gamma_l^{(t)}$ for $l = 1, \ldots, k-1, n-k+2, \ldots, n$ and for $t = 1, 2$, and the choice of $\beta_k^{(1)}$, $\beta_k^{(2)}$. Fixing a $(\beta_{n-k+1}^{(1)}, \gamma_{n-k+1}^{(1)})$, finally, the condition

$$(4.15) \qquad (\beta_{n-k+1}^{(2)}, \gamma_{n-k+1}^{(2)}) \in \tilde{S}_k^{(1,2),back} \cap \tilde{S}_k^{(2,1),\leftarrow} \cap \tilde{S}_k^{(2,2),\leftarrow}.$$

depends on $\beta_l^{(t)}$ and $\gamma_l^{(t)}$ for $l = 1, \ldots, k-1, n-k+2, \ldots, n$, $t = 1, 2$, and $\beta_k^{(1)}$, $\beta_k^{(2)}$, $\gamma_{n-k+1}^{(1)}$. Combining these four statements leads to the first assertion.

The second assertion is proven analogously to the proof of Lemma 4.11; we sketch the necessary ingredients here. Fixing an arbitrary choice of

$$\left\{ \beta_i^{(t)}, \gamma_i^{(t)}, v_i^{(t)} : i = 1, \ldots, k-1, t = 1, 2 \right\},$$

note that Condition 4.12 is asking the alignment of $\beta_k^{(1)}$ with 3 other fixed isometries. The property of the Schottky set $S^{(1)}$ (cf. Definition 2.8) tells us that such alignments are realized by all but at most 3 choices of $\beta_k^{(1)} \in S^{(1)}$. This implies that at most $3\#S^{(1)}$ choices in $(S^{(1)})^2 \supseteq \tilde{S}^{(1)}(v_k^{(1)})$ violate the condition. This implies that

$$\mathbb{P}\left((\beta_k^{(1)}, \gamma_k^{(1)}) \in \tilde{S}^{(1)}(v_k^{(1)}) \text{ violates Condition } 4.12 \,\middle|\, \beta_i^{(t)}, \gamma_i^{(t)}, v_i^{(t)} : i = 1, \ldots, k-1, t = 1, 2 \right)$$

$$\leq \frac{3\#S^{(1)}}{(\#S^{(1)})^2 - 2\#S^{(1)}} \leq \frac{5\#S^{(1)}}{(\#S^{(1)})^2}.$$

For a similar reason, we can prove that the probabilities of violating Condition 4.13, 4.14 and 4.15 are at most $5/\#S^{(2)}$, $5/\#S^{(1)}$ and $5/\#S^{(2)}$, respectively. Combining these, we can deduce

$$\mathbb{P}\left(\text{Condition 4.10 violated at step } k \,\middle|\, \text{Condition 4.10 violated till } k-1 \right) \leq \frac{10}{\#S^{(1)}} + \frac{10}{\#S^{(2)}}.$$

By induction, we can conclude the second assertion. $\square$

Recall that we have implemented pivotal times for random walks in Corollary 4.13. Following this idea, and also by combining Lemma 4.17 and Lemma 4.11, we deduce the following corollary:

Corollary 4.19. *Let (X, G, o) be as in Convention 1.1 and $(Z_n^{(1)})_{n>0}, (Z_n^{(2)})_{n>0}$ be two independent random walks generated by a non-elementary measure μ on G. Then there exists $K > 0$ such that the following holds outside a set of probability $Ke^{-n/K}$.*

The n-th step isometries $Z_n^{(1)}$, $Z_n^{(2)}$ arising from two random walks generate a free group of order 2. Moreover, the orbit map is a quasi-isometric embedding of $\langle Z_n^{(1)}, Z_n^{(2)} \rangle$ into a quasi-convex subset of X.

It is not difficult to consider k independent random walks; the arguments are identical. Hence, we conclude Theorem D.

5. Effective estimates for translation length

In this section, we establish an effective version of Corollary 4.13 that will be used for the counting problem.

We will employ Schottky sets to construct the generating set S''. For this we introduce a notation. Given a set of sequences $S \subseteq G^n$, we define

$$\Phi(S^4) := \{\Pi(s_1)\Pi(s_2)\Pi(s_3)\Pi(s_4) : s_1, s_2, s_3, s_4 \in S\}.$$

Applying the map Φ to S^4 does not erase any information because:

Lemma 5.1. *Let S be a fairly long Schottky set. Then the map Φ is a bijection between S^4 and $\Phi(S^4)$.*

Proof. This is a consequence of Lemma 5.2 below. $\square$

One technicality is that $\Phi(S^4)$ cannot be a symmetric subset of G for any Schottky set S. (This is due to our choice of Property (4) of Schottky sets.) To get around this technicality, we introduce another notation. Given a sequence $\alpha = (a_1, \ldots, a_n) \in G^n$, let us denote the sequence $(a_n^{-1}, \ldots, a_1^{-1})$ by α^{-1}. For a set $S \subseteq G^n$, we define

$$\check{S} := \{\alpha^{-1} : \alpha \in S\} = \left\{ \left(a_n^{-1}, \ldots, a_1^{-1}\right) : \left(a_1, \ldots, a_n\right) \in S \right\}.$$

Lemma 5.2. *Let $k \in \mathbb{Z}_{>0}$ and let S be a fairly long Schottky set. Let $\alpha_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for $i = 1, \ldots, k$. Suppose that there does not exist i such that $\alpha_i = \alpha_{i+1}$ and $\epsilon_i \epsilon_{i+1} = -1$ simultaneously holds. Then:*

(1) the sequence of Schottky axes

$$\left(\Gamma(\alpha_1^{\epsilon_1}), \quad \Pi(\alpha_1^{\epsilon_1})\Gamma(\alpha_2^{\epsilon_2}), \quad \ldots, \quad \Pi(\alpha_1^{\epsilon_1})\cdots\Pi(\alpha_{k-1}^{\epsilon_k})\Gamma(\alpha_k^{\epsilon_k})\right)$$

is D_0-aligned, and
(2) $\Pi(\alpha_1^{\epsilon_1})\cdots\Pi(\alpha_k^{\epsilon_k})$ is a nontrivial isometry.

Proof. Let $\alpha \in S$ and $\epsilon \in \{\pm 1\}$. Then

$$\mathrm{diam}\left(\pi_{\Gamma(\alpha^\epsilon)}(\Pi(\alpha^\epsilon)o) \cup o\right) = \mathrm{diam}\left(\Pi(\alpha^\epsilon)o \cup o\right) \geq 10E_0 > K_0$$

holds, which means that

$$\alpha \notin \left\{\beta \in S : \left(\Pi(\alpha)o, \Gamma(\beta)\right) \text{ is } K_0\text{-aligned}\right\},$$

$$\alpha \notin \left\{\beta \in S : \left(\Gamma(\beta), \Pi(\beta)\Pi(\alpha^{-1})o\right) \text{ is } K_0\text{-aligned}\right\}.$$

By Property 4 of Schottky sets (cf. Definition 2.8), we deduce that $\left(\Gamma(\beta), \Pi(\beta)\Pi(\alpha^\epsilon)o\right)$ and $\left(\Pi(\alpha^\epsilon)o, \Gamma(\beta)\right)$ are K_0-aligned sequences for each $\beta \in S \setminus \{\alpha\}$. The latter statement means that $\left(\Gamma(\beta^{-1}), \Pi(\beta^{-1})\Pi(\alpha^\epsilon)o\right)$ is K_0-aligned. $(*)$

Now for each i, we have the following cases.

(I) $\alpha_i \neq \alpha_{i+1}$. In this case, $\left(\Gamma(\alpha_i^{\epsilon_i}), \Pi(\alpha_i^{\epsilon_i})\Pi(\alpha_{i+1}^{\epsilon_{i+1}})o\right)$ is K_0-aligned due to $(*)$. Moreover, $\left(\Pi(\alpha_i^{\epsilon_i})o, \Pi(\alpha_i^{\epsilon_i})\Gamma(\alpha_{i+1}^{\epsilon_{i+1}})\right)$ is 0-aligned. This is because $\Pi(\alpha_i^{\epsilon_i})o$ projects onto $\Pi(\alpha_i^{\epsilon_i})\Gamma(\alpha_{i+1}^{\epsilon_{i+1}}) \ni \Pi(\alpha_i^{\epsilon_i})o$ at itself.

Now by Lemma 2.5, $\left(\Gamma(\alpha_i^{\epsilon_i}), \Pi(\alpha_i^{\epsilon_i})\Gamma(\alpha_{i+1}^{\epsilon_{i+1}})\right)$ is D_0-aligned.

(II) $\alpha_i = \alpha_{i+1}$: then $\epsilon_i = \epsilon_{i+1}$, and the sequence

$$\left(\Gamma(\alpha_i^{\epsilon_i}), \Pi(\alpha_i^{\epsilon_i})\Gamma(\alpha_i^{\epsilon_i})\right)$$

is D_0-aligned thanks to Property 5 of Schottky sets.

Combining these stepwise D_0-alignment, we conclude Item (1).

For convenience, we denote $\Pi(\alpha_1^{\epsilon_1}) \cdots \Pi(\alpha_k^{\epsilon_k})$ by w. By Lemma 2.6, $[o, wo]$ contains disjoint subsegments that $0.1E_0$-fellow travel with suitable translates of $\Gamma(\alpha_1^{\epsilon_1})$, ..., $\Gamma(\alpha_k^{\epsilon_k})$, respectively. Recall that these Schottky axes have diameter at least $10E_0$. Hence we have

$$d(o, wo) \geq \left[\sum_{i=1}^{k} d\left(o, \Pi(\alpha_k^{\epsilon_k})o\right)\right] - 0.2(k-1)E_0 \geq 5E_0 k.$$

In particular, w is nontrivial and Item (2) follows. $\square$

Lemma 5.3. *Let $S \subseteq G^n$ be a set of sequences in G and let $K_0 > 0$. Then the following hold.*

(1) $\Phi(\check{S}^4)$ consists of the inverses of elements of $\Phi(S^4)$.

(2) S is a fairly long K_0-Schottky set if and only if $\check{S}$ is.

(3) If S is a fairly long Schottky set, then $\Phi(\check{S}^4)$ and $\Phi(S^4)$ are disjoint.

Proof.   (1) This follows from the definition of $\check{S}$ and $\check{S}^4$.

(2) Let S be a fairly long K_0-Schottky set. Note the following equivalence for each $\alpha \in G^n$ and $x \in X$:

$$\Gamma(\alpha) \text{ is } K_0\text{-contracting} \Leftrightarrow \Gamma(\alpha^{-1}) \text{ is } K_0\text{-contracting},$$

$$d(o, \Pi(\alpha)o) > 10E_0 \Leftrightarrow d(o, \Pi(\alpha^{-1})o) > 10E_0,$$

$$\left(x, \Gamma(\alpha)\right), \left(\Gamma(\alpha), \Pi(\alpha)x\right) \text{ are } K_0\text{-aligned} \Leftrightarrow \left(x, \Gamma(\alpha^{-1})\right), \left(\Gamma(\alpha^{-1}), \Pi(\alpha^{-1})x\right) \text{ are } K_0\text{-aligned}$$

$$\left(\Gamma(\alpha), \Pi(\alpha)\Gamma(\alpha)\right) \text{ is } K_0\text{-aligned} \Leftrightarrow \left(\Gamma(\alpha^{-1}), \Pi(\alpha^{-1})\Gamma(\alpha^{-1})\right) \text{ is } K_0\text{-aligned}.$$

This implies that changing an element $\alpha \in S$ with α^{-1} does not affect the Schottky-ness of S. By converting all the elements into their inverses, we conclude that $\check{S}$ is also K_0-Schottky.

(3) If $\mathbf{s} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in S^4$ and $\mathbf{s}' = (\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4) \in \check{S}^4$ have the same Φ-values, then we have

$$(5.1) \qquad id = \Phi(\mathbf{s}) \cdot \Phi(\mathbf{s}')^{-1} = \Pi(\alpha_1) \cdots \Pi(\alpha_4) \cdot \Pi(\check{\alpha}_4^{-1}) \cdots \Pi(\check{\alpha}_1^{-1}).$$

Here, α_i's and $\check{\alpha}_i^{-1}$'s are all elements of S. With this, Equation 5.1 contradicts Lemma 5.2. $\qquad\square$

We are now ready to state and proved modified versions of Lemma 4.6 and Corollary 4.13.

Lemma 5.4. *For each $0 < q < p < 1$ there exists a constant $\epsilon = \epsilon(p, q) > 0$ such that the following holds.*

Let $S \subseteq G^{M_0}$ be a fairly long K_0-Schottky set and let μ be a probability measure on G such that

$$\mu \geq p \cdot \left(\text{uniform measure on } \Phi(S^4) \cup \Phi(\check{S}^{(4)}) \right).$$

Then for each n there exist a probability space Ω_n, a measurable subset $B_n \subseteq \Omega_n$, a measurable partition $\mathcal{Q}_n$ of B_n and RVs

$$\{w_i : i = 0, \ldots, m(n)\} \in G^{m(n)+1},$$

$$\mathcal{S} \in \{S, \tilde{S}\}, \qquad \left(m(n) := \lfloor \epsilon n \rfloor \right)$$

$$\{\alpha_i, \beta_i, \gamma_i, \delta_i : i = 1, \ldots, m(n)\} \in \mathcal{S}^{4m(n)}$$

such that the following hold:

(1) $\mathbb{P}(B_n) \geq 1 - (1 - q)^n$ *holds for each n.*

(2) *On each equivalence class $\mathcal{F} \in \mathcal{Q}_n$, the isometries $(w_i)_{i=0}^{m(n)}$ and the Schottky set $\mathcal{S}$ are constant, and $(\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^{m(n)}$ are i.i.d.s distributed according to the uniform measure on $\mathcal{S}^4$.*

(3) *The RV*

$$w_0 \Pi(\alpha_1) \Pi(\beta_1) \Pi(\gamma_1) \Pi(\delta_1) w_1 \cdots \Pi(\alpha_{m(n)}) \Pi(\beta_{m(n)}) \Pi(\gamma_{m(n)}) \Pi(\delta_{m(n)}) w_{m(n)}$$

is distributed according to μ^{*n} on Ω_n.

Proof. Let μ_1 be the uniform measure on $\Phi(S^{(4)})$ and μ_2 be the uniform measure on $\Phi(\check{S}^{(4)})$. Recall that $\Phi(S^4)$ and $\Phi(\check{S}^4)$ are disjoint by Lemma 5.3.(3). Hence we have a decomposition

$$\mu = p \left(\frac{1}{2} \mu_1 + \frac{1}{2} \mu_2 \right) + (1 - p)\nu,$$

Consider:

- Bernoulli random variables $(\rho_i)_{i \geq 0}$ with average p,
- Bernoulli random variables $(\sigma_i)_{i \geq 0}$ with average $1/2$,
- G-valued random variables $(\eta_i)_{i \geq 0}$ with the law μ_1,
- G-valued random variables $(\check{\eta}_i)_{i \geq 0}$ with the law μ_2, and

- G-valued random variables $(\nu_i)_{i \geq 0}$ with the law ν,

all independent. We then define

$$g_k = \begin{cases} \nu_k & \text{when } \rho_k = 0, \\ \eta_k & \text{when } \rho_k = 1 \text{ and } \sigma_k = 0, \\ \check{\eta}_k & \text{when } \rho_k = 1 \text{ and } \sigma_k = 1. \end{cases}$$

Then g_k's are i.i.d. distributed according to μ. Setting $m(n)$ suitably, we define

$$C_n = \left\{ \omega : \sum_{k=1}^n \rho_k(1 - \sigma_k) \geq m(n) \right\}, \quad \check{C}_n = \left\{ \omega : \sum_{k=1}^n \rho_k \sigma_k \geq m(n) \right\}.$$

Note that C_n and $\check{C}_n$ are determined by the values of $\{\rho_k, \sigma_k\}_k$. Moreover, the values of $\{\rho_k, \sigma_k\}_{k>0}$ determine the indices $i \in \{1, \ldots, n\}$ such that $\rho_i = 1$ and $\sigma_i = 0$, and for each $\omega \in C_n$ we can gather the $m(n)$ smallest such positive indices and label them by $\vartheta(1) < \ldots < \vartheta(m(n))$. For each $\omega \in \check{C}_n$, we analogously define $\check{\vartheta}(1) < \ldots < \check{\vartheta}(m(n))$ to be the $m(n)$ smallest indices i such that $\rho_i = 1$ and $\sigma_i = 1$.

Now on C_n, we declare $\mathcal{S} := S$ and let $\mathcal{Q}_n^{(1)}$ be the measurable partition determined by the values of $\{\rho_k, \sigma_k, \nu_k\}_{k>0}$ as well as $\{\eta_k : k > \vartheta(m(n))\}$. We then define

$$w_{i-1} := g_{\vartheta(i-1)+1} \cdots g_{\vartheta(i)-1} \quad (i = 1, \ldots, m(n)),$$
$$w_{m(n)} := g_{\vartheta(m(n))+1} \cdots g_n,$$
$$(\alpha_i, \beta_i, \gamma_i, \delta_i) := \Phi^{-1}(g_{\vartheta(i)}) \quad (i = 1, \ldots, m(n)).$$

Then on each equivalence class in $\mathcal{Q}_n^{(1)}$, w_i's are fixed and $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ are uniformly distributed on $S^{4m(n)}$. Moreover, on B_n we have
(5.2)
$$g_1 \cdots g_n$$
$$= w_0 \Pi(\alpha_1) \Pi(\beta_1) \Pi(\gamma_1) \Pi(\delta_1) w_1 \cdots \Pi(\alpha_{m(n)}) \Pi(\beta_{m(n)}) \Pi(\gamma_{m(n)}) \Pi(\delta_{m(n)}) w_{m(n)}.$$

Similarly, on $\check{C}_n \setminus C_n$, we first declare $\mathcal{S} := \check{S}$ and let $\mathcal{Q}_n^{(2)}$ be the measurable partition determined by the values of $\{\rho_k, \sigma_k, \nu_k\}_{k>0}$ as well as $\{\check{\eta}_k : k > \check{\vartheta}(m(n))\}$. We then define

$$w_{i-1} := g_{\check{\vartheta}(i-1)+1} \cdots g_{\check{\vartheta}(i)-1} \quad (i = 1, \ldots, m(n)),$$
$$w_{m(n)} := g_{\check{\vartheta}(m(n))+1} \cdots g_n,$$
$$(\alpha_i, \beta_i, \gamma_i, \delta_i) := \Phi^{-1}(g_{\check{\vartheta}(i)}) \quad (i = 1, \ldots, m(n)).$$

Then on each equivalence class in $\mathcal{Q}_n^{(2)}$, w_i's are fixed and $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ become i.i.d. distributed according to the uniform measure on $\check{S}^4$. Moreover, Equation 5.2 still holds on $\check{C}_n$. These constructions realize Item (1) and Item (2) of the conclusion on the set $B_n := C_n \cup \check{C}_n$.

Finally, we define w_i's and $\{\alpha_i, \beta_i, \gamma_i, \delta_i\}$ arbitrarily on $(C_n \cup \check{C}_n)^c$ so that Equation 5.2 still holds. Since $g_1 \cdots g_n$ is distributed according to μ^n, we conclude Item (3) of the conclusion.

It remains to precisely choose ϵ and $m(n)$. Note that $\sum_{k=1}^{n} \rho_k < 2m(n)$ holds outside $C_n \cup \check{C}_n$. Chernoff-Hoeffding's inequality as in [Hoe63, Theorem 1] tells us that

$$\mathbb{P}\left(\sum_{k=1}^{n} \rho_k \leq 2\epsilon n \right) \leq \left(f(\epsilon) := \left(\frac{p}{2\epsilon} \right)^{2\epsilon} \cdot \left(\frac{1-p}{1-2\epsilon} \right)^{1-2\epsilon} \right)^n.$$

Since $\lim_{\epsilon \to 0} f(\epsilon) = 1 - p$ is smaller than $1 - q$, we can pick small ϵ so that $\mathbb{P}\left(\sum_{k=1}^{n} \rho_k \leq 2\epsilon n \right) \leq (1-q)^n$ holds. By setting $m(n) = \lfloor \epsilon n \rfloor$, we arrive at the desired conclusion. $\square$

Recall that we obtained Corollary 4.13 by combining Lemma 4.6, 4.8, 4.9 and 4.11. By replacing Lemma 4.6 with Lemma 5.4, we obtain the following:

Corollary 5.5. *For each $0 < q < p < 1$ there exists a constant $\epsilon = \epsilon(p, q) > 0$ and $M = M(p, q) > 0$ such that the following holds.*

Let $S \subseteq G^{M_0}$ be a fairly long K_0-Schottky set with cardinality $N_0 > M$. Let μ be a probability measure on G such that

$$\mu \geq p \cdot \left(\text{uniform measure on } \Phi(S^4) \cup \Phi(\check{S}^{(4)}) \right).$$

Then for each n there exist an integer $m(n)$, a probability space Ω_n with a measurable subset $A_n \subseteq \Omega_n$, a measurable partition $\mathcal{P}_n$ of A_n and RVs

$$Z_n \in G,$$
$$\mathcal{S} \in \{S, \check{S}\},$$
$$\{w_i : i = 0, \ldots, m(n)\} \in G^{m(n)+1},$$
$$\{\beta_i, \gamma_i : i = 1, \ldots, m(n)\} \in S^{2m(n)}$$

such that the following hold:

(1) $\mathbb{P}(A_n) > 1 - 3 \cdot (1-q)^n$ *and $m(n) > \epsilon n$ for eventual n.*
(2) *On A_n, $(w_i)_{i=0}^{m(n)}$ is a D_0-pre-aligned sequence in G and $w_{m(n)}^{-1} w_0$ is a D_0-pre-aligned isometry.*
(3) *On each $\mathcal{E} \in \mathcal{P}_n$, $(w_i)_{i=0}^{m(n)}$ and $\mathcal{S}$ are constant and $(\beta_i, \gamma_i)_{i=1}^{m(n)}$ are i.i.d.s distributed according to the uniform measure on $\tilde{\mathcal{S}}(id)$.*
(4) *Z_n is distributed according to μ^{*n} on Ω_n and*

$$Z_n = w_0 \Pi(\beta_1) v_1 \Pi(\gamma_1) w_1 \cdots \Pi(\beta_{m(n)}) v_{m(n)} \Pi(\gamma_{m(n)}) w_{m(n)}$$

holds on A_n.

Proof. First, given the inputs p and q, Lemma 5.4 provides the constant $\epsilon_1 = \epsilon(p, q)$, and we set $\epsilon = 0.01\epsilon_1$. Now for each n with $N_1 := \lfloor \epsilon_1 n \rfloor$, Lemma 5.4 also provides a probability space Ω_n with a subset $B = B_n$, a measurable partition $\mathcal{Q}_n$ of B_n, a random fairly long K_0-Schottky set

$\mathcal{S} \in \{S, \check{S}\}$ of cardinality N_0, random variables $Z_n \in G$, $(w_i)_{i=0}^{N_1} \in G^{N_1+1}$ and $(\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^{N_1} \in \mathcal{S}^{4N_1}$ such that the following holds:

(1) $\mathbb{P}_n(B) \geq 1 - (1-q)^n$.

(2) On each $\mathcal{F} \in \mathcal{Q}_n$, the sequence $(w_i)_{i=0}^{N_1}$ and the Schottky set $\mathcal{S}$ are constant and $(\alpha_i, \beta_i, \gamma_i, \delta_i)_i$ are i.i.d.s distributed according to the uniform measure on $\mathcal{S}^4$.

(3) Z_n is distributed according to μ^{*n} on Ω_n and

$$Z_n = w_0 \Pi(\alpha_1)\Pi(\beta_1)\Pi(\gamma_1)\Pi(\delta_1)w_1 \cdots \Pi(\alpha_{N_1})\Pi(\beta_{N_1})\Pi(\gamma_{N_1})\Pi(\delta_{N_1})w_{N_1}$$

holds on B.

Now, as we did in the proof of Corollary 4.13, we apply Lemma 4.8; this time, it is with the input probability Dirac_{id} in place of μ, with the input Schottky set $\mathcal{S}$ in place of S, and with the input integer N_1 in place of n. As a result we obtain $N_2 = 2\lfloor 0.25N_1 \rfloor$. Just as in the proof of Corollary 4.13, we obtain a measurable set B' with measurable partition $\mathcal{Q}'$ and RVs $(w_i', \beta_i', \gamma_i')_i$ such that the following holds:

(1) B' has probability at least $\left(1 - (3\sqrt[4]{4/N_0})^{N_1}\right) \cdot \mathbb{P}(B) \geq \left(1 - (3\sqrt[4]{4/N_0})^{N_1}\right) \cdot (1 - (1-q)^n)$,

(2) On B' we have

$$Z_n = w_0' \Pi(\beta_1')\Pi(\gamma_1')w_1' \cdots \Pi(\beta_{N_1}')\Pi(\gamma_{N_2}')w_{N_2}',$$

(3) On each $\mathcal{F}' \in \mathcal{Q}'$, $(w_i')_{i=0}^{N_2}$ is a fixed D_0-pre-aligned sequence in G for $\mathcal{S}$ and $(\beta_i', \gamma_i')_{i=1}^{N_2}$ are i.i.d.s distributed according to the uniform measure on $\tilde{\mathcal{S}}(id)$.

Finally, by setting $m(n) := 2^{\lfloor \log_2 N_2 \rfloor - 1} \geq N_2/4$ and by considering the refinement of $\mathcal{Q}'$ as in the proof of Corollary 4.13, we can realize Item (2), (3), (4) of the conclusion on a subset A_n of B such that

$$\mathbb{P}(A_n) \geq \left(1 - (6/N_0)^{N_2/4}\right) \cdot \mathbb{P}(B').$$

Moreover, note that $m(n) \geq N_2/4 \geq N_1/5 \geq \epsilon n$ for large enough n, which establishes the latter half of Item (1) of the conclusion.

At this point, by requiring

$$N_0 > M := \max\left(\left(\frac{3^4 \cdot 4}{(1-q)^4}\right)^{4/\epsilon_1}, \left(\frac{6}{1-q}\right)^{8/\epsilon_1}\right) + 1,$$

we conclude

$$\mathbb{P}(A_n) \geq 1 - (1-q)^n - (3\sqrt[4]{4/N_0})^{N_1} - (6/N_0)^{N_2/4} \geq 1 - 3 \cdot (1-q)^n$$

for large enough n. This settles Item (1) of the conclusion. $\qquad \square$

We now combine Corollary 5.5 with Lemma 4.10. Note that on A_n, $(w_i)_{i=0}^{m(n)}$ is always a D_0-pre-aligned sequence and $\phi_1 := w_{m(n)}^{-1} w_0$ is a D_0-pre-aligned isometry. Lemma 4.10 then implies the following on A_n:

$$Z_n = w_0 \Pi(\beta_1)\Pi(\gamma_1)w_1 \cdots \Pi(\beta_{m(n)})\Pi(\gamma_{m(n)})w_{m(n)}$$

is a contracting isometry, and moreover, there exist Schottky axes $\kappa_1, \ldots, \kappa_{2m(n)}$ such that

$$\left(\ldots, Z_n^{-1}\kappa_1, \ldots, Z_n^{-1}\kappa_{2m(n)}, \kappa_1, \ldots, \kappa_{2m(n)}, Z_n\kappa_1, \ldots, Z_n\kappa_{2m(n)}, \ldots \right)$$

is D_0-semi-aligned, and that $(o, \kappa_1, \kappa_{2m(n)}, Z_n o)$ is also D_0-semi-aligned. Then Lemma 2.6 implies that for each j, $[o, Z_n^j o]$ contains $2m(n) \cdot j$ disjoint subsegments, each $0.1E_0$-fellow traveling with some $Z_n^l \kappa_i$ ($l = 0, \ldots, j-1, i = 1, \ldots, 2m(n)$). This implies that

$$d(o, W_{m(n)}^j o) \geq \sum_{i=1}^{2m(n)} \sum_{l=0}^{j-1} \left(\mathrm{diam}(Z_n^l \kappa_i) - 0.2E_0 \right) \geq 5E_0 m(n).$$

As a consequence, we obtain the following corollary:

Corollary 5.6. *For each $0 < q < p < 1$ there exists a constant $\epsilon = \epsilon(p, q) > 0$ and $M = M(p, q) > 0$ such that the following holds.*
  Let $S \subseteq G^{M_0}$ be a fairly long K_0-Schottky set with cardinality $N_0 > M$. Let μ be a probability measure on G such that

$$\mu \geq p \cdot \left(\text{uniform measure on } \Phi(S^4) \cup \Phi(\check{S}^{(4)}) \right),$$

and let $(Z_n)_{n>0}$ be the random walk generated by μ. Then

$$\mathbb{P}\left(Z_n \text{ is a contracting element with } \tau(Z_n) \geq 5E_0 \epsilon n \right) \geq 1 - 3(1-q)^n$$

holds for large enough n.

6. COUNTING PROBLEM

We now present a quantitative version of the main theorem in [Cho24].

Theorem 6.1 (Translation length grows linearly)**.** *Let (X, G) be as in Convention 1.1. Then for each $\lambda > 1$, there exists $\lambda_0 > 0$ satisfying the following. Let G be a finitely generated non-elementary subgroup of $\mathrm{Isom}(X)$ and $S' \subseteq G$ be a finite symmetric generating set.*
  Then there exist a subset S'' of G containing S' such that $\#S'' \leq (1 + \lambda)\#S' + \lambda_0$, and a constant $K > 0$, such that for each large n we have

$$\frac{\#\{g \in B_{S''}(n) : g \text{ is not contracting or } \tau_X(g) \leq Kn\}}{\#B_{S''}(n)} \leq Ke^{-n/K}.$$

Proof. First note that we can replace any finite symmetric generating set S' with $S' \cup \{id\}$ and then find S'', at the cost of increasing λ_0 by 1. For this reason, we from now on assume that S' contains id.
  The assumption $\lambda > 1$ implies that $\frac{\lambda}{1+\lambda} > 1/2$, so we can take parameters q and p such that $1/2 < q < p < \frac{\lambda}{1+\lambda}$. Take $\epsilon = \epsilon(p, q)$ and $M = M(p, q)$ as in Corollary 5.6.

Note that the function $f(x) := \frac{x}{x+1}$ is strictly increasing on the positive reals and $\lim_{x \to +\infty} f(x) = 1$. Meanwhile, note the equivalence

$$p(1 + \lambda) < \lambda \Leftrightarrow p < \lambda(1 - p) \Leftrightarrow \frac{p}{\lambda(1 - p)} < 1.$$

Since we took $p < \frac{\lambda}{1+\lambda}$, there exists some $\lambda_0 > M^4$ such that $f(\sqrt[4]{\lambda_0}) > \sqrt[4]{p/\lambda(1-p)}$. Furthermore, since $1/2p$ is also smaller than 1, we can further require that $f(\sqrt[4]{\lambda_0} - 1) > \sqrt[4]{1/2p}$.

Using Lemma 2.9, we take a fairly long Schottky set S in G with cardinality

$$N_0 = \max \left(\left\lfloor \sqrt[4]{\frac{1}{2}\lambda \# S'} \right\rfloor, \sqrt[4]{\lambda_0} \right).$$

Let K_0 be the Schottky parameter of S and D_0, D_1, E_0 be the associated constants.

Note that

$$2N_0^4 \le 2 \max \left(\frac{1}{2}\lambda \# S', \lambda_0 \right) \le \lambda \# S' + \lambda_0.$$

Meanwhile, using the monotonicity of $f(x) = \frac{x}{x+1}$ we deduce

$$(6.1) \qquad \frac{N_0^4}{(N_0 + 1)^4} = \left(f(N_0)\right)^4 \ge \left(f\left(\sqrt[4]{\lambda_0}\right)\right)^4 \ge \frac{p}{\lambda(1 - p)},$$

$$(6.2) \qquad \frac{(N_0 - 1)^4}{N_0^4} = \left(f(N_0 - 1)\right)^4 \ge \left(f\left(\sqrt[4]{\lambda_0} - 1\right)\right)^4 \ge 1/2p.$$

Note that Inequality 6.1 implies

$$2N_0^4 = 2(N_0 + 1)^4 \cdot \frac{N_0^4}{(N_0 + 1)^4} \ge \lambda \# S' \cdot \frac{p}{\lambda(1 - p)} \ge \frac{p}{1 - p} \# S'.$$

With these estimates, we set $S'' = S' \cup \Phi(S^4) \cup \Phi(\check{S}^4)$, which is still a symmetric set. Moreover, we have

$$\#S'' \le \#S' + 2(\#S)^4 = \#S' + 2N_0^4 \le (1 + \lambda)\#S'^2 + \lambda_0$$

and

$$\frac{\# \left(\Phi(S^4) \cup \Phi(\check{S}^4)\right)}{\#S''} = \frac{2N_0^4}{\#S''} \ge \frac{2N_0^4}{2N_0^4 + \#S'} \ge \frac{1}{1 + \frac{1-p}{p}} = p.$$

This implies that the uniform measure μ on S'' satisfies the assumption of Corollary 5.6. At this moment, let

$$\mathcal{A}_n = \{g \in G : g \text{ is contracting and } \tau(g) \ge 5E_0\epsilon n\}.$$

Then Corollary 5.6 tells us that, for large enough n, the n-th step Z_n of the μ-random walk is in $\mathcal{A}_n$ except for probability at most $3(1 - q)^n$.

Meanwhile, each element $g \in B_{S''}(id, n)$ arises as an n-th step Z_n of the μ-random walk, with probability at least $(\#S'')^{-n}$. Hence, we have

$$\# \left(B_{S''}(id, n) \setminus \mathcal{A}_n\right) \le \frac{\mathbb{P}(Z_n \notin \mathcal{A}_n)}{(\#S'')^{-n}} \le 3(\#S'')^n(1 - q)^n.$$

Meanwhile, let $\alpha_1, \ldots, \alpha_{4n}$ of S be such that $\alpha_i \neq \alpha_{i+1}$ for $i = 1, \ldots, 4n - 1$, and let $\epsilon_1, \ldots, \epsilon_n \in \{1, -1\}$. Lemma 5.2.(2) guarantees that the isometries

$$\prod_{i=1}^{4n} \Pi(\alpha_i^{\epsilon_{\lceil i/4 \rceil}}) = \Pi(\alpha_1^{\epsilon_1}) \cdots \Pi(\alpha_4^{\epsilon_1}) \cdot \Pi(\alpha_5^{\epsilon_2}) \cdots \Pi(\alpha_8^{\epsilon_2}) \cdots \Pi(\alpha_{4n-3}^{\epsilon_n}) \cdots \Pi(\alpha_{4n}^{\epsilon_n})$$

are distinct for different choices of such $\big((\alpha_i)_{i=1}^{4n}, (\epsilon_j)_{j=1}^{n}\big)$'s. All these isometries are contained in $B_{S''}(id, n)$. The number of such elements is at least

$$(\#S - 1)^{4n} \cdot 2^n = 2^n (N_0 - 1)^{4n} = \left(\frac{2N_0^4}{\#S''} \cdot (\#S'') \cdot \left(\frac{N_0 - 1}{N_0} \right)^4 \right)^n$$

$$\geq \left(p \cdot \frac{1}{2p} \cdot (\#S'') \right)^n. \qquad (\because \text{Inequality } 6.2)$$

Hence, for large enough n we have

$$\frac{\#(B_{S''}(id, n) \setminus \mathcal{A}_n)}{\#B_{S''}(id, n)} \leq \frac{3(1-q)^n (\#S'')^n}{(1/2)^n (\#S'')^n} = 3 \cdot (2(1-q))^n$$

This decays exponentially since $1 - q < 1/2$ as desired. $\qquad \square$

Using a similar argument that involves pivoting for quasi-isometric embedding of k independent random walks (Lemma 4.17 and 4.18), we can deduce the following version of Theorem E:

Theorem 6.2. *For each $k \in \mathbb{Z}_{>0}$ and $\lambda > 1$, there exists $\lambda_0 > 0$ satisfying the following. Let G be a finitely generated non-elementary subgroup of $\mathrm{Isom}(X)$ and $S' \subseteq G$ be a finite symmetric generating set.*

Then there exists a set $S'' \supseteq S'$ of G with $\#S' \leq (1+\lambda)\#S' + \lambda_0$ such that for all k-tuples $(g_1, \ldots, g_k)$ of elements in $B_{S''}(n)$ except an exponentially decaying proportion, $\langle g_1, \ldots, g_k \rangle$ is q.i. embedded into a quasi-convex subset of X.

References

[ACGH19] Goulnara N. Arzhantseva, Christopher H. Cashen, Dominik Gruber, and David Hume. Negative curvature in graphical small cancellation groups. *Groups Geom. Dyn.*, 13(2):579–632, 2019.

[AK11] Yael Algom-Kfir. Strongly contracting geodesics in outer space. *Geom. Topol.*, 15(4):2181–2233, 2011.

[BCK23] Hyungryul Baik, Inhyeok Choi, and Dongryul Kim. Linear growth of translation lengths of random isometries on Gromov hyperbolic spaces and Teichmüller spaces. *Journal of the Institute of Mathematics of Jussieu*, page 1–45, 2023.

[BF09] Mladen Bestvina and Koji Fujiwara. A characterization of higher rank symmetric spaces via bounded cohomology. *Geom. Funct. Anal.*, 19(1):11–40, 2009.

[BQ16a] Yves Benoist and Jean-François Quint. Central limit theorem for linear groups. *Ann. Probab.*, 44(2):1308–1340, 2016.

[BQ16b] Yves Benoist and Jean-François Quint. Central limit theorem on hyperbolic groups. *Izv. Ross. Akad. Nauk Ser. Mat.*, 80(1):5–26, 2016.

[Cho22a]   Inhyeok Choi. Limit laws on Outer space, Teichmüller space, and CAT(0) spaces. *arXiv preprint arXiv:2207.06597v1*, 2022.

[Cho22b]   Inhyeok Choi. Random walks and contracting elements I: Deviation inequality and limit laws. *arXiv preprint arXiv:2207.06597v2*, 2022.

[Cho23]   Inhyeok Choi. Central limit theorem and geodesic tracking on hyperbolic spaces and teichmüller spaces. *Adv. Math.*, 431:109236, 2023.

[Cho24]   Inhyeok Choi. Pseudo-Anosovs are exponentially generic in mapping class groups. *Geom. Top.*, 28:1923–1955, 2024.

[DH18]   François Dahmani and Camille Horbez. Spectral theorems for random walks on mapping class groups and $\mathrm{Out}(F_N)$. *Int. Math. Res. Not. IMRN*, (9):2693–2744, 2018.

[Far06]   Benson Farb. Some problems on mapping class groups and moduli space. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 11–55. Amer. Math. Soc., Providence, RI, 2006.

[Gou22]   Sébastien Gouëzel. Exponential bounds for random walks on hyperbolic spaces without moment conditions. *Tunis. J. Math.*, 4(4):635–671, 2022.

[GS21]   Antoine Goldsborough and Alessandro Sisto. Markov chains on hyperbolic-like groups and quasi-isometries. *arXiv preprint arXiv:2111.09837*, 2021.

[Hoe63]   Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.

[Hor18]   Camille Horbez. Central limit theorems for mapping class groups and $\mathrm{Out}(F_N)$. *Geom. Topol.*, 22(1):105–156, 2018.

[HY21]   Suzhen Han and Wen-Yuan Yang. Generic free subgroups and statistical hyperbolicity. *Groups Geom. Dyn.*, 15(1):101–140, 2021.

[LB22]   Corentin Le Bars. Central limit theorem on CAT(0) spaces with contracting isometries. *arXiv preprint arXiv:2209.11648*, 2022.

[Mah12]   Joseph Maher. Exponential decay in the mapping class group. *J. Lond. Math. Soc. (2)*, 86(2):366–386, 2012.

[Min96]   Yair N. Minsky. Quasi-projections in Teichmüller space. *J. Reine Angew. Math.*, 473:121–136, 1996.

[MS19]   Joseph Maher and Alessandro Sisto. Random subgroups of acylindrically hyperbolic groups and hyperbolic embeddings. *Int. Math. Res. Not. IMRN*, (13):3941–3980, 2019.

[MT18]   Joseph Maher and Giulio Tiozzo. Random walks on weakly hyperbolic groups. *J. Reine Angew. Math.*, 742:187–239, 2018.

[MT21]   Joseph Maher and Giulio Tiozzo. Random walks, WPD actions, and the Cremona group. *Proc. Lond. Math. Soc. (3)*, 123(2):153–202, 2021.

[Sis13]   Alessandro Sisto. Projections and relative hyperbolicity. *Enseign. Math. (2)*, 59(1-2):165–181, 2013.

[Sis18]   Alessandro Sisto. Contracting elements and random walks. *J. Reine Angew. Math.*, 742:79–114, 2018.

[TT16]   Samuel J. Taylor and Giulio Tiozzo. Random extensions of free groups and surface groups are hyperbolic. *Int. Math. Res. Not. IMRN*, (1):294–310, 2016.

[Yan19]   Wen-yuan Yang. Statistically convex-cocompact actions of groups with contracting elements. *Int. Math. Res. Not. IMRN*, (23):7259–7323, 2019.

[Yan20]   Wen-yuan Yang. Genericity of contracting elements in groups. *Math. Ann.*, 376(3-4):823–861, 2020.

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