A Math Review: Sets, Functions, Permutations, Combinations, and Notation

A.1 Introduction

In this appendix we review basic results concerning set theory, relations and functions, combinations and permutations, and summation and integration notation. We also review the meaning of the terms *definition, axiom, theorem, corollary, and lemma*, which are labels that are affixed to a myriad of statements and results that constitute the theory of probability and mathematical statistics. The topics reviewed in this appendix constitute basic foundational material on which our study of mathematical statistics will be based. Additional mathematical results, often of a more advanced nature, are introduced throughout the text as the need arises.

A.2 Definitions, Axioms, Theorems, Corollaries, and Lemmas

The development of the theory of probability and mathematical statistics involves a considerable number of statements consisting of definitions, axioms, theorems, corollaries, and lemmas. These terms will be used for organizing the various statements and results we will examine into these categories:

1. descriptions of meaning;
2. statements that are acceptable as true without proof;

3. formulas or statements that require proof of validity;
4. formulas or statements whose validity follows immediately from other true formulas or statements; and
5. results, generally from other branches of mathematics, whose primary purpose is to facilitate the proof of validity of formulas or statements in mathematical statistics.

More formally, we present the following meaning of the terms:

Definition A statement of the meaning of a word, word group, sign, or symbol.

Axiom (or postulate) A statement that has found general acceptance, or is thought to be worthy thereof, on the basis of an appeal to intrinsic merit or self-evidence and thus requires no proof of validity.

Theorem (or proposition) A formula or statement that is deduced from other proved or accepted formulas or statements and whose validity is thereby proved.

Corollary A formula or statement that is immediately deducible from a proven theorem and that requires little or no additional proof of validity.

Lemma An auxiliary proposition that has been proved either by the user or elsewhere and that is stated for the expressed purpose of immediate use in the proof of another proposition.

Thus, in the development of the theory of probability and mathematical statistics, axioms are the fundamental truths that are to be accepted at face value and not proven. Theorems and their corollaries are statements deducible from the fundamental truths and other proven statements and thus are *derived truths*. Lemmas represent results that facilitate proofs of other statements that are of more fundamental interest.

We elaborate on the concept of a lemma, since our discussions will implicitly rely on lemmas more than any other type of statement, but we will generally choose not to exhaustively catalogue lemmas in the discussions. What constitutes a lemma and what does not depends on the problem context or on one's point of view. A fundamental integration result from calculus could technically be referred to as a lemma when used in a proof of a statement in mathematical statistics, while in the study of calculus, it might be referred to as a theorem to be proved in and of itself. Since our study will require numerous auxiliary results from algebra, calculus, and matrix theory, exhaustively cataloguing these results as lemmas would be cumbersome and, more importantly, not necessary given the prerequisites assumed for this course of study, namely, a familiarity with the basic concepts of algebra, univariate and multivariate calculus, and an introduction to matrix theory. We will have occasion to state a number of lemmas, but we will generally reserve this label for more exotic mathematical results that fall outside the realm of mathematics encompassed by the prerequisites.

A.3 Elements of Set Theory

In the study of probability and mathematical statistics, sets are the fundamental objects to which probability will be assigned, and it is important that the concept of a set, and operations on sets, be well understood. In this section we review some basic properties of and operations on sets. This begs the following question: What is meant by the term *set*? In modern axiomatic developments of set theory, the concept of a set is taken to be primitive and incapable of being defined in terms of more basic ideas. For our purposes, a more intuitive notion of a set will suffice, and we avoid the complexity of an axiomatic development of the theory (see Marsden,[1] Appendix A, for a brief introduction to the axiomatic development). We base our definition of a set on the intuitive definition originally proposed by the founder of set theory, Georg Cantor (1845–1918).[2]

Definition A.1
Set

A **set** is a collection of objects with the following characteristics:

1. All objects in the collection are *clearly defined*, so that it is evident which objects are members of the collection and which are not;
2. All objects are *distinguishable*, so that objects in the collection do not appear more than once;
3. *Order is irrelevant* regarding the listing of objects in the collection, so two collections that contain the same objects but are listed in different order are nonetheless the *same set*; and
4. *Objects in the collection can be sets themselves*, so that a *set of sets* can be defined.

The objects in the collection of objects comprising a set are its **elements**. The term **members** is also used to refer to the objects in the collection. In order to signify that an object belongs to a given set, the symbol $\in$, will be used in an expression such as $\mathbf{x} \in \mathbf{A}$, which is to be read "$x$ is an element (or member) of the set A." If an object is *not* a member of a given set, then a slash will be used as $x \notin A$ to denote that "x is not an element (or member) of the set A." Note that the slash symbol, /, is used to indicate negation of a relationship. The characteristics of sets presented in Def. A.1 will be clarified and elaborated upon in examples and discussions provided in subsequent subsections.

[1] J.E. Marsden, (1974), *Elementary Classical Analysis*, San Francisco: Freeman and Co.

[2] Cantor's original text reads: "Unter einer 'Menge' verstehen wir jede Zusammenfassung *M* von bestimmten wohlunterschiedenen Objekten *m* unserer Anschauung oder unseres Denkens (welche die 'Elemente' von *M* genannt werden) zu einem ganzen" (*Collected Papers*, p. 282). Our translation of Cantor's definition is, "By set we mean any collection, *M*, of clearly defined, distinguishable objects, *m* (which will be called elements of *M*), which from our perspective or through our reasoning we understand to be a whole."

Set-Defining Methods

Three basic methods are used in defining the objects in the collection constituting a given set: (1) **exhaustive listing**; (2) **verbal rule**; and (3) **mathematical rule**. An *exhaustive listing* requires that each and every object in a collection be individually identified either numerically, if the set is a collection of numbers, or by an explicit verbal description, if the collection is not of numbers. The object descriptions are conventionally separated by commas, and the entire group of descriptions is enclosed in brackets. The following are examples of sets defined by an exhaustive listing of the objects that are elements of the set:

Example A.1 $S_1 = \{\text{HEAD}, \text{TAIL}\}$

Here S_1 is the set of possible occurrences when tossing a coin into the air and observing its resting position. Note the set can be equivalently represented as $S_1 = \{\text{TAIL}, \text{HEAD}\}$. □

Example A.2 $S_2 = \{1, 2, 3, 4, 5, 6\}$

Here S_2 is the set of positive integers from 1 to 6. Note that the set S_2 can be equivalently represented by listing the positive integers 1 to 6 in any order. □

A *verbal rule* is a verbal statement of characteristics that only the objects that are elements of a given set possess and that can be used as a test to determine set membership. The general form of the verbal rule is $\{x$: *verbal statement*$\}$, which is to be read "the collection of all x for which *verbal statement* is true." The following are examples of sets described by verbal rules:

Example A.3 $S_3 = \{x : x \text{ is a college student}\}$

Here S_3 is the set of college students. An individual is an element of the set S_3 **iff** (if and only if) he or she is a college student. □

Example A.4 $S_4 = \{x : x \text{ is a positive integer}\}$

Here S_4 is the set of positive integers 1,2,3, ... A number is an element of the set S_4 iff it is a positive integer. □

A *mathematical rule* is of the same general form as a verbal rule, except the verbal statement is replaced by a mathematical expression of some type. The general form of the mathematical rule is $\{x$: *mathematical expression*$\}$, which is to be read "the collection of all x for which *mathematical expression* is true." The following are examples of sets described by mathematical rules:

Example A.5 $S_5 = \{x : x = 2k + 1, k = 0, 1, 2, 3, \ldots\}$

Here S_5 is the set of odd positive integers. A number is an element of the set S_5 iff the number is equal to $2k + 1$ for some choice of $k = 0, 1, 2, 3, \ldots$. □

Example A.6 $S_6 = \{x : 0 \leq x \leq 1\}$

Here S_6 is the set of numbers greater than or equal to 0 but less than or equal to 1. A number is an element of the set S_6 iff it is neither less than 0 nor greater than 1. □

The choice of method for describing the objects that constitute elements of a set depends on what is convenient and/or feasible for the case at hand. For example, exhaustive listing of the elements in set S_6 is impossible. On the other hand, there is some discretion that can be exercised, since, for example, a verbal rule could have adequately described the set S_5, say as $S_5 = \{x : x \text{ is an odd positive integer}\}$. A mixing of the basic methods might also be used, such as $S_5 = \{x : x = 2k+1, k \text{ is zero or a positive integer}\}$. One can choose whatever method appears most useful in a given problem context.

Note that although our preceding examples of verbal and mathematical rules treat x as inherently one-dimensional, a vector interpretation of x is clearly permissible. For example, we can represent the set of points on the boundary or interior of a circle centered at (0,0) and having radius 1 as

$$S_7 = \left\{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\right\},$$

or we can represent the set of input–output combinations associated with a two-input Cobb–Douglas production function as

$$S_8 = \left\{(y, x_1, x_2) : y = b_0 x_1^{b_1} x_2^{b_2}, x_1 \geq 0, x_2 \geq 0\right\}$$

for given numerical values of b_0, b_1, and b_2. Of course, the entries in the x-vector need not be numbers, as in the set

$$S_9 = \{(x_1, x_2) : x_1 \text{ is an economist}, x_2 \text{ is an accountant}\}.$$

Set Classifications

Sets are classified according to the number of elements they contain and whether the elements are countable. We differentiate between sets that have a finite number of elements and sets whose elements are infinite in number, referring to a set of the former type as a **finite set** and a set of the latter type as an **infinite set**. In terms of countability, sets are classified as being either countable or uncountable. Note that when we count objects, we intuitively place the objects in a one-to-one correspondence with the positive integers, i.e., we identify objects one by one and count "1, 2, 3, 4, ..." Thus, a **countable set** is one whose elements can be placed in a one-to-one correspondence with some or all of the positive integers–any other set is referred to as an **uncountable set**.

A finite set is, of course, always countable, and thus it would be redundant to use the term "countable finite set." Sets are thus either **finite, countably infinite**, or **uncountably infinite**. Of the sets S_1 through S_9 described earlier, S_1, S_2, S_3, and S_9 are finite, S_4 and S_5 are countably infinite, and S_6, S_7, and S_8 are uncountably infinite (why?).

Special Sets, Set Operations, and Set Relationships

We now proceed to a number of definitions and illustrations establishing relationships between sets, mathematical operations on sets, and the notions of the universal and empty sets.

Definition A.2
Subset

A is a **subset** of B, denoted as $A \subset B$ and read A *is contained in* B, iff every element of A is also an element of B.

Definition A.3
Equality of sets

Set A is equal to set B, denoted as $A = B$, iff every element of A is also an element of B, and every element of B is also an element of A, i.e., iff $A \subset B$ and $B \subset A$.

Definition A.4
Universal set

The set containing all objects under consideration in a given problem setting, and from which all subsets are extracted, is the **universal set.**

Definition A.5
Empty or null set

The set containing no elements, denoted by $\emptyset$, is called the **empty**, or **null set.**

Definition A.6
Set difference

Given any two subsets A and B of a universal set, the set of all elements in A that are *not* in B is called the **set difference** between A and B and is denoted by $A - B$. If $A \subset B$, then $A - B = \emptyset$

Definition A.7
Complement

Let A be a subset of a universal set, Ω. The **complement** of the set A is the set of all elements in Ω that are not in A, and is denoted by $\bar{A}$. Equivalently, $\bar{A} = \Omega - A$.

Definition A.8
Union

Let A and B be any two subsets of a universal set, Ω. Then the **union** of the sets A and B is the set of all elements in Ω that are in *at least one* of the sets A or B; it is denoted by $A \cup B$.

Definition A.9
Intersection

Let A and B be any two subsets of a specified universal set, Ω. Then the **intersection** of the sets A and B is the set of all elements in Ω that are *in both* sets A and B, it is denoted by $A \cap B$

Definition A.10
Mutually exclusive (or disjoint) sets

Subsets A and B of a universal set, Ω, are said to be **mutually exclusive** or **disjoint** sets iff they have no elements in common, i.e., iff $A \cap B = \emptyset$.

We continue to use the slash, /, to indicate negation of a relationship (recall that / was previously used to indicate the negation of $\in$). Thus, $A \not\subset B$ denotes that A is not a subset of B, and $A \neq B$ denotes that A is not equal to B. We note here (and we shall state later as a theorem) that it is a logical requirement that $\emptyset$ is a subset of any set A, since if $\emptyset$ does not contain any elements, it cannot be the case that $\emptyset \not\subset A$, since the negation of $\subset$ would require the existence of an element in $\emptyset$ that was not in A.

Example A.7 Let the universal set be defined as $\Omega = \{x\colon\ 0 \leq x \leq 1\}$, and define three additional sets as

$$A = \{x : 0 \leq x \leq .5\}, \qquad B = \{x : .25 \leq x \leq .75\} \qquad \text{and,} \qquad C = \{x : .75 < x \leq 1\}.$$

Then we can establish the following set relationships:

$$\bar{B} = \{x : 0 \leq x < .25 \text{ or } .75 < x \leq 1\},$$
$$A \cup C = \{x : 0 \leq x \leq .5 \text{ or } .75 < x \leq 1\},$$
$$\bar{B} \subset A \cup C,$$
$$C \cap A = C \cap B = \emptyset,$$
$$\bar{C} = A \cup B = \{x : 0 \leq x \leq .75\},$$
$$A - B = \{x : 0 \leq x < .25\},$$
$$A \cap B = \{x : .25 \leq x \leq .5\}.$$
□

Note that although our definitions of subset, equality of sets, set difference, complement, union, and intersection explicitly involve only two sets, A and B, it is implicit that the concepts can be applied to more complicated expressions involving an arbitrary number of sets. For example, since $A \cap B$ is itself a set, we can form its intersection with a set C as $(A \cap B) \cap C$, or form the set difference, $(A \cap B) - C$, or establish that $(A \cap B) =$ or $\neq C$, and so on. The point is that the concepts apply to sets, which themselves may have been constructed from other sets via various set operations.

Example A.8 Let Ω, A, B, and C be defined as in Ex. A.7. Then the following set relationships can be established:

$$A \cup B \cup C = \Omega = \{x : 0 \leq x \leq 1\},$$
$$(A \cup C) \cap B = \{x : .25 \leq x \leq .5\},$$
$$(A \cap B) \cap C = \overline{(A \cup B)} \cap \bar{C} = \emptyset,$$
$$(B \cup C) - A = \{x : .5 < x \leq 1\},$$
$$((A \cup B) - (A \cap B)) \subset \bar{C} \cap (\bar{A} \cup \bar{B}).$$

Can $\subset$ be replaced with $=$ in the last relationship? □

It is sometimes useful to conceptualize set relationships through illustrations called *Venn diagrams* (named after the nineteenth-century English logician, John Venn). In a Venn diagram, the universal set is generally denoted by a rectangle, with subsets of the universal set represented by various geometric shapes located within the bounds of the rectangle. Figure A.1 uses Venn diagrams to illustrate the set relationships defined previously.

Rules Governing Set Operations

Operations on sets must satisfy a number of basic rules. We state these basic rules as theorems, although we will not take the time to prove them here. The reader may wish to verify the plausibility of some of the theorems through the use of Venn diagrams. One of DeMorgan's laws will be proved to illustrate the formal proof method for the interested reader.

Theorem A.1 (Idempotency Laws)

$A \cup A = A \quad \textit{and} \quad A \cap A = A$

Theorem A.2 (Commutative Laws)

$A \cup B = B \cup A \quad \textit{and} \quad A \cap B = B \cap A$

Theorem A.3 (Associative Laws)

$(A \cup B) \cup C = A \cup (B \cup C) \quad \textit{and} \quad (A \cap B) \cap C = A \cap (B \cap C)$

Theorem A.4 (Distributive Laws)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \textit{and} \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Theorem A.5 (Identity Elements $\cap$ and $\cup$)

$A \cap \Omega = A$ (Ω *is the identity element for* $\cap$)

$A \cup \emptyset = A$ ($\emptyset$ *is the identity element for* $\cup$)

Theorem A.6 (Intersection and Union of Complements)

$A \cup \bar{A} = \Omega \quad \textit{and} \quad A \cap \bar{A} = \emptyset$

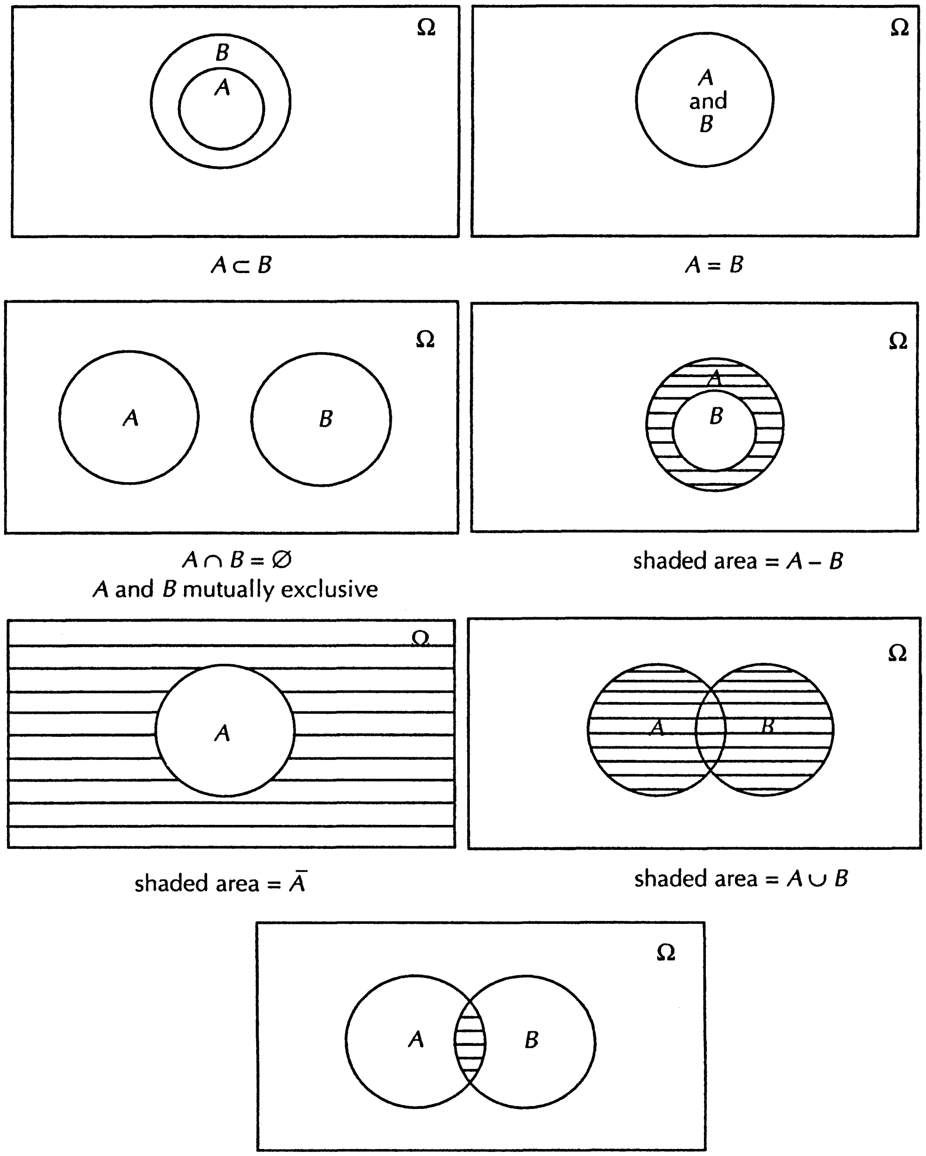
Theorem A.7 (Complements of Complements)

$(\bar{\bar{A}}) = A$

Theorem A.8 (Intersection with the Null Set)

$A \cap \emptyset = \emptyset$

Theorem A.9 (Null Set as a Subset) *If* A *is any set, then* $\emptyset \subset A$



$A \subset B$ $\quad$ $A = B$

$A \cap B = \varnothing$
A and B mutually exclusive

shaded area = $A - B$

shaded area = $\bar{A}$

shaded area = $A \cup B$

shaded area = $A \cap B$

Figure A-1
Venn diagrams illustrating set relationships.

Theorem A.10 (DeMorgan's Laws)

$$(\overline{A \cup B}) = \bar{A} \cap \bar{B} \quad \textit{and} \quad (\overline{A \cap B}) = \bar{A} \cup \bar{B}$$

Example A.9 Formal Proof of $(\overline{A \cap B}) = \bar{A} \cup \bar{B}$

By definition of the equality of sets, two sets are equal iff each is contained in the other. We first demonstrate that $\overline{A \cap B} \subset \bar{A} \cup \bar{B}$. By definition, $x \in \overline{A \cap B}$

implies that $x \notin A \cap B$. Suppose $x \notin \bar{A} \cup \bar{B}$. This implies $x \notin \bar{A}$ and $x \notin \bar{B}$, which implies $x \in A$ and $x \in B$, i.e., $x \in A \cap B$, a contradiction. Therefore, if $x \in \overline{A \cap B}$, then $x \in \bar{A} \cup \bar{B}$, which implies $\overline{A \cap B} \subset \bar{A} \cup \bar{B}$. We next demonstrate that $\bar{A} \cup \bar{B} \subset \overline{A \cap B}$. Let $x \in \bar{A} \cup \bar{B}$. Then $x \notin A \cap B$, for if it were, then $x \in A$ and $x \in B$, contradicting that x belongs to at least one of $\bar{A}$ and $\bar{B}$. However, $x \notin A \cap B$ implies $x \in \overline{A \cap B}$, and thus $\bar{A} \cup \bar{B} \subset \overline{A \cap B}$. □

We remind the reader that since the sets used in Theorems A.1–A.10. could themselves be the result of set operations applied to other sets, the theorems are extendable in a myriad of ways to involve an arbitrary number of sets. For example, in the first of DeMorgan's laws listed in Theorem A.10, if $A = C \cup D$ and $B = E \cup F$, then by substitution,

$$(\overline{C \cup D \cup E \cup F}) = (\overline{C \cup D}) \cap (\overline{E \cup F}).$$

Then by applying Theorem A.10 to both $(C \cup D)$ and $(E \cup F)$, we obtain a generalization of DeMorgan's law as

$$(\overline{C \cup D \cup E \cup F}) = \bar{C} \cap \bar{D} \cap \bar{E} \cap \bar{F}.$$

Given the wide range of extensions that are possible, Theorems A.1–A.10 provide a surprisingly broad conceptual foundation for applying the rules governing set operations.

Some Useful Set Notation Situations sometimes arise in which one is required to denote the union or intersection of a large number of sets. A convenient notation that represents such unions or intersections quite efficiently is available. Two types of notations are generally used, and they are differentiated on the basis of whether the union or intersection is of sets identified by a natural sequence of integer subscripts or whether the sets are identified by subscripts, say i's, that are elements of some set of subscripts, I, called an **index set**.

Definition A.11
Multiple union notation

a. $\cup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \ \ldots \ \cup A_n$.

b. $\cup_{i \in I} A_i =$ union of all sets A_i for which $i \in I$.

Definition A.12
Multiple intersection notation

a. $\cap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \ \ldots \ \cap A_n$.

b. $\cap_{i \in I} A_i =$ intersection of all sets A_i for which $i \in I$.

Example A.10 Let the universal set be defined as $\Omega = \{x\colon 0 \le x \le 1\}$, and examine the following subsets of $\boldsymbol{\Omega}$:

$$A_1 = \{x\colon 0 \le x \le .25\}, \qquad A_2 = \{x\colon 0 \le x \le .5\},$$

$$A_3 = \{x\colon 0 \le x \le .75\}, \qquad A_4 = \{x\colon .75 \le x \le 1\}.$$

Define the index sets I_1 and I_2 as

$$I_1 = \{1, 3\} \quad \text{and} \quad I_2 = \{1, 3, 4\}.$$

Then,

$\cup_{i=1}^4 A_i = \cup_{i=2}^4 A_i = \cup_{i=3}^4 A_i = \{x\colon 0 \le x \le 1\} = \Omega,$

$\cup_{i \in I_1} A_i = A_1 \cup A_3 = \{x\colon 0 \le x \le .75\},$

$\cup_{i \in I_2} A_i = A_1 \cup A_3 \cup A_4 = \{x\colon 0 \le x \le 1\} = \Omega,$

$\cap_{i=1}^4 A_i = \cap_{i=2}^4 A_i = \emptyset,$

$\cap_{i=3}^4 A_i = \{.75\},$

$\cap_{i \in I_1} A_i = A_1 \cap A_3 = \{x\colon 0 \le x \le .25\},$

$\cap_{i \in I_2} A_i = A_1 \cap A_3 \cap A_4 = \emptyset.$ □

Whenever a set A is an interval subset of the *real line* (where the **real line** refers to all of the numbers between $-\infty$ and ∞), the set can be indicated in abbreviated form by the standard notation for intervals, stated in the following definition.

Definition A.13
Interval set notation

Let a and b be two numbers on the real line for which $a < b$. Then the following four sets, called intervals with endpoints a and b, can be defined as

a. **Closed interval:**

$[a, b] = \{x\colon a \le x \le b\},$

b. **Half-open (or half-closed) intervals:**

$(a, b] = \{x\colon a < x \le b\}$, and

$[a, b) = \{x\colon a \le x < b\},$

c. **Open interval:**

$(a, b) = \{x\colon a < x < b\}.$

Note that *weak inequalities*, $x \le$ or $\le x$, are signified by brackets] or [, respectively. *Strong inequalities*, $x <$ *or* $< x$, are signified by parentheses,) or (, respectively. Note further that whether the interval set contains its endpoints determines whether the set is closed.

As we have already done, (x, y) will also be used to denote coordinates in the two-dimensional plane. The context of the discussion will make clear whether we are referring to an open interval (a, b) or a pair of coordinates (x, y).

A.4 Relations, Point Functions, and Set Functions

The concepts of point function and set function are central to our discussion of probability and statistics. We will see that probabilities can be represented by set functions and that in a large number of cases of practical interest, set functions can in turn be represented by a summation or integration operation applied to point functions. While readers may be somewhat familiar with point functions from introductory courses in algebra, the concept of a set function may not be familiar. We will review both function concepts within the broader context of the theory of relations. The relations context facilitates the presentation of a very general definition of "function" in which inputs into and outputs from the function may be objects of any kind, including, but not limited to, numbers. The relations context also facilitates a demonstration of the significant similarities between the concept of a set function and the more familiar point function concept.

Cartesian Product

The concept of a relation can be made clear once we define what is meant by the Cartesian product of two sets A and B, named after the French mathematician René Descartes (1596–1650).

Definition A.14 *Cartesian product of A and B*

> Let A and B be two sets. Then the **Cartesian product** of A and B, denoted as $A \times B$, is the set of ordered pairs
>
> $A \times B = \{(x, y): x \in A, y \in B\}$.

In words, $A \times B$ is the set of all possible pairs (x, y) such that x is an element of the set A and y is an element of the set B. Note carefully that the pairs are *ordered* in the sense that the first object in the pair must come from set A and the second object from set B.

Example A.11 Let $A = \{x: 1 \leq x \leq 2\}$ and $B = \{y: 2 \leq y \leq 4\}$. Then $A \times B = \{(x, y): 1 \leq x \leq 2$ and $2 \leq y \leq 4\}$ (see Figure A.2). □

Example A.12 Let $A = \{x: x$ is a man$\}$ and $B = \{y: y$ is a woman$\}$. Then $A \times B = \{(x, y): x$ is a man and y is a woman$\}$, which is the set of all possible man–woman pairings. □

We will have use for a more general notion of Cartesian product involving more than just two sets. The extension is given in the next definition.

Definition A.15 *Cartesian product (general)*

> Let $A_1, \ldots, A_n$ be n sets. Then the Cartesian product of $A_1, \ldots, A_n$ is the set of ordered n-tuples
>
> $\times_{i=1}^{n} A_i = A_1 \times A_2 \times \ldots \times A_n = \{(x_1, \ldots, x_n): x_i \in A_i, i = 1, \ldots, n\}$.

In words, $\times_{i=1}^{n} A_i$ is the set of all possible n-tuples $(x_1, \ldots, x_n)$ such that x_1 is an element of set A_1, x_2 is an element of set A_2, and so on. Note that should the

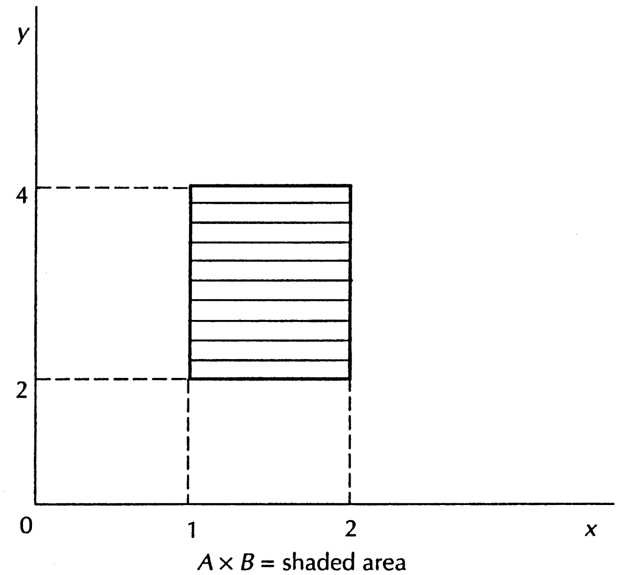


Figure A-2
$A \times B$ = shaded area.

need arise, a general Cartesian product of sets could also be represented by the notation $\times_{i \in I} A_i$, where here the product is taken over all sets having subscript i in the index set I (recall Def. A.11 and Def. A.12, and the use of index set notation).

In certain cases, we may be interested in forming a Cartesian product of a set A with itself. While we might represent such a Cartesian product by the notation

$$\times_{i=1}^{n} A = \{(x_1, \ldots, x_n) : x_i \in A, i = 1, \ldots, n\},$$

such a Cartesian product is generally denoted by A^n, and so, for example, $A^2 = A \times A$.

Relation (Binary)

We now define what we mean by the term *binary relation.*

Definition A.16
Binary Relation

Any subset of the Cartesian product $A \times B$ is a **binary relation from A to B.**

Note that the adjective *binary* signifies that only two sets are involved in the relation. Relations that involve more than two sets can be defined, but for our purposes the concept of a binary relation will suffice.[3] Henceforth, we will use the word *relation* to mean binary relation. We should also mention that in the case where $B = A$, we will simply remain consistent with Def. A.16 and

[3]A higher-order relation could be defined by taking a subset of the Cartesian product $\times_{i=1}^{n} A_i$, for example.

refer to a subset of $A \times A$ as a *relation from A to A*, although in this special case some authors prefer to call the subset of $A \times A$ a relation *on* A.

Now let $S \subset A \times B$. Thus, by definition, S is a relation from A to B. (We emphasize at this point that the choice of the letter S is quite arbitrary, and we could just as well have chosen any other letter to represent a subset of $A \times B$ defining a relation from A to B.) If $(x, y) \in S$, we say that **x is in the relation S to y** or that **x is S-related to y**. An alternative notation for $(x, y) \in S$ is xSy. Also, we use $S\colon A \to B$ as an abbreviation for "the relation S from A to B."

As it now stands, the concept of a relation no doubt appears quite abstract. However, in practice, it is the context provided by the definition of the subset S and the definitions of the sets A and B that provide intuitive meaning to xSy. That is, x will be S-related to y because of some property satisfied by the (x, y) pair, the property being indicated in the set definition of S. The real-world objects being related will be clearly identified in the set definitions of A and B. Some examples will clarify the intuitive side of the relation concept.

Example A.13 Let $A = [0, \infty)$, and form the Cartesian product $A^2 = \{(x, y)\colon x \in A \text{ and } y \in A\}$. The set A^2 can thus be interpreted as the nonnegative (or first) quadrant of the Euclidean plane. Then $S = \{(x, y)\colon x \geq y, (x, y) \in A^2\}$ is a relation from A to A representing the set of points in the nonnegative quadrant for which the first coordinate has a value greater than or equal to the value of the second coordinate. The defining property of the relation S is "$\geq$." This is displayed in Figure A.3. □

Example A.14 Let $A = \{x\colon x \text{ is an employed U.S. citizen}\}$ and $B = \{y\colon y \text{ is a U.S. corporation}\}$. Then $A \times B = \{(x, y)\colon x \text{ is an employed U.S. citizen and } y \text{ is a U.S. corporation}\}$ is the set of all possible pairings of employed U.S. citizens with U.S. corporations.

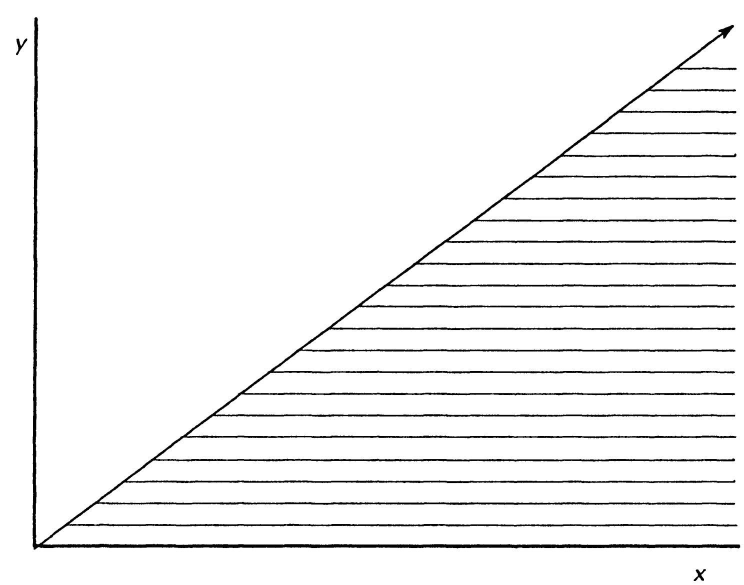


Figure A-3
xSy in shaded area.

The relation $S = \{(x, y): x \text{ is employed by } y, (x, y) \in A \times B\}$ from A to B is the collection of U.S. citizens who are employed by U.S. corporations paired with their respective corporate affiliation. The defining property of the relation is the phrase "is employed by," and xSy iff x is a U.S. citizen employed by a U.S. corporation, and y is his or her corporate affiliation. □

Function

We are now in a position to define what is meant by the concept of a function. As indicated in the following definition, a function is simply a special type of relation. In the definition we introduce the symbol $\forall$, which means **for every** or **for all**, and the symbol $\exists$, which means **there exists**.

Definition A.17 ***Function***

> A function from A to B is a relation $S: A \rightarrow B$ such that $\forall\ a \in A\ \exists$ one unique $b \in B$ such that $(a, b) \in S$.

A relation satisfying the above condition will often be given a special symbol to distinguish the relation as a function. A popular symbol used for this purpose is "f," where $f: A \rightarrow B$ is a common notation for designating the **function f from A to B**. As we had remarked when choosing a symbol to represent a relation, the choice of the letter f is arbitrary, and when it is convenient or useful, any other letter or symbol could be used to depict a subset of $A \times B$ that represents a function from A to B. In the text we will often have occasion to use a variety of letters to designate various functions of interest.

The unique element $b \in B$ that the function $f: A \rightarrow B$ associates with a given element $x \in A$ is called **the image of x under f** and is represented symbolically by the notation $\boldsymbol{f(x)}$. If $f(x)$ is a real number, the image of x under f is alternatively referred to as **the value of the function f at x**. In the following example, we use $\boldsymbol{R}$ to denote the set of real numbers $(-\infty, \infty)$, i.e., R stands for the **real line**. Furthermore, the nonnegative subset of the real line is represented by $R_{\geq 0}, = [0, \infty)$.

Example A.15 Let $f: R \rightarrow R_{\geq 0}$ be defined by $f = \{(x, y): y = x^2, x \in R\}$. The *image of* -2 *under* f is $f(-2) = 4$. The *value of the function* f *at* 3 is $f(3) = 9$. □

Associated with a given function, f, are two important sets called the **domain** and **range** of the function.

Definition A.18 ***Domain and range of a function***

> The **domain** of a function $f: A \rightarrow B$ is defined as $D(f) = A$. The **range** of f is defined as $R(f) = \{y: y = f(x), x \in A\}$.

Thus, the domain of a function $f: A \rightarrow B$ is simply the set A of the Cartesian product $A \times B$ associated with the function. The range of f is the collection of all elements in B that are images of the elements in A under the function f. It

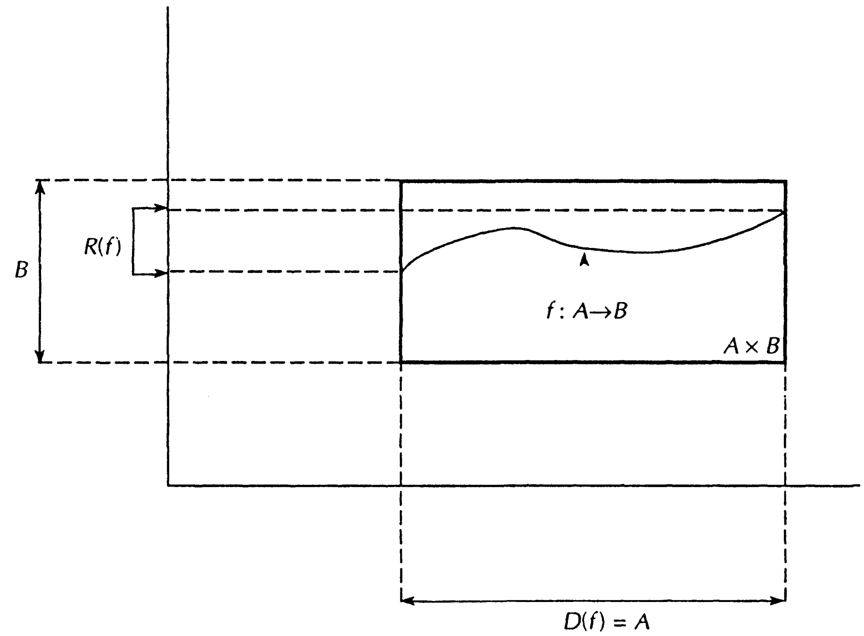


Figure A-4
Function with domain $D(f)$ and range $R(f)$

follows that $R(f) \subset B$. Figure A.4 provides a pictorial example of a function, including its *domain* and *range*.

Note that the concept of a function is completely general regarding the nature of the elements of the sets $D(f) = A$, $R(f)$, and B. The elements can be numbers, or other objects, or the elements can be sets themselves. For our work, it will suffice to deal only with *real-valued functions*, meaning that $R(f)$ is a set of real numbers.

Definition A.19
Real-valued function

A function $f\colon A \to B$ such that $R(f) \subset R$ is called a **real-valued function.**

The function defined in Ex. A.15 is a real-valued function. The following is another example.

Example A.16 Examine the Cobb–Douglas production function $f\colon R^2_{\geq 0} \to R_{\geq 0}$ defined as $f = \{((x_1, x_2), y)\colon y = 10x_1^2 x_2, (x_1, x_2) \in R^2_{\geq 0}\}$. Interpreting (x_1, x_2) as inputs into a production process, and y as the output of the process, we see that the domain of the production function is $D(f) = R^2_{\geq 0}$, i.e., any nonnegative level of the input pair (x_1, x_2) is an admissible input level. The associated range of the production function is $R(f) = R_{\geq 0}$, i.e., any nonnegative level of output, y, is possible. Since $R(f) = [0, \infty) \subset R$, the production function is a real-valued function. □

In some cases the relation from A to B that defines the function $f\colon A \to B$ also defines a function from B to A. This relates to the concept of an **inverse**

function. In particular, if for each element $y \in B$ there exists precisely one element $x \in A$ whose image under f is y, then such an *inverse function* exists.

Definition A.20
Inverse function of f

Let $f\colon A \to B$ be a function from A to B. If $R(f) = B$ and $\forall\, y \in B\ \exists$ a unique $x \in A$ such that $y = f(x)$, then the relation $\{(y, x)\colon y = f(x), y \in B\}$ is a function from B to A called the **inverse function of f** and denoted by $f^{-1}\colon B \to A$.

Note that neither of the functions in Ex. A.15 or Ex. A.16 is such that an inverse function exists. In Ex. A.15, the uniqueness condition of Def. A.20 is violated since $\forall\, y \neq 0$ there exist two values of x for which $y = x^2$, namely $x = \pm\sqrt{y}$. For example, when $y = 4$, $x = 2$ and -2 are each such that $y = x^2$. The reader can verify that Ex. A.16 also violates the uniqueness condition where an infinite number of (x_1, x_2) values satisfy $y = 10x_1^2 x_2$ for a fixed value of y (defining level sets or isoquants of the production function). Also, note that an inverse function does not exist for the function illustrated in Figure A.4.

As an example of a function for which an inverse function does exist, consider the following.

Example A.17 Let $f\colon R \to R_+$ be defined as

$$f = \{(x, y)\colon y = e^x, x \in R\}, \quad \text{where } R_+ = (0, \infty).$$

Note that the *inverse function* can be represented as

$$f^{-1} = \{(y, x)\colon x = \ln y, y \in R_+\},$$

so that $f^{-1}\colon R_+ \to R$. It is clear that $\forall\, y \in R_+$, $\exists$ one and only one $x \in R$ such that $x = \ln y$. □

The final concept concerning functions that we will review here is the **inverse image** of $y \in R(f)$ or of $H \subset R(f)$. The inverse image of y is the set of domain elements $x \in D(f)$ such that $y = f(x)$, i.e., the collection of all x values in the domain of f whose image under the function f is y. The inverse image of y can be represented as the set $\{x\colon f(x) = y\}$, and when the inverse function exists, the inverse image of y can be represented as the value $f^{-1}(y)$. In Ex. A.17, the inverse image of 5 is $f^{-1}(5) = \ln(5) = 1.6094$; in Ex. A.15, the inverse image of 4 is $\{-2, 2\}$; and in Ex. A.16, the inverse image of 3 is the isoquant $\{(x_1, x_2)\colon 3 = 10x_1^2 x_2, (x_1, x_2) \in R^2_{\geq 0}\}$. Similarly, the inverse image of $H \subset R(f)$ is the set of x values in the domain of f whose images under f equal some $y \in H$, i.e., the inverse image of H is $\{x\colon f(x) = y, y \in H\}$, and if the inverse function exists, $\{x\colon x = f^{-1}(y), y \in H\}$.

Real-Valued Point Versus Set Functions

Two types of functions—point functions and set functions—are utilized extensively in modern discussions of probability and mathematical statistics. The reader should already have considerable experience with the application of real-valued point functions, since this type of function is the one that appears in elementary algebra and calculus courses and is central to discussions of utility, demand, production, and supply that the reader has encountered in his or her study of economic theory. Specifically, a **real-valued point function** is a real-valued function whose domain consists of a collection of points, where points are represented by coordinate vectors in R^n. We have encountered examples of this type of function previously in Ex. A.15 through Ex. A.17. A typical ordered pair associated with a real-valued point function is of the form $(\mathbf{x}, y)$, where $\mathbf{x}$ is a vector in R^n and y is a real number in R.

A set function is more general than a point function in that its domain consists of a collection of *sets* rather than a collection of points.[4] A typical ordered pair belonging to a **real-valued set function** would have the form (A, y), where A is a set of some type of objects and y is a real number in R. If the sets in the domain of the set function are contained in R^n, i.e., they are sets of real numbers, then a real-valued set function assigns a real number to each *set* of points in its domain, whereas a real-valued point function assigns a real number to each *point* in its domain. A pictorial illustration of a set function contrasted with a point function is given in Figure A.5.

Examples of set functions are presented below.

Example A.18 Let $\Omega = \{1, 2, 3\}$, and let A be the collection of all of the subsets of Ω, i.e., $A = \{A_1, A_2, \ldots, A_8\}$, where $A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}, A_4 = \{1, 2\}, A_5 = \{1, 3\}$, $A_6 = \{2, 3\}$, $A_7 = \{1, 2, 3\}$, and $A_8 = \emptyset$.

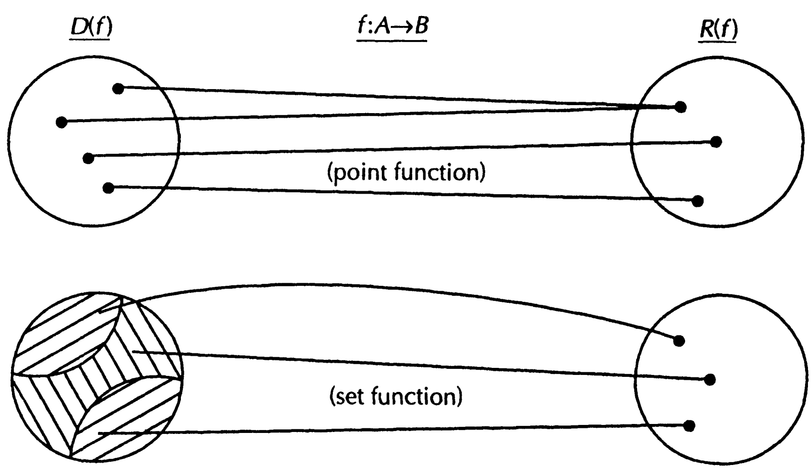


Figure A-5
Point versus set function.

[4]Note that, in a sense, a point function can be viewed as a special case of a set function, since points can be interpreted as singleton (single-element) sets. The set function concept is introduced to accommodate the case where one or more sets in its domain are *not* singleton.

The following is a real-valued set function $f\colon A \to R$:

$$f = \{(A_i, y)\colon y = \sum_{x \in A_i} x, A_i \subset A\},$$

where $\sum_{x \in A_i} x$ signifies the sum of the numerical values of all of the elements in the set A_i, and $\sum_{x \in \emptyset} x$ is defined to be zero. The range of the set function is $R(f) = \{0, 1, 2, 3, 4, 5, 6\}$, and the domain is the set of sets $D(f) = A$. The function can be represented in tabular form as follows:

A_i	$f(A_i)$
A_1	1
A_2	2
A_3	3
A_4	3
A_5	4
A_6	5
A_7	6
A_8	0

□

Example A.19 Let $A = \{A_r\colon A_r = \{(x, y)\colon x^2 + y^2 \leq r^2\}, r \in [0, 1]\}$, so that A is a set of sets, where the typical element A_r represents the set of points in R^2 that are on the boundary and in the interior of a circle centered at (0,0) with radius r. The following is a real-valued set function $f\colon A \to R$:

$$f = \{(A_r, y)\colon y = \pi r^2, A_r \subset A\}.$$

Note that the set function assigns a real number representing the area to each set, A_r. The assignment is made for circles having a radius anywhere from 0 to 1. The range of the set function is given by $R(f) = [0, \pi]$, and the domain is the *set of sets* $D(f) = A$. □

A special type of set function called the size-of-set function will prove to be quite useful.

Definition A.21
Size of set function

Let A be any set of objects. The **size-of-set function**, N, is the set function that assigns to the set A the number of elements that are in set A, i.e., $N(A) = \sum_{x \in A} 1$.[5]

Applying the size-of-set function in Ex. A.18, note that $N(A) = 8$. In Ex. A.19, note that $N(A) = \infty$.

Another special (point) function that will be useful in our study is the **indicator function**, defined as follows.

[5]Note that $\sum_{x \in A} 1$ signified that a collection of 1's are being summed together, the number in the collection being equal to the number of elements in the set A. If $A = \emptyset$, effectively no 1's are being added together, and thus $N(\emptyset) = 0$.

Definition A.22 *Indicator function*

Let A be any subset of some universal set Ω. The indicator function, denoted by I_A, is a real-valued function with domain Ω and range $\{0, 1\}$ such that

$$I_A(x) = \begin{Bmatrix} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{Bmatrix}.$$

Note that the indicator function *indicates* the set A by assigning the number 1 to any x that is an element of A, while assigning zero to any x that is not an element of A. The main use of the indicator function is notational efficiency in defining functions, as the following example illustrates.

Example A.20 Let the function $f\colon R \to R$ be defined by

$$f(x) = \begin{bmatrix} 0 & \text{for } x \in (-\infty, 0] \\ x & \text{for } x \in (0, 2] \\ 3 - x & \text{for } x \in (2, 3] \\ 0 & \text{for } x \in (3, \infty) \end{bmatrix}.$$

Utilizing the indicator function, we can alternatively represent $f(x)$ as

$$f(x) = xI_{(0,2]}(x) + (3 - x)I_{(2,3]}(x).$$ □

As a final note on the use of functions, we (as do the vast majority of other authors) will generally use a shorthand method for defining functions by simply specifying the relationship between elements in the domain of a function and their respective images in the range of the function. For example, we would define the function in Ex. A.19 by $f(A_r) = \pi r^2$ for $A_r \subset A$, or define the function in Ex. A.15 by $f(x) = x^2$ for $x \in R$. In all cases, the reader should remember that a function is a set of ordered pairs $(x, f(x))$, or $(A, f(A))$. The reader will sometimes find in the literature phrases like **the function $f(x)$** or **the set function $f(A)$**. Literally speaking, such phrases are inconsistent, because $f(x)$ and $f(A)$ are not functions but rather *images* of elements in the domain of the respective functions. In fact, the reader should *not* take such phrases literally, but rather interpret these phrases as shorthand for phrases such as *the function whose values are given by* $f(x)$ or *the set function whose values are given by* $f(A)$.

A.5 Combinations and Permutations

In a number of situations involving probability assignments, it will be useful to have an efficient method for counting the number of *different* ways a group of r objects can be selected from a group of n distinct objects, $n \geq r$. Obviously, if two groups of r objects do not contain the same r objects, they must be considered *different*. But what if two groups of r objects do contain the same objects, except the objects in the group are arranged in different orders? Are the two groups to be considered *different*? If difference in order constitutes difference in groups, then we are dealing with the notion of **permutations**. On the other hand, if the order of listing the objects in a group is not used as a basis for distinguishing between groups, then we are dealing with the notion of **combinations**.

In order to establish a formula for determining the number of permutations of n distinct objects taken r at a time, the following example is suggestive.

Example A.21 Examine the number of different ways a group of three letters can be selected from the letters a, b, c, d, where difference in order of listing is taken to mean difference in groups. Note that the first letter can be chosen in four different ways. After we have chosen one of the letters for the first selection, the second selection can be any of the remaining three letters. Finally, after we have chosen two letters in the first two selections, there are then two letters left to be potentially chosen for the third selection. Thus, there are $4 \cdot 3 \cdot 2 = 24$ different ways of selecting a group of 3 letters from the letters a, b, c, d if difference in the order of listing constitutes difference in groups. (The reader should attempt to list the 24 different groups.) □

The logic of the preceding example can be applied to establish a general formula for determining the number of permutations of n distinct objects taken r at a time:

$$(n)_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1),$$

where ! denotes the **factorial operation**, i.e.,

$$n! = n(n-1)(n-2)(n-3)\ldots 1.^{6}$$

Thus, for example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. In Ex. A.21, $n = 4$ and $r = 3$, so that $(4)_3 = 4!/1! = 24$.

In order to establish a formula for determining the number of combinations of n distinct objects taken r at a time, we return to Ex. A.21.

Example A.21 (continued)

Examine the number of different ways a group of three letters can be selected from the letters a, b, c, d, where difference in order of listing does *not* imply the groups are different. Recall that we discovered that there were 24 permutations of the 4 letters a, b, c, d selected 3 at a time. Now note that any 3 letters, say a, b, c, can be arranged in $(3)_3 = 6$ different orders, which represents "overcounting" from the combinations point of view. Reducing the number of permutations by the degree of "overcounting" results in the number of combinations, i.e., there are $24/6 = 4$ combinations of the 4 letters taken 3 at a time, namely (a, b, c), (a, b, d), (a, c, d), and (b, c, d). □

In the preceding example, the number of permutations of n $(= 4)$ objects taken r $(= 3)$ at a time was reduced by a factor of $r!$ $(= 3!)$, where the latter value

[6] By definition, we take $0! = 1$.

represents the number of possible permutations of r objects. This suggests the general formula for the number of combinations of n objects taken r at a time:

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n!}{(n-r)!r!}.$$

In Ex. A.21, we have

$$\binom{4}{3} = \frac{4!}{1!3!} = 4$$

as the appropriate number of combinations.

The concept of combinations is useful in determining the number of subsets that can be constructed from a finite set A. Note that in counting the number of subsets, changes in the order of listing set elements do not produce a different set, e.g., the sets $\{a, b, c\}$ and $\{c, a, b\}$ are the same set of letters (recall the definition of a set). Then the total number of subsets of a set A containing n elements is given by the number of different subsets defined by taking no elements (i.e., the null set) plus the number of different subsets defined by taking one element, plus the number of different subsets defined by taking two elements, ..., and finally, the number of different subsets defined by taking all n elements (i.e., the set A itself). Thus, the total number of different subsets of A can be written as

$$\sum_{r=0}^{n} \binom{n}{r} = \sum_{r=0}^{n} \frac{n!}{(n-r)!r!}.$$

This sum can be greatly simplified by recalling that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}, \quad n = 1, 2, \ldots,$$

which is the **binomial theorem**. Then letting $x = y = 1$, we have that

$$2^n = \sum_{r=0}^{n} \binom{n}{r},$$

so that 2^n is the number of different subsets contained in a set A that has n elements.

Example A.22 In Ex. A.18, recall that we identified a total of eight subsets of the set $\Omega = \{1, 2, 3\}$. This is the number of subsets we would expect from our discussion above, i.e., since $n = 3$, there are $2^3 = 8$ subsets of Ω. □

It should be noted that $\binom{n}{r}$ is *defined* to be 0 whenever $n < r$, or whenever n and/or $r < 0$ or are not integer valued. The rationale for $\binom{n}{r} = 0$ in each of these cases is that there is no way to define subsets of size r from a collection of n objects for the designated values of n and r.

When n is large, the calculation of $n!$ needed in the previous formulas pertaining to numbers of permutations or combinations can be quite formidable.

A result known as Stirling's formula can provide a useful approximation to $n!$ for large n:

Definition A.23 *Stirling's formula*[7]

$n! \approx (2\pi)^{1/2} n^{n+.5} e^{-n}$ for large n.[8]

A logical question to ask regarding the use of Stirling's formula is how large is "large n"? Stirling's formula invariably underestimates $n!$, but the percentage error $\leq 1\%$ for $n \geq 10$, and it monotonically decreases as $n \to \infty$.

A.6 Summation, Integration and Matrix Differentiation Notation

We will use a number of variations on summation and integration notation in this text. The meaning of the various types of notation are presented in Table A.1.

We illustrate the use of some of the notation in the following example.

Example A.23 Summation Notation

Let $A_1 = \{1,2,3\}$, $A_2 = \{2,4,6\}$, $A = A_1 \times A_2$, $B = \{(x_1, x_2): x_1 \in A_1, x_2 \in \{x_1, x_1+1, \ldots, 3x_1\}\}$, $y = (y_1, y_2, \ldots, y_n)$, and $f(x_1, x_2) = x_1 + 2x_2^2$. Then

$$\sum_{x \in A_1} x = 1 + 2 + 3 = 6, \qquad \sum_{x \in A_2} x^2 = 2^2 + 4^2 + 6^2 = 56,$$

$$\sum_{i \in A_2} y_i = y_2 + y_4 + y_6, \qquad \sum_{i \in A_1} y_i = \sum_{i=1}^{3} y_i = y_1 + y_2 + y_3,$$

$$\sum_{x_1 \in A_1} \sum_{x_2 \in A_2} f(x_1, x_2) = \sum_{x_1 \in A_1} \sum_{x_2 \in A_2} (x_1 + 2x_2^2) = 354,$$

$$\sum\sum_{(x_1, x_2) \in A} f(x_1, x_2) = \sum_{x_1 \in A_1} \sum_{x_2 \in A_2} f(x_1, x_2) = 354,$$

$$\sum\sum_{(x_1, x_2) \in B} f(x_1, x_2) = \sum_{x_1=1}^{3} \sum_{x_2=x_1}^{3x_1} (x_1 + 2x_2^2) = 802. \qquad \square$$

Example A.24 Integration Notation

Let $A_1 = [0,3]$, $A_2 = [2,4]$, $A = A_1 \times A_2$, $B = \{(x_1, x_2): x_1 \in A_1, 0 < x_2 < x_1^2\}$, and $f(x_1, x_2) = x_1 x_2^2$. Then

$$\int_{x \in A_1} 2x\,dx = \int_0^3 2x\,dx = \frac{2x^2}{2} \Big|_0^3 = 9,$$

---

[7] See W. Feller (1968), *An Introduction to the Theory of Probability and Its Applications*, 3rd ed. pp. 52–54.

[8] Note that $\approx$ means "approximately equal to."

Table A.1 Summation and Integration Notation

Notation	Definition
$\sum_{i=\ell}^{n} x_i$	Sum the values of $x_\ell, x_{\ell+1}, \ldots, x_n$, i.e., $x_\ell + x_{\ell+1} + \cdots + x_n$.
$\sum_{i \in I} x_i$	Sum the values of the x_i's, for $i \in I$.
$\sum_{x \in A} x$	Sum the values of $x \in A$.
$\sum_{x=a}^{b} x$	Sum the values of x in the sequence of integers from a to b, i.e., $a + (a+1) + (a+2) + \cdots + b$.
$\sum_{i=\ell}^{n} \sum_{j=k}^{m} x_{ij}$	Sum the values of the x_{ij}'s for $i = \ell, \ell+1, \ldots, n$ and $j = k, k+1, \ldots, m$.
$\sum_{i \in I} \sum_{j \in J} x_{ij}$ or $\sum \sum_{(i,j) \in A} x_{ij}$	Sum the values of the x_{ij}'s for $i \in I$ and $j \in J$, or for $(i, j) \in A$.
$\sum_{x_1 \in A_1} \cdots \sum_{x_n \in A_n} f(x_1, \ldots, x_n)$	Sum the values of $f(x_1, \ldots, x_n)$ for $x_i \in A_i$, $i = 1, \ldots, n$.
$\sum \cdots \sum_{(x_1, \ldots, x_n) \in A} f(x_1, \ldots, x_n)$	Sum the values of $f(x_1, \ldots, x_n)$ for $(x_1, \ldots, x_n) \in A$.
$\sum_{x_1=a_1}^{b_1} \cdots \sum_{x_n=a_n}^{b_n} f(x_1, \ldots, x_n)$	Sum the values of $f(x_1, \ldots, x_n)$ for x_i in the sequence of integers a_i to b_i, $i = 1, \ldots, n$.
$\int_a^b f(x)dx$	Integral of the function $f(x)$ from a to b (a can be $-\infty$ and/or b can be ∞).
$\int_{x \in A} f(x)dx$	Integral of the function $f(x)$ over the set of points A.
$\int_{x_1 \in A_1} \cdots \int_{x_n \in A_n} f(x_1, \ldots, x_n)dx_n \ldots dx_1$	Iterated integral of the function $f(x_1, \ldots, x_n)$ over the points $x_i \in A_i$, $i = 1, \ldots, n$.
$\int \cdots \int_{(x_1, \ldots, x_n) \in A} f(x_1, \ldots, x_n)dx_1 \ldots dx_n$	Multiple integral of the function $f(x_1, \ldots, x_n)$ over the points $(x_1, \ldots, x_n) \in A$.
$\int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)dx_n \ldots dx_1$	Iterated integral of the function $f(x_1, \ldots, x_n)$ for x_i in the (open, half open-half closed, or closed) interval a_i to b_i, for $i = 1, \ldots, n$.

$$\int_{x \in A_1 \cap A_2} x^2\, dx = \int_2^3 x^2\, dx = \frac{x^3}{3} \Big|_2^3 = \frac{19}{3},$$

$$\begin{aligned}
\int_{x_1 \in A_1} \int_{x_2 \in A_2} f(x_1, x_2) dx_2\, dx_1 &= \int_0^3 \int_2^4 f(x_1, x_2) dx_2\, dx_1 \\
&= \int_0^3 \frac{x_1 x_2^3}{3} \Big|_2^4 dx_1 = \int_0^3 \frac{56}{3} x_1\, dx_1 \\
&= \frac{56 x_1^2}{6} \Big|_0^3 = 84,
\end{aligned}$$

$$\iint_{(x_1, x_2) \in A} f(x_1, x_2) dx_1\, dx_2 = \int_{x_1 \in A_1} \int_{x_2 \in A_2} f(x_1, x_2) dx_2\, dx_1 = 84,$$

$$\iint_{(x_1,x_2)\in B} f(x_1,x_2)dx_1\,dx_2 = \int_0^3 \int_0^{x_1^2} f(x_1,x_2)dx_2\,dx_1$$

$$= \int_0^3 \frac{x_1 x_2^3}{3} \Big|_0^{x_1^2} dx_1 = \int_0^3 \frac{x_1^7}{3} dx_1$$

$$= \frac{x_1^8}{24} \Big|_0^3 = 273.375. \qquad \square$$

Regarding matrix differentiation notation, we utilize the following conventions. Let $g(\mathbf{x})$ and $\mathbf{y}(\mathbf{x})$ be a scalar and $n \times 1$ vector function of the $k \times 1$ vector $\mathbf{x}$, respectively. Then

Derivative	Matrix Dimension	(i, j)th Entry
$\frac{\partial g(\mathbf{x})}{\partial \mathbf{x}}$	$k \times 1$	$\frac{\partial g}{\partial x_i}$
$\frac{\partial g(\mathbf{x})}{\partial \mathbf{x}'} = \frac{\partial g(\mathbf{x})'}{\partial \mathbf{x}}$	$1 \times k$	$\frac{\partial g}{\partial x_j}$
$\frac{\partial g(\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}'}$	$k \times k$	$\frac{\partial^2 g}{\partial x_i \partial x_j}$
$\frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}}$	$k \times n$	$\frac{\partial y_j}{\partial x_i}$
$\frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}'} = \frac{\partial \mathbf{y}(\mathbf{x})'}{\partial \mathbf{x}}$	$n \times k$	$\frac{\partial y_i}{\partial x_j}$

Key Words, Phrases, and Symbols

definition
axiom (or postulate)
theorem (or proposition)
corollary
lemma
set
element
$\in$
negation, /
exhaustive listing
verbal rule
mathematical rule
finite set
iff (if and only if)
infinite set
countable set
uncountable set
subset
equality of sets, =
universal set
empty or null set, $\emptyset$
set difference, $-$
complement, $\bar{A}$
union, $\cup$
intersection, $\cap$
mutually exclusive (disjoint)
Venn diagram
idempotency laws
commutative laws
associative laws
distributive laws
identity elements
intersection and union of complements
complements of complements
intersection with null set
null set as a subset
DeMorgan's laws
index set
multiple union notation
multiple intersection notation
real line
interval set notation
closed, open, half-open-half-closed intervals
Cartesian product
binary relation from A to B, $S: A \to B$
xSy
$\forall$ (for every)
$\exists$ (there exists)

function from A to B, $f: A \to B$
real-valued function
$A^2 = A \times A$
$R_{\geq 0}$, R_+
image of x under f
inverse function, $f^{-1}: B \to A$
inverse image

point function
set function
size-of-set function
indicator function, $I_A(x)$
"the function $f(x)$"
"the set function $f(A)$"

permutations, $(n)_r$
combinations, $\binom{n}{r}$
binomial theorem
Stirling's formula
summation notation
integration notation

Problems

1. Using either an exhaustive listing, verbal rule, or mathematical rule, define the following sets:

a. the set of all senior citizens receiving social security payments in the United States;

b. the set of all positive numbers that are positive integer powers of the number 10 (i.e., 10^1, 10^2, etc.);

c. the set of all possible outcomes resulting from rolling a red and a green die and calculating the values of $y - x$, where y = number of dots on the red die and x = number of dots on the green die;

d. the set of all two-tuples (x_1, x_2) where x_1 is any real number and x_2 is related to x_1 by raising the number e to the power x_1.

2. Label the sets you have identified in Problem 1. as being either finite, countably infinite, or uncountably infinite, and explain your choice.

3. For each set below, state whether the set is finite, countably infinite, or uncountably infinite.

a. $S = \{x\colon x$ is a U.S. citizen who has purchased a Japanese car during the past year$\}$.

b. $S = \{(x, y)\colon y \leq x^2, x$ is a positive integer, $y \in R_{\geq 0}\}$.

c. $S = \{p\colon p$ is the price of a quart of milk sold at a retail store in the United States on Friday, September 13, 1991$\}$.

d. $S = \{x\colon x = 2y, y$ is a positive integer$\}$.

4. Let the universal set be $\Omega = [0, 10]$, and define the following subsets of Ω:

$A = [0, 2)$, $B = [2, 7]$, $C = [5, 6]$, $D = \{2\}$,

$E = \{x\colon x = y^{-1}, y$ is an even positive integer $\geq 4\}$.

a. Define the following sets:

$A \cup B$, $A \cap B$, $\overline{A \cup C}$, $(A \cup D) \cap B$,

$B - C$, $A \cap E$, $\bar{D} \cap B$.

b. For each of the sets in (a), indicate whether the set is finite, countably infinite, or uncountably infinite.

5. Let the universal set be defined by $\Omega = [-5, 5]$, and define the following subsets of Ω:

$A_1 = [-2, 1)$,

$A_2 = (1, 2)$,

$A_3 = [2, 5]$,

$A_4 = [-5, -2]$.

Also, define an index set $I = \{1, 3, 4\}$.

a. Define $\cup_{i \in I} A_i$.

b. Define $\cup_{i=1}^4 A_i$.

c. Define $A_1 \cap A_2$.

d. Define $A_4 - A_1$.

e. Define $\bar{A}_4$.

6. Define the universal set, Ω, as

$\Omega = \{x\colon 0 \leq x \leq 5$ or $10 \leq x \leq 20\}$,

and define the following subsets of Ω as

$A_1 = \{x\colon 0 \leq x < 2.5\}$,

$A_2 = \{x\colon 15 < x \leq 20\}$,

$A_3 = \{x\colon 2.5 \leq x \leq 5$ or $10 \leq x \leq 20\}$,

$A_4 = \{x\colon 0 \leq x \leq 5$ or $10 \leq x \leq 15\}$.

In addition, define the following two index sets as

$I_1 = \{1, 3\}$, $I_2 = \{1, 4\}$.

Define the following sets:

a. $\cup_{i \in I_1} A_i$

b. $\cap_{i=1}^4 A_i$

c. $\cap_{i \in I_2} A_i$

d. $\cap_{i=1}^2 A_i$

e. $A_1 - A_2$
f. $A_4 - A_3$
g. $\bar{A}_3$
h. $A_2 - \cup_{i \in I_1} A_i$

7. In each situation below indicate whether the relation is a function. If so, determine the domain and range of the function.

a. $A = [0, 10]$, $B = [0, \ln(11)]$, $S = \{(x, y): y = \ln(1 + x), (x, y) \in A \times B\}$.

c. $A = R^2_{\geq 0}$, $B = [0, \infty)$, $S = \{((x_1, x_2), y): y = 5x_1x_2^2, ((x_1, x_2), y) \in A \times B\}$.

8. For each relation below, state whether the relation is a function, and state whether an inverse function exists. Explicitly define the inverse function if it exists.

a. Let $P = \{\$.01, \$.02, \ldots, \$1.00, \$1.01, \ldots\}$ represent a set of possible prices for a given commodity, and let $Q = [0, \infty)$ represent possible levels of quantity demanded. Define $S: P \rightarrow Q$ as

$$S = \{(p, q): q = 20p^{-1.5}, (p, q) \in P \times Q\}.$$

b. Let $A = \{D: D = \{(x_1, x_2): x_1 \in [a_1, b_1], x_2 \in [a_2, b_2]\}, a_1 < b_1, a_2 < b_2\}$ be a set of rectangular sets. Define $S: A \rightarrow R$ as

$$S = \{(D, y): y = \text{ area of } D, (D, y) \in A \times R_+\}.$$

9. Let $A = (0, \infty)$, and examine the following relation on A:

$$S = \{(x, y): y = 2 + 3x, (x, y) \in A^2\}.$$

a. Is $2S8$?
b. Is S a function?
c. Does an inverse function exist?
d. If you can, define $f(2)$ and $f^{-1}(5)$.
e. What is $D(f)$? What is $R(f)$?

10. Define a universal set as

$$\Omega = \{x: 0 \leq x \leq 5\},$$

and consider the set function

$$P(A) = .5 \int_{x \in A} x dx + 12.5,$$

where the domain of the set function is all subsets $A \subset \Omega$ of the form $A = [a, b]$, for $0 \leq a \leq b \leq 5$.

a. What is the image of the set $A = [0, 2]$ under P?
b. What is the image of Ω under P?
c. What is the image of the set $A = [3, 3]$ under P?
d. What is the range of the set function?
e. What is the inverse image of 2?

11. Define a set function that will assign the appropriate area to all rectangles of the form $[x_1, x_2] \times [y_1, y_2]$, $x_2 \geq x_1$ and $y_2 \geq y_1$, contained in R^2. Be sure to identify the domain and range of the set function.

12. A statistics class has 20 students in attendance in a room where 25 desks are available for the students. How many different ways can the students leave 5 desks unoccupied?

13. There are 15 students in an econometrics class that you are attending.

a. How many different ways can a three-person committee be formed to give a class report?
b. Of the number of possible three-person committees indicated in (a), how many involve you?

14. Competing for the title of Miss America are 50 contestants from each of the 50 states plus 1 contestant from the District of Columbia. How many different ways can the contestants be assigned the titles of Miss America, 1st runner up, ..., 4th runner up?

15. Let $A_1 = \{x: x \text{ is a positive integer}\}$, $A_2 = \{1, 2, 3, 4, 5\}$, $B = \{(x_1, x_2): (x_1, x_2) \in A_1 \times A_2, x_1 \leq x_2\}$, and $y_i = i^2$. Calculate the values of the following sums.

a. $\sum_{x \in A_2} x$
b. $\sum_{i \in A_2} y_i$
c. $\sum_{x_1 \in A_1} \sum_{x_2 \in A_2} (1/2)^{x_1} x_2^2$
d. $\sum_{x \in A_1 - A_2} (1/3)^x$
e. $\sum \sum_{(x_1, x_2) \in B} (x_1 + x_2)$

16. Let $A_1 = [0, \infty)$, $A_2 = [1, 10]$, and $B = \{(x_1, x_2): (x_1, x_2) \in A_1 \times A_2, x_2 > x_1\}$. Calculate the values of the following integrals.

a. $\int_{x \in A_1} (1/2) e^{-x/2} dx$
b. $\int_{x_1 \in A_1} \int_{x_2 \in A_2} x_2 e^{-x_1} dx_2\, dx_1$
c. $\iint_{(x_1, x_2) \in B} (x_1 + x_2) dx_1\, dx_2$
d. $\int_0^2 \int_{x_2 \in A_1 \cap A_2} x_1 x_2^2\, dx_2\, dx_1$

B Useful Tables

Table B.1 Cumulative Normal Distribution $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9830	.9834	.9838	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9978	.9979	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

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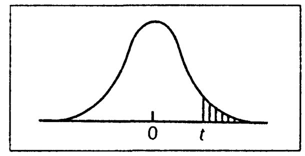
Table B.2 Student's t Distribution
The first column lists the number of degrees of freedom (ν). The headings of the other columns give probabilities (P) for t to exceed the entry value. Use symmetry for negative t values.



ν \ P	.10	.05	.025	.01	.005
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
40	1.303	1.684	2.021	2.423	2.704
60	1.296	1.671	2.000	2.390	2.660
120	1.289	1.658	1.980	2.358	2.617
∞	1.282	1.645	1.960	2.326	2.576

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Table B.3 Chi-square Distribution
The first column lists the number of degrees of freedom (ν). The headings of the other columns give probabilities (P) for the χ^2_ν random variable to exceed the entry value.

ν \ P	0.995	0.990	0.975	0.950	0.900	0.750
1	392704×10^{-10}	157088×10^{-9}	982069×10^{-9}	393214×10^{-8}	0.0157908	0.1015308
2	0.0100251	0.0201007	0.0506356	0.102587	0.210720	0.575364
3	0.0717212	0.114832	0.215795	0.351846	0.584375	1.212534
4	0.206990	0.297110	0.484419	0.710721	1.063623	1.92255
5	0.411740	0.554300	0.831211	1.145476	1.61031	2.67460
6	0.675727	0.872085	1.237347	1.63539	2.20413	3.45460
7	0.989265	1.239043	1.68987	2.16735	2.83311	4.25485
8	1.344419	1.646482	2.17973	2.73264	3.48954	5.07064
9	1.734926	2.087912	2.70039	3.32511	1.16816	5.89883
10	2.15585	2.55821	3.24697	3.94030	4.86518	6.73720
11	2.60321	3.15347	3.81575	4.57481	5.57779	7.58412
12	3.07382	3.57056	4.40379	5.22603	6.30380	8.43842
13	3.56503	4.10691	5.00874	5.89186	7.04150	9.29906
14	4.07468	4.66043	5.62872	6.57063	7.78953	10.1653
15	4.60094	5.22935	6.26214	7.26094	8.54675	11.0365
16	5.14224	5.81221	6.90766	7.96164	9.31223	11.9122
17	5.69724	6.40776	7.56418	8.67176	10.0852	12.7919
18	6.26481	7.01491	8.23075	9.39046	10.8649	13.6753
19	6.84398	7.63273	8.90655	10.1170	11.6509	14.5620
20	7.43386	8.26040	9.59083	10.8508	12.4426	15.4518
21	8.03366	8.89720	10.28293	11.5613	13.2396	16.3444
22	8.64272	9.54279	10.3923	12.3380	14.0415	17.2396
23	9.26042	10.19567	11.6885	13.0905	14.8479	18.1373
24	9.88623	10.8564	12.4011	13.8484	15.6587	19.0372
25	10.5197	11.5240	13.1197	14.6114	16.4734	19.9393
26	11.1603	12.1981	13.8439	15.3791	17.2919	20.8434
27	11.8076	12.8786	14.5733	16.1513	18.1138	21.7494
28	12.4613	13.5648	15.3079	16.9279	18.9392	22.6572
29	13.1211	14.2565	16.0471	17.7083	19.7677	23.5666
30	13.7867	14.9535	16.7908	18.4926	20.5992	24.4776
40	20.7065	22.1643	24.4331	26.5093	29.0505	33.6603
50	27.9907	29.7067	32.3574	34.7642	37.6886	42.9421
60	35.5346	37.4848	40.4817	43.1879	46.4589	52.2938
70	43.2752	45.4418	48.7576	51.7393	55.3290	61.6983
80	51.1720	53.5400	57.1532	60.3915	64.2778	71.1445
90	59.1963	61.7541	65.6466	69.1260	73.2912	80.6247
100	67.3276	70.0648	74.2219	77.9295	82.3581	90.1332

Table B.3 (*continued*)

ν \ P	0.500	0.250	0.100	0.050	0.025	0.010	0.005
1	0.454937	1.32330	2.70554	3.84146	5.02389	6.63490	7.87944
2	1.38629	2.77259	4.60517	5.99147	7.37776	9.21034	10.5966
3	2.36597	1.10835	6.25139	7.81473	9.34840	11.3449	12.8381
4	3.35670	5.38527	7.77944	9.48773	11.1433	13.2767	14.8602
5	4.35146	6.62568	9.23635	11.0705	12.8325	15.0863	16.7496
6	5.34812	7.84080	10.6446	12.5916	14.4494	16.8119	18.5476
7	6.34581	9.03715	12.0170	14.0671	16.0128	18.4753	20.2777
8	7.34412	10.2188	13.3616	15.5073	17.5346	20.0902	21.9550
9	8.34283	11.3887	14.6837	16.9190	19.0228	21.6660	23.5893
10	9.34182	12.5489	15.9871	18.3070	20.4831	23.2093	25.1882
11	10.3410	13.7007	17.2750	19.6751	21.9200	24.7250	26.7569
12	11.3403	14.8454	18.5494	21.0261	23.3367	26.2170	28.2995
13	12.3398	15.9839	19.8119	22.3621	24.7356	27.6883	29.8194
14	13.3393	17.1170	21.0642	23.6848	26.1190	29.1413	31.3193
15	14.3389	18.2451	22.3072	24.9958	27.4884	30.5779	32.8013
16	15.3385	19.3688	23.5418	26.2962	28.8454	31.9999	34.2672
17	16.3381	20.4887	24.4690	27.5871	30.1910	33.4087	35.7185
18	17.3379	21.6049	25.9894	28.8693	31.5264	34.8053	37.1564
19	18.3376	22.7178	27.2036	30.1435	32.8523	36.1908	38.5822
20	19.3374	23.8277	28.4120	31.4104	34.1696	37.5662	39.9968
21	20.3372	24.9348	29.6151	32.6705	35.4789	38.9321	41.4010
22	21.3370	26.0393	30.8133	33.9244	36.7807	40.2894	42.7956
23	22.3369	27.1413	32.0069	35.1725	38.0757	41.6384	44.1813
24	23.3367	28.2412	33.1963	36.4151	39.3641	42.9798	45.5585
25	24.3366	29.3389	34.3816	37.6525	40.6465	44.3141	46.9278
26	25.3364	30.4345	35.5631	38.8852	41.9232	45.6417	48.2899
27	26.3363	31.5284	36.7412	40.1133	43.1944	46.9630	49.6449
28	27.3363	32.6205	37.9159	41.3372	44.4607	48.2782	50.9933
29	28.3362	33.7109	39.0875	42.5569	45.7222	49.5879	52.3356
30	29.3360	34.7998	40.2560	43.7729	46.9792	50.8922	53.6720
40	39.3354	45.6160	51.8050	55.7585	59.3417	63.6907	66.7659
50	49.3349	56.3336	63.1671	67.5048	71.4202	76.1539	79.4900
60	59.3347	66.9814	74.3970	79.0819	83.2976	88.3794	91.9517
70	69.3344	77.5766	85.5271	90.5312	95.0231	100.425	104.215
80	79.3343	88.1303	96.5782	101.879	106.629	112.329	116.321
90	89.3342	98.6499	107.565	113.145	118.136	124.116	128.299
100	99.3341	109.141	118.498	124.342	129.561	135.807	140.169

(Source: Reprinted, by permission of the Biometrika Trustees from C. M. Thompson, "Tables of Percentage Points of the χ^2 Distribution," *Biometrika* 32 (1941): 188–189.)

Table B.4 *F*-Distribution: 5% Points
The first column lists the number of denominator degrees of freedom (v_2). The headings of the other columns list the numerator degrees of freedom (v_1). The table entry is the value of c for which $P(F_{v_1,v_2} \geq c) = .05$.

v_2 \ v_1	1	2	3	4	5	6	7	8	9
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385
3	10.128	9.5521	9.2766	9.1172	9.0135	8.9406	8.8868	8.8452	8.8123
4	7.7086	6.9443	6.5914	6.3883	6.3560	6.1631	6.0942	6.0410	5.9988
5	6.6079	5.7861	5.4095	5.1922	5.0503	4.9503	4.8759	4.8183	4.7725
6	5.9874	5.1433	4.7571	4.5337	1.3874	4.2839	4.2066	4.1468	4.0990
7	5.5914	4.7374	4.3468	4.1203	3.9715	3.8660	3.7870	3.7257	3.6767
8	5.3177	4.4590	4.0662	3.8378	3.6875	3.5806	3.5005	3.4381	3.3881
9	5.1174	4.2565	3.8626	3.6331	3.4817	3.3738	3.2927	3.2296	3.1789
10	4.9646	4.1028	3.7083	3.4780	3.3258	3.2172	3.1355	3.0717	3.0204
11	4.8443	3.9823	3.5874	3.3567	3.2039	3.0946	3.0123	2.9480	2.8962
12	4.7472	3.8856	3.4903	3.2592	3.1059	2.9961	3.9134	2.8486	2.7964
13	4.6672	3.8056	3.4105	3.1791	3.0254	2.9153	2.8321	2.7669	2.7144
14	4.6001	3.7389	3.3439	3.1122	2.9582	2.8477	2.7642	2.6987	2.6458
15	4.5431	3.6823	3.2874	3.0556	2.9013	2.7905	2.7066	2.6408	2.5876
16	4.4940	3.6337	3.2389	3.0069	2.8524	2.7413	2.6572	2.5911	2.5377
17	4.4513	3.5915	3.1968	2.9647	2.8100	2.6987	2.6143	2.5480	2.4943
18	4.4139	3.5546	3.1599	2.9277	2.7729	2.6613	2.5767	2.5102	2.4563
19	4.3808	3.5219	3.1274	2.8951	2.7401	2.6283	2.5435	2.4768	2.4227
20	4.3513	3.4928	3.0984	2.8661	2.7109	2.5990	2.5140	2.4471	2.3928
21	4.3248	3.4668	3.0725	2.8401	2.6848	2.5727	2.4876	2.4205	2.3661
22	4.3009	3.4434	3.0491	2.8167	2.6613	2.5491	2.4638	2.3965	2.3419
23	4.2793	3.4221	3.0280	2.7955	2.6400	2.5277	2.4422	2.3748	2.3201
24	4.2597	3.4028	3.0088	3.7763	2.6207	2.5082	2.4226	2.3551	2.3002
25	4.2417	3.3852	2.9912	2.7587	2.6030	2.4904	2.4047	2.3371	2.2821
26	4.2252	3.3690	2.9751	2.7426	2.5868	2.4741	2.3883	2.3205	2.2655
27	4.2100	3.3541	2.9604	2.7278	2.5719	2.4591	2.3732	2.3053	2.2501
28	4.1960	3.3404	2.9467	2.7141	2.5581	2.4453	2.3593	2.2913	2.2360
29	4.1830	3.3277	2.9340	2.7014	2.5454	2.4324	2.3463	2.2782	2.2229
30	4.1709	3.3158	2.9223	2.6896	2.5336	2.4205	2.3343	2.2662	2.2107
40	4.0848	3.2317	2.8387	2.6060	2.4495	2.3359	2.2490	2.1802	2.1240
60	4.0012	3.1504	2.7581	2.5252	2.3683	2.2540	2.1665	2.0970	2.0401
120	3.9201	3.0718	2.6802	2.4472	2.2900	2.1750	2.0867	2.0164	1.9588
∞	3.8415	2.9957	2.6049	2.3719	2.2141	2.0986	2.0096	1.9354	1.8799

Table B.4 *F*-Distribution: 5% Points (*continued*)

v_2 \ v_1	10	12	15	20	24	30	40	60	120	∞
1	241.88	243.91	245.95	248.01	249.05	250.09	251.14	252.20	253.25	254.32
2	19.396	19.413	19.429	19.446	19.454	19.462	19.471	19.479	19.487	19.496
3	8.7855	8.7446	8.7029	8.6602	8.6385	8.6166	8.5944	8.5720	8.5494	8.5265
4	5.9644	5.9117	5.8578	5.8025	5.7744	5.7459	5.7170	5.6878	5.6581	5.6281
5	4.7351	4.6777	4.6188	4.5581	4.5272	4.4957	4.4638	4.4314	4.3984	4.3650
6	4.0600	3.9999	3.9381	3.8742	3.8415	3.8082	3.7743	3.7398	3.7047	3.6688
7	3.6365	3.5747	3.5108	3.4445	3.4105	3.3758	3.3404	3.3043	3.2674	3.2298
8	3.3472	3.2840	3.2184	3.1503	3.1152	3.0794	3.0428	3.0053	2.9669	2.9276
9	3.1373	3.0729	3.0061	2.9365	2.9005	2.8637	2.8259	2.7872	2.7475	2.7067
10	2.9782	2.9130	2.8450	2.7740	2.7372	2.6996	2.6609	2.6211	2.5801	2.5379
11	2.8536	2.7876	2.7186	2.6464	2.6090	2.5705	2.5309	2.4901	2.4480	2.4045
12	2.7534	2.6866	2.6169	2.5436	2.5055	2.4663	2.4259	2.3842	2.3410	2.2962
13	2.6710	2.6037	2.5331	2.4589	2.4202	2.3803	2.3392	2.2966	2.2524	2.2064
14	2.6021	2.5342	2.4630	2.3879	2.3487	2.3082	2.2664	2.2230	2.1778	2.1307
15	2.5437	2.4753	2.4035	2.3275	2.2878	2.2468	2.2043	2.1601	2.1141	2.0658
16	2.4935	2.4247	2.3522	2.2756	2.2354	2.1938	2.1507	2.1058	2.0589	2.0096
17	2.4499	2.3807	2.3077	2.2304	2.1898	2.1477	2.1040	2.0584	2.0107	1.9604
18	2.4117	2.3421	2.2686	2.1906	2.1497	2.1071	2.0629	2.0166	1.9681	1.9168
19	2.3779	2.3080	2.2341	2.1555	2.1141	2.0712	2.0264	1.9796	1.9302	1.8780
20	2.3479	2.2776	2.2033	2.1242	2.0825	2.0391	1.9938	1.9464	1.8963	1.8432
21	2.3210	2.2504	2.1757	2.0960	2.0540	2.0102	1.9645	1.9165	1.8657	1.8117
22	2.2967	2.2258	2.1508	2.0707	2.0283	1.9842	1.9380	1.8895	1.8380	1.7831
23	2.2747	2.2036	2.1282	1.0476	2.0050	1.9605	1.9139	1.8649	1.8128	1.7570
24	2.2547	2.1834	2.1077	2.0267	1.9838	1.9390	1.8920	1.8424	1.7897	1.7331
25	2.2365	2.1649	2.0889	2.0075	1.9643	1.9192	1.8718	1.8217	1.7684	1.7110
26	2.2197	2.1479	2.0716	1.9898	1.9464	1.9010	1.8533	1.8027	1.7488	1.6906
27	2.2043	2.1323	2.0558	1.9736	1.9299	1.8842	1.8361	1.7851	1.7307	1.6717
28	2.1900	2.1179	2.0411	1.9586	1.9147	1.8687	1.8203	1.7689	1.7138	1.6541
29	2.1768	2.1045	2.0245	1.9446	1.9005	1.8543	1.8055	1.7537	1.6981	1.6377
30	2.1646	2.0921	2.0148	1.9317	1.8874	1.8409	1.7918	1.7396	1.6835	1.6223
40	2.0772	2.0035	1.9245	1.8389	1.7929	1.7444	1.6928	1.6373	1.5766	1.5089
60	1.9926	1.9174	1.8364	1.7480	1.7001	1.6491	1.5943	1.5343	1.4673	1.3893
120	1.9105	1.8337	1.7505	1.6587	1.6084	1.5543	1.4952	1.4290	1.3519	1.2539
∞	1.8307	1.7522	1.6664	1.5705	1.5173	1.4591	1.3940	1.3180	1.2214	1.0000

(Source: Reprinted, by permission of the Biometrika Trustees from M. Merrington and C. M. Thompson, "Tables of Percentage Points of the Inverted Beta(*F*) Distribution," *Biometrika* 33 (1943): 80–81.)

Table B.5 *F*-Distribution: 1% Points
The first column lists the number of denominator degrees of freedom (v_2). The headings of the other columns list the numerator degrees of freedom (v_1). The table entry is the value of c for which $P(F_{v_1,v_2} \geq c) = .01$.

v_2 \ v_1	1	2	3	4	5	6	7	8	9
1	4052.2	4999.5	5403.3	5624.6	5763.7	5859.0	5928.3	5981.6	6022.5
2	98.503	99.000	99.166	99.249	99.299	99.332	99.356	99.374	99.388
3	34.116	30.817	29.457	28.710	28.237	27.911	27.672	27.489	27.345
4	21.198	18.000	16.694	15.977	15.522	15.207	14.976	14.799	14.659
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158
6	13.745	10.925	9.7795	9.1483	8.7459	8.4661	8.2600	8.1016	7.9761
7	12.246	9.5466	8.4513	7.8467	7.4604	7.1914	6.9928	6.8401	6.7188
8	11.259	8.6491	7.5910	7.0060	6.6318	6.3707	6.1776	6.0289	5.9106
9	10.561	8.0215	6.9919	6.4221	6.0569	5.8018	5.6129	5.4671	5.3511
10	10.044	7.5594	6.5523	5.9943	5.6363	5.3858	5.2001	5.0567	4.9424
11	9.6460	7.2057	6.2167	5.6683	5.3160	5.0692	4.8861	4.7445	4.6315
12	9.3302	6.9266	5.9526	5.4119	5.0643	4.8206	4.6395	4.4994	4.3875
13	9.0738	6.7010	5.7394	5.2053	4.8616	4.6204	4.4410	4.3021	4.1911
14	8.8616	6.5149	5.5639	5.0354	4.6950	4.4558	4.2779	4.1399	4.0297
15	8.6831	6.3589	5.4170	4.8932	4.5556	4.3183	4.1415	4.0045	3.8948
16	8.5310	6.2262	5.2922	4.7726	4.4374	4.2016	4.0259	3.8896	3.7804
17	8.3997	6.1121	5.1850	4.6690	4.3359	4.1015	3.9267	3.7910	3.6822
18	8.2854	6.0129	5.0919	4.5790	4.2479	4.0146	3.8406	3.7054	3.5971
19	8.1850	5.9259	5.0103	4.5003	4.1708	3.9386	3.7653	3.6305	3.5225
20	8.0906	5.8489	4.9382	4.4307	4.1027	3.8714	3.6987	3.5644	3.4567
21	8.0166	5.7804	4.8740	4.3688	4.0421	3.8117	3.6396	3.5056	3.3981
22	7.9454	5.7190	4.8166	4.3134	3.9880	3.7583	3.5867	3.4530	3.3458
23	7.8811	5.6637	4.7649	4.2635	3.9392	3.7102	3.5390	3.4057	3.2986
24	7.8229	5.6131	4.7181	4.2184	3.8951	3.6667	3.4959	3.3629	3.2560
25	7.7689	5.5680	4.6755	4.1774	3.8550	3.6272	3.4568	3.3239	3.2172
26	7.7213	5.5263	4.6366	4.1400	3.8183	3.5911	3.4210	3.2884	3.1818
27	7.6767	5.4881	4.6009	4.1056	3.7848	3.5580	3.3882	3.2558	3.1494
28	7.6356	5.4529	4.5681	4.0740	3.7539	3.5276	3.3581	3.2259	3.1195
29	7.5976	5.4205	4.5378	4.0449	3.7254	3.4995	3.3302	3.1982	3.0920
30	7.5625	5.3904	4.5097	4.0179	3.6990	3.4735	3.3045	3.1726	3.0665
40	7.3141	5.1785	4.3126	3.8283	3.5138	3.2910	3.1238	2.9930	2.8876
60	7.0771	4.9774	4.1259	3.6491	3.3389	3.1187	2.9530	2.8233	2.7185
120	6.8510	4.7865	3.9493	3.4796	3.1735	2.9559	2.7918	2.6629	2.5586
∞	6.6349	4.6052	3.7816	3.3192	3.0173	2.8020	2.6393	2.5113	2.4073

Table B.5 *F*-Distribution: 1% Points (*continued*)

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	6055.8	6106.3	6157.3	6208.7	6234.6	6260.7	6286.8	6313.0	6339.4	6366.0
2	99.399	99.416	99.449	99.458	99.458	99.466	99.474	99.483	99.491	99.501
3	27.229	27.052	26.872	26.690	26.598	26.505	26.411	26.316	26.221	26.125
4	14.546	14.374	14.198	14.020	13.929	13.838	13.745	13.652	13.558	13.463
5	10.051	9.8883	9.7222	9.5527	9.4665	9.3793	9.2912	9.2020	9.1118	9.0204
6	7.8741	7.7183	7.5590	7.3958	7.3127	7.2285	7.1432	7.0568	6.9690	6.8801
7	6.6201	6.4691	6.3143	6.1554	6.0743	5.9921	5.9084	5.8236	5.7372	5.6495
8	5.8143	5.6668	5.5151	5.3591	5.2793	5.1981	5.1156	5.0316	4.9460	4.8588
9	5.2565	5.1114	4.9621	4.8080	4.7290	4.6486	4.5667	4.4831	4.3978	4.3105
10	4.8492	4.7059	4.5582	4.4054	4.3269	4.2469	4.1653	4.0819	3.9965	3.9090
11	4.5393	4.3974	4.2509	4.0990	4.0209	3.9411	3.8596	3.7761	3.6904	3.6025
12	4.2961	4.1553	4.0096	3.8584	3.7805	3.7008	3.6192	3.5355	3.4494	3.3608
13	4.1003	3.9603	3.8154	3.6646	3.5868	3.5070	3.4253	3.3413	3.2548	3.1654
14	3.9394	3.8001	3.6557	3.5052	3.4274	3.3476	3.2656	3.1813	3.0942	3.0040
15	3.8049	3.6662	3.5255	3.3719	3.2940	3.2141	3.1319	3.0471	2.9595	2.8684
16	3.6909	3.5527	3.4089	3.2588	3.1808	3.1007	3.0182	2.9330	2.8447	2.7528
17	3.5931	3.4552	3.3117	3.1615	3.0835	3.0032	2.9205	2.8348	2.7459	2.6530
18	3.5082	3.3706	3.2273	3.0771	2.9990	2.9185	3.8354	2.7493	2.6597	2.5660
19	3.4338	3.2965	3.1533	3.0031	2.9249	2.8442	2.7608	2.6742	2.5839	2.4893
20	3.3682	3.2311	3.0880	2.9377	2.8594	2.7785	2.6947	2.6077	2.5168	2.4212
21	3.3098	3.1729	3.0299	2.8796	2.8011	2.7200	2.6359	2.5484	2.4568	2.3603
22	3.2576	3.1209	2.9780	2.8274	2.7488	2.6675	2.5831	2.4951	2.4029	2.3055
23	3.2106	3.0740	2.9311	2.7805	2.7017	2.6202	2.5355	2.4471	2.3542	2.2559
24	3.1681	3.0316	2.8887	2.7380	2.6591	2.5773	2.4923	2.4035	2.3099	2.2107
25	3.1294	2.9331	2.8502	2.6993	2.6203	2.5383	2.4530	2.3667	2.2695	2.1694
26	3.0941	2.9576	2.8150	2.6640	2.5848	2.5026	2.4170	2.3273	2.2325	2.1315
27	3.0618	2.2956	2.7827	2.6316	2.5522	2.4699	2.3840	2.2938	2.1984	2.0965
28	3.0320	2.8959	2.7530	2.6017	2.5223	2.4397	2.3535	2.2629	2.1670	2.0642
29	3.0045	2.8685	2.7256	2.5742	2.4946	2.4118	2.3253	2.2344	2.1378	2.0342
30	2.9791	2.8431	2.7002	2.5487	2.4689	2.3680	2.2992	2.2079	2.1107	2.0062
40	2.8005	2.6648	2.5216	2.3689	2.2880	2.2034	2.1142	2.0194	1.9172	1.8047
60	2.6318	2.4961	2.3523	2.1978	2.1154	2.0285	1.9360	1.8363	1.7263	1.6006
120	2.4721	2.3363	2.1915	2.0346	1.9500	1.8600	1.7628	1.6557	1.5330	1.3805
∞	2.3209	2.1848	2.0385	1.8783	1.7908	1.6964	1.5923	1.4730	1.3246	1.0000

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Index