Solution to the Differential Equation $\ddot{x} + \beta\dot{x} = g$

Solution

To solve the differential equation $\ddot{x} + \beta\dot{x} = g$, where g is a constant, $\dot{x} = \frac{dx}{dt}$, and $\ddot{x} = \frac{d^2x}{dt^2}$, we recognize it as a second-order linear ordinary differential equation with constant coefficients and a constant forcing term. Below is a step-by-step solution.

Step 1: Rewrite the Equation

The equation is:

$$\ddot{x} + \beta\dot{x} = g \tag{1}$$

This can be written in standard form:

$$\ddot{x} + \beta\dot{x} - g = 0 \tag{2}$$

The left-hand side involves derivatives, and the right-hand side is a constant, indicating a non-homogeneous linear ODE.

Step 2: Solve the Homogeneous Equation

First, solve the associated homogeneous equation:

$$\ddot{x} + \beta\dot{x} = 0 \tag{3}$$

Assume a solution of the form $x(t) = e^{rt}$. Substituting gives the characteristic equation:

$$r^2 + \beta r = 0 \tag{4}$$

$$r(r + \beta) = 0$$

The roots are:

$$r = 0 \quad \text{or} \quad r = -\beta$$

Since the roots are real and distinct, the homogeneous solution is:

$$x_h(t) = c_1 + c_2 e^{-\beta t} \tag{5}$$

where c_1 and c_2 are arbitrary constants.

Step 3: Find a Particular Solution

For the non-homogeneous equation $\ddot{x} + \beta\dot{x} = g$, since g is a constant and $r = 0$ is a root of the characteristic equation, try a particular solution of the form:

$$x_p(t) = At \tag{6}$$

Then:

$$\dot{x}_p = A, \quad \ddot{x}_p = 0$$

Substitute into the differential equation:

$$0 + \beta \cdot A = g$$

$$A = \frac{g}{\beta} \quad (\beta \neq 0)$$

Thus, the particular solution is:

$$x_p(t) = \frac{g}{\beta}t \tag{7}$$

Step 4: General Solution

The general solution is the sum of the homogeneous and particular solutions:

$$x(t) = x_h(t) + x_p(t) = c_1 + c_2e^{-\beta t} + \frac{g}{\beta}t \tag{8}$$

Step 5: Consider the Case $\beta = 0$

If $\beta = 0$, the equation becomes:

$$\ddot{x} = g \tag{9}$$

Integrate twice:

$$\dot{x} = \int g\,dt = gt + c_1$$

$$x = \int (gt + c_1)\,dt = \frac{1}{2}gt^2 + c_1t + c_2$$

So, for $\beta = 0$:

$$x(t) = \frac{1}{2}gt^2 + c_1t + c_2 \tag{10}$$

Step 6: Interpret the Solution

- For $\beta \neq 0$, the solution $x(t) = c_1 + c_2e^{-\beta t} + \frac{g}{\beta}t$ includes:
  - A constant term c_1.
  - An exponential term $c_2e^{-\beta t}$, which decays if $\beta > 0$ or grows if $\beta < 0$.
  - A linear term $\frac{g}{\beta}t$, representing the effect of the constant forcing g.
- For $\beta = 0$, the solution is quadratic, indicating unbounded growth.
- Constants c_1 and c_2 are determined by initial conditions (e.g., $x(0)$ and $\dot{x}(0)$).

Final Answer

The solution to the differential equation $\ddot{x} + \beta\dot{x} = g$ is:

- If $\beta \neq 0$:
$$x(t) = c_1 + c_2 e^{-\beta t} + \frac{g}{\beta} t \tag{11}$$

- If $\beta = 0$:
$$x(t) = \frac{1}{2} g t^2 + c_1 t + c_2 \tag{12}$$

where c_1 and c_2 are constants determined by initial conditions.