Solution to the Differential Equation $mg - \beta\dot{x}^2 - k(x - x_0) = m\ddot{x}$

Solution with Initial Conditions $x(0) = x_0$, $\dot{x}(0) = v_0$

Given the differential equation $mg - \beta\dot{x}^2 - k(x - x_0) = m\ddot{x}$ with initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$, where m, g, β, k, and x_0 are constants, $\dot{x} = \frac{dx}{dt}$, and $\ddot{x} = \frac{d^2x}{dt^2}$, we solve as follows. The equation is nonlinear due to the $\beta\dot{x}^2$ term, complicating analytical solutions.

Step 1: Rewrite and Simplify

The equation is:

$$m\ddot{x} + \beta\dot{x}^2 + k(x - x_0) = mg \tag{1}$$

Substitute $y = x - x_0$, so $x = y + x_0$, $\dot{x} = \dot{y}$, $\ddot{x} = \ddot{y}$:

$$m\ddot{y} + \beta\dot{y}^2 + ky = mg \tag{2}$$

The initial conditions become:

$$y(0) = x(0) - x_0 = x_0 - x_0 = 0, \quad \dot{y}(0) = \dot{x}(0) = v_0$$

Step 2: Equilibrium Analysis

Set $\ddot{y} = \dot{y} = 0$:

$$ky = mg \implies y = \frac{mg}{k}$$

Thus, the equilibrium position is $x = x_0 + \frac{mg}{k}$, where the spring force balances gravity.

Step 3: Attempt Analytical Solution

The nonlinear term $\beta\dot{y}^2$ makes analytical solutions challenging. We explore homogeneous and particular solutions.

Homogeneous Equation

For $mg = 0$ and $\beta = 0$:

$$m\ddot{y} + ky = 0 \tag{3}$$

$$\ddot{y} + \omega^2 y = 0, \quad \omega = \sqrt{\frac{k}{m}}$$

The solution is:

$$y_h(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \tag{4}$$

However, the $\beta\dot{y}^2$ term prevents direct use.

Particular Solution

Try $y_p = A$ (constant):

$$\ddot{y}_p = 0, \quad \dot{y}_p = 0$$

$$kA = mg \implies A = \frac{mg}{k}$$

$$y_p(t) = \frac{mg}{k}, \quad x_p(t) = x_0 + \frac{mg}{k}$$

Verify:

$$m \cdot 0 + \beta \cdot 0 + k \cdot \frac{mg}{k} = mg$$

This satisfies the equation.

Step 4: Phase-Plane Analysis

Set $\dot{y} = v$, so $\ddot{y} = \frac{dv}{dt} = v\frac{dv}{dy}$:

$$mv\frac{dv}{dy} + \beta v^2 + ky = mg \tag{5}$$

$$v\frac{dv}{dy} = \frac{mg - \beta v^2 - ky}{m}$$

Or in time:

$$\frac{dv}{dt} = \frac{mg - \beta v^2 - ky}{m} \tag{6}$$

With $\frac{dy}{dt} = v$, we have:

$$\frac{dy}{dt} = v \tag{7}$$

$$\frac{dv}{dt} = \frac{mg - \beta v^2 - ky}{m} \tag{8}$$

Initial conditions: $y(0) = 0$, $v(0) = v_0$. This system is nonlinear and typically requires numerical methods.

Step 5: Apply Initial Conditions

The particular solution $y_p = \frac{mg}{k}$ gives $y(0) = \frac{mg}{k} \neq 0$, which doesn't satisfy the initial condition. The nonlinear term requires a dynamic solution.

Step 6: Terminal Velocity

Assume $\ddot{y} = 0$, $\dot{y} = v_t$:

$$\beta v_t^2 + ky = mg$$

At equilibrium $y = \frac{mg}{k}$:

$$\beta v_t^2 = 0 \implies v_t = 0$$

This suggests no terminal velocity at equilibrium.

Step 7: Linear Approximation

For small β, approximate $\beta \dot{y}^2 \approx 0$:

$$m\ddot{y} + ky \approx mg \tag{9}$$

$$\ddot{y} + \omega^2 y = g, \quad \omega = \sqrt{\frac{k}{m}}$$

Particular solution: $y_p = \frac{mg}{k}$. Homogeneous solution:

$$y_h = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

General solution:

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{mg}{k} \tag{10}$$

Apply initial conditions:

$$y(0) = 0 = c_1 + \frac{mg}{k} \implies c_1 = -\frac{mg}{k}$$

$$\dot{y}(t) = -c_1\omega \sin(\omega t) + c_2\omega \cos(\omega t), \quad \dot{y}(0) = v_0 = c_2\omega \implies c_2 = \frac{v_0}{\omega}$$

$$y(t) = -\frac{mg}{k} \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) + \frac{mg}{k}$$

$$x(t) = x_0 + \frac{mg}{k} (1 - \cos(\omega t)) + \frac{v_0}{\omega} \sin(\omega t), \quad \omega = \sqrt{\frac{k}{m}}$$

Step 8: Numerical Solution

The nonlinear system:

$$\frac{dy}{dt} = v \tag{11}$$

$$\frac{dv}{dt} = \frac{mg - \beta v^2 - ky}{m} \tag{12}$$

with $y(0) = 0$, $v(0) = v_0$ requires numerical methods (e.g., Runge-Kutta) for an exact solution.

Final Answer

The differential equation $mg - \beta\dot{x}^2 - k(x - x_0) = m\ddot{x}$ with $x(0) = x_0$, $\dot{x}(0) = v_0$ is nonlinear. In terms of $y = x - x_0$:

$$m\ddot{y} + \beta\dot{y}^2 + ky = mg, \quad y(0) = 0, \quad \dot{y}(0) = v_0 \tag{13}$$

- **Approximate solution** (small β):

$$x(t) \approx x_0 + \frac{mg}{k}\left(1 - \cos\left(\sqrt{\frac{k}{m}}t\right)\right) + \frac{v_0}{\sqrt{\frac{k}{m}}}\sin\left(\sqrt{\frac{k}{m}}t\right) \tag{14}$$

- **Exact solution**: Solve the system:

$$\frac{dy}{dt} = v \tag{15}$$

$$\frac{dv}{dt} = \frac{mg - \beta v^2 - ky}{m} \tag{16}$$

  numerically with $y(0) = 0$, $v(0) = v_0$. The behavior depends on m, β, k, g, and v_0.