Fair Division

Cake Cutting Algorithms: Be Fair if You Can

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Overview

Meeting 1

Agenda

- Introduction
- Fair Division for n Players
    - Banach Knaster
    - Dubins Spanier
    - Even Paz

Introduction

Imagine two people want to share this cake.

Introduction

- The cake is complicated
- The two people may value different parts of the cake differently

Introduction

- The cake is complicated
- The two people may value different parts of the cake differently
- Can we come up with an algorithm where both people are happy?

Cut and Choose

Procedure

1. Player 1 cuts the cake into what they believe is half
2. Player 2 chooses the piece which they think is better

Proof of Correctness

- Player 1 recieves $\frac{1}{2}$ of the cake
- Player 1 values Player 2's allocation to also be worth $\frac{1}{2}$

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- Player 1 recieves $\frac{1}{2}$ of the cake
- Player 1 values Player 2's allocation to also be worth $\frac{1}{2}$
- Player 2 recieved the piece which they thought was better
- Player 2 must value their piece to be at least $\frac{1}{2}$ of the cake

Banach-Knaster Last Diminisher

Procedure

1. Player 1 cuts $\frac{1}{n}$ of the cake
2. Player 2 through n
    - If they believe the piece is worth $> \frac{1}{n}$ of the cake, they may trim it
    - If they believe the piece is worth $\leq \frac{1}{n}$ of the cake, they may pass it to the next person

Procedure

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2. Player 2 through n
   - If they believe the piece is worth $> \frac{1}{n}$ of the cake, they may trim it
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3. The last person to trim the piece recieves it and drops out

Procedure

1. Player 1 cuts $\frac{1}{n}$ of the cake
2. Player 2 through n
   - If they believe the piece is worth $> \frac{1}{n}$ of the cake, they may trim it
   - If they believe the piece is worth $\leq \frac{1}{n}$ of the cake, they may pass it to the next person
3. The last person to trim the piece recieves it and drops out
4. Repeat until no players remain

Proof of Correctness

- Cutting a piece to be $> \frac{1}{n}$ can cause further division to be limited to $< \frac{1}{n}$ of the cake
- This is most easily seen with an extreme example

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- Cutting a piece to be $> \frac{1}{n}$ can cause further division to be limited to $< \frac{1}{n}$ of the cake
- This is most easily seen with an extreme example
    1. Person 1 cuts 98% of the cake, with the goal of taking it for themselves.
    2. After passing the cake around, the last diminisher has only cut the piece down to 97% of the value of the cake
    3. Person 1 now cannot receive more than 3% of the cake.

Dubins-Spanier Moving Knife

Procedure

- Rather than having many cuts, a "moving knife" can be used to allocate chunks of cake.
    1. A knife moves over the cake continuously from one side to the opposite side (for example from left to right)
    2. When a person thinks that the portion remaining from the starting side/previous cut is worth $\frac{1}{n}$, then they may say "Cut", and they will take the portion on the left side.

Proof of Correctness

- The same person who said "Cut" at any given point would have been the last diminisher in in the Banach-Knaster Last Diminisher Method.

Proof of Correctness

- The same person who said "Cut" at any given point would have been the last diminisher in in the Banach-Knaster Last Diminisher Method.
- On a surface level, this seems to take n-1 cuts, but this is incorrect. Instead, it takes an infinite number of cuts perpendicular to the direction of movement.

Even-Paz Divide and Conquer

Procedure

1. Players 1...n-1 cut the cake in half
2. Player n compares the cake to the left and to the right of middle cut and chooses the piece which they think is bigger.

Procedure

1. Players 1...n-1 cut the cake in half
2. Player n compares the cake to the left and to the right of middle cut and chooses the piece which they think is bigger.
3. Player n and the players on the side n chose repeat the procedure on that side
4. The remaining players repeat the procedure on the other side

Meeting 2

Agenda

- Stromquist Envy-Free Moving Knife
- Austin's Perfect Division for n=2
- Aziz-Mackenzie Envy-Free Procedure

Stromquist Envy Free Moving Knife

Procedure

Austin's Perfect Division for n=2

Defining Perfect Division

Procedure

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Aziz-Mackenzie Envy-Free Procedure

Aziz-Mackenzie Envy-Free Procedure for n

https://youtu.be/fvM8ow6zNw4?si=AGr0GF7vSZSGt4QK&t=711

Meeting 3

Agenda

- HOWTO: Reading Papers
- Unequal Division Naïvely
- Cutting into 1-sized parts
- Ramsey Partitions
- Near-Halves

Next Week: Finish Chapter 3 & Chapter 4

Reading Papers

Let's be honest

Most papers are dryer than the Sahara Desert

Reading Papers

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Most papers are dryer than the Sahara Desert

So why read papers?

Reading Papers

- Understanding the field better
-

Fundamentally, we are learning: But what are we trying to learn?

Reading Papers

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Reading Papers

But what are we trying to learn?
Our goal when reading a paper is to contextualize that paper's findings in the field.

Reading Papers

Thankfully, scientists are aware of this, so they write about it.

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Parts of a Paper

- Title & Authors

Reading Papers

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Parts of a Paper

- Title & Authors
- Abstract

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Parts of a Paper

- Title & Authors
- Abstract
- Introduction

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Parts of a Paper

- Title & Authors
- Abstract
- Introduction
- Related Works

Reading Papers

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Parts of a Paper

- Title & Authors
- Abstract
- Introduction
- Related Works
- Content

Reading Papers

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Parts of a Paper

- Title & Authors
- Abstract
- Introduction
- Related Works
- Content
- Discussion/Conclusion

Reading Papers

`https://matt.might.net/articles/phd-school-in-pictures/IllustratedGuidePhD-Matt-Might.pdf`

Dividing Camels

First son gets $\frac{1}{2}$. Second son gets $\frac{1}{3}$. Third son gets $\frac{1}{9}$