Fair Division

Cake Cutting Algorithms: Be Fair if You Can

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Overview

Meeting 1

Agenda

- Introduction
- Fair Division for n Players
  - Banach Knaster
  - Dubins Spanier
  - Even Paz

Introduction

Imagine two people want to share this cake.

Introduction

- The cake is complicated
- The two people may value different parts of the cake differently

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- The cake is complicated
- The two people may value different parts of the cake differently
- Can we come up with an algorithm where both people are happy?

Cut and Choose

Procedure

1. Player 1 cuts the cake into what they believe is half
2. Player 2 chooses the piece which they think is better

Proof of Correctness

- Player 1 recieves $\frac{1}{2}$ of the cake
- Player 1 values Player 2's allocation to also be worth $\frac{1}{2}$

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- Player 1 recieves $\frac{1}{2}$ of the cake
- Player 1 values Player 2's allocation to also be worth $\frac{1}{2}$
- Player 2 recieved the piece which they thought was better
- Player 2 must value their piece to be at least $\frac{1}{2}$ of the cake

Banach-Knaster Last Diminisher

Procedure

1. Player 1 cuts $\frac{1}{n}$ of the cake
2. Player 2 through n
   - If they believe the piece is worth $> \frac{1}{n}$ of the cake, they may trim it
   - If they believe the piece is worth $\leq \frac{1}{n}$ of the cake, they may pass it to the next person

Procedure

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   - If they believe the piece is worth $> \frac{1}{n}$ of the cake, they may trim it
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3. The last person to trim the piece recieves it and drops out

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1. Player 1 cuts $\frac{1}{n}$ of the cake
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3. The last person to trim the piece recieves it and drops out
4. Repeat until no players remain

Proof of Correctness

- Cutting a piece to be $> \frac{1}{n}$ can cause further division to be limited to $< \frac{1}{n}$ of the cake
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- This is most easily seen with an extreme example
  1. Person 1 cuts 98% of the cake, with the goal of taking it for themselves.
  2. After passing the cake around, the last diminisher has only cut the piece down to 97% of the value of the cake
  3. Person 1 now cannot receive more than 3% of the cake.

Dubins-Spanier Moving Knife

Procedure

- Rather than having many cuts, a "moving knife" can be used to allocate chunks of cake.
  1. A knife moves over the cake continuously from one side to the opposite side (for example from left to right)
  2. When a person thinks that the portion remaining from the starting side/previous cut is worth $\frac{1}{n}$, then they may say "Cut", and they will take the portion on the left side.

Proof of Correctness

- The same person who said "Cut" at any given point would have been the last diminisher in in the Banach-Knaster Last Diminisher Method.

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- The same person who said "Cut" at any given point would have been the last diminisher in in the Banach-Knaster Last Diminisher Method.
- On a surface level, this seems to take n-1 cuts, but this is incorrect. Instead, it takes an infinite number of cuts perpendicular to the direction of movement.

Even-Paz Divide and Conquer

Procedure

1. Players 1...n-1 cut the cake in half
2. Player n compares the cake to the left and to the right of middle cut and chooses the piece which they think is bigger.

Procedure

1. Players 1...n-1 cut the cake in half
2. Player n compares the cake to the left and to the right of middle cut and chooses the piece which they think is bigger.
3. Player n and the players on the side n chose repeat the procedure on that side
4. The remaining players repeat the procedure on the other side

Meeting 2

Agenda

- Stromquist Envy-Free Moving Knife
- Austin's Perfect Division for n=2
- Aziz-Mackenzie Envy-Free Procedure

Stromquist Envy Free Moving Knife

Procedure

Austin's Perfect Division for n=2

Defining Perfect Division

Procedure

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Aziz-Mackenzie Envy-Free Procedure

Aziz-Mackenzie Envy-Free Procedure for n

https://youtu.be/fvM8ow6zNw4?si=AGrOGF7vSZSGt4QK&t=711