Fair Division

Cake Cutting Algorithms: Be Fair if You Can

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Overview I

Overview II

Meeting 1

Agenda

- Introduction
- Fair Division for n Players
  - Banach Knaster
  - Dubins Spanier
  - Even Paz

Introduction

Imagine two people want to share this cake.

Introduction

- The cake is complicated
- The two people may value different parts of the cake differently

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- The two people may value different parts of the cake differently
- Can we come up with an algorithm where both people are happy?

Cut and Choose

Procedure

1. Player 1 cuts the cake into what they believe is half
2. Player 2 chooses the piece which they think is better

Proof of Correctness

- Player 1 recieves $\frac{1}{2}$ of the cake
- Player 1 values Player 2's allocation to also be worth $\frac{1}{2}$

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- Player 1 values Player 2's allocation to also be worth $\frac{1}{2}$
- Player 2 recieved the piece which they thought was better
- Player 2 must value their piece to be at least $\frac{1}{2}$ of the cake

Banach-Knaster Last Diminisher

Procedure

1. Player 1 cuts $\frac{1}{n}$ of the cake
2. Player 2 through n
   - If they believe the piece is worth $> \frac{1}{n}$ of the cake, they may trim it
   - If they believe the piece is worth $\leq \frac{1}{n}$ of the cake, they may pass it to the next person

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3. The last person to trim the piece recieves it and drops out

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4. Repeat until no players remain

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  1. Person 1 cuts 98% of the cake, with the goal of taking it for themselves.
  2. After passing the cake around, the last diminisher has only cut the piece down to 97% of the value of the cake
  3. Person 1 now cannot receive more than 3% of the cake.

Dubins-Spanier Moving Knife

Procedure

- Rather than having many cuts, a ”moving knife” can be used to allocate chunks of cake.
    1. A knife moves over the cake continuously from one side to the opposite side (for example from left to right)
    2. When a person thinks that the portion remaining from the starting side/previous cut is worth $\frac{1}{n}$, then they may say ”Cut”, and they will take the portion on the left side.

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- The same person who said "Cut" at any given point would have been the last diminisher in in the Banach-Knaster Last Diminisher Method.
- On a surface level, this seems to take n-1 cuts, but this is incorrect. Instead, it takes an infinite number of cuts perpendicular to the direction of movement.

Even-Paz Divide and Conquer

Procedure

1. Players 1...n-1 cut the cake in half
2. Player n compares the cake to the left and to the right of middle cut and chooses the piece which they think is bigger.

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3. Player n and the players on the side n chose repeat the procedure on that side
4. The remaining players repeat the procedure on the other side

Meeting 2

Agenda

- Stromquist Envy-Free Moving Knife
- Austin's Perfect Division for n=2
- Aziz-Mackenzie Envy-Free Procedure

Stromquist Envy Free Moving Knife

Austin's Perfect Division for n=2

Defining Perfect Division

Perfect Division is the allocation where

$$\forall_i V_i(A_i) = \frac{1}{n}$$

No bounded algorithm exists for perfect division

Procedure

1. Knife moves from left to right
2. A player may call stop

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3. A second knife is placed on the left edge
4. Both knives move parallely
5. The other player calls stop

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5. The other player calls stop
6. Assign pieces arbitrarily

Aziz-Mackenzie Envy-Free Procedure

Aziz-Mackenzie Envy-Free Procedure for n

https://youtu.be/fvM8ow6zNw4?si=AGrOGF7vSZSGt4QK&t=711

Meeting 3

Agenda

- HOWTO: Reading Papers
- Unequal Division Naïvely
- Cutting into 1-sized parts
- Ramsey Partitions
- Halving

Next Week: Finish Chapter 3 & Chapter 4

Reading Papers

Let's be honest

Most papers are dryer than the Sahara Desert

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So why read papers?

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- Understanding the current research better

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Fundamentally, we are learning: But what are we trying to learn?

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Our goal when reading a paper is to contextualize that paper's findings in the field.

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- Related Works

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- Content
- Discussion/Conclusion

PhD

https://matt.might.net/articles/phd-school-in-pictures/IllustratedGuidePhD-Matt-Might.pdf

Dividing Camels

First son gets $\frac{1}{2}$. Second son gets $\frac{1}{3}$. Third son gets $\frac{1}{9}$

Naïve Method

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- Duplicate each player proportional to their ratio.
- We will say that
  - Player 1 recieves A_1 allocation
  - Player 2 recieves A_2 allocation
- Using the Even-Paz method, $\theta((\sum A)\log(\sum A))$ cuts are required.

- We can do better!

Cutting Ones

- Similar to the cut-and-choose algorithm

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- Player 1 divides the cake into 1-sized parts
- Player 2 chooses A_2 of the 1-sized parts

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- We can observe this finding $R(3,3)$
- The same property seen in Ramsey Theory can also be used to help partition

Ramsey Partitions

- Assume $A_1 < A_2$
- Player 1 cuts what they percieve to be A_1 of the cake.
- The smaller piece can be called X_1 and the larger piece can be called X_2
- If $\mu_2(X_2) < A_2$ (If the person thinks they got less than they were supposed to)
- Player 2 takes X_1 and division continues in the ratio $A_2 - A_1 : A_1$ until no cake remains

Ramsey Partitions

- This can be simplified using ramsey partitions
- Person 1 cuts the Ramsey Partitions of $\sum A$
- Person 2 marks all pieces $\mu_2(X_j) > \mu_1(X_j)$ (Everything they are willing to accept)

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  1. If the sum of a subset of the marked pieces $= A_2$, we can give those pieces to Player 2 and give the rest to Player 1
  2. Otherwise, Player 1 may choose pieces summing to A_1 of the unmarked pieces.
  3. This works because of the Ramsey Partitioning!

Meeting 4

Agenda

- Review
- Halving
- The Serendipity of Disagreement
- Summer

Halving

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- Why were ramsey partitions more effective than Cutting Ones?
- They allow us to allocate more of the cake at once
- We can get rid of more of the cake at once by cutting closer to $\frac{1}{2}$ of the cake at once.

Halving

- Take the example of dividing 13 with a 8:5 ratio

Halving

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- Let's draw the tree

Halving

- Take the example of dividing 13 with a 8:5 ratio
- Let's draw the tree
- It can be shown that this is at least as good as Ramsey Partitioning.

Disagreement

- Sometimes it may feel as if differences of opinion cause conflict
- But through the existence of envy-free division, we can see this may actually lead to more social good

Disagreement

- Each person divides the cake into n parts (n-1 lines)
- Each person can receive a piece of cake which they think is at least $\frac{1}{n}$

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- Each person can receive a piece of cake which they think is at least $\frac{1}{n}$
- If at least 1 person disagrees, there will be a way to allocate with excess

Meeting 5

Agenda

- The Serendipity of Disagreement
- Strong Fair Division
- Classes of Fair Division

Strong Fair Division

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- It is clear to see that there may not always exist a strongly fair allocation

Strong Fair Division

- All players think they got more than $\frac{1}{n}$ of the cake
- It is clear to see that there may not always exist a strongly fair allocation
- If $\exists_{i,j} \mu_i \neq \mu_j$, we can always create a strongly fair allocation

This is the value of disagreement!

Strong Fair Division

- Imagine a piece of cake $A \subseteq X$
- If $\mu_1(A) > \mu_2(A)$, we may continue

Strong Fair Division

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- If $\mu_1(A) > \mu_2(A)$, we may continue
  - Let $\mu_1(A) = a$
  - Let $\mu_2(A) = b$
- Let $X - A = B$

Strong Fair Division$_{n=2}$

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- $1 - b > 1 - \frac{p}{q} = \frac{q-p}{q}$

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- Player 2 cuts B into $q - p$ equal parts, each with value $> \frac{1}{q}$
- They each disagreed with the other person initially $\implies$
- $\exists_i \ \mu_2(A_i) < \frac{1}{q} \wedge \exists_j \ \mu_1(B_j) < \frac{1}{q}$
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- Cut and choose all other piece of cake $(X - A_i - B_i)$
- Give A_i to 1 and B_j to 2

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  - Each of them think the other person got $< \frac{1}{q}$ for some piece
- Cut and choose all other piece of cake $(X - A_i - B_i)$
- Give A_i to 1 and B_j to 2
- q-1 cuts

Strong Fair Division$_{n=3}$

- We will take a bottom-up approach to adding a 3_{rd} person
- Without loss of generality, we want $\frac{2k-1}{3k-1} * \mu_1(X_1) > \frac{1}{3}$

Strong Fair Division$_{n=3}$

- We will take a bottom-up approach to adding a 3_{rd} person
- Without loss of generality, we want $\frac{2k-1}{3k-1} * \mu_1(X_1) > \frac{1}{3}$
- Remember what we did for the bottom-up method: We cut the X_1 into $3k-1$ equal pieces
- Player 3 chooses k pieces
  - $1 - \frac{2k-1}{3k-1} = \frac{k}{3k-1}$
- Player 1 must be content
- Player 3 must also be content

Strong Fair Division

Let's look at the number of cuts needed -

- The number of cuts is dependent on k
- $\frac{1}{2} + \frac{\epsilon}{2} \quad 0 < \epsilon < 1$
- We must find k s.t $\frac{(2k-1)}{2} * (1+\epsilon) > \frac{3k-1}{3}$
- $\epsilon(6k-1) > 1$
- k can grow to become quite large depending on ϵ. Since we do not know ϵ beforehand, we must choose a large number of cuts.

Classes of Fair Division

- Finite vs Infinite
  - Irrational Unequal Shares

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Classes of Fair Division

- Finite vs Infinite
  - Irrational Unequal Shares
- Bounded vs Unbounded (Upper Bounding)
- Continuous vs Discrete (Moving Knife?)

Meeting 7

Agenda

- Motivations - Indivisibility
- Bipartite Matching
- Pareto Optimality in cake

Bipartite Matching

- What is a bipartite matching?

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Bipartite Matching

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- Conditions for a matching to exist – Hall's marriage theorem
- Cake as a bipartite matching
  - What are the two sets?
  - When do we draw an edge?
  - The set of players and the set of cake
  - When a piece is acceptable to a person

Bipartite Matching

- Envy Freeness
  - The value of the piece is at least as good as every other piece
  - $\mu_i(X_j) \geq \mu_i(X_k)$

Bipartite Matching

- Envy Freeness
  - The value of the piece is at least as good as every other piece
  - $\mu_i(X_j) \geq \mu_i(X_k)$
- Fairness
  - The value of the piece is more than $\frac{1}{n}$ of the cake
  - $\mu_i(X_j) \geq \frac{1}{n}$

Bipartite Matching

- Checking for a Perfect Matching
  - Check all subsets to see if a perfect matching exists.
  - For n players, 2^n checks necessary when directly applying Hall's Marriage Theorem

Division with Bipartite Matching

- Find the smallest subset for which a perfect matching can be found by Hall's Marriage Theorem (k pieces and k players)
- Have the remaining n-k players divide the remaining pieces of cake

Pareto Optimality

- Someone would object in the case of a different allocation

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- $n!$ permutations to brute force
- Maximum flow algorithm $O((n+m)^{\epsilon})$
- Greedy solutions
    - Repeatedly minimize worst disappointment
    - Repeatedly maximize most satisfaction