# Fair Division

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# Dividing a cake

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- Imagine two people want to share this cake...
- But we have some problems

# Dividing a cake

- This cake is complicated
- Person 1 and Person 2 value different parts of the cake differently
- Can we come up with an algorithm where both people are happy?

# What is happiness anyways?

- Fairness
  - Every person believes that they received at least $\frac{1}{n}$ of the cake
  - $\mu_i(X_i) \geq \frac{1}{n}$
- Envy-Freeness
  - Every person believes that they received at least a good a piece of the cake as everyone else.
  - $\forall_{i,j} \mu_i(X_i) \geq \mu_i(X_j)$
- Fair and envy-free divisions are always guaranteed to exist!

## An ideal algorithm for n=2

- When n=2, an extremely simple algorithm exists
  - This solution is extremely old - it was first presented in the Torah
- One person can cut the cake, and the other person can choose their preferred piece
- Therefore, this algorithm is called "Cut & Choose"

# Why does this work?

- The cutter gets exactly $\frac{1}{2}$ of the cake
- The cutter knows that the other piece is also worth $\frac{1}{2}$ of the cake
- The chooser gets the piece which they think is better
- One of those pieces must be worth at least $\frac{1}{2}$ of the cake
- The cutter is always incentivized to tell the truth - a concept in mechanism design called *DISC (Dominant Strategy is Incentive Compatible)*

# The need for generality

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- Sadly, the envy-free problem is extremely non-trivial to generalize...
- So let's first focus on fairness, which can be both tractable straightforward.

For n people, there are many algorithms for fair division

# Banach-Knaster Last Diminisher

- Banach-Knaster was the first proposed solution for fairness
- for i = 1...n-1
- Person i cuts $\frac{1}{n}$ of the cake and passes it to person i+1
- Person i+1 cuts if they believe that the piece is $> \frac{1}{n}$ of the cake, then passes to person i+2, etc.
- The last person to cut the cake now believes that the piece is worth $\frac{1}{n}$ of the cake, and they may take the piece.

# The price of lying

- If someone wants to lie, they do so at significant risk.
- The only way they can lie is to not cut when they believe the piece of cake is worth $> \frac{1}{n}$.
- To make it clear why this doesn't work, let's show an extreme example
  - Person 1 cuts 98% of the cake, with the goal of taking it for themselves.
  - After passing the cake around, the last diminisher has only cut the piece down to 97% of the value of the cake!
  - Because Person 1 is not the last diminisher, they must continue playing, however they cannot receive more than 3% of the cake.

# A $\sigma$-algebra of crumbs

We want to measure the number of cuts needed to cut the cake

- In the worst case, all people cut at every turn.
- First n people cut, then one player drops out
- Then n-1 people cut, then n-2 people, etc.
- Thus, the number of cuts needed is $\sum_{i=1}^{n} i = \frac{n*(n+1)}{2} = O(n^2)$
- In the best case, the first cutter is also the last diminisher so n cuts are sufficient.

# Dubins-Spanier Moving Knife

- Here we can see another style of fair cake cutting solutions.
- Rather than having many cuts, a "moving knife" can be used to allocate chunks of cake.
  - A knife moves over the cake continuously from one side to the opposite side (for example from left to right)
  - When a person thinks that the portion remaining from the starting side/previous cut is worth $\frac{1}{n}$, then they may say "Cut", and they will take the portion on the left side.

## Does this work?

- This works, because the same person who said "Cut" at any given point would have been the last diminisher in in the Banach-Knaster Last Diminisher Method.
- On a surface level, this seems to take n-1 cuts, but this is incorrect. Instead, it takes an infinite number of cuts perpendicular to the direction of movement.