Fair Division

Cake Cutting Algorithms: Be Fair if You Can

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Overview

1. Introduction to Fair Division

2. Cut and Choose

3. Fair Division for n

3.1 Banach-Knaster Last Diminisher
3.2 Dubins-Spanier Moving Knife
3.3 Even-Paz Divide and Conquer
3.4 Stromquist Envy-Free Moving Knife
3.5 Austin's Perfect Division for n=2
3.6 Aziz-Mackenzie Envy-Free Procedure

Meeting 1

Agenda

- Introduction
- Fair Division for n Players
  - Banach Knaster
  - Dubins Spanier
  - Even Paz

Introduction

Imagine two people want to share this cake.

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- The cake is complicated
- The two people may value different parts of the cake differently

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- The two people may value different parts of the cake differently
- Can we come up with an algorithm where both people are happy?

Cut and Choose

Procedure

1. Player 1 cuts the cake into what they believe is half
2. Player 2 chooses the piece which they think is better

Proof of Correctness

- Player 1 recieves $\frac{1}{2}$ of the cake
- Player 1 values Player 2's allocation to also be worth $\frac{1}{2}$

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- Player 2 recieved the piece which they thought was better
- Player 2 must value their piece to be at least $\frac{1}{2}$ of the cake

Banach-Knaster Last Diminisher

Procedure

1. Player 1 cuts $\frac{1}{n}$ of the cake
2. Player 2 through n
   - If they believe the piece is worth $> \frac{1}{n}$ of the cake, they may trim it
   - If they believe the piece is worth $\leq \frac{1}{n}$ of the cake, they may pass it to the next person

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3. The last person to trim the piece recieves it and drops out

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3. The last person to trim the piece recieves it and drops out
4. Repeat until no players remain

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- This is most easily seen with an extreme example
  1. Person 1 cuts 98% of the cake, with the goal of taking it for themselves.
  2. After passing the cake around, the last diminisher has only cut the piece down to 97% of the value of the cake
  3. Person 1 now cannot receive more than 3% of the cake.

Dubins-Spanier Moving Knife

Procedure

- Rather than having many cuts, a "moving knife" can be used to allocate chunks of cake.
    1. A knife moves over the cake continuously from one side to the opposite side (for example from left to right)
    2. When a person thinks that the portion remaining from the starting side/previous cut is worth $\frac{1}{n}$, then they may say "Cut", and they will take the portion on the left side.

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- The same person who said "Cut" at any given point would have been the last diminisher in in the Banach-Knaster Last Diminisher Method.
- On a surface level, this seems to take n-1 cuts, but this is incorrect. Instead, it takes an infinite number of cuts perpendicular to the direction of movement.

Even-Paz Divide and Conquer

Procedure

1. Players 1...n-1 cut the cake in half
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3. Player n and the players on the side n chose repeat the procedure on that side
4. The remaining players repeat the procedure on the other side

Meeting 2

Agenda

- Stromquist Envy-Free Moving Knife
- Austin's Perfect Division for n=2
- Aziz-Mackenzie Envy-Free Procedure

Stromquist Envy Free Moving Knife

Procedure

Austin's Perfect Division for n=2

Defining Perfect Division

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Aziz-Mackenzie Envy-Free Procedure

Aziz-Mackenzie Envy-Free Procedure for n

https://youtu.be/fvM8ow6zNw4?si=AGrOGF7vSZSGt4QK&t=711

Meeting 3

Agenda

- HOWTO: Reading Papers
- Unequal Division Naïvely
- Cutting into 1-sized parts
- Ramsey Partitions
- Halving

Next Week: Finish Chapter 3 & Chapter 4

Reading Papers

Let's be honest

Most papers are dryer than the Sahara Desert

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So why read papers?

Reading Papers

- Understanding the field better
-

Fundamentally, we are learning: But what are we trying to learn?

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Our goal when reading a paper is to contextualize that paper's findings in the field.

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Parts of a Paper

- Title & Authors

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- Title & Authors
- Abstract
- Introduction
- Related Works

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- Title & Authors
- Abstract
- Introduction
- Related Works
- Content

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- Title & Authors
- Abstract
- Introduction
- Related Works
- Content
- Discussion/Conclusion

PhD

`https://matt.might.net/articles/phd-school-in-pictures/`
`IllustratedGuidePhD-Matt-Might.pdf`

Dividing Camels

First son gets $\frac{1}{2}$. Second son gets $\frac{1}{3}$. Third son gets $\frac{1}{9}$

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- We will say that
  - Player 1 recieves A_1 allocation
  - Player 2 recieves A_2 allocation
- Using the Even-Paz method, $\theta((\sum A)\log(\sum A))$ cuts are required.
- We can do better!

Cutting Ones

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- Player 1 divides the cake into 1-sized parts
- Player 2 chooses A_2 of the 1-sized parts

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The same property seen in Ramsey Theory can also be used to help partition

Ramsey Partitions

- Assume $A_1 < A_2$
- Player 1 cuts what they percieve to be A_1 of the cake.
- The smaller piece can be called X_1 and the larger piece can be called X_2
- If $\mu_2(X_2) < A_2$ (If the person thinks they got less than they were supposed to)
- Player 2 takes X_1 and division continues in the ratio $A_2 - A_1 : A_1$ until no cake remains

Ramsey Partitions

- This can be simplified using ramsey partitions
- Person 1 cuts the Ramsey Partitions of $\sum A$
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    1. If the sum of a subset of the marked pieces $= A_2$, we can give those pieces to Player 2 and give the rest to Player 1
    2. Otherwise, Player 1 may choose pieces summing to A_1 of the unmarked pieces.
    3. This works because of the Ramsey Partitioning!