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Children and Arithmetic

Peter Bryant

Abstract—The development of children's understanding of mathematical relations and of their grasp of the number system is described. It is discussed that children easily recognise one-way part–part relations but that the number system at first causes them difficulty. Children's relational understanding allows them to solve addition and subtraction problems fairly well when these deal with simple increases and decreases in quantity. It also helps them to make proportional judgements when these involve part–part relations. However, problems that involve relations between parts and wholes are at first extremely difficult. The review also deals with the effects of context and shows the considerable aptitudes that are handed on to children in informal settings.

Keywords: Mathematics, logic, cognitive development

Introduction

Relations are the stuff of mathematical development. Children must learn what it means to say that two numbers are equal or that one is more than the other. They have eventually to learn to deal with part–part relations (ratios) and with part–whole relations (fractions, for example). They must be able to co-ordinate separate relational judgements in order to measure and understand measurement ($A = B$, $B = C$, therefore $A = C$) and to understand an ordinal series ($A > B$, $B > C$, therefore $A > C$) properly. They have to learn about multiplicative relations. On top of all this they have to learn about the connections between these different relationships, if they are to acquire a coherent picture of the mathematics that they are taught at school and the mathematics that they use in their everyday life.

The research that psychologists have done on children's understanding of these mathematical relations falls quite neatly into three separate branches. Each of these asks its own distinctive questions, each has its own kind of theory and each employs its own empirical paradigms. Although all three approaches have made a great deal of progress, particularly in recent years, the subject as a whole has suffered from a certain lack of connection between the three. There is no compelling theoretical or practical reason for this separation, and one of my main arguments

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Requests for reprints to: Peter Bryant, Department of Experimental Psychology, University of Oxford, South Parks Road, Oxford, OX1 3UD, U.K.

in this paper is that there is now an urgent need (by means mainly of longitudinal research and intervention studies) for us to bring the three together in order to understand how children learn mathematics as well as they do and why they learn no better than they do.

The Three Approaches

(1) *Universal intellectual development and mathematical achievement*

The first approach is concerned with intellectual development in general and in particular with the growth of the understanding of universal logical principles. The central idea entertained by those who adopt this approach is that young children's abilities increase and improve with age, and that this sequence of development is universal. It is the same everywhere, whether or not children go to school and are taught mathematics. Not only is the development universal, but so also are the truths that are understood as a result of this development. The link between children's intellectual development and their mathematical progress is simply that they need to reach a certain intellectual level in order to understand certain mathematical principles. Young children, it is argued, must surmount certain intellectual barriers before they can make any progress in mathematics.

Since the focus of interest here is on events which precede mathematics learning, the experimental paradigms that people have adopted to answer the questions posed by the first approach are usually a far cry from the sums and the word problems that children have to face in their mathematics lessons.

Piaget's theory (1952) and his experiments are the clearest example of this approach, but there are others too. In Piaget's view logic is the essential requirement and also at first a formidable barrier for the child learning mathematics. He argued that children's understanding of the counting system, their success with addition and subtraction and later with multiplication and division, their dealings with proportions, with measurement and with geometry, all depend on their ability to make various logical moves. So he and his colleagues, and later countless followers, gave children one-to-one correspondence, seriation and conservation tasks in order to see if they were ready for number, and transitivity tasks to see if they were in principle able to understand measurement.

(2) *Mathematical achievements and mathematical difficulties*

The second approach tackles the child's experiences in the classroom head on. The main concern of those concerned with this approach is to find out what is happening when children solve mathematical problems, how they do so and why they make mistakes. Not surprisingly the empirical paradigms used here are tasks which are strikingly similar to those that the child is given in the classroom. The most common form of mathematical problem used by this group of researchers has been the word problem, in which sums are given in the form of brief and meaningful cameos ("Bill has three oranges: Sam has eight. How many more does Sam have than Bill?"), but there does not seem to be any particular theoretical reason for this preference, and it probably reflects the increasing use of word problems in mathematics lessons and in the work books which school children have to work through nowadays.

Although the hypotheses that different people have produced about children doing sums and solving various other mathematical problems are diverse, the form of the data that interest them is rather consistent. By and large they are interested either in the speed with which a child reaches a conclusion or in the mistakes that s/he makes on the way to a solution of a mathematical problem. Their concentration on actual mathematical problems has meant that they have little to say about the link between the way that a child solves a mathematical problem and the way that s/he solves any other problem. Their interests seem to stop at the classroom door, and thus the contrast between the two approaches that I have mentioned so far is stark. One group of researchers concentrates on abilities which affect many aspects of a child's life and can only speculate on their mathematical relevance: the other produces a great deal of data on children's mathematical achievements but cannot say anything for certain about the relevance of these to the child's intellectual life in general.

(3) *Mathematics and culture*

The third approach concentrates on the transmission of knowledge. According to this view a considerable part of children's intellectual development is a direct result of the information that they are given and the intellectual tools that they acquire with the help of other people more experienced than themselves. What they learn about mathematics from other people is a valid part of this intellectual development, and it is valid in two ways. One is that the progress that children make in mathematics depends not only on the stage of intellectual development that they happen to have reached, but also quite considerably on the kind of mathematical instruction and mathematical experiences that they have both in and out of school. The second is that what they learn about mathematics may have in turn a formidable effect on their understanding of the environment. There is the possibility of a two-way connection: the children's intellectual level will affect their success in mathematics, but their growing mathematical success will play a part in their mathematical development.

Most of the psychologists who adopt this approach acknowledge Vygotsky (Vygotsky, 1986; Vygotsky & Luria, 1993) as their leader. It was he who proposed the idea of the "zone of proximal development", whereby children eventually manage to do on their own what at first they could only do with the help of an adult, and also the notion of the "cultural tool". He argued that people have devised effective tools (such as systems of measurement) over the centuries to help them to study and to control their environment, and have then handed these on across the generations, either in the classroom or in less formal settings: when a child masters such a tool not only is s/he more effective in an intellectual sense than before, but also their intellectual processes are transformed at the same time.

The First Approach

(1) *Cardinal number*

Piaget's influence (Piaget, 1952; Piaget & Inhelder, 1974; Piaget, Inhelder &

Szeminska, 1960; Inhelder & Piaget, 1958) on our ideas about mathematical development is sometimes unacknowledged in current reviews of the subject, but it is still very strong. He posed most of the important questions which still dominate work on children's mathematics, and some of his answers are still quite plausible.

He can be counted as a sceptic as far as early mathematics is concerned. The idea of the development of reversibility—his central theoretical tenet—led him to the belief that young children acquire the appearance of being mathematical when in fact they have no real understanding of what they are doing and will not for several years.

Reversibility is the ability to perceive a change and at the same time to cancel it out subjectively by imagining the opposite change. In Piaget's theory reversibility lies at the heart of the understanding of all logic, and therefore, of all mathematics. A child whose intellectual processes are not reversible, according to Piaget, cannot understand the cardinal and ordinal properties of number, has no notion of the additive composition of numbers, is quite unable to reason about multiplication or division and cannot measure.

We can start with principle of cardinal number. This is that any set of a given number of objects will have the same quantity of objects in it as any other set with the same number in it. This is easy to show, because there is one-to-one correspondence between any two sets of the same number. For each object in one set there is an equivalent object in the other set and vice versa.

Suppose that the child is shown two parallel rows of objects and is asked to compare them numerically. A good way to accomplish the task would be to count the two rows. Another just as respectable way would be to compare the rows object by object, which is a strategy called one-to-one correspondence: see whether for every object in one row there is an equivalent object in the other and, for example, if there is and there are still some left over in the second row conclude that the second row is the more numerous one. A third, but thoroughly unreliable, way of making the comparison would be to judge the two rows on the basis of their lengths.

Young children adopt the third way, the unreliable way. Their spontaneous reliance on length is one of the most often repeated and most reliable phenomena in the history of empirical psychology. Children even choose this untrustworthy cue in the face of considerable inducement not to do so: there are two studies in which the one-to-one correspondence cues were emphasised by lines between the objects in each row, and even here many of the children still disregarded one-to-one correspondence and went for length.

Reversibility, according to Piaget, is the key to the child's eventual discovery that one-to-one correspondence is a good cue and length a very bad one in comparisons of number. If a child who has reversible cognitive processes sees one of the rows being spread out s/he can work out that this has no effect on the actual number of objects in the row: they can now cancel out this change for themselves by imagining the inverse change and thus can realise that there has been no real change in number. Furthermore they can also see that changing the length of one of the rows does not alter the one-to-one correspondence between the two rows.

The child who thinks about quantity in an irreversible manner (the "irreversible" child) cannot do these things, and this leads him into some surprising errors. The most striking of these was illustrated first in some observations by Piaget himself (1952), and later in a provocative (though complex) experiment by Greco (1962). Greco noted that children still made the mistake of relying on length when they could already count objects quite well. He asked children both to count and to compare the quantity of two rows,one of which was longer than the other, though the two contained an equal number of objects. He found that many of the children counted the rows correctly, and said for example that there were five objects in one and five in the other, but nevertheless asserted that the longer row was more numerous.

Piaget and his colleagues reached two momentous conclusions on the basis of this and similar results. One was that children do not at first understand the cardinal properties of number: if they use length rather than one-to-one correspondence as a cue for quantity comparisons, and if they say that two rows both contain five objects and yet that one is more numerous than the other, they have no idea that two rows with the same number will be in one-to-one correspondence with each other and thus must be equal numerically. The other conclusion was that children at first do not know what they are doing when they count: a child who says that there are more objects in one row of five objects than in another row of five objects does not know the correct meaning of the word "five".

These two conclusions are difficult to fault. The evidence for the young child's mistaken reliance on the length cue and reluctance to turn to one-to-one correspondence with displays such as those we have described is hard and fast, and there is, as far as I know, no good evidence to refute Greco's claim that initially what the child says when s/he counts ('quotite') has nothing to do with what s/he thinks about the numerical quantities involved ('quantite').

However, there is at least one good reason for hesitating about the negative conclusion on one-to-one correspondence. It is that children often share, and sharing is an activity which on the face of it seems to depend on one-to-one correspondence. Three studies (Miller, 1984; Desforges & Desforges, 1980; Frydman & Bryant, 1988) have shown that children as young as 4 share out numbers of things equally between two or more recipients rather successfully, and they usually do so on a repetitive "one for A, one for B" basis. Is this not a temporal form of one-to-one correspondence?

The answer must be "Yes", but then we have to consider the possibility that they share on a one-to-one basis without any idea why this is the right thing to do. Sharing, like counting, is a common activity which young children must witness quite often and may very well imitate. They may know that sharing in a one-to-one way is the appropriate action without having any good idea about the reason for it.

Olivier Frydman and I tried to find out more about children's understanding of sharing by devising a task in which children who share on a rote basis without understanding what they are doing would behave in one way, and children who understand the basis of one-to-one sharing would respond quite differently. We gave the children "chocolates" which were either single or double chocolates: in fact these were plastic unifix bricks, all of the same colour, which could be stuck together. We asked the children to share the chocolates out to two recipients, so that each recipient ended up with the same total amount. But we also told the children that one of the recipients only accepted doubles and the other only singles. So the child's problem was to work out that for every double that he gave one recipient, he now had to give two singles to the other one. We reasoned that a child who had shared on a one-to-one basis in a rote fashion would not be able to make this adjustment, whereas a child who understood the basis for one-to-one sharing would see the reason for changing to a one (double) for A, two (singles) for B pattern.

This study produced a very sharp developmental difference. Most of the 4-year-old children did not make the adjustment, and in fact the majority ended up giving the recipient who accepted doubles twice as many chocolates in all as the recipient who accepted singles. This was because these children continued sharing in a one-to-one manner, which meant that for every single that they gave one recipient they handed out a double to the other. In contrast, most of the 5-year-olds did manage to make the necessary adjustment. These children usually gave the double to one recipient and then immediately two singles to the other, and so on. The reason for the difference between the two age groups is unclear to us, but at the very least the study establishes that 5-year-old children have a clear and flexible understanding of the mathematical basis of one-to-one sharing.

What about the 4-year-olds? They had certainly hit a barrier in our new version of the sharing task, but we still had no idea how formidable that barrier was for them. So, in a later study we devised a new version of the singles/doubles task in which we introduced bricks of different colours. Our aim was to use colour cues to emphasise one-to-one correspondence. In this new task each double consisted of a yellow and a blue brick joined together, and half the singles were blue and half yellow. This was the only change, and yet it had a dramatic effect. Nearly all the 4-year-old children solved the problem, and they did so because they could now see how to use one-to-one correspondence to solve the problem. The typical pattern of sharing was to give a double (consisting of course of one yellow and one blue brick) to one recipient and then to give a yellow and a blue single to the other one. They adapted the one-to-one strategy successfully when the one-to-one cues were emphasised. They also learned a great deal from this experience, because when later on we gave the same children the single/doubles task with bricks of one colour only (as in the original experiment) these children did extremely well. They had surmounted the barrier that we identified in the first study, and we conclude from this that even 4-year-old children have a basic understanding of the reason why one-to-one sharing leads to equal quantities. It follows that they do have a respectable understanding of one-to-one correspondence and therefore, a basis for understanding the cardinal properties of number.

But do they extend this understanding to number words? We looked at this question in another study. In this we took a group of 4-year-old children who could share quite well, and we asked them to share out some "sweets" (again unifix bricks) between two recipients. When this was done we counted out aloud the number of sweets that the child had given to one recipient, and then asked him or her how many had been given to the other recipient. None of the children straightaway made the correct inference that the other recipient had the same number of sweets even though they had meticulously shared the sweets out on a one-to-one basis: instead all of the children tried to count the second lot of sweets. We stopped them doing so, and asked the question again. Even then less than half the children made the correct inference about the second recipient's sweets.

Thus many 4-year-old children fail to extend their considerable understanding of sharing to counting. We conclude from this that young children do grasp the cardinality of number and yet do not at first apply this understanding to number words. Here, it could be said, is an example of quantite without quotite. The children do have quite a good grasp of one-to-one correspondence, but do not apply this knowledge to number words.

(2) *Counting*

This last conclusion takes us directly to the question of young children's counting. The first point to be made about this extraordinary phenomenon is that it happens and that it happens at a very young age. From roughly the age of 2 years children begin to count. They count objects and actions and on occasion their counting is 'abstract' in the sense that they simply produce their version of the number sequence without attaching the numbers to anything in particular. At first they make many mistakes and it is interesting to note that many of these seem to stem from their attempts to get to grips with the decade structure. Erroneous sequences like "twenty-nine, twenty-ten, twenty-eleven" are common (Fuson, 1988) and show that it takes some time at least for children to grasp how the number sequence is punctuated by decades and, at a higher level, by hundreds and thousands.

The irregularity of our number words also seems for a while to be a serious problem. Most of the teen words "twelve, thirteen" and some of the decade words ("thirty", "fifty") are highly irregular in the sense that the words themselves do not denote their position in the decades structure at all well. Contrast the spoken number "one hundred and ten" with "eleven" and it is easy to see that in linguistic terms the former is much more clearly placed than the latter. The evidence that linguistic irregularities are stumbling blocks for children comes partly from data which show that teen numbers, where these irregularities are most striking, are particularly difficult for children who are learning to count, and partly from comparisons with children learning to count in other languages, like Chinese, in which the number words are not so capricious. I shall describe some of these studies in more detail later on in this article.

Nevertheless, even English speaking children do count before they go to school, and by the time they get there, their knowledge of number words, though still imperfect, is quite considerable. What, we must now ask, is the significance of this achievement?

Two starkly different answers have been given to this question. The first is that pre-school counting is vacuous, that at first children mouth number words without understanding what these mean and that several years elapse between their first acquaintance with the number sequence and their understanding what this sequence actually means. Piaget (1952) took the most extreme version of this position. For him the evidence that children fail in one-to-one correspondence tasks and his consequent conclusion that they do not understand cardinality meant that they had a strikingly incomplete idea of the meaning of the number words which they so readily produce. A child who says that there are five counters in row A and five in row B and yet that row A is more numerous, does not know the correct meaning of the word "five".

A milder version of this position comes from the work by Frydman and myself on sharing. Our idea is that children do at first have a very poor idea of the meaning of number words, but that the stumbling block is the number system itself and not the principle of cardinality which, as the work on sharing shows, causes them few serious problems.

The starkly contrasting view (to Piaget's and to ours) of children's counting is that they grasp its essential principles right from the start. This is the conclusion reached by Gelman and Gallistel in their 1978 book on young children's counting and by Gelman and Meck (1983) in a later study of children's judgements of the correctness of a puppet's counting.

Gelman and her colleagues argued that children must grasp five basic principles in order to understand what they are doing when they count, and the main conclusion of these research workers is that children as young as 2 and 3 years do have this understanding even though they make many mistakes. The name that Gelman gives to this theoretical position is "principles before skills". Children understand the basic principles of counting right from the start, she argues: their mistakes are merely failures to put these principles into practice all the time, and this rests on skills which it takes them some time to acquire.

There are two main empirical bases for this theoretical position. One is some research, much of it done by Gelman's colleagues, which apparently shows that even babies can discriminate between numbers (Starkey & Cooper, 1980; Strauss & Curtis, 1981, 1984; Antell & Keating, 1983; Starkey, Spelke & Gelman, 1983; Cooper, 1984; Starkey, Spelke & Gelman, 1990). The other is Gelman's own work on counting in children between the ages of 2 and 5 years. In this paper I shall deal only with the second set of studies.

The first three of the five principles are "how to count" principles. One is "the one-to-one principle". Here the requirement is that the child understands that he or she must count all the objects in a set once and once only: each one must be given just one number tag. This is quite a different requirement to Piaget's. For him one-to-one correspondence was about the relationship between members of different sets. For Gelman and her colleagues the one-to-one principle is about how to count one set of objects.

Another "how to count" principle is "the stable order principle", which means that one counts in a set order, and in the same set order each time. This principle is the nearest that Gelman and Gallistel get to ordinality, but again their requirement is different from and much less demanding than Piaget's because they are only concerned with the children's appreciation that they must count in a consistent sequence and not with their understanding that the sequence is a sequence of increasing magnitude.

The final how to count principle is "the cardinal principle", but despite its name it is a far cry from Piaget's notion of cardinality. In Gelman's terms the cardinal principle involves the understanding that the last number counted represents the value of the set. So a child who counts a set "one, two, three, four" must understand that "four", the last of these numbers, represents the number of objects in that set. The requirement falls short of Piaget's because it concerns the value of a single set and not the relation between sets of the same number.

Gelman's requirements involve two other principles. One is "the abstraction principle" which states that the number in a set is quite independent of any of the qualities of the members in that set: the rules for counting a heterogeneous set of objects are the same as for counting a homogeneous one. The other is "the order irrelevance principle": the point here is that the order in which members of a set are counted makes no difference, and anyone who counts a set, for example, from left to right will come to the same answer as someone else who counts it from right to left.

The main aim of Gelman's research on counting was to show that even very young children understand these principles at the time that they begin to count, and, therefore, know from the start what counting means. Gelman and Gallistel observed young children counting sets of objects and recorded whether the children always counted the same order and always counted each object once, and also whether they seemed to recognise that the last number counted signified the number of the set. Gelman and Meck (1983) also asked children to make judgements about a puppet which they saw counting: this puppet occasionally violated the one-to-one principle and the cardinal principle, and the aim of these experiments was to see whether the children could spot these violations. By and large the results of these studies supported Gelman's contention that the children did have some understanding of the three "how to count" principles as these were set out in her model.

So we end up with two entirely different answers to the question "Do children understand what they are doing when they count?", and it is worth spending some time considering what is the nature of this striking disagreement. It is certainly not a dispute about evidence, since the actual research of the two groups took an entirely different form. The fact is that Piaget and Gelman did different kinds of experiments because the criteria that each of them used for understanding counting were also quite different. Piaget opted for ordinality and cardinality, and Gelman for her five principles.

We have already noted that in some ways Gelman's requirements are less demanding than Piaget's, and this means that we should start by considering two possibilities. One is that Piaget's requirements are too strong, the other that Gelman's are too weak.

The first of these possibilities seems much more plausible than the second. Both models, for example, include cardinality, but they treat it quite differently. Piaget concentrated on the relationship between sets of the same number, and it is impossible to dispute his claim that a child will only understand the meaning of the word "six" if he or she also understands that a set of six objects is equal in number to any other set of six objects. Gelman, on the other hand, concentrated on the question of the child understanding that the last number counted represents the number of the set. It is true of course that the child must realise this, but it is also true that a child could grasp the fact that the last number is the important one without really understanding the quantitative significance of this number, at any rate as far as its relationship to other sets is concerned.

There is some striking evidence that young children who count quite proficiently still do not know how to use numbers to compare two different sets. Both Michie (1984) and Saxe (1979) have reported a remarkable reluctance in young children who have been asked to compare two sets of objects quantitatively to count the two sets. They knew the number sequence and so they were in principle capable of counting the objects and it would have been the right thing to have done, but they did not do it.

This reluctance to use number as a comparative measure was demonstrated even more clearly in a remarkable experiment by Sophian (1988) in which she asked 3 and 4-year-old children to judge whether a puppet who counted was doing the right thing. This puppet was given two sets of objects and was told in some trials to compare the two sets and in others to find out how many objects there were in front of it altogether. So in the first kind of trial the right thing to do was to count the two sets separately while in the second it was to count them together. Sometimes the puppet got it right but at other times it mistakenly counted all the objects together when it was asked to compare the two sets and counted them separately when it was asked how many objects there were altogether.

The results of this experiment were largely negative. The younger children did particularly badly (below chance) in the trials in which the puppet was asked to compare two different sets. They clearly had no idea that one must count two rows separately in order to compare them, and this suggests that they had not yet grasped the cardinal properties of the numbers that they counted.

This empirical evidence only serves to underline a conceptual point that Gelman's requirement for the understanding of cardinality was too undemanding. It is in principle possible for a child to understand that the last number counted is important and still have no idea about its quantitative significance. Fuson (1988) and Baroody (1992) have made the same point.

One of the most obvious differences between the two approaches is that Piaget is concerned above all with relations between sets while Gelman concentrates on children counting one set at a time. But there is even some evidence about young children counting single quantities that casts further doubt on Gelman's claims that children understand the cardinality of number words as soon as they begin to count.

An interesting study by Shipley and Shepperson (1990) shows that children find it quite difficult to count objects when these are broken up into different physical entities. When they were asked to count the number of forks and the forks themselves were physically separate entities, young children tended to count the physical bits, and not the actual number of complete forks. Also several studies (Wynn, 1990; Frye, Braisby, Lowe, Maroudas & Nicholls, 1989; Fuson, 1988) have shown that when children are asked to give someone a certain number of objects ("Give me five bricks") they often fail to count and simply grab a handful of objects and the number that they hand over is for the most part wrong. So, even when only a single set is involved, young children do not seem to understand the significance of counting. They may realise, when they do count, that the last number is the important one, but the fact that they do not seem to know exactly when to count suggests that they have no idea why counting is important. They have not grasped the cardinal properties of the number words that they know so well. Their performance fits the Piagetian picture of quotite without quantite very closely.

Much the same point can be made about the other two "how to count" principles. Children certainly have to know that they should count each object once and only once, but this is not the only form of one-to-one correspondence that they must understand. Piaget's point that children must also understand one-to-one correspondence between sets if they are to understand the quantitative significance of number words is surely right, and yet this is not part of Gelman's one-to-one principle.

Nor does Gelman's stable-order principle go nearly far enough. Numbers come in a certain order and is certainly true that children have to understand this. But the reason for the order is that numbers are arranged in increasing magnitude and it is quite possible that a child who always produces numbers in the same sequence realises the quantitative significance of this sequence. Gelman's evidence is no help on this particular point, but much the same goes for the work of other interested psychologists. Piaget's evidence as we have seen is remarkably indirect since it concerns only continuous quantities and not number, and there does not seem to be any other clear work on young children's understanding of the ordinal properties of number. We simply cannot say whether children understand the ordinality of the number sequence or not.

My conclusions on this controversy about children's understanding of counting are simple. Of the two sets of requirements Piaget's are better than Gelman's. Children will only understand the quantitative significance of the number words that they learn when they have grasped both the cardinal and the ordinal properties of the number sequence. Most of the evidence suggests that children at first do not understand cardinality and Gelman's own work on the cardinal principle, as she defines it, throws no doubt on this suggestion. Our knowledge about children's understanding of ordinality is much less advanced, but again Gelman's work on the stable order principle does not in any way show that children understand the ordinal relations in the number sequence. There are good reasons for thinking that at first children are practising little more than a verbal routine when they count.

(3) *Ordinal number, inferences and measurement*

The order of the words in the number sequence represents their magnitude (10 is more than 9, 23 more than 22), and as we have seen the difference between Piaget's assessment of children's understanding and Gelman's are as great over ordinality as over cardinality. We have seen that Gelman's more optimistic assessment is based on the children's evident ability to produce the same number words in the same order on different occasions, at any rate with small numbers.

The evidence to which Piaget appeals is typically a great deal less direct. Piaget's position is that young children are incapable of grasping the nature of a series that increases in quantity. He attributes this intellectual gap once again to a lack of reversibility. He argues that a child who does not possess this intellectual property is, as a result, incapable of handling the quantitative relations in even a simple series. So, faced with the series A > B > C, the "irreversible" child will be able to take in at one time that A > B and at another that B > C, but simply cannot grasp these two relations at the same time. He cannot do this because it is beyond him to understand that B can simultaneously be smaller than one quantity and larger than another.

The actual empirical studies which Piaget offers as evidence for this proposition are his well known seriation and transitivity studies. In the seriation experiment children are asked to put sticks of different lengths in ordered series—ordered that is by their size. Young children fail to do this. The youngest behave randomly and slightly older children simply form two piles, one of the smaller and the other of the larger sticks, a pattern of responding which, according to Piaget, shows that they can take in single relations such as A > B at the same time, but have greater difficulty with a series in which a particular quantity is smaller than one and larger than another. More recently this position has received considerable support from an experiment by Perner and Mansbridge (1983) on children's memory for length. The task that they gave children was to learn about and remember the relations between three pairs of sticks of different colours and lengths. One group was shown a series of four sticks (A > B, B > C, C > D), and the other three pairs of disparate sticks (A > B, C > D, E > F), over a series of trials. The second group were much the more successful of the two groups: they learned better and they made fewer mistakes. Perner and Mansbridge quite plausibly attributed this difference to the fact that the first group had to deal with two way relations (B larger in one pair and smaller in the other), and it is easy to see that this conclusion is at one with Piaget's view of children's difficulties with ordinality.

If the problem that children have with an ordered series is to be traced back to an inability to handle two-way relations, they should be equally out of their depth with transitive inferences. The premises in a transitive inferences take the form of two or more quantitative relations (e.g. A > B, B > C) and the inference involves combining these in order to answer a question about the relation between the two quantities which are not directly compared (A?C). The idea that young children cannot make such an inference has more than one implication for their understanding of mathematics. The claim is relevant to the question of ordinality, of course, but it also implies that they should be unable to measure or to understand other people's measurements. If they cannot connect A and C by virtue of both having been compared to a common value (measure) B, then they will not see the point of taking a ruler to compare two lengths which cannot be compared directly.

The question about transitive inferences is one of the most vexed in studies of cognitive development. This is largely because the question imposes some formidable empirical problems for the researcher. In order to be sure that a mistake in a transitive inference task is a genuinely logical one (i.e. a failure to combine two premises to make an inferential judgement), one must be sure that the child can recall the two premises at the time that they are asked the inferential question (i.e. can they remember that A > B and B > C when asked the A?C question). This was pointed out some time ago by Bryant and Trabasso (1971) and ever since we did so the most common empirical solution has been to make sure that the children learn the premises thoroughly before they have to face the empirical question.

But this leads to a new problem which was originally pointed out by Perner and Mansbridge (1983). It is that the experimenter might unwittingly be teaching the child something about ordinal relations during the learning period. If a child finds it difficult to remember that A > B and that B > C because he cannot appreciate that A can have different relations to different values, maybe repeated experience with these two pairs will eventually teach him that such two way relations are possible.

The problem intensifies when one considers the empirical connotations of another requirement for transitive inference tasks for which Bryant and Trabasso (1971) were also responsible. We argued that such a task must involve at least five values rather than a minimum of three. We made the claim that an A > B, B > C (three value) task is inadequate. The child, we argued, could answer the eventual A?C inferential question in such tasks by remembering that A was larger when he last saw it or that C was the smaller. Thus the child could answer the question correctly, but illogically, merely by repeating one or both of these remembered values. If, however, one has a task with four premises (A > B, B > C, C > D, D > E) three of the quantities, B, C and D) are the smaller value in one of these pairs and the larger in another. Inferential judgements based on these quantities cannot be dismissed as mere parroting.

This requirement is now generally accepted, but unfortunately it makes the problem of ensuring that children remember the initial premises a much more daunting one. It is, as one might expect from Perner and Mansbridge's results, quite difficult for a 4-year-old child to learn and remember an A > B, B > C, C > D, D > E series. When we (Bryant & Trabasso) originally gave 4 and 5-year old children four such pairs to learn, we found that it took most of them many trials to do so. Thus our discovery that the children who remembered these pairs then made transitive inferences successfully is quite interesting, but we cannot rule out the possibility that we taught the children about ordinality in the first part of our experiment. Of course if that were true, it would be quite interesting too: one needs to know if it is so easy to teach young children about a logical principle in one single experimental session.

One way of getting round this empirical difficulty is to present children with the premises at the same time as they are asked the transitive question, but this is not so easy to do without at the same time providing so much information that the need for the inference actually disappears. Ros Pears and I (Pears and Bryant, 1990) have managed an inferential task not with length but with relative position (up–down) in which no learning at all was necessary because the children could see the premises (pairs of different coloured bricks, one on top of the other) at the same time as they were asked the inferential question (the relative position of two of these bricks in a tower of five or six bricks) and we found that even 4-year-old children can make respectable transitive inferences, but since this is not a dimension of much importance in children's mathematics I will not dwell on the study any further.

We must pass on instead to the second mathematical implication of work on transitive inferences—measurement. If children need to understand transitive inferences in order to measure, then evidence that children can measure is evidence too that they can make transitive inferences. There is such evidence.

We ourselves (Bryant & Kopytynska, 1976) gave 5-year-old children a simple measurement task in which they were faced with two blocks of wood each with a hole at the top, and were asked to compare the depths of the holes. The children also had a stick, and they used it systematically to measure these depths. Three different experiments of ours confirmed this result, and more recently Miller (1989) has reported an equivalent success with a similar but more meaningful task (working out which hole Snoopy must be hiding in). It is hard to see how young children could manage as well as they do in these tasks unless they understood the significance of transitive inferences.

Yet, we must still be cautious. Piaget's criteria for logical understanding were demanding, and the hardest criterion of all was his insistence on children grasping logical necessity. As Smith (1993) in a recent stimulating monograph on logical necessity has clearly shown, Piaget's position was that no one can be said to be making a genuinely logical judgement, even when it is the correct solution to a logical problem, unless that person also understands the properties which make this judgement necessarily true. "Necessary properties lay down both why something is, and has to be, what it is, and why it is not, and cannot be, anything else" (Smith, 1993, p. 2). Piaget also thought that the only way that a person can show that he understands the necessity of a logical judgement is by justifying it logically. His position was that "a true, unjustified belief never amounts to knowledge" (Smith, 1993, p. 65). The natural upshot of this is that children could not be said to understand transitivity, even when they solve transitivity problems, unless they can justify their inferences subsequently, and it needs to be said straightaway that none of the studies that I have just described fits that bill.

We are faced with an empirical question, which is how to establish not only the presence, but also the absence, of the understanding of logical necessity. It is perfectly plausible that someone who appeals to the logical necessity of a correct solution to a logical problem really does understand logical necessity, but one cannot, in my view, conclude that a person who fails to produce such a judgement therefore lacks this understanding. A child may have grasped logical necessity without being able to put it into words.

My rather hesitant conclusion about Piaget's hypothesis on ordinality and transitivity is that it is in the end rather unconvincing. Children may fumble in the seriation task, but they still seem to be able to work out that a quantity can have two values and to use this information in a measuring task. However, we certainly need more data on how they justify what they do in such tasks.

(4) *Addition and subtraction*

The work that we have discussed so far shows that young children are able, at the very least, to take in simple quantitative relations like "same", "greater" and "smaller". It seems quite possible, therefore, that they can also cope with simple transformations which increase a quantity or decrease it (make it greater or smaller).

In fact children do naturally have a great deal of experience with increasing and decreasing quantities, and there is growing evidence that they may have some idea of the nature of addition and subtraction before they go to school.

Indeed psychologists' attempts to demonstrate the pervasiveness of the first two operations have now reached the cradle. Karen Wynn (1992) set out to see whether babies of 8 months could work out the results of addition and subtraction. She used a measure of surprise (babies look longer at events which go against their expectations). She showed the babies either one or two Mickey Mouse toys which she then hid behind a screen. Then in full view of the child she either added a Mickey Mouse (1+1) to the one behind the screen or subtracted one from the two that were there (2–1). Finally she removed the screen. In every case either one or two Mickey Mouses were revealed: in half the trials the number there was the appropriate one (two Mickey Mouses in the 1+1 condition and one in the 2–1 condition), and in the other half it was not (one in the 1+1 condition and two in the 2–1 condition).

Wynn found that the babies looked longer at the inappropriate displays than at the appropriate ones, and she concluded that they could work out the results of simple additions and subtractions. However, she was quite rightly concerned that the babies might have simply responded as they did on the basis of expecting a change from the original number and finding no change in the appropriate displays (e.g. one in the 1+1 displays represents no change). So she ran another experiment in which she repeated the 1+1 condition with one alteration. When she finally removed the screen the babies either saw two Mickey Mouses (appropriate) or three (inappropriate). Thus both the appropriate and the inappropriate displays represented a change in number from the original amount.

Here too the babies looked at the inappropriate display for a longer time than at the appropriate one, and it is difficult to see how this impressive result can be interpreted in any other way than as evidence that even babies can work out the results of a 1+1 addition. [Whether they are as capable with subtraction is not yet certain (Bryant, 1992).]

Given this striking result, it should not surprise us that other research with children who are considerably older, though still well within the preschool period, has also shown that they can work out the consequences of simple additions and subtractions. Starkey's (1983) study is a case in point. He gave children aged between 24 and 35 months two, three or four objects to put in a container. Then he either added or subtracted some objects himself or left the container untouched. The child was asked to remove all the objects from the container which was built in such a way that the child could only take out one object at a time.

Starkey's question was whether the children would reach into the box the right number of times (e.g. three times in a 3–1 trial), and he found that on the whole they did, at any rate when numbers less than four were involved. When the final sum was amounted to three or less the children were correct more often than would be expected by chance. Starkey concluded that pre-school children can work out the results of simple additions and subtractions, and, given Wynn's results, it is easy to accept this suggestion.

These, of course, are nonverbal tasks. Wynn's subjects knew no words: Starkey did not count or mention number in his experiment, and his measure (the number of reaches) was a nonverbal one. When number words are introduced, young children begin to make serious mistakes, but there is some interesting evidence that these errors are more frequent in some conditions than in others. In a rather similar task to Starkey's, Hughes (1986) also gave children (2–4 years in age) some bricks in a container and then added further bricks or subtracted some. Then he asked the children about the consequent number in the box. In fact the children managed quite well in this task but had considerable difficulty in another one which contained no reference to concrete material: the children were asked, for example, "what does one and one make?". This was a far more difficult task, and Hughes' contention, that children understand addition and subtraction, so long as these involve concrete material, seems a plausible one.

(4) *Conclusions—the first approach*

It is, I think, a fair summary of the data on mathematics in the preschool period that it shows that children understand and use simple mathematical relations, and that they begin to learn about the number sequence, but that they have difficulty in combining these two very different types of mathematical achievement. One consequence of this disconnection is that young children often make mistakes about the meaning of the number words that they are learning. Piaget's doubts about their understanding of number words are justifiable: his reservations about their understanding of cardinal and ordinal relations, however, are far too pessimistic.

My thesis poses a causal question. What makes children so quick to grasp and use quantitative relations and yet so slow to come to terms with the basic meaning of number words? One possible explanation may lie in the informal instruction that they receive at home. A number of observational studies (Durkin, 1993; Durkin, Shire, Riem, Crowther & Rutter, 1986; Saxe, Guberman & Gearhart, 1987) show how difficult it is for parents not to give ambiguous verbal information to their young children about actual numbers. However a study by Riem (cited by Durkin, 1993) suggests that mothers find it quite easy to encourage one-to-one correspondence, at any rate as far as counting is concerned, and sharing, as I have remarked, is an activity which is probably quite easy to imitate.

The Second Approach

The second approach is mainly sums. The people studied in this research are mostly school children, but in some cases they are adults. The mathematical problems that are given in these studies are school-like and they are often quite hard. They take the form mainly of word problems, which are problems embedded in a meaningful sentence or a passage of prose.

(1) *Counting and addition and subtraction*

Hughes' conclusion, with which we ended the last section, is an encouragement to look again at counting and at number words. Abstract additions and subtractions which evidently cause young children so much difficulty are usually couched in number words. In fact, it is possible to reduce an addition to a simple count. You can solve 2+5 sum by counting up to 2 and then counting on by 5 from there. This is the most basic solution of all, and it is known as counting-all. It is, however, a relatively uneconomic way to solve an additional problem. After all, why go to the trouble of counting the 2? Why not start with 2 and count on from there? This justifiable shortcut is called the counting-on strategy and it is far more economic. But there is a still more sophisticated option which is (rather clumsily) called counting-on-from-larger. Given 2 + 5 the child could start with the larger number 5 and simply count on by 2 from there. To start with the larger number has the effect of reducing the amount that has to be added and thus makes the operation quicker and probably less prone to error.

There is ample evidence that children progress from counting-all to counting-on to counting-on-from-larger during their first 2 years at school. Furthermore, Groen and Resnick (1977) have shown that it is possible to hurry on this development by giving children concentrated experience with additions.

The apparently spontaneous realisation by young schoolchildren that the order of the addends is immaterial implies quite strongly that they have also understood one of the main properties of addition—its commutativity. To treat 2 + 5 in the same way as 5 + 2 is surely to demonstrate an understanding that 5 + 2 and 2 + 5 are equivalent. So it seems quite likely that children who consistently count-on-from-larger will also understand the commutativity principle. Yet the evidence on this apparently highly plausible connection is rather disappointing. Baroody and Gannon (1984) claim that they have found children who have adopted the counting-all-from-larger strategy and yet do not seem to have any idea of the commutativity of addition. However, our own results (Turner & Bryant, submitted) fail to confirm this claim. In our sample all the children who counted-on-from-larger appeared to understand commutativity, though not vice versa.

One further point needs to be made about these counting strategies. They can be used as well for subtraction, and there is evidence (Woods, Resnick & Groen, 1975) that children sometimes count down and sometimes count up to solve such problems. Indeed Woods *et al.* claim that experienced children use whichever of the two strategies is the more economical: given a sum like 9 – 2 where there is a small subtrahend and quite a large gap between the two numbers they count down (by 2, in this case): but with a sum like 9 – 7 where the subtrahend is larger and the gap smaller they count up (again by 2).

There is some debate, however, about the frequency and the value of these strategies. Woods *et al.* report that children count down frequently in subtraction problems, and that the strategy serves them well. Siegler (1987) on the other hand claims that the strategy is relatively infrequent and that it is error-prone. He reached this conclusion on the basis of a study in which he found that a large proportion of the children's subtraction errors occurred in problems in which they counted down.

Nevertheless the general readiness of young children to use their knowledge of counting to solve addition and subtraction problems is good evidence that, in spite of the initial difficulties which they have in understanding the nature of the number system in their first years at school, they are still able to use the counting system in an impressively flexible way to solve mathematical problems. These data also suggest that children understand a great deal about one property involved in addition—its commutativity. However, they tell us nothing about children's understanding of another essential mathematical property—the additive composition of number. For direct evidence on that we have to turn to another source of information about children's mathematical understanding—word problems.

(2) *Word problems*

Word problems are sums that are embedded either in stories or at least in meaningful sentences. They play a familiar part in the mathematical exercise books that school children have to work their way through at school as they do in psychological studies of children's mathematical skills. Broadly speaking there are four kinds of addition and subtraction word problems, and these are:

(a) *Compare problems*
Mary has six books, and John has four. How many more books does Mary have than John?
(b) *Equalise problems*
John has five comics, and Mary has two. How many comics does Mary need to have the same as John?
(c) *Change problems*
John has four marbles and is given two more. How many does he have now?
(d) *Combine problems*
Mary has four apples, and John has two. How many do they have altogether?

The great interest of problems such as these is that different kinds of problem can contain exactly the same mathematical sums as each other but in a different context (Fayol, 1992; de Corte, Verschaffel & de Win, 1985; de Corte & Verschaffel, 1987, 1988; Stern, 1993). The different kinds of word problem simply represent the sum in different ways and in different contexts. Therefore, if one type of problem turns out to be much more difficult than another, the difference between the two should tell us something about the way in which children represent mathematical tasks to themselves. Here are some examples of such differences.

Compare problems are consistently harder than Equalise problems (Carpenter & Moser, 1982; Riley, Greeno & Heller, 1983; Cividanes-Lago, 1993) even though, a glance at the two examples above will demonstrate, that in mathematical terms the two are the same. The difference is quite pervasive too, for Cividanes-Lago (1993) has shown that it is almost as strong when continuous material (length) is used as when the problems are given in their usual numerical form. So we are dealing with a question about the representation of quantity in general, and not just of numbers.

One possibility is that it is a matter of young children's difficulty in understanding the additive composition of number. The most direct way to solve the compare problem given above is to realise that the larger number—6—consists of the smaller number 4 plus some other number, which must, therefore, be $6 - 4$. To carry out this line of thought the child must realise that if $6 - 4 = 2$ then $4 + 2 = 6$. This in turn depends on an understanding (1) of part–whole relationships (6 consists of $4 + 2$ or $3 + 3$ or $1 + 5$) and (2) of the inverse relation between addition and subtraction, and it is worth noting that Piaget's theory predicts that both these forms of understanding are difficult, if not impossible, for children whose understanding of quantity is irreversible.

There is good evidence that the stumbling block in the difficult Compare problems is indeed the need to plot part–whole relationships. When Hudson (1983) managed to make these relationships more obvious, the performance of the young children that he was testing improved dramatically. His ingenious manoeuvre was to tell children about a certain number of birds who were looking for worms, when only a limited number of worms were available. How many birds, he asked, were unable to find a worm? The children found this version of the task considerably easier than a more typical Compare problem which involved exactly the same material. Evidently the new task was easier because the children were given a rationale for breaking up the larger number into two parts.

On the other hand the context of the Equalise problems probably helps the child by encouraging him to start with the smaller quantity and to count up to the larger. The context, therefore, helps the child to avoid the part–whole problem: the child does not have to break up the larger number: he simply has to increase the smaller one by a certain amount. Much the same analysis of the Equalise/Compare difference can be found in Riley, Greeno and Heller's theory about word problems.

(4) *Multiplication and division*

It is generally recognised that multiplication and division are more difficult operations and are generally understood a great deal later by young school children than addition and subtraction. Indeed the relation between the easier and the more difficult operations is often portrayed in terms of conflict. There is considerable evidence that children often wrongly use an additive strategy when a multiplicative strategy is the appropriate one, and on the other hand Miller and Paredes (1990) have claimed that for a while children actually "learn to add worse" as a result of coming to grips with multiplication. However, it is also clear that children use their knowledge of the easier operations to help them with the more difficult ones. Children often resort to repeated addition to solve multiplication problems and to repeated subtraction to do division (Nunes, Schliemann & Carraher, 1993).

It is often claimed that multiplicative relations are particularly difficult for children. The reason offered for the difficulty is that in multiplication the transformation depends on the size not just of the multiplier but also of the multiplicand. If you add 2 to 4 and 2 to 6, the change is the same in both cases: both numbers increase by 2. But if you multiply each of those numbers by 2 the transformation is quite different in either case. This particular difference between the two operations leads naturally to the question whether it is harder for children to understand multiplicative relations than additive ones.

There is no doubt at all that the answer is that it is a great deal harder. Multiplicative relations pose great difficulties even in late childhood in tasks devised by Inhelder and Piaget (1958) and reworked by others (Siegler, 1976; 1978). It has also been claimed that children often treat a multiplicative relation as an additive one (Anderson & Cuneo, 1978; Wilkening, 1979; Wilkening & Anderson, 1982).

The best example of the paradigm which has been used most frequently to support this claim is an experiment by Wilkening (1981). In this, children had to make judgements about time, speed and distance. They were given three animals who ran at quite different speeds (turtle, guinea pig, cat) and in different trials each of these animals ran for a certain period of time (the length of time that a dog barked). There were three periods of different lengths and that made nine trials in each task. In one of these tasks the child had to judge how far each animal would have run given the amount of time the dog barked at it. In another the child had to judge how long the dog had barked given the distance each animal had run. The question that Wilkening asked was whether the children's judgements would reflect the multiplicative relations in these tasks. For example, in the distance task would the children's judgements show that they realised that increasing the time would change the distance covered by the fast animal far more than the distance travelled by the slow one?

In fact in the Distance task the children's judgements did appear to be multiplicative, but Wilkening argued that this was probably an artefact. He observed that children tended to move their eyes along the track while the dog barked, and he guessed that they moved them at different speeds for the different animals. This is a perfectly respectable practical solution, but one that Wilkening dismisses as unmathematical. He laid more store by the results of the Time task which in the case of the younger children were essentially additive. The children recognised that the slower animals would take longer to travel a certain distance than a faster animal, and that each animal took more time to travel a long than a short distance. But their judgements did not show that they had any idea that the difference in time taken to travel a short and a long distance was far greater for the slow than for the fast animal. On the other hand, the pattern of the older children's judgements appeared to represent the multiplicative relationship quite well. So, Wilkening concluded that younger children adopt an additive strategy for such tasks but, as they grow older, begin to recognise the multiplicative nature of situations like these.

These and other experiments which employ a similar design are elegant and their results are compelling in many ways. But there is a need for caution. After all, the children are not being asked to solve mathematical problems: the question is whether they recognise a multiplicative relation and it is quite possible that they may not see that this or that situation requires multiplication even though their judgements fall into the pattern that conforms with multiplicative relations.

Perhaps one should turn instead to word problems and to the kind of multiplication problems that children have to solve at school. Greer (1992) has proposed 10 different kinds of multiplication word problem and, though his list of categories is quite a complex one, it has the advantage of raising the possibility that different problems may mean quite different things to school children. Vergnaud (1983) has argued that this is the case but that there are in effect only two broad categories of multiplication problem, which, he claims, children treat in different ways. One type involves "isomorphism of measures", the other "product of measures".

Isomorphism of measures problems concern only two variables, and the proportion between them, such as: "If one box holds eight oranges, how many oranges would 18 boxes hold?" The question in isomorphism of measures problems is always about one of the original two variables (in this example oranges). In product of measures problems, two measures are multiplied to produce a third measure. Area problems are one example ("How many square inches is a table top with a length of 36 inches and a width of 18 inches?"), and Cartesian Product measures ("If a boy has seven different trousers and eight different shirts, what is the number of different outfits (trouser–shirt combinations) that he could wear?) are another.

Vergnaud's own research suggests that isomorphism of measures problems are a great deal easier for children than product of measures problems are. This may mean that isomorphism of measures problems are quite easy to represent, and particularly in terms of repeated addition.

There is, however, one known exception to the rule that isomorphism measures problems are easier than product of measure problems. Nunes and Bryant (1992) have shown that 10-year-old children from England and Brazil are more likely to grasp the principle of the commutativity of multiplication ($8 \times 3 = 3 \times 8$) with product of measures problems. In one study we posed several multiplication problems to each child and at the same time we made it possible for them to use a calculator if they wanted to. In the commutativity problems the children were readier not to use the calculator and to rely on the commutativity principle when the problems involved product of measures problems. Again this seems to be a matter of representation. It is quite easy to imagine changing a rectangular area measuring 8×3 feet to one measuring 3×8 feet. One simply needs to rotate the rectangle. On the other hand it is quite hard to make the imaginary transformation of eight oranges each in three boxes to three oranges each in eight boxes: that would involve an awful lot of juggling of imaginary oranges.

So, one important constraint in these multiplication problems is in the way that the children represent the multiplications. Another might be the kind of relations that they can cope with. This seems to be the case with proportional problems too. Proportions are pervasive—how full a glass is, how much of the day has passed, how much red and white paint to mix to make a particular pink. Yet most of the work on proportions seems to show that children under the age of 10 are completely at sea in proportional tasks and some demonstrate that many older children and even adults also find such tasks extremely hard. Bruner and Kenney's (1966) and Siegler and Vago's (1978) well known studies of children's judgements about the relative fullness and emptiness of different sized glasses produced a dismally low performance in children below the age of 10, and Noelting's (1980a, b) attempt to find suitably proportional judgements about the ratio on concentrated orange juice to water ended in a similarly low result with children in this age range. Karplus and Peterson (1970) and Hart (1981) have found consistent errors in their proportional task in children as old as 16 years.

The most common explanation for these striking difficulties was originally provided by Inhelder and Piaget (1958). It is that proportional problems involve second order relations ("rapports de rapports"). To work out for example that one glass is proportionately fuller than another one has to look first at the proportion of liquid in each glass, and then to compare these two proportions. This last comparison—of a relation between relations—is what causes the difficulty according to Piaget and his colleagues.

There is another possibility. It is that children may be able to use their relational skills to work out some proportions but not others. In this case their difficulty would be in plotting individual proportions and not in comparing them. According to this analysis some proportions should be easier to work out than others.

A paper by Spinillo and Bryant (1991) took this view. The starting point was the evidence that children can take in and remember simple relations like larger and smaller, more and less. This means, we argued, that if an object is clearly divided into two parts the child should be able to register something about the relations between the two parts: for example, he should be able to work out that the two parts are equal in size, when they are, or that one part is bigger and takes up more room than the other. So the child should be able to compare quantities on the basis of their internal proportions in some cases. For example, if he remembers that there is more liquid than empty space in one glass (it's more than half full) that he has seen and then compares to another which has more empty space than liquid (less than half full) he should be able to work out that the relation between liquid and empty space is different in each container.

Notice that in this hypothetical example one container is more than half and the other less than half full. In fact the half boundary plays a crucial role in this analysis. Children should be able to make proportional discriminations which cross the half boundary, (one glass more than half and the other less than half full) than when they do not (e.g. both containers more than half full, even though one is proportionally fuller than the other).

Our experiments with 5–8-year-old children supported this hypotheses. The task they were given was to recognise which of two boxes of white and blue bricks was represented in a much smaller picture. In the crucial comparison the bricks were in a different arrangement; the arrangement of the two sections in the picture was different from the arrangement in the two boxes (vertical vs. horizontal dividing line between the blue and white sections), which meant that the children could not solve the task on the basis of any shape or size cues. Yet the children did particularly well in judgements which crossed the half boundary (3/8 blue vs 5/8 blue) and rather badly in comparisons which did not cross this boundary (e.g. 5/8 blue vs 7/8 blue).

As far as we can see, the successful judgements were genuinely proportional ones, and the justifications which we recorded for these successful judgements bear this conclusion out. But it is worth noting that our idea is of part–part comparisons not of part–whole ones. The children, according to our hypothesis, were able to take in the relation between the blue and the white segments, but not between the blue section and the whole container. Otherwise they would be as successful in the comparisons that do not cross the half boundary as in those that do. Children of this age, it should be noted, would be at sea with the demand in Parrat-Dayan's task (Parrat-Dayan, 1980, 1985; Parrat-Dayan & Voneche, 1992): "Give me half of the five apples." That task also involves part–whole relations.

What Does the Culture Provide?

In the last decade there has been a remarkable shift in emphasis in psychologists' accounts of mathematical development. In the past most psychologists who dealt with the subject argued about children's abilities to do this and that, and paid very little attention to where these abilities came from. By and large they assumed that the culture in which the child lived played very little part in the development of these abilities. Piaget, for example, looked mainly to the child's informal interactions with his environment for the source of mathematical skills. Gelman argued that the basic 'principles' behind counting, for example, are innate. These are disparate hypotheses, but both of them exclude the possibility of cultural influences.

There are now two lines of evidence which suggest that we should consider the role of culture very seriously. One concerns the nature of the number system, and the other the plain fact that the context in which a child deals with a mathematical problem can have a remarkable effect on the way that he tries to solve it.

Many of the psychologists who study children's counting, seem to treat the number system as a simple sequence. But our number system is not like that. It is a hierarchial structure based on decades. The decade structure makes it possible to count generatively. One does not have to remember that the next number after 1119 is 1120. We can generate these numbers on the basis of our knowledge of the structure of 10s, 100s and 1000s.

In the best, or at any rate the most regular, of all possible linguistic worlds one should be able to generate all the numbers up to 99 by knowing the rules and remembering the names for the numbers from 1 to 10. Chinese is actually one such language: the number words are entirely regular: thus the word for 11 is the equivalent of 'ten–one' and for 24 'two–ten–four' and so on. However, there is a degree of irregularity in English number words, and particularly in the English teen words. The word 'eleven', for example, gives nothing away, and 'fifteen' is hardly better. Nevertheless the majority of English words for numbers greater than 20 do represent their position in the decade structure quite clearly.

This structure plays an important part in all but the simplest arithmetical operations: it is essential, of course, for the system of decimals: and it is also a necessary part of our daily lives since it is the basis for our currency and also for many of the measures that we use and talk about all the time. Not surprisingly, people who live in cultures which do not have this structure or any equivalent to it find it quite hard to do any but the simplest of mathematical calculations (Saxe, 1991).

This structure, which lies at the heart of our mathematical lives, is a cultural invention. It was invented relatively late in the history of mankind, and it is not to be found in all cultures (Saxe, 1981, 1991; Saxe & Posner, 1983). It is not something that children will learn about spontaneously. It is handed on from generation to generation, and it would disappear if it were not taught either formally or informally to successive generations. These are important points for a developmental psychologist, because they mean that the decade structure perfectly fits Vygotsky's idea of a cultural tool. Cultural tools, Vygotsky argued, are inventions which increase the human intellectual power, and also transform humans' intellectual processes. The alphabet is one example, systems of measurement another, and the number system an obvious third.

Yet we still know relatively little about the way in which children learn about the decade system or about the effects that this learning has on their mathematical understanding. The best evidence is cross-linguistic. Miller and Stigler (1987) compared the way in which 4, 5 and 6-year-old Taiwanese and American children counted and found quite striking differences. For the most part the Taiwanese children did a great deal better at abstract counting (i.e. just producing the numbers in the correct sequence) and there was a striking difference between the two groups in the counting of the teens which gave the American children a great deal more difficulty than it did the children from Taiwan. When the two groups counted objects, there was absolutely no difference between them in terms of their success in counting each object once (Gelman and Gallistel's one-to-one principle), but again the Taiwanese children did a great deal better in producing the right number words in the right order.

Miller and Stigler attribute the differences to the regularity of the Chinese system. One cannot rule out the possibility of differences in other factors, such as motivation, playing a part, but the Miller and Stigler explanation looks plausible and receives considerable support from subsequent comparisons by Miura, Kim, Chung and Okamoto (1988) of Japanese and American children's performance in simple mathematical tasks and by reports from Fuson and Kwon (1992a, b) of the considerable achievements of Korean children in complex addition tasks (the Japanese and the Korean number words are a great deal more regular than the English ones.).

The differences originally reported by Miller and Stigler go far beyond success in counting. We (Lines, Nunes & Bryant unpublished data) recently compared Taiwanese and British children in a shop task which involved money. This shop task was originally devised by Carraher and Schliemann (1990), who asked children to buy certain objects and charged them certain amounts of money. In some cases the children could pay in one denomination (ones or tens), and in others they had to mix denominations (ones and tens) in order to reach the right sum. The condition which mixed denominations was easily the harder of the two, and Carraher and Schliemann rightly argued that this demonstrated that the children were having some difficulty in using the decade structure to solve mathematical problems, at any rate as far as money is concerned.

The Carraher/Schliemann study made an interesting developmental point about growth in the understanding of the decade structure, and our more recent project (the one by Lines, Nunes and Bryant) suggests that the nature of the linguistic system may have a considerable effect on the way that children become able to use the decade system. For we found not only that British children were worse at counting than Taiwanese children (a replication of Miller and Stigler) but also that, in the shop task, the Taiwanese/British difference in the mixed denominations condition was particularly pronounced. The Taiwanese were no better than the British children when the task was to pay for the purchases in ones, and not much better than the British group when they had to pay in tens. But when the children had to pay in a mixture of tens and ones (10p and 1p or $10 and $1) the superiority of the Taiwanese children was very striking indeed. It seems that the linguistic advantage helps the Chinese speaking children not just to count more proficiently but also to grasp the relations between different levels of the decade structure and to use these relations to solve simple problems. The number system becomes a cultural tool far earlier for them than for English speaking children.

So the nature of the cultural tool affects the way that children learn about it, and so does the context in which they learn about this tool. Children learn about the decade structure at school but also outside it. The fact that money and other measures are organised in decades means that all children are bound to receive a significant amount of informal instruction about decades outside the classroom.

It is now clear that children can pick up quite different forms of mathematical knowledge in these two environments. The clearest evidence of this is the work by Carraher, Carraher and Schliemann (1985) with children who work in the informal economy in Brazil. These psychologists bought food from children who worked on market stalls, thus setting them problems which involved addition and multiplication (working out the price) and also subtraction (working out the change). Later Carraher *et al.* gave the same children equivalent problems in a more formal setting. These were either word problems or straight sums. The children were a great deal more successful in the market stall transactions than with the more formal problems. But the most striking result of the study was a qualitative difference between the calculations that the children made in the market and with the more formal calculations. In the market the children were remarkably flexible: they moved from addition to multiplication, in the same calculation, with no difficulty. In the more formal tasks they tended to stick to one algorithm, even when they realised that the answer that it produced for them was quite implausible.

The proficiency of children working in the informal economy in Brazil was later confirmed in a series of studies by Saxe (1988; 1991). Nunes, Schliemann and Carraher (1993) themselves went on to investigate the mathematical achievements in farmers, building foremen and fishermen who had received very little education, but had to carry out quite complex mathematical calculations in their work. These studies, and others of a similar genre (Gay & Cole, 1967; Lave, Murtaugh & la Rocha, 1984; Lave, 1988; Posner, 1982; Scribner, 1986; Schliemann & Nunes, 1990; Nunes, 1992) leave no doubt at all about the need to consider the situation in which people acquire mathematical knowledge and then use it. Ironically, mathematical achievements outside the classroom encourage us to think more about what is taught at schools. At the very least we must consider the reason for the psychological barrier, identified so clearly by Carraher *et al.* (1985) and by Nunes *et al.* (1993), that clearly exists between children's formal and informal mathematical experiences. Why do children make so little connection between the two?

Conclusions

The study of children's mathematics is the study of their understanding of mathematical relations. This review has shown that very early on in their lives and well within the pre-school period children are able to detect and to remember simple one-way relations, and that their ability to do so allows them to do things which are genuinely mathematical. They share and they measure. These are relational activities: one involves a simple, one-to-one correspondence, relation between two quantities, and the other rests on a combination of two separate relational judgements. Furthermore, very early on they seem to grasp the way in which quantities can increase and decrease by addition and subtraction. The thesis that children's initial entry into the mathematical world is through their understanding of simple quantitative relations is actually quite an old one (Lawrenson & Bryant, 1972; Bryant, 1974).

In contrast, their pre-school encounters with numbers may not at first be of much importance, as far as their understanding of mathematics is concerned. This is probably because a true understanding of the number system almost certainly rests on a thorough grasp of the decade structure, and this must, in one way or another, be taught.

The relational understanding which children bring with them to school helps them in some ways, but is insufficient in others. It allows them to understand addition and subtraction fairly well, when these involve simple increases and decreases in quantity, and they soon are able to involve their growing understanding of counting to help them estimate the effects of these transformations by counting on or counting down. But problems which require some attention to the relations between parts and wholes continue to stump them. Much the same can be said about children's understanding of proportions. Their fairly rapidly acquired sensitivity to "half" is based on their ability to make part–part comparisons. Proportional judgements which involve part–whole calculations continue to elude them even in late adolescence.

For much the same reason multiplicative relations, as Piaget claimed a long time ago, are plainly much harder than additive relations for children to grasp, although we still need to find out a great deal more about this aspect of children's mathematical understanding.

Finally, the context in which children learn about and use their mathematics is a matter of great importance and great interest too. In particular the Brazilian work on "street mathematics" has demonstrated not only significance of the considerable achievements which are handed on to children in informal settings, but also the worrying subjective barriers between children's formal and informal learning of mathematics. It is as if they live in two separate mathematical worlds with hardly a connection between the two. Surely we should find ways for them to bridge that gap.

There are other gaps for us to bridge. Although the broad outline of mathematical development is reasonably clear, or so it seems to me, yet there are still some formidable gaps in our research on the topic. In particular, we know very little about the connections between one mathematical achievement and another. There is a striking contrast here with work on children's reading, which has demonstrated, for example, many strong connections between children's pre-school abilities and their success in reading later on (Goswami & Bryant, 1990). We have no such research in mathematics, and I cannot see why.

There are testable hypotheses here. One is that children's early experiences with relational comparisons, but not with the counting, are strongly connected to the progress that they make in learning about the four arithmetical operations at school. Another is that the real breakthrough in understanding the number system comes not through learning to count but through understanding the structure of the decade system. A third is that experience with part–part relations leads to an eventual understanding of part–whole relations, and hence to full proportional reasoning.

These are ideas which can be tested by a combination of longitudinal research and intervention studies, in just the same way as similar causal connections have been pursued and eventually discovered in work on children's reading. This should be our next move.

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