Duckie
And The Search For The Golden Goose

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Contents

All was quiet in the goose village. The autumn breeze scented the air within the forest. The sun shone brightly, casting light onto the main path. To an outsider looking at the village, it would have seemed idyllic. Undoubtedly, it was the sort of place where parent geese could look out and feel their goslings were safe. At any other time, it would have been a pleasant evening - a tranquil picture of peace. Yet that fall evening, there was a sense of despair in the village. Each goose knew that their quiet evenings would soon be over, and that they would have to leave for migration.

“Enough is enough.” Duckie thought. For as long as he could remember, he had been afraid of the long and dangerous journey. Between the icy mountain peaks and vast exposing stretches of forest, the journey left him and the other geese vulnerable to the unwelcome dangers of the wilderness. It was no wonder as to why fewer geese returned every year. Duckie looked up the main road at city hall, where the goose authorities stood watching the city.

Created to protect the village from danger, the goose authorities created and enforced the rules village. To ensure the the citizens followed their rules, they created the elite corps of goose police. In the past years, the goose authorities had begun to abuse their power. They had closed down the schools, burned the village’s books, and closed off the village from the outside world. But what could Duckie do? The goose authorities, the leaders of the village, had controlled migrations for as long as anyone in the village could remember, and he would soon have to embark on the dangerous journey.

Duckie thought back to his childhood.

When Duckie was young, he loved stories. He would spend hours at the library reading stories of the great heroes of the village. Duckie especially loved tales of the golden goose of legend, who lived on the highest mountain’s peak.

Duckie reached into his pocket and looked at a charred page. It was all that remained of the library. A scrap of stained paper with an inscription, smeared with years of use. It read

...

Duckie knew what he needed to do.

Many geese had tried to seek the Golden Goose, but it was a dangerous journey, and none had ever returned.

“No. I have no choice.”

“Remember the legend Duckie. Remember the golden goose.”

Chapter 1

Arithmetic

That night, Duckie couldn't sleep. He knew something had to be done. Looking out his window, he saw the border of the village and heard the goosepolice patrolling. He would try to find the golden goose, and would help create the great changes the hero was meant to bring. Hopefully then, he could save the village. He knew he would have to break the goose authority's rules and search for the hero. Duckie carefully crept out of his house. The authorities had made it difficult to leave the village. Not only was trying to leave heavily punished, it was also difficult. Duckie would have to be stealthy.

Duckie waited until everyone had gone to sleep, and started on his journey.

He began to sneak towards the village border. As he approached, he saw two guards snoring loudly. He heard the fluttering of the wind, and could feel his feathers being lifted by the cold winds of early winter. He tiptoed past, and all seemed quiet for a moment.

"Phew", he whispered as he crossed the border.

But just as he turned his back to the village, he heard alarms ringing.

Honk! Honk! Honk!

They droned loudly. Duckie turned around and saw the two guards he had walked past coming towards him. The goose authorities had been alerted of his mission, and he would have to run.

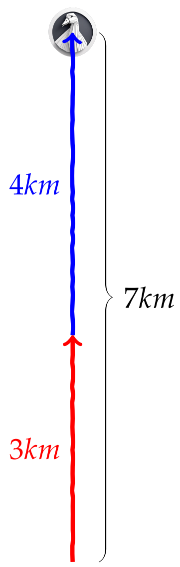
"Hold it right there!"

With a rustle of feathers and beating of wings, Duckie took off and began to fly towards the mountain as fast as he could. As he looked back, he saw the goose police on his tail.

1.1 Addition

The first night of the journey, Duckie felt the breeze blowing beneath his wings as he flew away from the village. A thick fog had blanketed the forest surrounding the village, making it hard to fly. “Keep flying. Find the golden goose”, Duckie said to himself, looking at the lights coming from the village houses.

To get away from the village, Duckie first flew $3km$. He turned around, hoping that he had shaken the goose authorities off the trail. Unable to see clearly through he fog, he decided to land on the forest brush to regain his strength, but just as he had settled down, he heard the nearing squawks of the goosepolice. With a spurt of energy, Duckie took off from the forest floor, and flew another $4km$. As the sun rose, Duckie decided to form a plan.



He needed to know how much he had traveled in the direction of the mountain. Where was Duckie?

Solution

Duckie first flew $3km$, then changed this amount by 4. Duckie’s position, 3 kilometers, changed by 4 kilometers. This is $3 + 4 = 7km$.

Definition

Addition, or adding, is the most basic way of using numbers. It represents changing one number by another to form a single, combined number. It represents taking two separate quantities and combining them together to find a new quantity

1.2 Multiplication

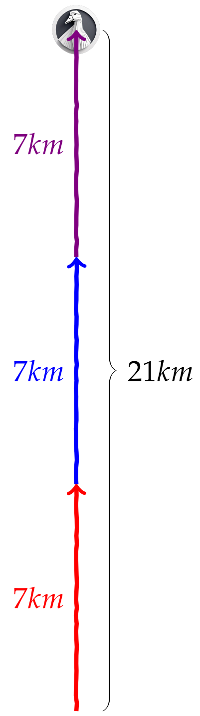
"Wow, they fly so quickly. I'm going to have to keep going." Duckie said aloud to himself. He heard the faint shouts of the goose police in the distance.

"Keep an eye out. You know what the goose authorities will do if we don't find him." they clamored.

Duckie continued his journey toward the mountain, with the goose authorities following closely behind.

For the first 3 days, the same cycle would repeat. Duckie would fly 7 km. Each day, he stopped, hoping to find some rest. His muscles ached from exhaustion, but the thought of the golden goose forced him to keep going.

Duckie wanted to know how much he had flown towards the mountain. Where was Duckie?



Solution

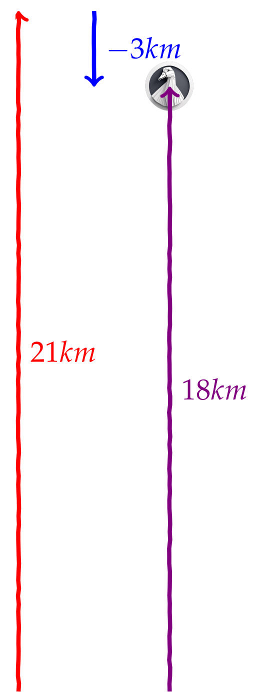
Duckie flew 7 km, 3 times. This means the amount he flew was triple what he flew on the first day. This is $7 * 3$, or $21km$.

Definition

Multiplication, or multiplying, is the second basic way of using numbers. It represents changing a number by a certain scale. For example, doubling, tripling, etc. $A * B$ is A times more than B.

1.3 Negative Numbers

On the morning of the fourth day, Duckie had traveled $21km$. His journey had gone successfully so far, but he knew he could not avoid the goosepolice much longer. He decided he had to take evasive maneuvers to try to confuse the goose police.



Their training to be the fastest and best in the village meant they would inevitably catch up with him if he kept flying without rest. To escape, he decided do the last thing they expected: go back towards the village. By the time Duckie had finished taking evasive maneuvers, he had traveled in the opposite direction of the mountain for $3km$.

Duckie wanted to know how far he was from the mountain. Where was Duckie?

Solution

This is called $-3km$.
– means opposite direction. He first flew $21km$, then flew $-3km$. This is$21 - 3$, or $18km$.

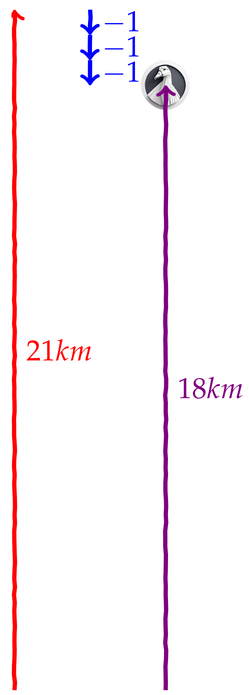
Definition

A number less than 0 is called "negative", and is in the opposite direction.

1.4 Multiples of a negative

As Duckie performed evasive maneuvers, he decided to be strategic. He knew he could not travel quickly, especially after having traveled so much. He needed to gather his strength for the difficult journey ahead, so he decided to take a break after every kilometer. He traveled -1 kilometer 3 times.

How far did Duckie travel while taking evasive maneuvers?



Solution

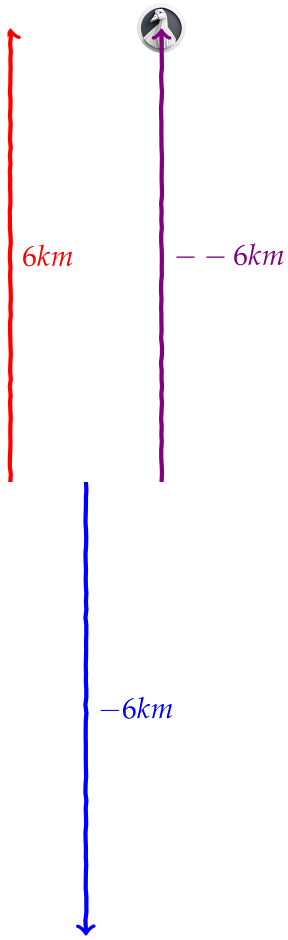
Duckie traveled -1 km 3 times. This is $-1 * 3$, or -3 km.

Definition

Multiplying by a negative number shows stretching or scaling a number by some amount, but in the opposite direction.

1.5 Negative of a Negative

By the fifth day, Duckie could no longer heard the police in the distance. In a way, he felt bad for them - they had been forced to fight him, whether or not the believed in the prophecy of the golden goose. Feeling he had avoided the goose police, Duckie began to head back towards the mountain.



“I’ve got to make up for lost time. The goose police are really tough.”

Duckie turned away from the village, and began to fly as long and far as he could. After 6 km, Duckie felt faint. Flying so far was not an easy task. He had flown in the opposite direction of the negative direction for 6 km, or $-(-6)$.

Duckie wanted to know how far he was from the mountain. Where was Duckie?

Solution

Initially, Duckie was at 18 km. He then traveled $-(-6)$ km. Duckie traveled 6 km towards the mountain. he had flown $18 + 6$ km, or 24 km towards the mountain.

Definition

The opposite direction of the opposite or negative direction of a number is in the same direction of the number. It is written as -(-number), which is the same as the number itself.

1.6 Multiplication Table

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

Observations about the Multiplication Table :

0 Any number multiplied by 0 is 0. This makes sense, because any number repeated 0 times is the same as not having it at all.

1 Any number multiplied by 1 is itself. This makes sense, because any number repeated one time is itself. This has a special name: Identity

10 Any number multiplied by 10 is shifted to the left by one digit.

Symmetry The table is identical across the diagonal. This shows that $4*2$ is the same as $2*4$. This property also has a special name: Commutativity.

Chapter 2

Properties

Duckie was exhausted. It had been five days since Duckie had had a proper rest. He glanced over his shoulder. The fog had cleared, and Duckie could see the vast expanse of wilderness behind him. To Duckie, it seem light - unecumbered by the signt of the the goosepolice in the distance .Looking backwards, Duckie began to descend to rest for the first time since beginning his journey.

As he sailed to towards the forest floor “Squawk!”. He ran headfirst into another goose.

“Hold it right there! You need to come back with me to the village!” the goose demanded. Duckie blinked. As his eyes cleared, he saw that the other goose was weasing the blue cap of the goose police. “It’s dangerous in the outside world. the policegoose demanded, equally dazed. This is for your own good!”

Duckie examined the policegoose for a moment. He wore the bright double arrow badge of a lieutenant. He knew he could not overpower him, so he attempted to reason, hoping the other goose was reasonable. “I know it is dangerous officer, but what can I do?” Duckie explained his fears for the village “I have to save the village!” he ended. The policegoose had also heard of the legends, but his time in the goose authorities had left him with a less-than-imaginative disposition. “That’s just a myth.” He replied, ruffling his feathers. “Stop risking your well-being for a ridiculous dream, and come back to the village. Haven’t the goose police kept you safe all this time?”

“Its true. Some of the rules do keep us safe, but too many take away our liberty. I think they are afraid of what will happen if I find the golden goose. If I can find her, then they may not be able to stay in power. You joined the goose police to help people, and if you let me go, you can do just that!”

“You might be right, but I can’t betray the village. If I just let you go without at least trying to stop you, I won’t be able to look myself in the mirror, much less be seen in the village! I have no interest in fighting, so I'll tell you what. I will have a set of challenges with you. If you beat me, I will let you go. If not, you have to let me take you back. Sound fair?"

Duckie was reluctant, but knew that he had no other options. "Fine. Let's do the challenges."

2.1 Associativity

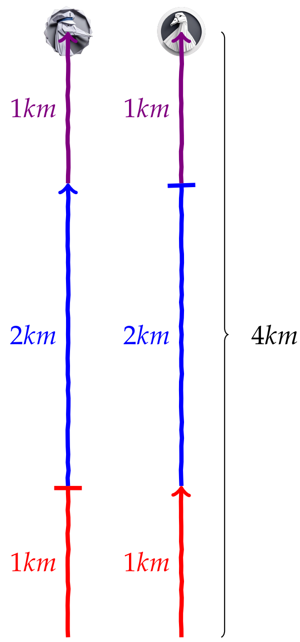
Güs and Duckie both felt that they were good fliers.

"To begin, let's do a flying contest in 3 parts." Güs announced. "Whoever flies the furthest wins."

Duckie thought about this. "Fine, but to make sure no one gets an unfair advantage, let's allow one break in between."

The first leg of the race was 1 km, the second leg was 2 km, and the third leg was 3 km.

Duckie and Güs both wanted to win, so created strategies. Güs decided to travel the first leg (1 km) of the race, took his break then travel the second and third legs (2 km and 3 km). Duckie instead traveled the first leg, (1 km) of the race and the second leg (2 km) of the race, then took his break, and traveled the third leg (3 km). Did they tie by traveling the same amount?



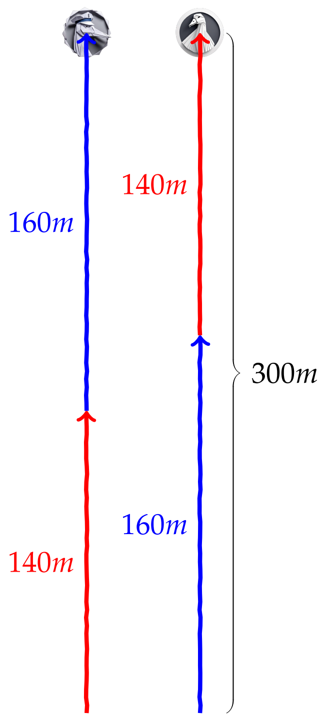
Solution

Yes! Duckie traveled $(1\ km + 2\ km) + 3\ km$, or 6 km. Güs traveled $1\ km + (2\ km + 3\ km)$, or 6 km. Duckie and Güs both traveled 6 km.

Definition

Addition can be grouped in any way. In other words: (A+B)+C = A+(B+C).

2.2 Additive Commutativity



Duckie and Güs sat down, tired. They had each expected to win, and had surprised each other.

"What should we do? Will you let me go?" Duckie asked hopefully.

Güs thought about this. "No, but I will let you try to beat me again"

Because geese are water animals, swimming is a very important skill, so they decided to swim for their next competition. They each had a different strategy. Güs swam 140 m, dried off, then swam 160 m. Duckie instead swam 160 m, dried off, then swam 140 m. Did they tie?

Solution

Yes! Güs flew $140\ m + 160\ m = 300\ m$. Duckie instead flew $160\ m + 140\ m = 300\ m$. They swam the same amount.

Definition

Order doesn't matter when adding. In other words: A+B = B+A.

2.3 Distributivity

Duckie and Güs were starting to get frustrated.

"How are we going to get around this?" Güs asked.

Duckie thought for a moment. He knew Güs could only compete for so long before becoming tired, and the more they flew, the better chance he would have. "How about a flying contest? We can do four parts instead of just three. Maybe then one of us will win."

Güs agreed. His training from the goosepolice had given him excellent stamina, and was confident he could beat Duckie in long distance flying.

In the third contest, Güs traveled 2 *km*, then 3 *km*, and repeated that pattern 2 times. Duckie instead traveled 2 *km* 2 times, then 3 *km* 2 times. Did they tie?

Solution

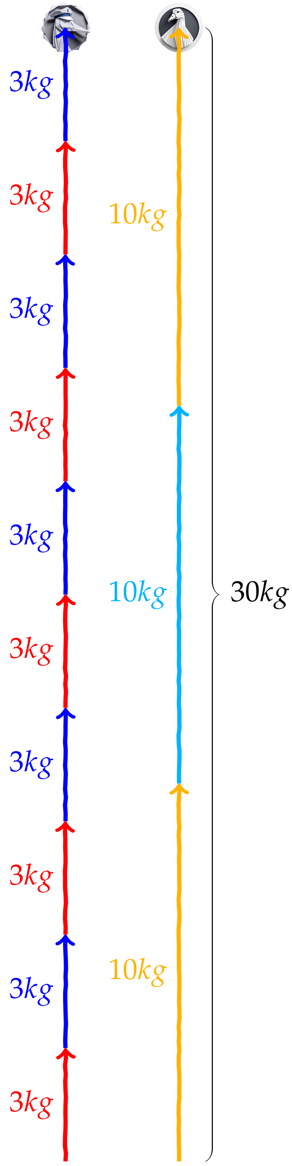
Yes! Duckie flew $2 * (2\ km + 3\ km)$, or 10 *km*, and Güs flew $2 * 2\ km * 2 * 3\ km$, or 10 *km*.

Definition

Multiplying with an expression is the same as multiplying with each part of the expression. In other words: A∗(B+C) = (A∗B) + (A∗C).

2.4 Multiplicative Commutativity

At this point, Güs realized that Duckie was a very capable goose. When he first challenged Duckie, he expected winning to be a cakewalk, but Duckie ended up posing quite a challenge. Seeing Duckie's perseverance, Güs began to respect Duckie. He decided to give him another chance. He decided that the next contest should be a weightlifting competition, where whoever lifted the most amount of total weight would win. Duckie lifted 10 kg 3 times, whereas Güs lifted only 10 kg but did it 10 times. Did they tie?



Solution

Yes! Duckie lifted 10 $kg * 3 = 30$ kg, and Duckie lifted 3 $kg * 10 = 30$ kg. They lifted the same amount.

Definition

When multiplying, order does not matter. In other words: A∗B = B∗A.

Duckie and Güs landed on the forest floor, exhausted. They had realized that, no matter what the competition was, they were equally matched. They hadn't beat each other, but they had won each other's respect.

"Why don't you join me?. You can make a difference!" asked Duckie.

"I can't!" Güs cried. "What about my life in the goose police?"

"Well, do you believe the golden goose is out there?"

"Honestly, I'm not sure. Do you really think it will make the village a better place?"

Duckie knew that there was the possibility the golden goose was a hoax, but knew the danger of disobeying the goose authorities, but knew he could help the entire village.

"You joined the goose police to help people", he reasoned. "You are clearly capable of making the dangerous journey, and if you can join me, you can do exactly that."

Güs knew he was right. While he was afraid of what could happen along the journey, deep down, knew the tyranny of the authorities had gone too far. He decided to be brave and join Duckie.

"All right. I will join you!" he said resolutely.

Chapter 3

Essential Algebra

Duckie and Güs's various competitions made the goose authorities a bit confused. Because of Duckie and Güs's competitions, they had lost track of Duckie's location. They decided that they needed to review their records, and try to figure out where their activities fifth day had landed them.

3.1 Equality

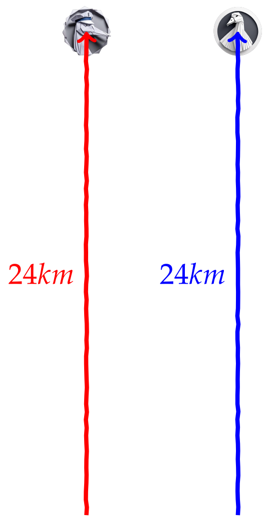
Duckie and Güs had tied in each of competitions. Because of this, their positions were always the same.

Solution

Güs's position = Duckie's position. Duckie's distance from the village was 24 km, so Güs's position was also 24 km.

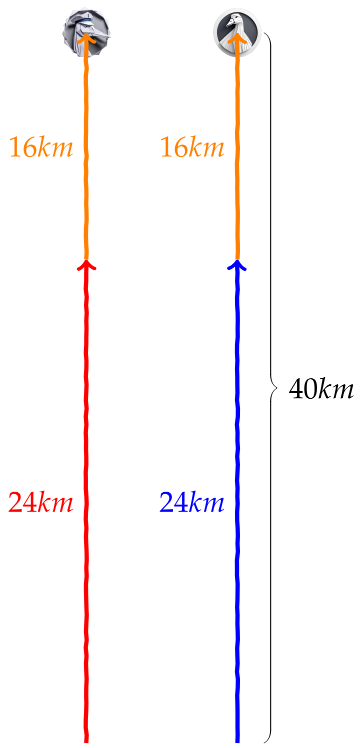
Definition

When two things are the same, we write this in mathematics as "=", or "equals".



3.2 The Equality Rule

The goose authorities, knew Güs was a fast flier. His training as a policegoose had made him a fearsome foe. While Duckie and Güs were traveled together, they realized that every time Güs flew ahead of Duckie, Duckie had to increase his position by the same amount to keep up.



Solution

Duckie's position + Δ position = Güs's position + Δ position.
$24\ km + 16\ km = 40\ km$ [1]

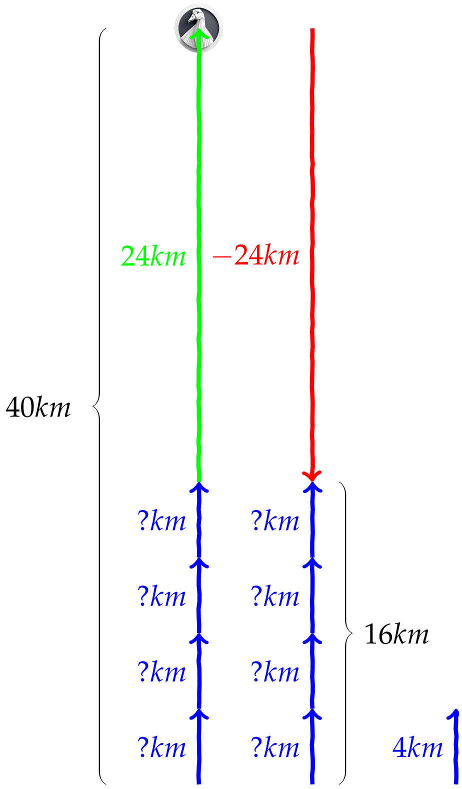
Definition

What happens to one side of an equation must happen to the other side so that they are still the same.

[1] Δ means "change in"

3.3 Variables

The goose authorities were still a bit confused about Duckie and Güs's progress. They knew that the two had 4 competitions, and that Duckie and Güs began competing when they were 24 km away from the village. They also knew that Duckie and Güs stopped competing at 40 km. How much did they travel on average in each competition?



Solution

$$\begin{gathered} 4x + 24 = 40 \\ 4x + 24 - 24 = 40 - 24 \\ 4x = 16 \\ \frac{4x}{4} = \frac{16}{4} \\ x = 4 \end{gathered}$$

Definition

A letters or symbol can be used to represent a number which isn't already known, such as "x".
$4x + 24 = 40$ means that $4*$(some number)$+24 = 40$.

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Chapter 4

Ratios and Fractions

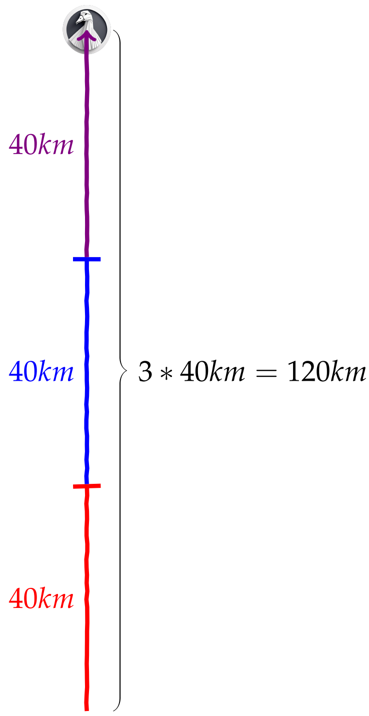
It took a lot of work, but the authorities finally figured out where Duckie and Güs were. They noticed that the pair had made quite a bit of progress, and that they needed to take the threat seriously. Duckie and Güs's position information was know extremely valuable to stopping them. Güs, a prominent policegoose's defection, showed them that sending other policegeese would only undo their efforts in building a loyal army.

In their past migrations, they had worked with the worst geese in the land to help control the village. As a result, they knew many bounty hunters along the way, and believed that they could have them set traps.

They quickly called them all, and sent them to bring back Duckie and Güs, dead or alive. These famously clever bounty hunters each decided to strategy to help them catch Duckie and Güs.

4.1 Ratios

The first bounty hunter decided to look at the entire journey, and set up traps every 4 km. He guessed that the entire journey was 120 km long. Duckie having guessed the first hunter's (fairly obvious) scheme, knew that if he knew the number of traps, he could avoid them. For every 4 km in the journey, there was 1 trap. How many traps were there be on the journey?



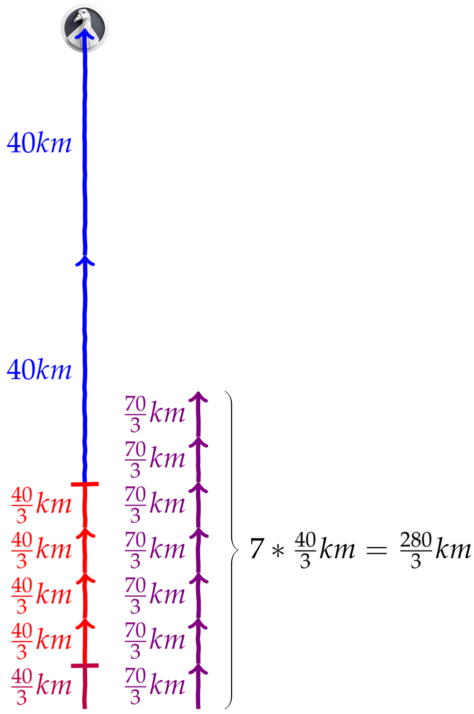
Solution

For every 4 km Duckie had to travel out of 4 km overall, there was one trap. That meant that there was a trap in every $\frac{120}{4}$ km of the journey. For every one km in 4 km, there were 3 km in the journey, which meant that there were 3 lengths of size 4 km within 120 km. There were $\frac{3}{1}$, or 3 traps.

Definition

A ratio represents the quantity of A compared to the quantity of B, and is written as $\frac{A}{B}$, or A for every B. This is also the same as breaking up A into B equal parts.

4.2 Multiplying with Fractions



Using his knowledge of mathematics, Duckie figured out how to avoid the first bounty hunter's traps. Never underestimating his opponent, the second bounty hunter decided to prepare better. He realized that km are a very large unit for geese, so decided to start measuring the distance Duckie and Güs flew in a unit he created called "goosemeters" (gm). This meant that he needed to change Duckie and Güs's position from kilometers into goosemeters. There are exactly $7gm$ in every $3km$. To help beat the second bounty hunter, Duckie and Güs decided to also measure using gm. How many gms are in $40km$?

Solution

$\frac{7gm}{3km} * \frac{120km}{1} * \frac{1}{3} = \frac{280}{3}gm.$

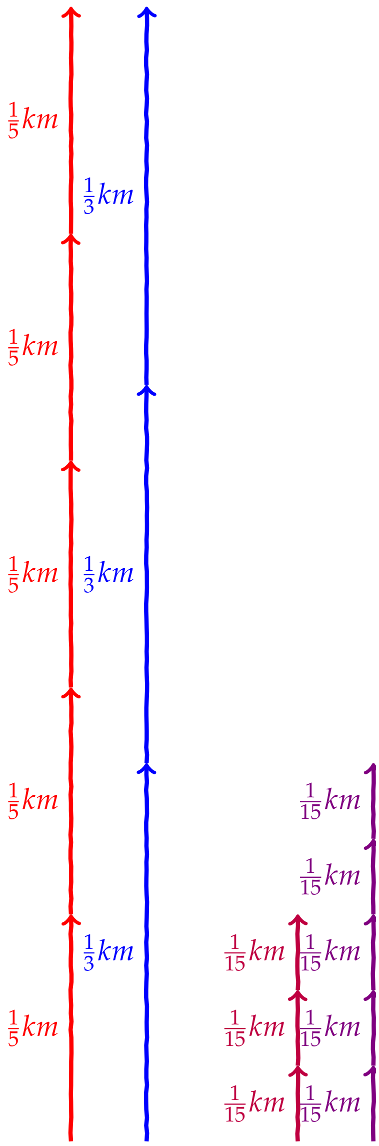
Definition

Multiplying by a ratio also represents stretching or scaling by a certain amount. It represents scaling the number P so that $P * \frac{A}{B}$ is its new size. This is the same as breaking up $P * A$ into B equal parts, or $\frac{P}{B}$ A times (breaking up P into B parts and taking each part A times).

Notice that that the size of $\frac{A}{B}$ can be in between 0, 1, 2, etc.

4.3 Comparing Fractions

Duckie knew that he could avoid the bounty hunters if he flew far enough in a single day. On the eighth day, Duckie and Güs traveled $\frac{1}{5}$ of the journey. If Duckie traveled more on the eighth day than the previous days ($\frac{1}{3}$ of the journey), the bounty hunter would stop chasing them. Did Duckie manage to confuse the bounty hunter?



Solution

Duckie realized that

$$\frac{5}{5} = 1$$
$$\frac{1}{3} * 1 = \frac{1}{3}$$
$$\frac{1}{3} * \frac{5}{5} = \frac{1*5}{3*5} = \frac{5}{15}$$

Which means $\frac{1}{3}$ is same as $\frac{5}{15}$

similarly $\frac{1}{5} = \frac{1}{5} * \frac{3}{3} = \frac{3}{15}$

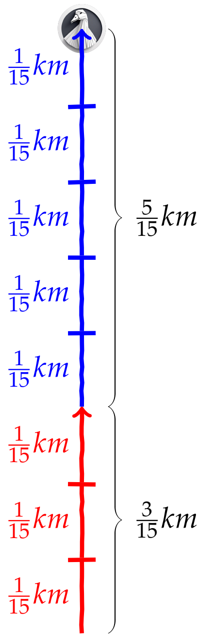
Now, we can compare the equal size parts together. $\frac{3}{15} < \frac{5}{15}$. Duckie traveled less on the eighth day than the other days combined.

Definition

We can only deal with fractions together using same-size parts. We can chop up one fraction by multiplying it by another fraction equal to one. If the denominators (the bottom number of the fractions) are the same, the fractions have same sized parts.

4.4 Adding Fractions

Despite Duckie's best efforts, he was unfortunately not able to confuse the bounty hunter. He was starting to get worried, but Güs had a great idea. If they could find out where they were along the journey, they might have been able find a new route, which would take them away from the hunter's grasp! They traveled $\frac{1}{3}$, then they traveled $\frac{1}{5}$. How much of the journey have Duckie and Güs traveled?



Solution

Duckie and Güs have traveled $\frac{1}{3} + \frac{1}{5}$
$= \frac{3+5}{15}$
$= \frac{8}{15}$ of the journey.

Definition

When adding fractions, break up each fraction so that they are in terms of same-size parts (or having a common denominator). Then, add the numerators as usual.

Duckie and Güs sighed in relief. After running for bounty hunters for two days and two nights, they were exhausted. They decided to travel slower for a few days and enjoy the scenery.

"I can't believe how persistent the goose authorities are." Duckie sighed. "I knew this journey would be tough, but I wasn't expecting bounty hunters!"

Güs considered this. He had worked with the goose authorities, and knew their patterns. He realized how unusual this was. "I agree, the authorities must really be desparate".

Just as Duckie opened his beak to reply, he felt a sudden weight on his shoulders, and began spiraling out of control. It was a net! Another bounty hunter must have set a trap! He started falling down fast, gaining speed as he went. Crash!. Duckie fell to the ground with a thud. Güs hurriedly landed next to him.

"Are you alright Duckie?" he cried in alarm.

"I'm all right, but I think I need to rest my wing for a few days", Duckie winced. He felt that he hadn't broken a wing, but saw some bruising, and knew he would need time to heal.

"The forest is dangerous though! We can't stay here in the open with all the predators!"

"Last year, during migration, I made a friend, named Bessie the cow. I think her farm is close enough to walk to from here," Duckie said. "I think she will be able to help keep us safe from the goose authorities."

Together, they waddled to Bessie's farm.

Chapter 5

Rectangles, Perimeter, and Area

After walking all afternoon, Duckie and Güs finally arrived at Bessie's farm. Despite the bad luck with the net, it was a beautiful afternoon, and the pair couldn't help but feel a sense of calm as the walked up to Bessie's barn.

Bessie was sitting outside, chewing on some grass.

"Hey there Duckie! How's it going? Who's your friend there?" she moo'd.

"Hi Bessie", Duckie greeted. "This is my friend Güs. We are looking for the golden goose of legend, but I need some help."

"Of course! What do you need?" asked Bessie.

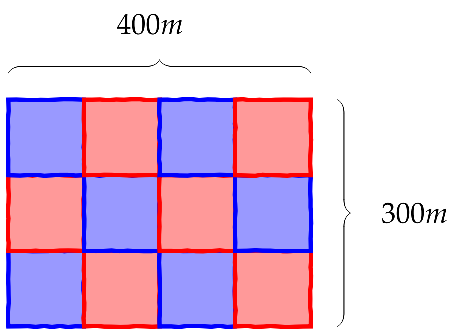
"I hurt my wing, and I need to rest here for a few days. The goose authorities sent some bounty hunters after me, and I need you to keep me safe from them," Duckie asked hesitantly.

"Wow, that sounds serious. Feel free to stay as long as you like" Bessie turned around to look at Güs. "While Duckie's is getting better, I could use some help on the farm. It's grass-planting season, and it would be much appreciated if you could help me out on the farm."

Güs, enthusiastically agreed. "That sounds great!"

5.1 Area of Rectangles

On the ninth day of the journey, Güs began working on the farm. His first task was to fly around and scatter grass seeds on the farm. As he went into the store room to find the bags of seeds, he realized he needed to know how many bags of seeds he needs to bring with him. Bessie told him that each bag of seeds can cover a square of grass that is 1 m wide and 1 m across. Güs also knows that the farm is $300m$ high and $400m$ wide. How many bags of seeds does Güs need to carry?



Solution

There are 300 rows of grass, where each row is made of 400 1 by 1 squares. This means that there are $300 * 400 = 300 * 400 meters^2$ of grass, and that Güs needs to carry $300 * 400$ bags.

Definition

Area is how much space an object takes up. When measuring a 2d shape, we find how many 1 by 1 squares the object fills. When measuring the area of a 3d shape, we find how many 1 by 1 by 1 cubes the object fills. Multiplying the length of each side together for rectangular gives us the area. To simplify, the repeated multiplication, we can write the number of multiplications above the number. For example, $m = m^1$. $m * m = m^2$. $m * m * m = m^3$

5.2 Area of Square Edge Triangles

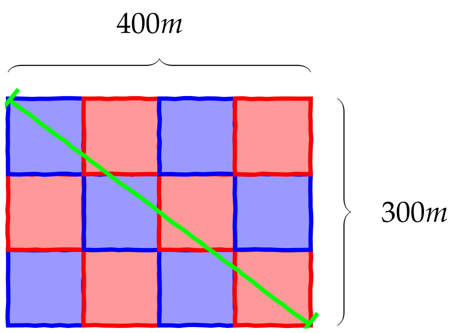
As Güs took off, he noticed that the farm was split into two equal parts along the diagonal to make space for Bessie's pet human Farmer John. He realized that here, he could not plant grass, as Farmer John needed it for corn. How many bags of grass should Güs plant?

Solution

The farm is divided in half, so Güs only needs to plant half of his bags.
$\frac{1}{2} * 300m * 400m$
$= 1200m^2 * \frac{1}{2}$
$= 600m^2$.

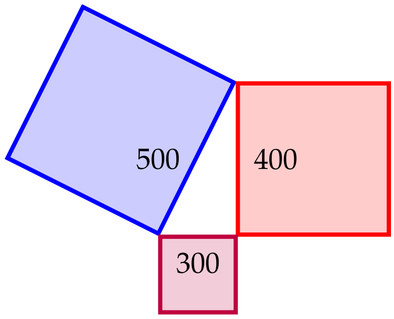
Definition

Right triangles (triangles with a square edge), are rectangles that have been split in half along the diagonal. This means that the area of a right triangle with side lengths x and y has an area of $\frac{x*y}{2}$.



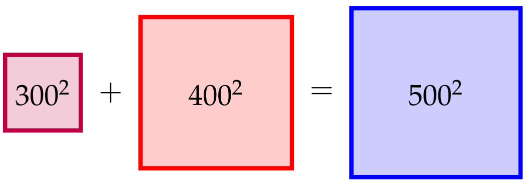
5.3 Pythagorean Theorem

Güs started planting grass. He remembered from school that grass spread rapidly, and so decided that to keep Farmer John's corn safe, he would make a small fence along the diagonal. How long of a fence does Güs need to buy?



Solution

$300m * 300m + 400m * 400m = 500m * 500m$. Güs needs to buy $500m$ of fence.



Definition

In a right triangle, the lengths of sides are related to one another. In such a triangle, a∗a + b∗b = c∗c, where c is the diagonal length in the triangle. This relationship is called the pythagorean theorem. [1]

[1] A proof of the theorem is found here: `https://www.mathsisfun.com/geometry/pythagorean-theorem-proof.html`

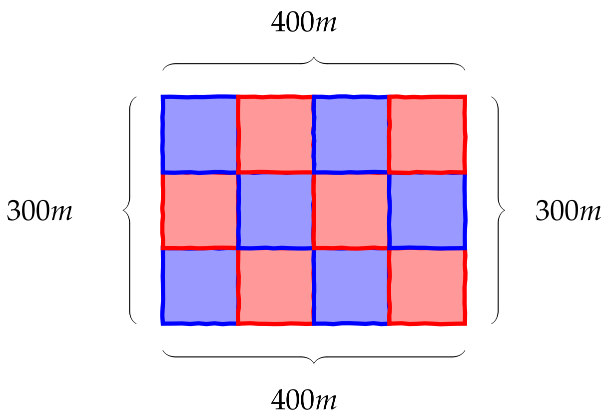
5.4 Perimeter of a Shape

After Güs finished planting grass, he decided to take a break for lunch.

"Aaaah," he sighed, contented. As he looked out at the farm, he noticed a discontented look on Bessie's face.

"What's the issue?" he asked.

"It's this fence." Bessie replied. "It's falling apart and we have to look at it all day." Güs decided that to thank her for letting them stay for the night, he would buy her a new fence. How many meters of fence does Güs need to buy?



Solution

Two sides of the fence are $300m$ long, and two sides of the fence are $400m$ long. That means that there are $300m + 300m + 400m + 400m =$ $2 * 300m + 2 * 400 =$ $2 * (300m + 400m) = 2 * 700 = 1400m$ of fence.

Definition

Surface, or perimeter, is the size of the edge of an object. When measuring a 2d shape, we find how many lines of length 1 can fit around the object. When measuring a 3d shape, we find how many 1 by 1 squares fit around the object. (We will discuss this further with integrals). The perimeter is the sum of all of the side lengths. Because 2 of the side length of the sides of a rectangle are always the same, 2x+2y is its perimeter, which can also be written as 2(x+y).

5.5 Shoelace Theorem

The shoelace formula is a tool which lets us find the area of any polygon. Go around the vertex's of the polygon, where the current vertex's coordinates are x_i, y_i and the next vertex's coordinates are x_{i+1}, y_{i+1}
$A := \frac{1}{2}\sum_{cyc}(x_i y_{i+1} - x_{i+1} y_i)$.

Chapter 6

Circles and Angles

On the tenth day, Bessie had an issue: Farmer John was bored and kept causing trouble on the farm. Rather than letting her and the other cows graze, he was trying to cut down their grass. To keep him entertained and away from the grass, Bessie decided to create crop circles in the corn field.

“Duckie!” she called. “Want to help make some crop circles?”

“Sure! I think it will be great for helping my wing recover!” Duckie replied.

.

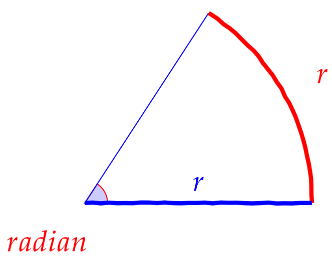
6.1 Angles

Bessie and Duckie came up with a plan to draw their circles. They decided that Duckie would fly around Bessie, staying exactly $7m$ away from Bessie at each point. To make sure Duckie is exactly the same distance, Bessie would spin around and look at him at any given point. Bessie knows she will get dizzy if she spins too much, so decides to keep track of how much she has turned at any moment.

Solution

Definition

The number of meters between the Duckie and Bessie is called r, or the “radius”. When Duckie has flown r meters around Bessie, the amount Bessie has turned is a “radian”.



6.2 Drawing a single Circle

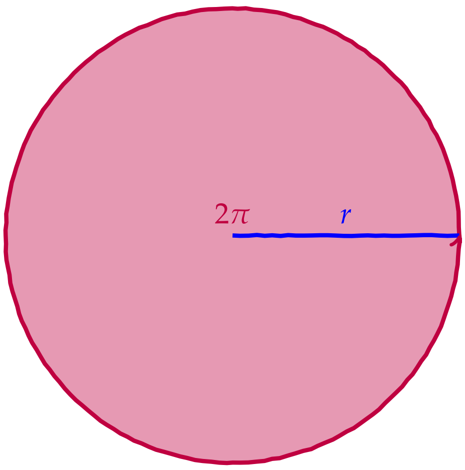
Bessie and Duckie began to draw the first circle, but ran into an issue. She didn't know when to stop spinning! How many radians does Bessie have to turn so that Duckie makes a full circle (and she looks in the same place she started)?

Solution

Definition

When Duckie has flown a full circle around Bessie and Bessie was looking in the same direction where she started, she had turned around $2 * \pi$ radians. This amount Bessie turned cannot be written as a fraction, and so is called "irrational". This specific irrational number has the name "pi".

For convenience, approximate π to be $\frac{22}{7}$ (the actual value of pi is slightly smaller).



6.3 Perimeter of a Circle

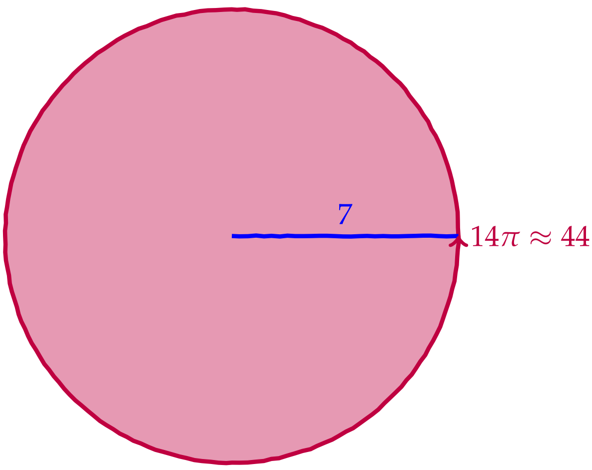
After some effort calculating, Duckie drew one full circle around Bessie. But, because of his injured wing, he began to feel a bit tired. He decided that in the full day, he would only fly a little so that his wing could recover properly. How much has Duckie flown?

Solution

For every radian Bessie turned, Duckie flew $7m$. Because Bessie turned $2*\pi$ radians, Duckie have must turned $7*2*\pi)m$. Using $\frac{22}{7}$ as π, we get $\frac{2*22*7}{7} = \frac{2*22}{*1} = 44m$

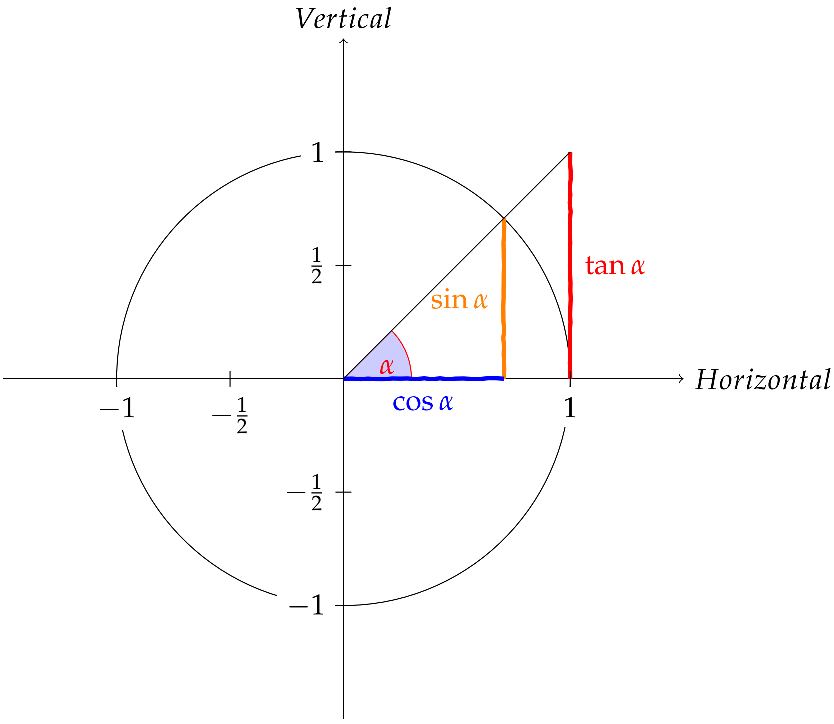
Definition

The perimeter of a circle is always $2*\pi*r$, where r is radius of the circle.



6.4 Trigonometric Functions

After resting for a few minutes, Duckie and Bessie started drawing their next circle. They decided that, to help give Farmer John some contrast, they would make it have a radius of 1. While creating the circle, tragedy struck! Duckie lost track of where he was! He sees that Bessie turned $\frac{\pi}{4}$ radians. What is Duckie's vertical position? What is Duckie's horizontal position?



Solution

When Bessie rotated $\frac{\pi}{2}$ radians, Duckie's vertical position and horizontal positions are $\frac{\pi}{2}$.

Definition

Take a line of length 1. As we rotate the line, we can see it has both a height and a width. When it starts, it has width but not height. When it rotates $\frac{\pi}{2}$ radians it has height but not width. When x is the angle rotated, the width at each angle (or the adjacent side/the diagonal side) is $cos(x)$. The height at each angle (or the opposite side/the diagonal side) is $sin(x)$. The ratio between these, $\frac{sin(x)}{cos(x)}$ is called $tan(x)$.[1]

[1] These functions, and other trigonometric functions can be viewed here. `https://visualize-it.github.io/trig_functions/simulation.html`

6.5 Trigonometric Identities

Duckie, tired, but with a healed wing, sat down. He and Bessie were ready to take the day off, and he was ready to start flying the next day. Güs, who had been planting grass, walked over to them.

"Hey Guys! How was your work in the cornfield?"

"It was pretty good!" Bessie replied. "We were drawing circles for Farmer John while you planted grass."

Güs looked at Bessie in surprise. "That's what you were doing? I thought you were tracing the point of an arrow where the width×the width plus the height×the height equaled one!" The trio looked at each other in confusion. Who is right?

Solution

They were all right! The width of the arrow is cos(x), and the height of the arrow is sin(x). Through Pythagorean Theorem, the length of the arrow must be 1, which is also the radius that Duckie flew.

Definition

$sin(x) * sin(x) + cos(x) * cos(x) = 1 * 1 = 1.$

Chapter 7

Sets and Functions

It was time for Duckie and Güs to continue on their journey. They had enjoyed spending time with Bessie, but they still needed to get to the Golden Goose.

“What are we going to do about the Bounty Hunters?” Güs asked Duckie nervously. They both were concerned about the bounty hunters’ traps.

“I’m not sure” Duckie replied, “But we will have to keep moving. Hopefully we find something or someone along the way who can help”.

As the duo began to fly once more that evening, they saw an approaching flock of geese behind them.

“Quick! Land!” Duckie cried. In a flash, in a rustle of feathers, the pair landed and waited, hoping to go unnoticed.

There was the squaking of the the voices of the other group of geese as they landed beside Duckie and Güs.

“Hi, how’s it going?” one of the geese asked. This surprised Duckie. He was expecting to be taken away to the goose village, but he wasn’t expecting a greeting.

“Hello...” Duckie said cautiously. “Who are you? Have the goose authorities sent you?”

“We are just a gaggle from the north. My name is Snow. What are the goose authoriies?”. Duckie relaxed. Clearly, these geese wouldn’t take them away. Güs, having had experience from journeying with the Goose Police, knew the value of traveling with a group.

“Oh, I guess if you don’t know who they are, they aren’t that important to you,” Güs said to Snow. “We are trying to find the golden goose, and it seems like you are heading in the same direction as us. Mind if we join you?”

“Alright!” Snow said. “But you’ll have to join our dance!”.

7.1 Cartesian Product

That night, the new flock decided to have their traditional nightly dance. It was a partners dance, with the following rules: geese with red feet were one group, and the geese with blue feet were another group. Each of the red geese partnered with each of the blue geese, where a red goose lead. The group of red geese in the dance was {Güs, Snow, Grey, Blue}. The group of blue geese was {Duckie, Barnie, Swan}. How many ways could the geese have formed pairs? What are those pairs?

Solution

All the pairs that the Geese could have formed were {(Güs, Duckie), (Güs, Barnie), (Güs, Swan), (Snow, Duckie), (Snow, Barnie), (Snow, Swan), (Grey, Duckie), (Grey, Barnie), (Grey, Swan), (Blue, Duckie), (Blue, Barnie), (Blue, Swan)}. There were 4 geese with red feet, and 3 geese with blue feet. There were $\begin{array}{r} 4 \\ \times\ 3 \\ \hline 1\ 2 \end{array}$ ways to form pairs

Definition

A set is an unordered collection of things. If there are two sets, A and B, the cartesian product (AxB) are the pairs (a, b) where $a \in$ [1] A and $b \in B$.

[1] $\in$, means "is in the set"

7.2 Relations

Maggie, one of the blue footed geese and, the organizer of the dance, looked over Güs's shoulder to see what he was doing.

"Hi there! Whatcha doin'?" she asked inquisitively.

"Nothing much, I'm just trying to figure out how the dance works. Can you take a look?"

"Of course!" Maggie considered his drawings. "It seems like you are on the right track, but I think Snow and Barnie like to dance different styles. Maybe they shouldn't be a possible pair?"

Güs removed their pairing from his plans. "So all of the pairs in the new set are in the original set."

How many sets of pairs could Maggie and Güs make out of the original cartesian product?

Solution

Maggie and Güs can quite a few sets. These include {}, the AxB itself, all sets with only a single pair, all the sets with only two pairs, etc.

Definition

If a set A (also called the domain) is a subset of set B (also called the codomain), then every element in set A is also in set B. For every set A, there are $2^{|A|}$ possible subsets which can be made [2]. A subset of a cartesian product called is a relation.

[2] |A| means size of A

7.3 Functions

Güs and Maggie continued to plan, and decided to come up with the list of pairs of geese.

"Hmm..." Maggie thought, looking at the list of relations. "There are not enough blue geese for each goose to have their own partner. How can we make sure each goose has their own partner?"

Güs thought for a moment. "Maybe some of the blue geese can take turns switching between partners?"

"I'm not sure, but maybe it would work. Let's try it out!" Come up with a possible set of pairs dance partners Maggie and Güs could have formed.

Solution

Maggie and Güs eventually decided on the following set of dance partners {(Güs, Barnie), (Snow, Barnie), (Grey, Swan), (Blue, Duckie)} with Maggie sitting aside

Definition

A function is a special type of relation between the sets A and B. In a function, in every pair (a, b) (where $a \in A$ and $b \in B$), a only maps to a single output. This allows us to say that the $f(a) = b$, or that the value a maps to the value b.

7.4 Types of Functions

Definition

A function where every element in the domain has a different mapping is called “one-to-one”. A function where every element in the codomain is mapped to is called “onto”. A function which is both one-to-one and onto is called “bijective”.

As the dance began, Maggie watched the other geese dance. Güs noticed her watching, and came over.

“Why don’t you come over and dance with us?” Güs asked.

“Ah, you don’t want me to dance, I don’t to mess up the fun. Besides, this way, the blue geese don’t have to worry about switching partners.”

Güs looked over. “Don’t worry about that, it’s fun! Besides, now there aren’t enough red geese. C’mon!”.

Maggie hesitated, “Well, let’s see how they do on their own for now.”

“You’ll do great!”

“Alright! Let’s dance!”

Güs and Maggie went back to the dance floor together. What does the new function mapping red geese with blue geese look like?

Solution

With all the geese dancing, the red geese and new geese were mapped so that the dancing partners were {(Güs, Maggie), (Snow, Barnie), (Grey, Swan), (Blue, Duckie)}

7.5 Inverse of Functions

As the first dance came to a close, Maggie went to the makeshift stage to begin the next dance.

"All right everyone!" she announced happily. "Same dance, but this time, blue geese lead!"

The geese cheered, and began the dance again with the blue geese taking the lead. Was this new pairing a function? If so, what did it look like?

Solution

The new pairing was {(Maggie, Güs), (Barnie, Snow), (Swan, Grey), (Duckie, Blue)}. This new pairing was also a function, because every first element only mapped to a single output. For example, originally, f(Güs) =Maggie, but now f^{-1}(Maggie) =Güs

Definition

The inverse of a function $f(x)$ is written as $f^{-1}(x)$. The input of of f is the output of f^{-1}, and the output of f is f^{-1}. In other words, if $f(a) = b$, then $f^{-1}(b) = a$. In order for there to be an inverse function for $f(x)$, $f(x)$ must be bijective.

7.6 Continuous Functions

As Güs and Maggie danced with each other, they realized that they could also explain the movement of the dance as a function.
$f(t) = -t * t + 10$, where t was the number of minutes since the dance began, and $f(t)$ represented how far they stood from the stage. What was Güs and Maggie's position when they had danced for 3 minutes?

Solution

$f(3) = -3 * 3 + 10$
$f(3) = 1$.
When they had danced for three minutes, Güs and Maggie were standing 1 meter from the stage.

Definition

Functions can be written as pair explicitly, or with a formula. When making a function using a formula, the pairs are $(x, f(x))$

Chapter 8

Polynomials

The next morning, Duckie and Güs looked out of their campground.

“Looks like we are almost there. I’ve been talking to Maggie and I think she wants to come with us to find the Golden Goose” Güs remarked.

“That’s great!” Duckie said. “We shouldn’t count our geese before they hatch though. Even though we are almost there, we are going to start the most difficult part of the journey. Look at the mountain!”

The looked at the mountain and saw it towering above them.

“I agree. I think we need to prepare.”

As the rest of the geese woke up, Duckie and Güs stood up and announced their departure.

“Hello everyone. Thank you for your hospitality. I want to make sure we can get there quickly so that we can save our village, so we need to travel fast. We need to continue on our journey. Would anyone like to come with us?”

Maggie stepped forward. “I’ll join!”

“Great!” Duckie said.

8.1 Polynomial Functions

Duckie, Maggie, and Güs began flying towards the mountain.

“Wow thats pretty steep. We should probably try to figure out where it is and what it looks like.” Maggie noted.

“That makes sense” Duckie said. “Looking at the mountain, it looks to me like it touches the ground at two points. But how will we figure out how the mountain looks?”

Maggie pondered this over this for a moment.

“What if we made a function? It can map every position on the ground to the height of the mountain at that point. Since it touches the ground at two points, h(x) = 0”

“That’s a good idea” Gus said. “But where does it touch the ground? Maybe we can say it touches the ground when x_1 = ? and x_2 = ?”

What is h(x)?

Solution

We can separate the function into two parts. We want the function to be 0 when x is 0. $x - x_1$ will be 0 when $x = x_1$. Similarly, $x - x_2$ will be 0 when $x = x_2$.

0∗any number is 0.

Because of this, we can set $h(x) = (x - x_1)(x - x_2)$.

Definition

A polynomial function is a function where f(x) is the sum of terms, such that each term is a constant multiplied with x raised to a natural number. In other words, $\sum a * x^k$ s.t. $a \in \mathbb{R}, k \in \mathbb{N}$.

8.2 Polynomial Functions

Duckie thought for a moment.

"I am not so sure about that actually. To me, the function looks like $ax^2 - bx + c$"

Maggie was hesitant "Won't this give us the incorrect ground locations?"

"I'm not sure. How can we make sure?"

Where would the mountain touch the ground if $h(x) = ax^2 - bx + c$?

Solution

$\frac{-b \pm \sqrt{b^2-4ac}}{2a} =$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a} =$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a} =$

Definition

The roots of a polynomial with degree two are $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

8.3 Polynomial Functions

The trio sped up as they approached the mountain. As they approached, it seemed to grow taller and taller and wider and wider. They could see the snowcapped peaks clearly, and realized they had made the mistake. The mountain was x meters taller than they had originally thought.

"What can we do?" Gus pondered aloud.

"Can we modify the function we made before?" Maggie asked.

Modify the height of the function so that every point is x points higher and y points to the right.

Solution

$h(x) = h(x) + x$

Definition

A function can be transformed in 3 ways.

1. It can be shifted up and down by adding a constant from the overall function
2. It can be shifted to the left and right by subtracting a constant from x
3. It can be compressed by multiplying x by a constant.

What is an exponent?
They have not seen the top of the mountain yet, so they assume that it is a quadratic function
What is a polynomial? They need to figure out what the mountain looks like so that they can reason about it. The function has a peak, and has height 0 when on the ground, its positions are (x_1, x_2)
In short, they must find a polynomial given the roots
How do we do it the other way around? - Finding the roots of a polynomial - Find roots given quadratic formula
As they fly around the mountain, they see the second peak. They guess the new roots to be x_3 and x_4 (and also x_1, x_2)
Transformations - they realize they guessed wrong as they approach the mountain, so must adjust the function
We must expand the story of the golden goose to include advice (which Duckie uses through they story like in that dragon book). The legend will speak of a riddlye which will unlock the secrets of the mountain a) I have to beef up the prologue b) The riddle will give some hits on exponentiation and will help provide information on the journey.

The prophecy:

The highest mountain
So high that it will make
The ground itself shake
It will double, double again
and on it the geese will rise
The humans near and far
And dangers to the geese
Will be saved

From the geese of yore
The golden goose will rise

The journey to the highest mountain
will save

And great peace will come to the lands of geese

Chapter 9

Calculus

9.1 Average Rate of Change

Duckie, Maggie, and Gus began to climb the mountain.

It was a hot day, and they felt the rivulets of sweat running around their neck and over their wings.

Panting and sweating Maggie stopped the group - “This is ridiculous.

This mountain is so steep!”

Gus, more experienced with physical labor, replied “You are right, this mountain is too steep to climb”.

He stopped, and the rest of the group paused.

“Why can’t we fly?”.

Duckie chimed in “We need to know how fast to fly up! If not, we would run straight into the mountain!”

The group had traveled p meters horizontally, and had as a result climbed up k meters vertically. How much did they climb for every meter?

Solution

Definition

9.2 Instantaneous Rate of Change

As they continued to climb the mountain, they realized that it was not changing constantly. They were either flying too far away from the ground or running into the mountain.

“What did we do wrong?” Duckie asked.

“It seemed like we did it correctly before” answered Maggie, confused

The mountain snow and ice which blanketed the mountain trembled as the flew past, revealing a deep chasm. Gus, watching the events unfold, made an observation.

Because the mountain was not straight, the speed at which the height decreased and increased stayed the same as well.

“But how make it more accurate? I mean, after all, we want to find the change at a single point - the next bit of land which we will go over.”

“Yeah, that amount of horizontal change is really small. Could we ’shrink’ our change in x?”.

Is Duckie correct? If so, why?

Solution

Definition

9.3 Maxima and Minima

As they flew higher up the mountain, they became increasingly exhausted. Even though flying helped, it was still a steep mountain, and took a lot of effort.

"Let's stop and rest"

"I agree, we really need to stop. But the mountain is so steep. Where is the top of the mountain? Then we can stop where it will be a flat."

"Great!"

Where are the top and bottom of the mountain?

Solution

Definition

They get to the base of the mountain
They bein climbing, and notice it is getting steeper. They have climbed δy amounts vertically, and δx amount horizontally.
What is average rate of change
They want to know about slope at each point. They can make it more accurate by having less δx. They shrink the x until it is infinitesimally small, and are able to get the slope.
Can make another function for this mountain which has the value of the slope at every point? Look at a simple case and work the way up
They are getting tired, and must rest. They also need to know where the golden goose is.
They find the max and min of this function. Duckie and Gus get to the top

of the mountain and see that there is no one there there are some golden feathers scattered but it doesn't seem like anyone has been there in a long time duckie and gus are ultimately dissapointed after all, they came all this way so they turn around, and look from the top of the mountain from their great height, they can barely make out the village in the distance and they come to the realization that they made it all this way made so many friends in making the effort to find the golden goose they had created a reliable system for they themselves to provide safe passage The math they learned along the way as duckie looked out the mouth of the cave, his feathers began to shimmer, and in the evening light, they almost seemed golden