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Review article

Conceptual knowledge of the associativity principle: A review of the literature and an agenda for future research

Joanne Eaves [a,*], Camilla Gilmore [b], Nina Attridge [c]

[a] University of Nottingham: University Park Campus, Psychology, Pharmacy, Life Sciences, East Drive, Nottingham, NG7 2RD, United Kingdom
[b] Centre for Mathematical Cognition, Loughborough University, LE11 3TU, United Kingdom
[c] Department of Psychology, Portsmouth University, PO1 2UP, United Kingdom

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ABSTRACT

Individuals use diverse strategies to solve mathematical problems, which can reflect their knowledge of arithmetic principles and predict mathematical expertise. For example, '6 + 38 − 35' can be solved via '38 − 35 = 3' and then '3 + 6 = 9', which is a shortcut-strategy derived from the associativity principle. The shortcut may be critical for understanding algebra, however approximately 50% of adults fail to use it. We review the research to consider why the associativity principle is challenging and highlight an important distinction between shortcut identification and execution. We also discuss how domain-specific skills and domain-general skills might play an important role in shortcut identification and execution, and provide an agenda for future research.

1. Introduction

Knowledge of arithmetic principles is widely regarded as important for success with mathematics [70] and for developing mathematical expertise [66]. In the last twenty years, the number of studies investigating individuals' understanding of arithmetic principles has increased [102] and as a result, seven arithmetic principles are now widely discussed: identity, negation, complementarity, commutativity, inversion, equivalence and associativity [110]. Table 1 describes these principles.

Studies have found that many individuals have an inadequate understanding of arithmetic principles, because they fail to use strategies that those principles encourage, or fail to understand the basis of the strategy that they choose [91]. One principle that individuals have particular difficulty with is associativity [89], the property that permits some operations to be performed in a different order from that in which they are presented. Education practitioners have called for individuals' understanding and use of arithmetic principles to improve, including associativity [71] because it may play an important role in the transition from basic to more advanced mathematics [6].

However, compared to other arithmetic principles there is much less research into associativity. Studies to date mostly come from cognitive psychology (see Table 2 for a summary of relevant literature) with a handful from the mathematics education literature [6,29,61,98] and those from cognitive psychology most often used associativity as a comparison for other principles rather than investigating it as a standalone concept. We aim to synthesise this research by first reviewing the history, importance and measurement issues associated with the principle, and summarising the evidence of how frequently individuals apply the principle when solving arithmetic problems. We then make an important distinction between identifying and executing arithmetic strategies, and suggest that the origin of individuals' low use of the associativity shortcut may lie in the skills required for these two processes. In the final section we outline priority questions for the field and make suggestions for how they could be investigated. The primary goal for research, we suggest, should be to understand why the principle is rarely applied during problem-solving and this review discusses the possible explanations that future research could explore.

This review draws on research into other principles that have richer research histories than associativity, including a) inversion, the principle that addition and subtraction, and multiplication and division, are opposite operations [8], b) equivalence, the principle that two sides of an equation are equal and interchangeable [56] and c) subtraction complementarity, which refers to the insight that if 'a − b = c' then 'a − c = b' [11,15]. Research into these principles could serve as a guide for future investigations into associativity.

* Corresponding author.
E-mail address: joanne.eaves2@nottingham.ac.uk (J. Eaves).

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Table 1
Description of seven arithmetic principles that are often discussed in the mathematical cognition literature.

Arithmetic principle	Description
Identity	If an arithmetic operation produces a given result, then repetition of the same operation will produce the same result.
Negation	Subtracting an integer gives the same result as adding its additive inverse, e.g. "2 – 3" = "2 + (−3)".
Complementarity	If a + b = c then c − b = a and c − a = b.
Commutativity	Some operands can be performed in any order, e.g. a + b = b + a.
Inversion	Addition and subtraction, and multiplication and division involving the same value result in no change.
Equivalence	Two sides of an equation are equal and interchangeable.
Associativity	Some problems can be solved by decomposing and recombining groups of operations.

2. Definition and history

The associativity principle (hereafter 'associativity') states that '(a + b) + c = a + (b + c)'. In other words, because some operations are related, problems can sometimes be solved by first decomposing and then recombining their problem sets [16] and the answer to a problem can be the same regardless of which group of operations is dealt with first (e.g. a + b − c = b − c + a). In psychology studies, the principle has been studied in different situations, such as with problems that contain only addition (e.g. a + b + c = b + c + a), addition and subtraction (e.g. a + b − c = b − c + a), only multiplication (e.g. a × b × c = b × c × a), or multiplication and division (e.g. a × b ÷ c = b ÷ c × a), [15,92]. In mathematics education studies, the principle has most often been studied in the context of addition-only and multiplication-only problems [6].

Research has made good progress in uncovering individuals' knowledge of some principles, for example, numerous studies have focused on inversion and equivalence. However, this has left other principles, such as associativity, underexplored. While inversion has been investigated by psychologists for nearly 70 years [77], associativity received little attention until the late 1990s. At this time, it was predominantly studied through addition problems (a + b + c = b + c + a) and problems that inferred conceptual knowledge from the strategies used across sequential trials (e.g. recognising the conceptual relationship between '3 + 6 + 2' and the subsequently presented problem '9 + 2'), [16–18]. This approach continues to be fruitful for inferring knowledge of associativity [23,44].

In the decade that followed, the inversion literature (which often used problems of the form 'a + b − b' or 'a × b ÷ b' to infer knowledge of the principle) began to include problems such as 'a + b − c' and 'a × b ÷ c' as part of their studies. These problems were labelled as 'standard' or 'control' problems for inversion. The inversion problems were expected to be solved through a shortcut strategy (where the individual avoids the computation 'b − b' or 'b ÷ b' and simply picks 'a') and the control problems were expected to be solved through a left-to-right strategy. Indeed, verbal reports, accuracy and response times confirmed that this was the case [13,41,42,93]. However, in 2000 Klein & Bisanz noticed that some individuals solved control problems using a 'right-to-left' strategy (e.g. solving '6 + 38 − 35' by '38 − 35 = 3' and then '3 + 6 = 9'), a strategy now known in the psychology literature as an 'associativity shortcut' [34,35,85,87,88,90]. The 'shortcut' strategy is therefore one way of solving 'a + b – c' problems; for '6 + 38 – 35' it is more efficient than a left-to-right strategy. Although there may be alternative ways in which some people solve 'a + b – c' problems, left-to-right approaches and right-to-left shortcuts are the main strategies that are discussed in the literature, and that we focus on in this review. After Klein & Bisanz's [58] discovery of the shortcut strategy, Robinson & Ninowski [92] and Robinson et al., [93] pushed for 'a + b − c' problems to be used more widely as a measure of associativity. Their call was acknowledged, and today problems with opposing operations (addition and subtraction or multiplication and division) are a dominant paradigm used to investigate how well individuals understand the principle.

3. The importance of associativity

Recently, education practitioners have called for greater research effort to develop students' conceptual knowledge of arithmetic principles and the use of strategies that they permit [22,70]. This is an issue of international relevance given that some countries (UK, USA and Canada) place little emphasis on arithmetic principles in their education curricula, while other countries (Germany, New Zealand) place a strong emphasis on them [29]. The call is warranted for associativity in particular because it is thought to be one of the principles that helps with students' transition from arithmetic to algebra [105], to help them understand how to use brackets in problems with multiple operations [48], and ultimately to help them progress from elementary to more advanced mathematics.

For example, one way in which understanding associativity may help individuals to progress in mathematics is by encouraging them to interpret algebra problems as 'generalised arithmetic' problems [2,19,59,105], where using principles on arithmetic problems can help individuals to infer the correct processes for solving abstract algebra problems. For example, in the algebraic problem '9x + 2 = 6x + 2y', an individual could be asked to make a letter the subject of the equation. Along with other principles such as equivalence, associativity could help individuals on this problem because it permits sub-expressions to be solved in a different order from that in which they are presented, and therefore builds an understanding of how to apply transformations to both sides of the equation and re-order groups of operations to obtain the answer.

Finally, associativity may also help individuals to understand when and how to use brackets appropriately in multi-term problems [48]. For example, in the problem 'a – b + c', a left-to-right strategy ('a – b' first) and a right-to-left strategy ('b + c' first) result in different answers and brackets are needed to resolve this ambiguity. However, brackets are not required on some problems, for example 'a + b – c', where both left-to-right and right-to-left strategies result in the same answer. Using brackets when they are not necessary, as would be the case for the problem 'a + b – c', has been found to have negative consequences on mathematical understanding such as understanding of the order of operations [48]. Knowledge of associativity may therefore reduce the reliance on instructive devices, such as brackets, for solving arithmetic problems.

4. Measurement

Measuring knowledge of arithmetic principles sparks animated debate [25,81], and associativity is no exception. Methods broadly divide into explicit and implicit techniques, and typically involve presenting individuals with arithmetic problems (e.g. 'a + b − c' or 'a × b ÷ c') that they are asked to evaluate, justify or compare (explicit measures), or solve themselves (implicit measures). For example, if an individual reports using a right-to-left shortcut strategy on 'a + b − c' or 'a × b ÷ c', the researcher has explicit evidence that they have applied their knowledge of associativity. However, while self-reports provide explicit evidence for the use of a strategy, they also require conscious awareness and verbal skills to describe the strategy used, making a reliance on them alone sub-optimal [25]. Implicit techniques, which infer strategy use from solution accuracy and response times, are therefore often used in conjunction with verbal reports ([85,86,88,89,92,93]).

For associativity, one recent development is the suggestion that shortcut use can be measured implicitly by comparing performance on problems that are 'conducive' to the principle to those that are not [36]. Conducive problems (e.g. '16 + 47 − 45') encourage the use of associativity because the shortcut yields a small positive number (47 − 45 =

Table 2
A summary of relevant studies that have used explicit and implicit measures of average performance on 'a + b − c' and 'a × b ÷ c' problems that are conducive to the associativity shortcut.

Study	Age (years)	Operation	Percent of trials on which the shortcut was used*	Speed (s)*	Accuracy (% correct)*
Robinson & Ninowski, [96]†	Adults	Addition-subtraction	58%	3.5	78.5%
		Multiplication-division	32.5%	7	76.5%
Robinson et al., [93] †	1112	Addition-subtraction	11.5%	8	85%
		Multiplication-division	0%	18	65.5%
	12	Addition-subtraction	23%	6	87.5%
		Multiplication-division	2.5%	11.5	78%
Robinson & Dubé, [86]	7		12%	17.5	50%
	8	Addition-subtraction	26%	(averaged	(averaged
	9		24%	across grade)	across grade)
Dubé & Robinson, [33]††	Adults	Multiplication-division	44%	5	98 − 74%
Robinson & Dubé, [87]	7		13%	12 (averaged	65% (averaged
	8	Addition-subtraction	33%	across grade)	across grade)
	9		16%		
Robinson & Dubé, [88]	8	Addition-subtraction	11%	5.5	21%
	9		8.5%	6.5	37.5%
	10		24.5%	5.5	50.5%
Edwards [36]	Adults	Multiplication-division	24.5%	2 − 8	93.5%
Dubé [30]	12	Multiplication-division	14%	10	72%
	13		29.5%	5.5	83.5%
	14		39.5%	5.5	89%
	Adults		54%	4.5	91.5%
Robinson & Beatch, (2016)†	Adults	Addition-subtraction	65%	2.5	85%
		Multiplication-division	75%	2.5	75%
Robinson et al., [90]	6	Multiplication-division	0%	2.5 (averaged across grade)	42.5% (averaged
	7		7.5%		across grade)
	8		14.5%		
Robinson et al., [89]	8		11%		35%
	9	Addition-subtraction	15%	Not reported	50%
	10		13%		58%
Robinson et al [94]	10	Addition-subtraction	16%	Not reported	57%
		Multiplication-division	2%		24%
	11	Addition-subtraction	15%		66%
		Multiplication-division	4%		40%
	12	Addition-subtraction	29%		76%
		Multiplication-division	15%		59%
Eaves et al., [34]	Adults	Addition-subtraction	17% (Study 1)**	Not reported	Not reported
			32% (Study 2)**		
			37% (Study 3)**		
Eaves et al., [35]	Adults	Addition-subtraction	57% (Study 1)	5.5 (Study 1)	92% (Study 1)
			51% (Study 2)	Not reported (Study 2)	Not reported (Study 2)

†Statistics were reported for conducive and non-conducive problems combined rather than separately.
††Some statistics were reported for different clusters of individuals, rather than the whole sample.
*Percent, speed and accuracy are approximate scores that are averaged over small and large problems. Self-reports are used to determine percent scores.
** The percent of individuals who self-reported using the shortcut to the first associativity problem presented.

2), which makes a right-to-left strategy less computationally demanding. Non-conducive problems (e.g. '36 + 27 − 45') are designed not to be easier to solve using shortcuts and instead encourage a left-to-right strategy. Studies on inversion sometimes used 'a + b – c' problems as control problems for inversion, which often varied in how conducive they were to an associativity shortcut. Here, we focus on conducive and non-conducive problems as a way to infer knowledge of associativity. For example, the digits in conducive problems can be selected such that the difference in efficiency between a left-to-right strategy and a shortcut strategy is substantial, and therefore incentivises individuals to use the shortcut. If accuracy and response times are better on conducive than non-conducive problems, it can be inferred that an individual is likely to have used the shortcut on the conducive problems.

Three studies have compared performance on conducive and non-conducive associativity problems [34–36]. Edwards [36] found that self-reported shortcut use was significantly higher on 'a × b ÷ c' problems that were conducive to a shortcut than those that were not conducive. Eaves et al., [34] found that self-reported users of the shortcut solved more conducive 'a + b – c' problems in a restricted timeframe than individuals who did not self-report using the shortcut, but that self-reported users and non-users did not differ on non-conducive 'a + b – c' problems. Comparing performance on conducive and non-conducive problems is therefore one way that researchers could investigate implicit associativity shortcut use independently from other concepts. It may also be a useful approach for researchers wanting to measure strategy use in an unbiased way. To the best of our knowledge, all but one previous study [35] that used self-reported solution strategies did so by asking participants to describe the strategy they used multiple times in a short timeframe, i.e. after every trial (e.g. see [30] and [85,86] with children and adolescents). For adults, this approach may be problematic because repeatedly asking them about strategy use could provide a hint that alternative strategies exist and influence how a person solves subsequent problems [49,101]. By comparing accuracy and reaction time on conducive and non-conducive problems, use of the associativity shortcut can be inferred without inadvertently encouraging it through repeated questioning.

We note that the inferences that can be drawn from both explicit and implicit measures, however, are not clear-cut because strategies and knowledge of arithmetic principles are not perfectly related [7,10,25,95]. Using a shortcut does not guarantee a 'deep' understanding of the principle [9,102], i.e. an understanding that because some operations are related (e.g. addition and subtraction), groups of operations can sometimes be solved in different orders. Instead, shortcut use could reflect 'superficial' understanding, where individuals follow a right-to-left approach due to previous experience or memorised procedures [7,10,55,87,101]. The reverse case is also plausible, where individuals have a deep understanding of associativity but fail to select a shortcut (see Sections 6 and 7). Scholars investigating the understanding of any arithmetic principle through strategy use should therefore remember that the strategy an individual uses does not just reflect how well an individual understands a principle, but also their the ability and willingness to apply their knowledge [99].

Furthermore, knowledge of a principle is not "all-or-none" ([41], p7); as individuals may understand some part(s) or form(s) of a principle, but not others [13,25] or be able to apply the principle in some situations but not others. Shortcut use on arithmetic problems is just one way of measuring knowledge of a principle and the failure to use a strategy does not imply a lack of understanding of all forms of a principle. Some researchers have therefore used multiple tasks to measure individuals' understanding, such as tasks where problems are presented with objects, tasks where they explain why a strategy is valid, and tasks where they evaluate and choose between different strategies (see [25] for an overview). Bisanz et al., [12] put forward a framework where these tasks were organised by a) how explicit/implicit they were and b) the breadth of understanding that could be inferred from them. For example, recognising the validity of a strategy on one task demonstrates implicit, narrow understanding of a principle, while explaining a principle on a variety of related tasks demonstrates explicit, broad understanding. To characterise the breadth and depth of an individual's understanding, they therefore need to complete a variety of related measures.

In the case of associativity, few studies have used multiple tasks to assess individuals' understanding (see [87], for an exception) and more are warranted to capture the extent to which individuals understand and apply their knowledge of the principle. In the cases of inversion, complementarity and equivalence for example, children demonstrate a better understanding when problems are presented in a concrete format with objects and pictures than an abstract format with digits [14,15,42,73]. No studies have quantitatively compared performance on concrete and abstract associativity problems, although Klein & Bisanz [58] anecdotally reported that on at least one occasion, 25% of pre-schoolers used associativity shortcuts on problems presented with poker chips. Asghari & Khosroshahi [4] observed similar behaviours in 5 − 6 year olds. Thus, it can be argued that adults are likely to have some form of understanding of associativity given that young children do, and that difficulties may arise when applying that knowledge to more abstract 'a + b − c' or 'a × b ÷ c' digit-based problems.

5. Low use of the associativity shortcut

Compared to simpler concepts such as inversion, children dislike associativity, and prefer to operate left-to-right rather than use a shortcut [87,90]. For example, on the problem '6 + 38 – 35', they prefer to perform the addition first ('6 + 38 = 44') and then the subtraction ('44 – 35 = 9'), rather than use a right-to-left shortcut ('38 – 35 = 3' and then '3 + 6 = 9'). At the age of 6 − 10 years, inversion shortcuts are used approximately 35 − 60% of the time [87,88], a rate that matches adults by the age of 14 years (approximately 70 − 75% on multiplication-division inversion problems and 90 − 95% on addition-subtraction inversion problems) [30,31,92]. In comparison, the use of associativity shortcuts lags behind (see Table 2 for a summary of relevant literature). Children aged 7 − 10 years use associativity shortcuts only 10 − 25% of the time [85], a rate that remains low (approximately 30%) in early adolescence, aged 11 − 14 years [93]. Even in adulthood (aged 18 years and over), there is substantial room for improvement, where associativity shortcut use hovers around 50 − 60% on addition-subtraction problems [84,92] and even some qualified teachers have a poor understanding of the principle [6].

These statistics are also interesting to compare to equivalence, because equivalence has a reputation for being the most poorly understood arithmetic principle by school-aged children [56]. Many children struggle to define the equals sign and make consistent errors on problems of the format 'a + b = c + ?' where they overlook 'c' [67]. Indeed, a recent study found that children applied the equivalence principle to problems in this format only 10 − 40% of the time [89]. However, in this same study the application of associativity to 'a + b − c' problems was even lower, at 10 − 15%. Greater recognition of the associativity principle and the fact that few individuals apply it on 'a + b – c' problems is therefore warranted.

We note that presentation format can influence the strategy an individual uses on two-term arithmetic problems [20]. Similarly, shortcuts on 'a + b – c' problems may be created in other ways, for example in '38 + 6 – 35' the location of the shortcut is split across the left and right of the problem (''38 – 35 = 3' and then '3 + 6 = 9'). However, we know of no studies that have investigated associativity shortcuts in this 'split' format (i.e. 'a – c', then '+ b'), and we therefore focus on shortcuts located on the right-hand side ('b – c' then '+a') for the purpose of this review.

Studies have also uncovered different groups or clusters of individuals based on how they solve 'a + b − c' associativity problems and 'a + b − b' inversion problems. These groups are often referred to as 'dual concept', 'inversion only', 'negation' and 'no concept' [30–32,86, 88,89,94], and represent individuals with different levels of understanding. Those in the dual concept cluster use shortcuts frequently on both inversion and associativity problems, those in the inversion cluster use them on only inversion problems, and those in the 'no concept' cluster use neither. 'Negation' is a term used in the literature to describe how some individuals solve 'a + b − b' inversion problems, where an individual initially uses a left-to-right strategy but then switches to a shortcut part-way through calculating, potentially after realising that the last two digits are the same. One notable finding is that these clusters exist in all age groups, children and adults alike [30,31,86] and that there is no cluster of individuals who understand associativity but not inversion. Individual differences in selecting shortcut strategies therefore exist at all stages of development and understanding inversion may be an important precursor to understanding associativity. Research studies need to investigate why these difficulties and individual differences arise if they are to improve knowledge of arithmetic principles [70]. We address this topic in Section 7.

6. Identifying vs executing shortcut strategies

Arithmetic strategies are often discussed in terms of four lower-level components and two higher-level components [63,104]. Lower-level components include repertoire (the number of strategies an individual knows), selection (the strategy an individual chooses from their repertoire to solve a problem), frequency (the number of times an individual uses a strategy) and efficiency (the accuracy and speed with which a strategy is performed). Higher-level components consist of flexibility and adaptivity, which refer to the ability to switch between strategies, and in a way that maximises efficiency, respectively. In this review, strategy selection on 'a + b – c' problems (left-to-right or a shortcut) is the primary interest. More specifically, this review focuses on what may be deemed a subcomponent of strategy selection, a process that we call *identification.*

Identifying an arithmetic strategy is distinct from executing it. Identification refers to noticing a strategy for the first time on a given task or within a particular context. In other words, it is the time-point when an individual first recognises the validity of a strategy for solving a set of problems, akin to what Shrager & Siegler [97] call 'strategy discovery'. Where the strategy is efficient (and more efficient than alternative strategies) identification is likely to be synonymous with the first-time use of it. Identification is distinct from executing a strategy, which refers to all of the processes involved in performing the strategy after it has been identified, i.e. the decision to use it and the process of calculating it. Execution differs from efficiency, where efficiency refers to the speed and accuracy with which the calculations (execution) have been performed. This distinction between identification and execution can be made apparent by comparing 'a + b − b' inversion problems to conducive 'a + b − c' associativity problems. To use a shortcut on 'a + b − b' inversion problems, identification and execution coincide, in the sense that if an individual has identified the inversion shortcut, they have also executed it. However, for 'a + b − c' associativity problems, an individual may identify the 'b − c' shortcut but choose not to execute it. Strategy execution has been well researched and is typically measured by averaging solution accuracy and response time to multiple problems that are solved using the same strategy. Measuring the point of identification is more difficult because it requires microgenetic techniques, techniques where performance is measured and compared at multiple points in a narrow timeframe. For example, in our earlier work [35] we developed a novel, implicit method for measuring identification; we recorded participants' solution times to conducive 'a + b − c' problems and compared them on a trial-by-trial basis to determine when participants' became more efficient in solving them. We referred to this as the 'identification point'. Studies investigating individuals' knowledge and application of arithmetic principles rarely mention the distinction between identification and execution. However, we judge that the distinction is important for scholars investigating individuals' understanding of arithmetic principles because it highlights that the reason why an individual may find a strategy difficult may be due to one process more than the other.

One consequence of making this distinction is that the two processes of strategy identification and strategy execution might draw on different sets of underlying cognitive skills (e.g. attention, working memory), or different levels of those skills. Individuals with different profiles of cognitive skills may therefore find either strategy identification or strategy execution more difficult. Improving the application of arithmetic principles [71] might therefore require an individualised approach, where individuals' levels of different skills are considered. For example, an individual might have a good understanding of the principle and the ability to compute the calculations involved in the shortcut ('b − c' and then '+ a') but fail to identify it due to difficulties, for example, with attention or inhibition (see Section 7). Alternatively, an individual may notice the relevance of 'b − c' and identify the shortcut but not execute it if they are sufficiently efficient in calculating left-to-right. In the following section we consider the skills that might be required in the processes of identification and execution in more detail.

We emphasise that this distinction is relevant to researchers investigating a variety of arithmetic principles, not just associativity. For example, the distinction can be applied to commutativity, which is sometimes measured through the 'tens strategy' where '4 + 37 + 6' is solved by '4 + 6 = 10' and then '10 + 37 = 47', or through a 'look back' strategy where solving '7 + 6' helps to subsequently solve '6 + 7' [47]. The calculations in both strategies should be straightforward for adults to execute, but they may be difficult to notice or identify. Equally, individuals may identify the strategy but fail to execute it if they incorrectly believe that looking back to a previous problem is not permitted [75,91].

7. Skills involved in identifying and executing shortcut-strategies

We theorise that a variety of skills could be required to notice and select a strategy for the first time (strategy identification), as well as to perform the computations involved in applying the strategy (strategy execution). These skills may be domain-specific or domain-general (see Fig. 1). Domain-specific skills apply to the context of interest (in this case, mathematics) while domain-general skills apply to a broad range of tasks. We suggest that varying levels of domain-specific and domain-general skills might be involved in each process.

Calculation skill and knowledge of the order of operations are two domain-specific skills that might be important for both identifying and executing the associativity shortcut. First, proficient calculation skills (accurate and quick calculations) could help individuals to identify the shortcut as it may give them time to solve a conducive problem through multiple strategies (e.g. left-to-right and a shortcut), compare the result of those strategies, and deduce that the shortcut is valid if both strategies return the same answer. However, proficient calculation skills might reduce the likelihood of executing the shortcut if the individual calculates a left-to-right strategy with similar accuracy and speed as the shortcut itself. These two possibilities could be investigated in future research.

Knowledge of the order of operations may also be important for both processes. The associativity literature rarely discusses the order of operations, but it is a factor that could be very relevant. It refers to the convention [108] that for problems with mixed operations (e.g. '2 + 4 × 5'), multiplication and division should be performed before addition and subtraction, but within multiplication and division and within addition and subtraction order does not alter the result. In the mathematics education literature, case studies, interviews and analyses of the errors individuals make when solving multi-term problems suggest that some individuals have misconceptions with the acronyms that are used to teach it [52]. For example, in the United States, the acronym 'PEMDAS' ('Parentheses, Exponents, Multiplication, Division, Addition,

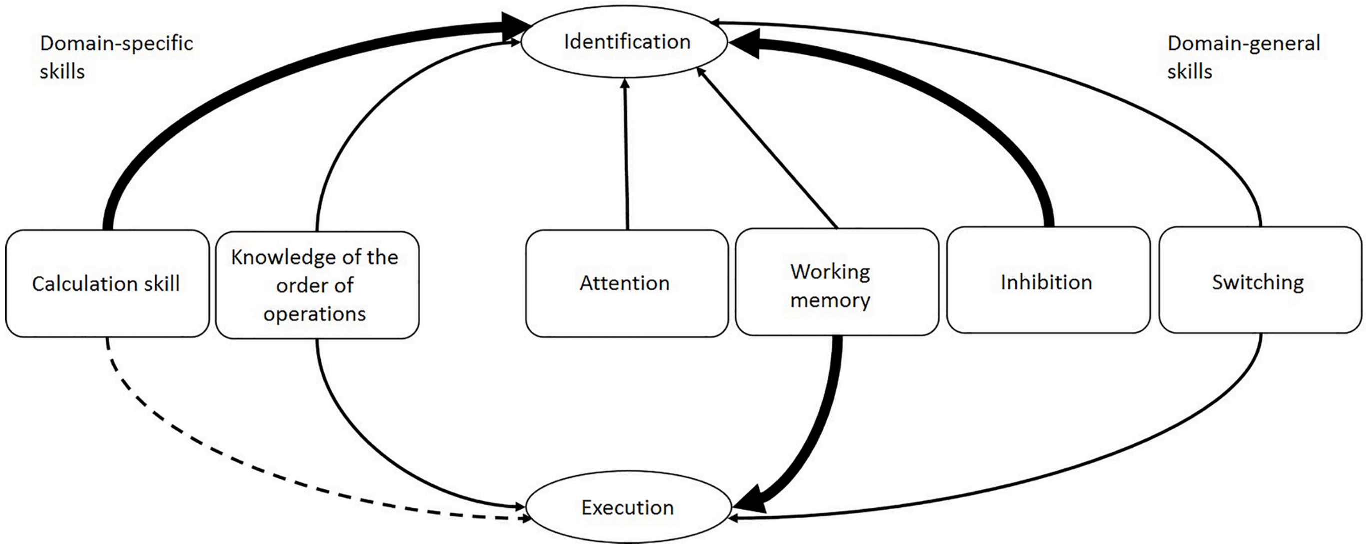

Fig. 1. Skills that might be required to identify and execute the associativity shortcut on conducive 'a + b − c' problems. Thicker lines are hypothesised to play a greater role in each process. Solid lines indicate a hypothesised positive relationship, dashed lines indicate a hypothesised negative relationship.

Subtraction') is used. Some individuals have a literal interpretation of this acronym and incorrectly believe that multiplication must be performed before division, and addition must be performed before subtraction [46]. In our research some individuals report having a 'left-to-right' interpretation, and incorrectly believe that if only multiplication and division are present in the problem, or only addition and subtraction, then the problem must be solved in a left-to-right manner [33]. Literal and left-to-right interpretations could hinder the search for the shortcut on conducive 'a + b − c' problems because a) the addition is earlier in the acronym and b) the addition is presented on the left-hand side of the problem, respectively. Similarly, for an individual who does notice and identify 'b − c', they may refrain from executing it if they have a strong literal or left-to-right misconception. This may be similar to, or even explain why, some children say that they see shortcuts as "cheating" [87].

Domain-general skills include attention, working memory, inhibition and switching, all of which are frequently suggested to be important for mental arithmetic. There are several reviews and meta-analyses of their role in executing calculations [28,39,74,79,107] and scholars have begun to hypothesise that they may be specifically important for solving three-term inversion and associativity problems [87,94]. In what follows we focus on identification, and discuss mechanisms for how domain-general skills might help individuals to identify arithmetic strategies rather than execute them. The roles of the skills are not necessarily mutually exclusive and may operate together.

First, attention is a domain-general skill that consists of different components (see [76] and [83] for more in-depth discussion); selective attention refers to the prioritised processing of certain stimuli [109] and spatial attention refers to the prioritised processing of information at a relevant location [57]. In the context of identifying the shortcut in conducive 'a + b − c' associativity problems, these components are likely to play a similar role of helping to direct visual resource toward the location of 'b − c' on the right-hand side. Indeed, in one model of strategy discovery, the Strategy Choice and Discovery Simulation model (SCADS*), attention is the first cognitive process hypothesised to be required for identifying the 'b − b' shortcut on 'a + b − b' inversion problems [100].

Second, working memory may help individuals to identify the associativity shortcut by enabling them to build 'mental models' of novel problems [36,62]. Mental models refer to visual images of problems that are spatially represented in memory. Being able to hold a mental image of 'a + b − c' problems in mind might allow individuals to move their attention around the problem, and from one group of operations ('a + b') to another ('b − c') in order to locate the shortcut. After locating the shortcut individuals might then need working memory to evaluate the subexpression 'b − c' (i.e. to judge whether it is easy or difficult), navigate back to the left-hand side, compare it to 'a + b', and select the strategy that they prefer. For a more in-depth discussion of the structure and role of working memory in arithmetic, please see Baddeley & Hitch, [5], Cowan [111] , Cragg et al [23], Gilmore [45], Edwards [36] and Rasmussen et al [80].

Inhibition refers to the ability to stop or override a mental process with or without instruction [65]. It may enable identification by helping individuals to resist interference from irrelevant information (e.g. the digit 'a' in the context of 'a + b − c' problems), help them to forget previously learnt material (e.g. rules or procedures they were taught at school), and help them to resist using pre-potent strategies (i.e. strategies that have become familiar through practice). In western societies, one would expect the pre-potent strategy on 'a + b − c' problems to be a left-to-right approach because classroom-based mathematics teaches and encourages individuals to solve problems in a left-to-right manner [103]. In order to identify the associativity shortcut, these strategies must therefore be suppressed and counteracted. For a more in-depth discussion of the structure and role of inhibition in arithmetic, please see Cyders & Coskunpinar, [26], Dempster [27], Dubé & Robinson [32], Friedman & Miyake [38], Gilmore et al., [44], Harnishfeger & Pope [50], Hasher et al., [51] and Nigg, [72].

Lastly, switching refers to changing from one mental set to another [107]. Switching might help individuals to identify the shortcut by allowing them to change from a (non-shortcut) strategy that they executed on a previous problem, to considering alternative strategies on the current problem. For example, in everyday settings and experimental studies, individuals are presented with a variety of arithmetic problems that can be solved through different strategies. To identify the shortcut, they therefore need to be able to change from one mindset or previously used approach, to another. This might involve shifting the focus of their attention, i.e. from the left to the right, or from one type of operation (addition first) to another (subtraction first). Research into switching skills in mathematics are still at a very early stage, however relevant papers for a more in-depth discussion include Andersson [3], Cragg et al., [23], Gilmore et al., [43], Watchorn et al., [106] and Yeniad et al. [107].

Most of the research that has investigated the role of domain-general skills in the understanding and application of arithmetic principles focuses on children, with problems that measure the application of simpler principles such as inversion, commutativity and complementarity (e.g. [23,40,43,44,80]). A handful of studies have included three-term associativity problems: collectively they provide some evidence that in children, inhibition might be related to the use of addition-subtraction inversion and associativity shortcuts [88] and that in adults, spatial skills may be important [36]. No studies have explored working memory and switching with associativity problems, although findings from the inversion literature indicate that they may play a role [112]. Attention is theoretically expected to play a role in solving inversion problems [100] but there is no empirical evidence that it is important on '$a + b - c$' and '$a \times b \div c$' problems [32,35]. Further research into the role of domain-specific and domain-general skills in strategy identification and execution is therefore warranted.

8. Future research

Only three published studies have investigated associativity as a standalone concept, rather than as a comparison for other principles [34–36]. Given the theoretical importance of associativity in algebra learning and the transition from basic to advanced mathematics, further research is warranted. Indeed, associativity may be as important as equivalence to progressing in mathematics, and of similar or even greater difficulty for individuals to understand and apply [89]. Associativity should therefore be investigated with as much rigour, and the equivalence literature could serve as a guide for how. Substantial research effort has been invested into equivalence, with studies investigating attention and perception, working memory, teaching materials, prior conceptions, its relation to subsequent algebra learning, and how it can be taught [1,24,29,40,60,68,69,113]. Associativity research could benefit by using that literature as a guide for the range of questions that are important to consider, and for generating ideas of methodologies that could be used to answer them, which we now discuss.

We propose a series of priority research questions for the field. The first question we need to address is "why are associativity shortcut strategies used so infrequently?" In this review we highlighted a selection of domain-specific and domain-general skills that might be important for identifying and executing the shortcut. Correlational and experimental studies in children, adolescents and adults could be conducted to investigate whether and to what extent their role is empirically supported. The studies in Section 7 offer a starting point for how they could be investigated. To understand how those skills help an individual select the shortcut strategy though, new methods might need to be created. For example, measures that can separate the process of strategy identification from strategy execution [35] might be more likely to capture the contribution of the skills involved because they target mechanisms more precisely (i.e. identification or execution).

The second question that we need to answer is whether knowledge of arithmetic procedures (i.e. the order of operations) and knowledge of the associativity principle conflict. We identified misconceptions of the order of operations as one reason why individuals might not execute the associativity shortcut. This offers a different perspective on the relationship between procedural and conceptual knowledge; most researchers converge on the view that they are positively and iteratively related [82], but our suggestion implies that this may not always be the case. In some situations, the two types of knowledge may conflict. This could be tested by mirroring the methodologies that have been used in the equivalence literature; for example, Crooks & Alibali [24] intentionally activated misconceptions of the equals sign in adults by exposing them to words such as 'total' and 'sum', and then measured the accuracy with which they reconstructed equivalence problems. For associativity, exposing adults to multi-term problems that contain brackets might activate misconceptions of acronyms such as PEMDAS. This might reduce the identification and execution of associativity shortcuts on subsequent problems, compared to a group without that activation.

The third question relates to our assumption that most adults have some level of understanding of associativity and that difficulties with the principle arise when applying that understanding to the context of digit-based problems. This suggestion could be tested by comparing individuals' strategies on conducive '$a + b - c$' and '$a \times b \div c$' associativity problems presented in different formats (e.g. digit, word problems and object-based formats), as has been done with inversion [42], equivalence [96], commutativity, complementarity [21] and addition-only associativity [15]. If individuals execute associativity strategies on '$a + b - c$' and '$a \times b \div c$' more frequently with concrete materials (word problems, objects) than abstract materials (digits), this would lend support to our suggestion.

The fourth question relates to the assumption that associativity serves an important function in higher-level mathematics and algebra [2,19,59,105]. Although it is a logical suggestion, no studies have empirically investigated whether such a link exists. Again, to mirror the equivalence literature [60] this could be explored by comparing algebra performance in a group of individuals with a good understanding of associativity to a group with a poorer understanding, after controlling for general mathematical achievement.

Perhaps the most challenging research priority is to devise ways that associativity can be optimally taught. This goal is challenging because the principle needs to be taught in a manner that is direct enough for individuals to a) understand the meaning of it, b) correct any misconceptions they may have and c) avoid any confusion with similar principles such as commutativity. This final point is relevant because people have been found to conflate ordering operands (commutativity) with ordering operations (associativity), which is thought to create difficulty when transitioning from arithmetic to algebra [6]. Additionally, the principle needs to be taught in a way that does not prescribe solution strategies and instead allows the individual to explore how it can be used in different contexts. The danger with directly teaching strategies is that it can cause some individuals to adopt 'fixed' mindsets, where they persevere with a taught strategy even when it is no longer relevant [37]. This phenomenon is known as the Einstellung effect [64] and in the case of associativity it could manifest as people applying the same strategy on problems where it is not valid (e.g. solving '$38 - 5 + 6$' by '$5 + 6 = 11$' and then '$38 - 11 = 27$'). Interventions that teach arithmetic principles must therefore be designed to encourage the use of strategies *only* when they are appropriate and efficient, and to switch to an alternative when they are not. Techniques that do not prescribe strategies and instead increase awareness of those available may be more likely to achieve this [53,54,78,114].

9. Summary

Associativity is a principle that many children, adolescents, adults and teachers struggle to apply on arithmetic problems such as '$a + b - c$' and '$a \times b \div c$'. To date, few researchers have investigated associativity independently of other concepts, with only a handful of studies from the psychology and mathematics education literature. However, the principle is thought to be important for the transition from elementary to more advanced mathematics, and studies that answer calls for further research into it are overdue [71]. We provide a number of priority research questions for the field along with suggestions for how they could be investigated. Furthermore, we suggest that researchers may have overlooked an important distinction between strategy identification and strategy execution, and that we need to consider this distinction when investigating why individuals may not use efficient or adaptive strategies. The distinction may help us to understand whether and how different domain-specific skills (calculation skill and knowledge of the order of operations) and domain-general skills (attention, working memory, inhibition and switching) are involved in applying conceptual knowledge to solve arithmetic problems. Ultimately, we suggest that associativity research could benefit from the lessons learnt in the equivalence literature, which is an equally important and similarly difficult principle with a much richer research history.

Declaration of Competing Interest

There are no conflicts of interest for the authors of this article.

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Ethical statement

Not applicable for this article

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