Duckie
And The Search For The Golden Goose

Iniyan Joseph

Introduction

Why Write this Book

Mathematics is an essential aspect of human life. There is a stereotype of people disliking and struggling with mathematics, which I believe is simply because we are, at a young age, often told to try to compute, rather than being challenged with fundamental questions of how to understand the world around us.

How to use this book

Learning and understanding mathematics takes more than reading this book purely for its story, but wrestling with new ideas and questions, and attempting to resolve those questions through reasoning. To this end, after reading the story on each page, have students attempt to develop an answer using their intuition and prior knowledge to build toward the correct answer. Only after they have worked towards the right answer without hints and engaged in discussion, read the solution.

Even beyond the scope of the book, encourage them to ask questions about the world around them. Reason through math in games they know and love, in stories, and in everyday life and present unintuitive problems which they can use the tools from this book and otherwise to solve in genuine scenarios.

Additional Resources

- MathForLove `https://mathforlove.com/`
- MathIsFun `https://www.mathsisfun.com/`

Contents

All was quiet in the goose village. It was a sort of place where parent geese could look out and feel their goslings were safe. But one fall evening, there was a sense of despair in the village. All the geese knew that migration was coming, and that they would have to embark on a dangerous journey.

”Enough is enough.” Duckie thought. But what could he do? The goose authorities, the leaders of the village, had controlled migrations for as long as anyone in the village could remember.

Created to protect the village from danger, they ruled the village and made decisions for the village. To ensure the others followed their rules, they created the elite corps of goose police. This system had worked for some time, but in the past years, they had abused their power. They had closed down the schools, burned the village’s books, and stopped all geese from leaving the village.

Growing up, Duckie had heard the stories of the fabled ”golden goose” of legend who lived on the highest mountain. According to the stories, the golden goose would bring great change to the village. That change would in turn lead to great change, and would save the village from tyranny, and lead to an era of peace. Many geese had tried to seek her, but it was a dangerous journey, and none had ever returned.

”No. I have no choice.”

Duckie decided then and there, that he would try to find the golden goose, and would help create the great change to society the hero was meant to bring. Hopefully then, he could save all the geese. He would have to break the goose authority’s rules and search for the hero.

Chapter 1

Arithmetic

That night, Duckie decided to leave the village. He knew he would leave secretly because of the tyrannical rules of the goose authorieties. They had tried so hard to keep their power, and the search for the Golden Goose risked everything they stood for. Duckie waited until everyone had gone to sleep, and started on his journey.

He began to sneak towards the village border, where he saw two guards snoring loudly. He tiptoed past, and all seemed quiet, but just as he turned his back to the village, he heard alarms ringing.

Honk! Honk! Honk!

They droned. Duckie turned around and saw the two guards he had walked past coming towards him. The goose authorities had been alerted of his mission, and he would have to run. He began flying, and as he looked back, saw the goose police on his tail.

1.1 Addition

1.1.1 Story

On the first night of his journey, Duckie flew **3***km* from the village towards the mountain with the goose police in pursuit. Looking back, he wouln'd see them behind him, so, unsure of how far the goose police would chase him, he decided to take a quick break to gain his strength, but in a flash, he saw the police on the horizon and began to fly again. From there, he flew another **4***km*. Duckie wanted to know how much he had traveled in the direction of the mountain. Where was Duckie?

1.1.2 Problem

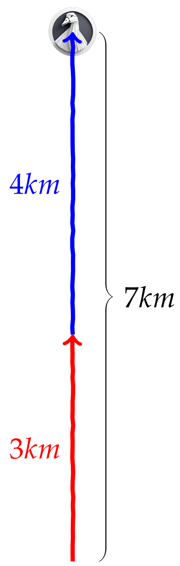
He first flew **3***km*, then changed this amount by **4**. Duckie's position, 3 kilometers, changed by 4 kilometers.

This is $\begin{array}{r} 3 \\ + \; 4 \\ \hline 7 \end{array} km$.

1.1.3 Definition

Addition is the most basic way of using numbers. It represents changing one number by another to form a single, combined number.

1.1.4 Graphic



1.2 Multiplication

1.2.1 Story

As Duckie began his journey towards the mountain, the goose authorities followed close behind. Every time he turned back, he saw them on the horizon. On each of the first three days, he flew **7km**. Duckie wanted to know how much he had flown towards the mountain. Where was Duckie?

1.2.2 Problem

Duckie flew $\boldsymbol{7}km$, $\boldsymbol{3}times$. This means the amount he flew was triple what he flew on the first day. This

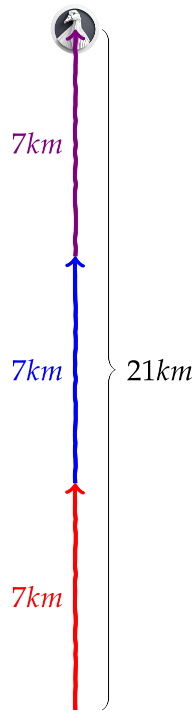
$$\begin{array}{r} 3 \\ \times\ 7 \\ \hline 2\ 1 \end{array}$$

is $21\,km$.

1.2.3 Definition

Multiplication is the second basic way of using numbers. It represents stretching a number by a certain scale. For example, doubling, tripling, etc. $A * B$ is A times more than B.

1.2.4 Graphic



1.3 Negative Numbers

1.3.1 Story

On the morning of the fourth day, after Duckie had traveled **21***km*, he decided he had to take evasive maneuvers to try to confuse the goose police. In order to do this, he decided he should do the one thing they would not expect: go towards the village. He traveled backwards **3***km*. Duckie wants to know how far he is from the village. Where is Duckie?

1.3.2 Problem

This is called $-\mathbf{3}km$.
– means opposite direction. He first flew $\mathbf{21}km$, then flew $-\mathbf{3}km$. This

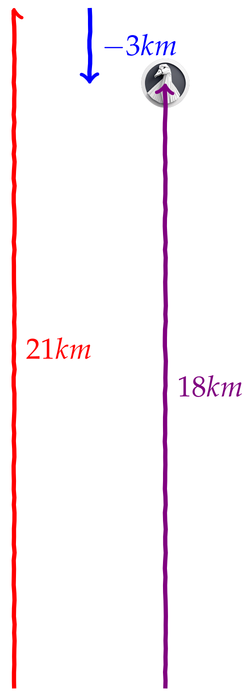
$$\begin{array}{r} 2\,1 \\ -\quad 3 \\ \hline \end{array}$$

is $1\,8\,km$.

1.3.3 Definition

A number less than 0 is called ”negative”, and is in the opposite direction.

1.3.4 Graphic



1.4 Multiples of a negative

1.4.1 Story

While Duckie was traveling in the negative direction, he took a break every $-\mathbf{1}km$. He flew that distance $\mathbf{3}$ times. How far does Duckie travel while taking evasive maneuvers?

1.4.2 Problem

Duckie traveled $-\mathbf{1}km$ $\mathbf{3}$ times. This

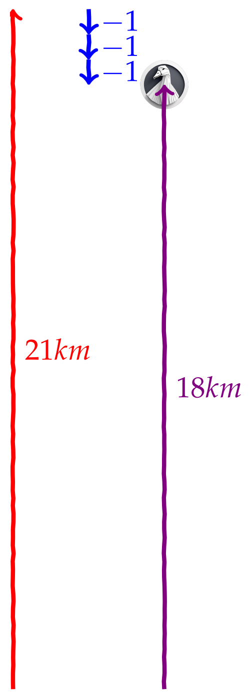
$$\begin{array}{r} 3 \\ \times\ 1 \\ \hline \end{array}$$

is - 3 km

1.4.3 Definition

Multiplying by a negative number shows stretching a number so its length another number times larger, but in the opposite direction.

1.4.4 Graphic



1.5 Negative of a Negative

1.5.1 Story

On the fifth day, Duckie thought he had confused the goose police, and that they were no longer following him. To get to the Golden Goose, he knew he would have to travel away from the village, and so turned around again and flew for $\mathbf{6}km$. Duckie traveled in the opposite direction of the negative direction by $\mathbf{6}$, or $-(-\mathbf{6})$. Duckie traveled $\mathbf{6}km$ forward. Duckie wants to know how far he is from the goose village. Where is Duckie?

1.5.2 Problem

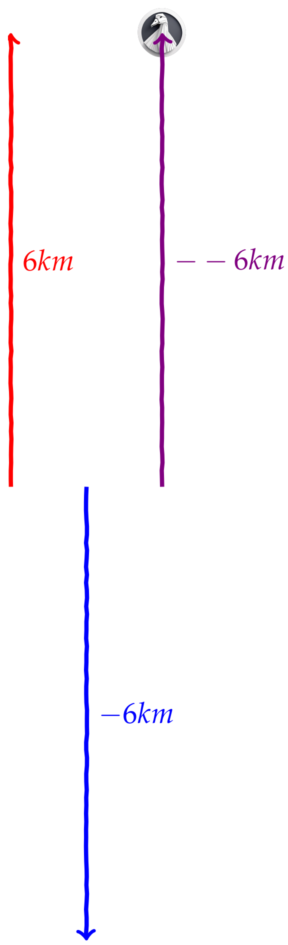
He was at $\mathbf{18}$ and then traveled $-(-\mathbf{6})$. This is

$$\begin{array}{r@{\,}r@{\,}r} & {\scriptstyle 1} & \\ & 1 & 8 \\ + & & 6 \\ \hline & 2 & 4 \end{array} km.$$

1.5.3 Definition

The opposite direction of the opposite direction of a number is in the same direction of that number. It is written as -(-number), which is the same as that number.

1.5.4 Graphic



1.6 Multiplication Table

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

Observations about the Multiplication Table :

0 Any number multiplied by 0 is 0. This makes sense, because any number repeated 0 times is the same as not having it at all.

1 Any number multiplied by 1 is itself. This makes sense, because any number repeated one time is itself. This has a special name: Identity

10 Any number multiplied by 10 is shifted to the left by one digit.

Symmetry The table is identical across the bolded diagonal. This shows that 4×2 is the same as 2×4. This property also has a special name: Commutativity.

Chapter 2

Properties

After five days of running from the goose police, Duckie was certain that he had escaped. He decided to stop for dinner, but just as he took off, Squak! He ran headfirst into another goose.

”Hold it right there! You need to come back with me to the village!” the policegoose demanded, slightly dazed.

”It’s dangerous in the outside world. This is for your own safety!”

Duckie knew that the policegoose, named Güs was rational and decide to reason with him. ”Of course it’s dangerous! But I have to do this. I am looking for the golden goose of legend. The one who is supposed to save us, change all of goose society and help us through our migrations.”

”But that’s against the rules! Why would you risk your life to break their rules? Don’t they keep us safe?” Güs asked.

”They are afraid of what will happen if I find the golden goose. If I can find her, then they may not be able to stay in power. You joined the goose police to help people, and if you let me go, you can do just that!”

”You might be right, but I can’t betray the village. If I just let you go without at least trying to stop you, I won’t be able to show my face in the village! I’ll tell you what. I will have a series of challenges with you. If you beat me in all the challenges, I will let you go. If not, you have to let me take you back. Sound fair?”

Duckie was reluctant, but knew that he had no other options. ”Fine. Let’s do the challenges.”

2.1 Associativity

2.1.1 Story

Güs and Duckie both knew that they were good fliers, so they decided to compete in a three-part flying distance competition. The first leg of the race was $1km$, the second leg was $\mathbf{2}km$, and the third leg was $\mathbf{1}km$. The distance flying competition rules stated that any goose would be allowed to take one break in between laps. They both wanted to win, so created strategies. Güs decided to travel the first leg ($\mathbf{1}km$) of the race, took his break then travel the second and third legs ($\mathbf{2}km$ and $\mathbf{1}km$). Duckie instead traveled the first leg, ($\mathbf{1}km$) of the race and the second leg ($\mathbf{2}km$) of the race, then took his break, and traveled the third leg ($\mathbf{1}km$). Did they tie by travelling the same amount?

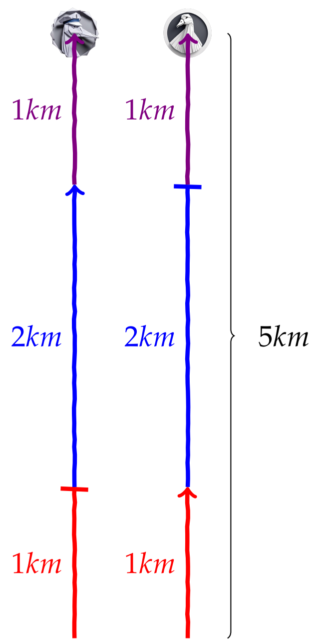
2.1.2 Problem

Yes! Duckie and Güs traveled the same amount.

2.1.3 Definition

When adding, calculation can be grouped in any way. In other words: (A+B)+C = A+(B+C).

2.1.4 Graphic



2.2 Additive Commutativity

2.2.1 Story

Duckie and Güs decided that to break the tie, they needed to do another competition.
Because geese are water animals, swimming is a very important skill, so they decided that that should be their next competition. They each had a different strategy. Güs swam **140**m, dried off, then swam **160**m. Duckie instead swam **160**m, dried off, then swam **140**m. Did they tie?

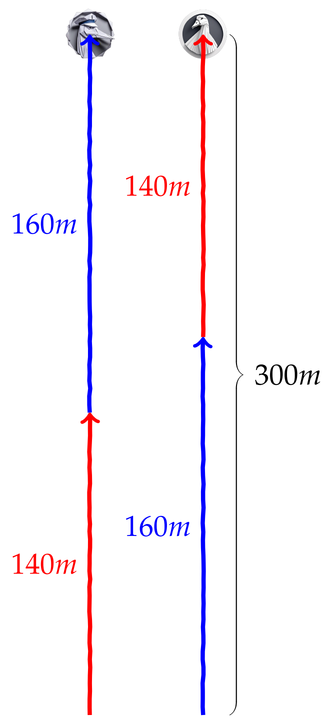
2.2.2 Problem

Yes! Duckie and Güs swam the same amount.

2.2.3 Definition

When adding, order doesn't matter.
In other words: A+B = B+A.

2.2.4 Graphic



2.3 Distributivity

2.3.1 Story

Duckie and Güs were starting to get frustrated. No matter what happened, the seemed to tie every time! Duckie had thought that if he competed long enough, Güs would get tired and he would be able to win, so challenged him to another flying contest. In the third contest, Güs traveled $\mathbf{2}km$, then $\mathbf{3}km$, and repeated that pattern $\mathbf{2}$ times. Duckie instead traveled $\mathbf{2}km\mathbf{2}$ times, then $\mathbf{3}km\mathbf{2}$ times. Did they tie?

2.3.2 Problem

Yes! Duckie and Güs flew the same amount.

2.3.3 Definition

Multiplying a number with a full expression is the same as multiplying the number with each part of the expression. In other words: A∗(B+C) = (A∗B) + (A∗C).

2.3.4 Graphic

2.4 Multiplicative Commutativity

2.4.1 Story

At this point, Güs realized that Duckie was a very capable goose. When he first challenged Duckie, he expected winning to be a cakewalk, but Duckie ended up posing quite a challenge. Being an honorable goose, this made him respect Duckie more. He decided that the next contest should be a weightlifting competition, where whoever lifted the most amount of total weight would win. Duckie lifted **10***kg***3** times, whereas Güs lifted only **3***kg* but did it **10** times. Did they tie?

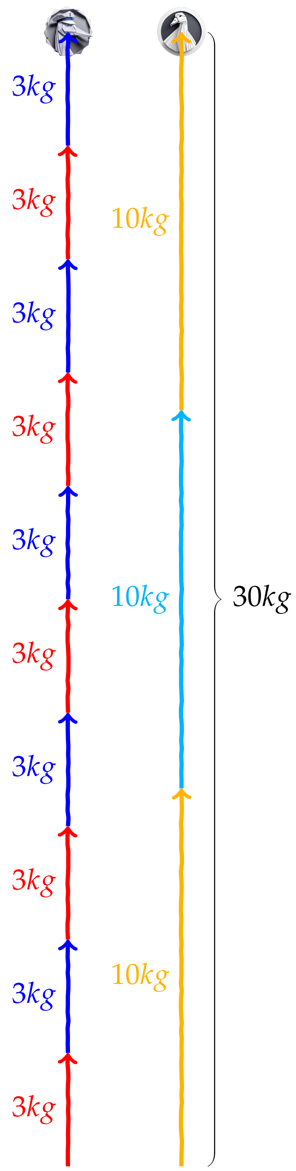
2.4.2 Problem

Yes! Duckie and Güs lifted the same amount.

2.4.3 Definition

When multiplying, order doesn't matter. In other words: A∗B = B∗A.

2.4.4 Graphic



After the competition, Duckie and Güs sat down, exhausted. They had both realized that, no matter what competition they had, they were equally matched. Neither one would be able to beat the other, and they had won each other's respect.

”Why don't you join me?” asked Duckie.

”I can't!” Güs cried. ”What about my life in the goose police?”

”Well, do you believe the golden goose is out there?”

”Honestly, I'm not sure, but if the stories are true, it could change the world.”

Duckie agreed. He knew that there was the possibility the golden goose was a hoax, but believed the positive implications outweighed the dangers of disobeying the goose authorities.

”You joined the goose police to help people”, he reasoned. ”You are clearly strong and capable of making the dangerous journey, and if you can join me, you could do exactly that.”

Güs knew he was right. While he was afraid of what could happen along the journey, he decided to be brave and join Duckie.

”All right. I will join you!” he said resolutely.

Chapter 3

Essential Algebra

Meanwhile, Duckie and Güs's various competitions made the goose authorities a bit confused. Because of Duckie and Güs's competitions, they lost track of where Duckie was. They decided that they needed to review their records of Duckie and Güs's locations, and try to figure out what they were doing on the fifth day.

3.1 Equality

3.1.1 Story

Because Duckie and Güs were travelling together, they were always at the same position at any time.

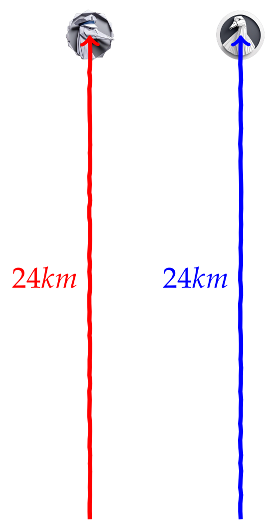
3.1.2 Problem

Güs's position = Duckie's position. Duckie's distance from the village was $24km$, so Güs's position was also $24km$.

3.1.3 Definition

When two things are the same, we write this in mathematics as "=", or "equals".

3.1.4 Graphic



3.2 The Equality Rule

3.2.1 Story

The goose authorities, having trained Güs in the police academy, knew he was a very fast flyer. In fact, he had won the village racing competition for the past three years! Because Duckie and Güs were travelling together, they realized that every time Güs had flown ahead of Duckie, Duckie had to increase his position by the same amount to keep up.

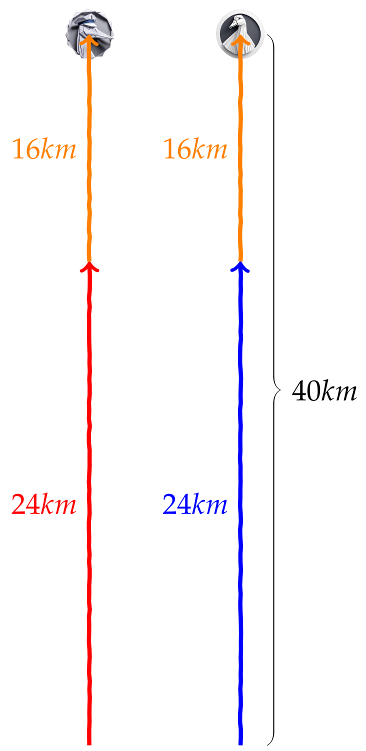
3.2.2 Problem

Duckie's position + Δ[1]x = Güs's position + Δx.

3.2.3 Definition

What happens to one side of an equation must happen to the other side so that they are still equal.

3.2.4 Graphic



[1] Δ means "change in"

3.3 Variables

3.3.1 Story

The goose authorities were still a bit confused about Duckie and Güs's progress. They knew that there were 4 competitions, and that they started at $24km$. They also knew that they ended at $40km$. How much did they travel on average in each competition?

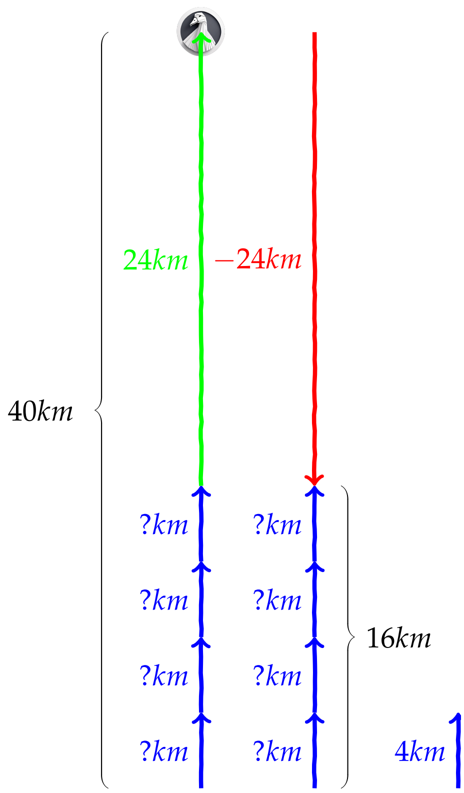
3.3.2 Problem

$$4x + 24 = 40$$
$$4x + 24 = 40 - 24$$
$$4x = 16$$
$$\frac{4x}{4} = \frac{16}{4}$$
$$x = 4$$

3.3.3 Definition

A letters or symbol can be used to represent a number which isn't already known, such as "x".
$4x + 16 = 40$ means that
$4 * (somenumber) + 16 = 40$.

3.3.4 Graphic



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Chapter 4

Ratios and Fractions

It took a lot of work, but the authorities finally figured out where Duckie was. They noticed that Duckie and Güs made quite a bit of progress, which was very bad news for them. They wanted to use this information to try to stop them. Because of Güs defecting to join Duckie's journey, they realized that sending other policegeese would make it hard to keep their power in the village.

Through their past migrations, they knew lots of bounty hunters along the way, and they believed that they could have them set traps. These bounty hunters were famously clever, and each of them decided to use specific strategies to help them catch Duckie and Güs.

4.1 Ratios

4.1.1 Story

The first bounty hunter decided to look at the entire journey, and set up traps every $40km$. He estimated that the entire journey was $120km$ long. Duckie having guessed this fairly obvious scheme, thought that if he knew how many traps there would be, he could avoid them. For every km in the entire journey, there would be a trap every $40km$. How many traps will there be on the journey?

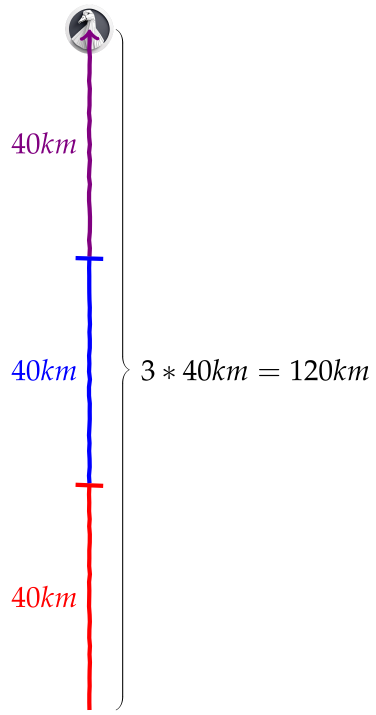
4.1.2 Problem

For every $40km$ Duckie has to travel out of $120km$ overall, there will be a trap. That means that there is a trap every $\frac{120}{40}km$ of the journey. For every one km in $40km$, there are $3km$ in the journey, which means that there are 3 lengths of size $40km$ within $120km$. There are $\frac{3}{1}$, or 3 traps.

4.1.3 Definition

A ratio is a certain amount for each of another amount, and is written as $\frac{A}{B}$. Break up A into B equal parts.

4.1.4 Graphic



4.2 Multiplying with Fractions

4.2.1 Story

Using this knowledge, Duckie figured out how to avoid the first bounty hunter's traps. The second bounty hunter decided to prepare better than the first hunter. He realized that km are a very large unit for geese, so decided to start measuring the distance Duckie and Güs flew in a unit he created called "goosemeters" (gm). This meant that he needed to change Duckie and Güs's position from kilometers into goosemeters. There are exactly $7gm$ in every $3km$. To help beat the second bounty hunter, Duckie and Güs decide to also measure using gm. How many gms are in $40km$?

4.2.2 Problem

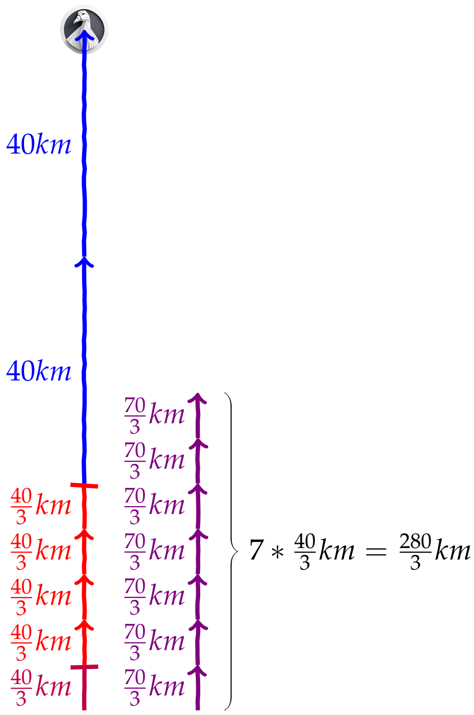
$\frac{7gm}{3km} * \frac{120km}{1} * \frac{1}{3} = \frac{280}{3}gm.$

4.2.3 Definition

Like any number, multiplying by a ratio represents scaling by a certain amount. Imagine a rubber band which is P long, which you can stretch and squish. Multiplication represents scaling the band such that $P * \frac{A}{B}$ is the new length of the band. This is the same as $\frac{P*A}{B}$, or $\frac{P}{B} * A$. This can be seen as dividing the band into B equal parts, then taking one of those parts A times.

Notice that that $\frac{A}{B}$ can create numbers in between 0, 1, 2, etc.

4.2.4 Graphic



4.3 Comparing Fractions

4.3.1 Story

Duckie thought that if he traveled enough in one day, the bounty hunter would become confused and leave them alone. On the eighth day, Duckie and Güs traveled an additional $\frac{1}{5}$ of the journey. If Duckie traveled more on the eighth day than the combined previous seven days ($\frac{1}{3}$ of the journey), the bounty hunter would become confused and stop chasing Duckie. Has Duckie managed to confuse the bounty hunter?

4.3.2 Problem

Duckie realizes that

$$\frac{5}{5} = 1$$
$$\frac{1}{3} * 1 = \frac{1}{3}$$
$$\frac{1}{3} * \frac{5}{5} = \frac{1*5}{3*5} = \frac{5}{15}$$

Which means $\frac{1}{3}$ is same as $\frac{5}{15}$

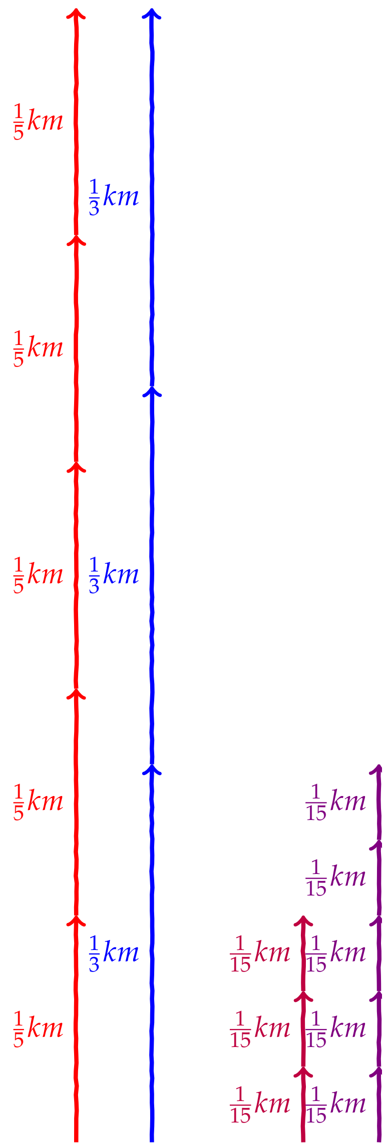
similarly $\frac{1}{5} = \frac{1}{5} * \frac{3}{3} = \frac{3}{15}$

Now, we can compare the equal size parts together. $\frac{3}{15} < \frac{5}{15}$. Duckie traveled less on the eighth day than the other days combined.

4.3.3 Definition

We can only deal with fractions together using same-size parts. We can chop up one fraction by multiplying it by another fraction equal to one. If the denominators (the bottom number of the fractions) are the same, the fractions have same sized parts.

4.3.4 Graphic



4.4 Adding Fractions

4.4.1 Story

Despite Duckie's best efforts, he was unfortunately not able to confuse the bounty hunter. He was starting to get worried, but Güs had a genius idea. If they could find out where they were along the journey, they might have been able find a new route, which would take them away from the hunter's grasp! They traveled $\frac{1}{3}$, then they traveled $\frac{1}{5}$. How much of the journey have Duckie and Güs traveled?

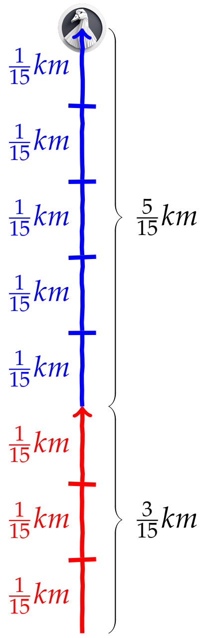
4.4.2 Problem

Duckie and Güs have traveled $\frac{1}{3} + \frac{1}{5}$
$= \frac{3+5}{15}$
$= \frac{8}{15}$ of the journey.

4.4.3 Definition

When adding fractions, break up each fraction so that they are in terms of same-size parts (or having a common denominator). Then, add the numerators as usual.

4.4.4 Graphic



Duckie and Güs sighed in relief. After running for bounty hunters for two days and two nights, they were exhausted. They decided to travel slower for a few days and enjoy the scenery.

”I can’t believe how persistent the goose authorities are.” Duckie sighed. ”I knew this journey would be tough, but I wasn’t expecting bounty hunters!”

Güs considered this. He had worked with the goose authorities, and knew their patterns. He realized how unusual this was. ”I agree, the authorities must really be desparate”.

Just as Duckie opened his beak to reply, he felt a sudden weight on his shoulders, and began spiraling out of control. It was a net! Another bounty hunter must have set a trap! He started falling down fast, gaining speed as he went. Crash!. Duckie fell to the ground with a thud. Güs hurriedly landed next to him.

”Are you alright Duckie?” he cried in alarm.

”I’m all right, but I think I need to rest my wing for a few days”, Duckie winced. He felt that he hadn’t broken a wing, but saw some bruising, and knew he would need time to heal.

”The forest is dangerous though! We can’t stay here in the open with all the predators!”

”I think my friend Bessie the cow’s farm is close enough to walk to from here,” Duckie said. ”I think she will be able to help keep us safe from the goose authorities.”

Together, they waddled to Bessie’s farm.

Chapter 5

Rectangles, Perimeter, and Area

After walking all afternoon, Duckie and Güs finally arrived at Bessie's farm. Despite the bad luck with the net, it was a beautiful afternoon, and the pair couldn't help but feel a sense of calm as the walked up to Bessie's barn.

Bessie was sitting outside, chewing on some grass.

"Hey there Duckie! Hello Güs! How's it going?" she moo'd.

"Hi Bessie", Duckie greeted. "I'm looking for the golden goose of legend, and I need some help."

"Of course! What do you need?" asked Bessie.

"I hurt my wing, and I need to rest here for a few days. The goose authorities sent some bounty hunters after me, and I need you to keep me safe from them," Duckie asked hesitantly.

"Wow, that sounds serious. Feel free to stay as long as you like" Bessie turned around to look at Güs. "While Duckie's is getting better, I could use some help on the farm. It's grass-planting season, and it would be much appreciated if you could help me out on the farm."

Güs, enthusiastically agreed. "That sounds great!"

5.1 Area of Rectangles

5.1.1 Story

On the ninth day of the journey, Güs began working on the farm. His first task was to fly around and scatter grass seeds on the farm. As he went into the store room to find the bags of seeds, he realized he needed to know how many bags of seeds he needs to bring with him. Bessie told him that each bag of seeds can cover a square of grass that is 1 m wide and 1 m across. Güs also knows that the farm is $300m$ high and $400m$ wide. How many bags of seeds does Güs need to carry?

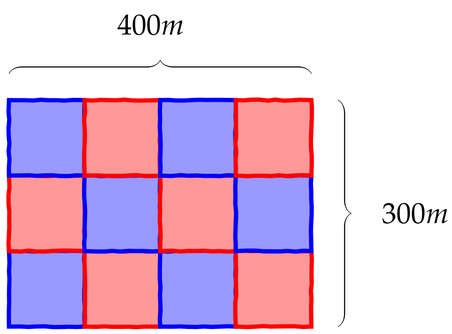
5.1.2 Problem

There are 300 rows of grass, where each row is made of 400 1 by 1 squares. This means that there are $300 * 400 = 300 * 400 meters^2$ of grass, and that Güs needs to carry $300 * 400$ bags.

5.1.3 Definition

Area is how much space an object takes up. When measuring a 2d shape, we find how many 1 by 1 squares the object fills. When measuring the area of a 3d shape, we find how many 1 by 1 by 1 cubes the object fills. Multiplying the length of each side together for "square" shapes gives us the area. (We will discuss area further with integrals).

5.1.4 Graphic



5.2 Area of Square Edge Triangles

5.2.1 Story

As Güs took off, he noticed that the farm was split into two equal parts along the diagonal to make space for Bessie's pet human Farmer John. He realized that here, he could not plant grass, as Farmer John needed it for corn. How many bags of grass should Güs plant?

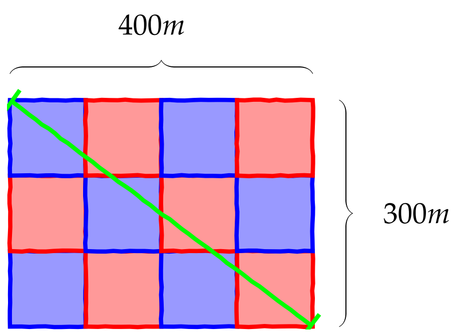
5.2.2 Problem

The farm is divided in half, so Güs only needs to plant half of his bags. $\frac{1}{2} * 300 * 400 = 600m^2$.

5.2.3 Definition

Right triangles (triangles with square edges), are rectangles that have been split in half along the diagonal. This means that the area of a right triangle with side lengths x and y has an area of $\frac{400*300}{2}$.

5.2.4 Graphic

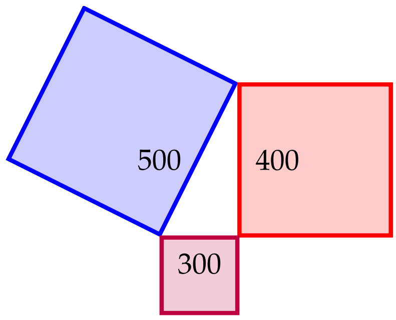


5.3 Pythagorean Theorem

5.3.4 Graphic

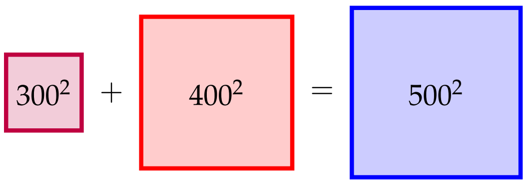
5.3.1 Story

Güs started planting grass. He remembered from school that grass spread rapidly, and so decided that to keep Farmer John's corn safe, he would make a small fence along the diagonal. How long of a fence does Güs need to buy?



5.3.2 Problem

$300 * 300 + 400 * 400 = 500 * 500$.
Güs needs to buy $500m$ of fence.



5.3.3 Definition

In a right triangle, the lengths of sides are related to one another. In such a triangle, a∗a + b∗b = c∗c, where c is the diagonal length in the triangle. This relationship is called the pythagorean theorem. [1]

[1] A proof of the theorem is found here: `https://www.mathsisfun.com/geometry/pythagorean-theorem-proof.html`

5.4 Perimeter of a Shape

5.4.1 Story

After Güs finished planting grass, he decided to take a break for lunch.

”Aaaah,” he sighed, contented. As he looked out at the farm, he noticed a discontented look on Bessie’s face.

”What’s the issue?” he asked.

”It’s this fence.” Bessie replied. ”It’s falling apart and we have to look at it all day.” Güs decided that to thank her for letting them stay for the night, he would buy her a new fence. How many meters of fence does Güs need to buy?

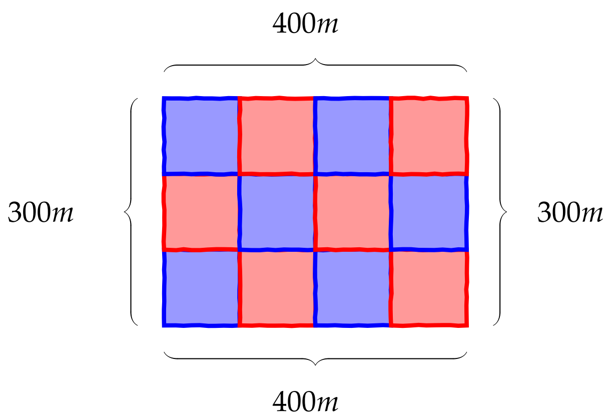
5.4.2 Problem

Two sides of the fence are $300m$ long, and two sides of the fence are $400m$ long. That means that there are $300m + 300m + 400m + 400m = 2 * 300m + 2 * 400 = 2 * (300m + 400m) = 2 * 700 = 1400m$ of fence.

5.4.3 Definition

Surface is the size of the edge of an object. When measuring a 2d shape, we find how many 1 long lines fit around the object. When measuring a 3d shape, we find how many 1 by 1 squares fit around the object. (We will discuss this further with integrals) The perimeter is the sum of all of the side lengths. Because 2 of the side length of the sides of a rectangle are always the same, 2x+2y is its perimeter, which can also be written as 2(x+y).

5.4.4 Graphic



5.5 Shoelace Theorem

The shoelace formula is a tool which lets us find the area of any polygon. Go around the vertex's of the polygon, where the current vertex's coordinates are x_i, y_i and the next vertex's coordinates are x_{i+1}, y_{i+1} $A := \frac{1}{2}\sum_{cyc}(x_i y_{i+1} - x_{i+1} y_i)$.

Chapter 6

Circles and Angles

On the tenth day, Bessie had an issue: Farmer John was bored and kept causing trouble on the farm. Rather than letting her and the other cows graze, he was trying to cut down their grass. To keep him entertained and away from the grass, Bessie decided to create crop circles in the corn field.

”Duckie!” she called. ”Want to help make some crop circles?”

”Sure! I think it will be great for helping my wing recover!” Duckie replied.

.

6.1 Angles

6.1.1 Story

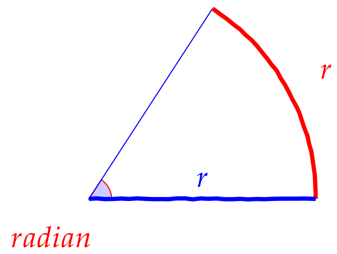
Bessie and Duckie came up with a plan to draw their circles. They decided that Duckie would fly around Bessie, staying exactly $\mathbf{7}m$ away from Bessie at each point. To make sure Duckie is exactly the same distance, Bessie would spin around and look at him at any given point. Bessie knows she will get dizzy if she spins too much, so decides to keep track of how much she has turned at any moment.

6.1.2 Problem

6.1.3 Definition

The number of meters between the Duckie and Bessie is called r, or the "radius". When Duckie has flown r meters around Bessie, the amount Bessie has turned is a "radian".

6.1.4 Graphic



6.2 Drawing a single Circle

6.2.1 Story

Bessie and Duckie began to draw the first circle, but ran into an issue. She didn't know when to stop spinning! How many radians does Bessie have to turn so that Duckie makes a full circle (and she looks in the same place she started)?

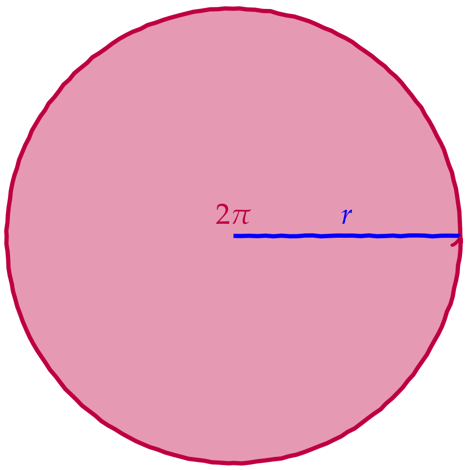
6.2.2 Problem

6.2.3 Definition

When Duckie has flown a full circle around Bessie and Bessie was looking in the same direction where she started, she had turned around $2 * \pi$ radians. This amount Bessie turned cannot be written as a fraction, and so is called "irrational". This specific irrational number has the name "pi".

For convenience, approximate π to be $\frac{22}{7}$ (the actual value of pi is slightly smaller).

6.2.4 Graphic



6.3 Perimeter of a Circle

6.3.1 Story

After some effort calculating, Duckie drew one full circle around Bessie. But, because of his injured wing, he began to feel a bit tired. He decided that in the full day, he would only fly W m so that his wing would recover properly. How much has Duckie flown?

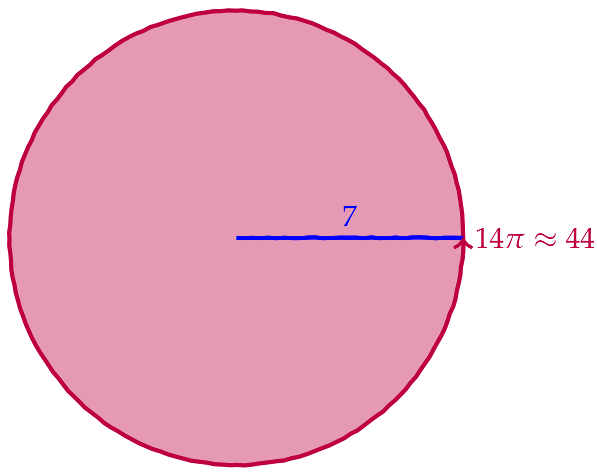
6.3.2 Problem

For every radian Bessie turned, Duckie flew $7m$. Because Bessie turned $2 * \pi$ radians, Duckie have must turned $7 * 2 * \pi)m$. Using $\frac{22}{7}$ as π, we get $\frac{2*22*7}{7} = \frac{2*22}{*1} = 44m$

6.3.3 Definition

The perimeter of a circle is always $2 * \pi * r$, where r is radius of the circle.

6.3.4 Graphic



6.4 Trigonometric Functions

6.4.1 Story

After resting for a few minutes, Duckie and Bessie started drawing their next circle. They decided that, to help give Farmer John some contrast, they would make it have a radius of 1. While creating the circle, tragedy struck! Duckie lost track of where he was! He sees that Bessie turned $\frac{\pi}{4}$ radians. What is Duckie's vertical position? What is Duckie's horizontal position?

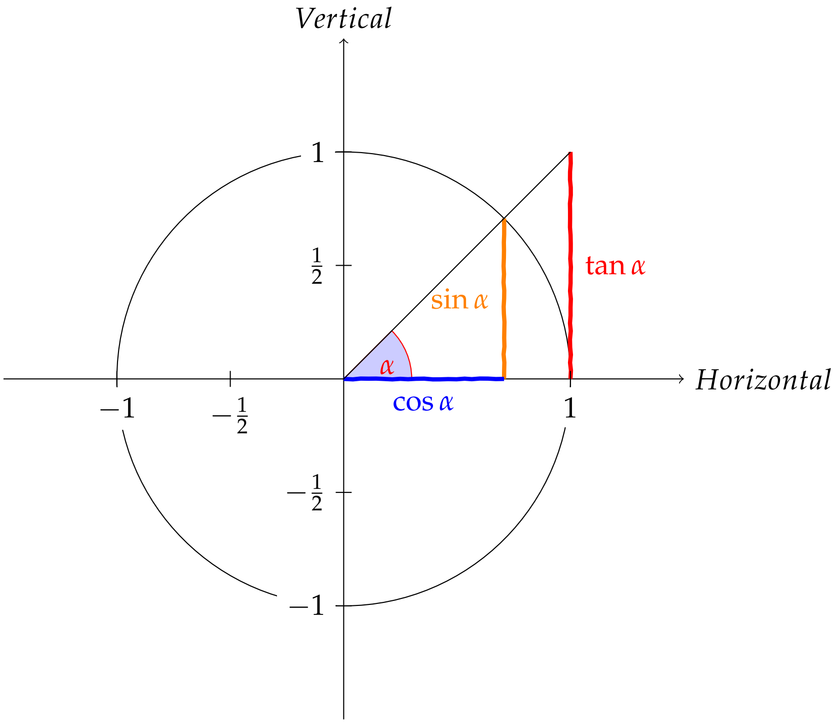
6.4.2 Problem

When Bessie rotated $\frac{\pi}{2}$ radians, Duckie's vertical position and horizontal positions are $\frac{\pi}{2}$.

6.4.3 Definition

Take a line of length 1. As we rotate the line, we can see it has both a height and a width. When it starts, it has width but not height. When it rotates $\frac{\pi}{2}$ radians it has height but not width. When x is the angle rotated, the width at each angle (or the adjacent side/the diagonal side) is $cos(x)$. The height at each angle (or the opposite side/the diagonal side) is $sin(x)$. The ratio between these, $\frac{sin(x)}{cos(x)}$ is called $tan(x)$.[1]

6.4.4 Graphic



[1]These functions, and other trigonometric functions can be viewed here. `https://visualize-it.github.io/trig_functions/simulation.html`

6.5 Trigonometric Identities

6.5.1 Story

Duckie, tired, but with a healed wing, sat down. He and Bessie were ready to take the day off, and he was ready to start flying the next day. Güs, who had been planting grass, walked over to them.

"Hey Guys! How was your work in the cornfield?"

"It was pretty good!" Bessie replied. "We were drawing circles for Farmer John while you planted grass."

Güs looked at Bessie in surprise. "That's what you were doing? I thought you were tracing the point of an arrow where the width×the width plus the height×the height equaled one!" The trio looked at each other in confusion. Who is right?

6.5.2 Problem

They were all right! The width of the arrow is cos(x), and the height of the arrow is sin(x). Through Pythagorean Theorem, the length of the arrow must be 1, which is also the radius that Duckie flew.

6.5.3 Definition

$sin(x) * sin(x) + cos(x) * cos(x) = 1 * 1 = 1.$

6.5.4 Graphic

Chapter 7

Sets and Functions

It was time for Duckie and Gus to continue on their journey. They had enjoyed spending time with Bessie, but they still needed to get to the Golden Goose.

”What are we going to do about the Bounty Hunters?” Gus asked Duckie nervously. They both were concerned about the bounty hunters’ traps.

”I’m not sure” Duckie replied, ”But we will have to keep moving. Hopefully we find something or someone along the way who can help”.

As the duo began to fly once more that evening, they saw an approaching flock of geese behind them.

”Quick! Land!” Duckie cried. In a flash, in a rustle of feathers, the pair landed and waited, hoping to go unnoticed.

There was the squaking of the the voices of the other group of geese as they landed beside Duckie and Gus.

”Hi, how’s it going?” one of the geese asked. This surprised Duckie. He was expecting to be taken away to the goose village, but he wasn’t expecting a greeting.

”Hello...” Duckie said cautiously. ”Who are you? Have the goose authorities sent you?”

”We are just a gaggle from the north. My name is Snow. What are the goose authoriies?”. Duckie relaxed. Clearly, these geese wouldn’t take them away. Gus, having had experience from journeying with the Goose Police, knew the value of traveling with a group.

”Oh they aren’t anyone important,” Gus said to Snow. ”We are trying to find the golden goose, and it seems like you are heading in the same direction as us. Mind if we join you?”

”Alright!” Snow said. ”But you’ll have to join our dance!”.

7.1 Cartesian Product

7.1.1 Story

That night, the new flock decided to have their traditional nightly dance. It was a partners dance, with the following rules: geese with red feet were one group, and the geese with blue feet were another group. Each of the red geese partnered with each of the blue geese, where a red goose lead. The group of red geese in the dance was {Gus, Snow, Grey, Blue}. The group of blue geese was {Duckie, Barnie, Swan}. How many ways could the geese have formed pairs? What are those pairs?

7.1.2 Problem

All the pairs that the Geese could have formed were {(Gus, Duckie), (Gus, Barnie), (Gus, Swan), (Snow, Duckie), (Snow, Barnie), (Snow, Swan), (Grey, Duckie), (Grey, Barnie), (Grey, Swan), (Blue, Duckie), (Blue, Barnie), (Blue, Swan)}. There were 4 geese with red feet, and 3 geese with blue feet.

There were $\begin{array}{r} 4 \\ \times\ 3 \\ \hline 1\ 2 \end{array}$ ways to form pairs

7.1.3 Definition

A set is an unordered collection of things. If there are two sets, A and B, the cartesian product (AxB) are the pairs (a, b) where $a \in$ [1] A and $b \in B$.

[1] ∈, means ”is in the set”

7.1.4 Graphic

7.2 Relations

7.2.1 Story

Maggie, one of the blue footed geese and, the organizer of the dance, looked over Gus's shoulder to see what he was doing.

"Hi there! Whatcha doin'?" she asked inquisitively.

"Nothing much, I'm just trying to figure out how the dance works. Can you take a look?"

"Of course!" Maggie considered his drawings. "It seems like you are on the right track, but I think Snow and Barnie like to dance different styles. Maybe they shouldn't be a possible pair?"

Gus removed their pairing from his plans. "So all of the pairs in the new set are in the original set."

How many sets of pairs could Maggie and Gus make out of the original cartesian product?

7.2.2 Problem

Maggie and Gus can quite a few sets. These include {}, the AxB itself, all sets with only a single pair, all the sets with only two pairs, etc.

7.2.3 Definition

If a set A (also called the domain) is a subset of set B (also called the codomain), then every element in set A is also in set B. For every set A, there are $2^{|A|}$ possible subsets which can be made [2]. A subset of a cartesian product called is a relation.

[2] |A| means size of A

7.2.4 Graphic

7.3 Functions

7.3.1 Story

Gus and Maggie continued to plan, and decided to come up with the list of pairs of geese.

"Hmm..." Maggie thought, looking at the list of relations. "There are not enough blue geese for each goose to have their own partner. How can we make sure each goose has their own partner?"

Gus thought for a moment. "Maybe some of the blue geese can take turns switching between partners?"

"I'm not sure, but maybe it would work. Let's try it out!" Come up with a possible set of pairs dance partners Maggie and Gus could have formed.

7.3.2 Problem

Maggie and Gus eventually decided on the following set of dance partners {(Gus, Barnie), (Snow, Barnie), (Grey, Swan), (Blue, Duckie)} with Maggie sitting aside

7.3.3 Definition

A function is a special type of relation between the sets A and B. In a function, in every pair (a, b) (where $a \in A$ and $b \in B$), a only maps to a single output. This allows us to say that the $f(a) = b$, or that the value a maps to the value b.

7.3.4 Graphic

7.4 Types of Functions

7.4.1 Story

As the dance began, Maggie watched the other geese dance. Gus noticed her watching, and came over.

"Why don't you come over and dance with us?" Gus asked.

"Ah, you don't want me to dance, I don't to mess up the fun. Besides, this way, the blue geese don't have to worry about switching partners."

Gus looked over. "Don't worry about that, it's fun! Besides, now there aren't enough red geese. C'mon!".

Maggie hesitated, "Well, let's see how they do on their own for now."

"You'll do great!"

"Alright! Let's dance!"

Gus and Maggie went back to the dance floor together. What does the new function mapping red geese with blue geese look like?

7.4.2 Problem

With all the geese dancing, the red geese and new geese were mapped so that the dancing partners were {(Gus, Maggie), (Snow, Barnie), (Grey, Swan), (Blue, Duckie)}

7.4.3 Definition

A function where every element in the domain has a different mapping is called "one-to-one". A function where every element in the codomain is mapped to is called "onto". A function which is both one-to-one and onto is called "bijective".

7.4.4 Graphic

7.5 Inverse of Functions

7.5.1 Story

As the first dance came to a close, Maggie went to the makeshift stage to begin the next dance.

”All right everyone!” she announced happily. ”Same dance, but this time, blue geese lead!”

The geese cheered, and began the dance again with the blue geese taking the lead. Was this new pairing a function? If so, what did it look like?

7.5.2 Problem

The new pairing was {(Maggie, Gus), (Barnie, Snow), (Swan, Grey), (Duckie, Blue)}. This new pairing was also a function, because every first element only mapped to a single output. For example, originally, f(Gus) =Maggie, but now f^{-1}(Maggie) =Gus

7.5.3 Definition

The inverse of a function $f(x)$ is written as $f^{-1}(x)$. The input of of f is the output of f^{-1}, and the output of f is f^{-1}. In other words, if $f(a) = b$, then $f^{-1}(b) = a$. In order for there to be an inverse function for $f(x)$, $f(x)$ must be bijective.

7.5.4 Graphic

7.6 Continuous Functions

7.6.1 Story

As Gus and Maggie danced with each other, they realized that they could also explain the movement of the dance as a function.
$f(t) = -t * t + 10$, where t was the number of minutes since the dance began, and $f(t)$ represented how far they stood from the stage. What was Gus and Maggie's position when they had danced for 3 minutes?

7.6.2 Problem

$f(3) = -3 * 3 + 10$
$f(3) = 1$.
When they had danced for three minutes, Gus and Maggie were standing 1 meter from the stage.

7.6.3 Definition

Functions can be written as pair explicitly, or with a formula. When making a function using a formula, the pairs are $(x, f(x))$

7.6.4 Graphic